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THERMAL SHOCK PROBLEM IN A HOMOGENEOUS ISOTROPIC HOLLOW CYLINDER WITH ENERGY DISSIPATION

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In this work, we constructed the equations of generalized thermoelasticity of a homogeneous isotropic hollow cylinder. The formulation is applied in the context of the Green and Naghdi theory of types II and III. The material of the cylinder is assumed to be homogeneous isotropic both mechanically and thermally. The problem has been solved numerically using a finite-element method. Numerical results for the temperature distribution, displacement, radial stress, and hoop stress are represented graphically. Comparisons are made with the results predicted by the types II and III. The results obtained in this paper can be used to design various homogeneous thermoelastic elements under thermal load to meet special engineering requirements.

Keywords: Hollow cylinder, finite-element method, Green–Naghdi theory, thermal shock.

1. Introduction

During the second half of the Twentieth Century, nonisothermal problems of the theory of elasticity became increasingly important. This is due to their many applications in widely diverse fields. First, the high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure. Second, in the nuclear field, the extremely high temperature and temperature gradients originating inside nuclear reactors influence their design and operations (Nowinski [1]).

The classical theory of thermoelasticity as exposed, for example, in Carlson's article [2] has found generalizations and modifications into various thermoelastic models that run under the label hyperbolic thermoelasticity; see the survey of Chandrasekharaiah [3] and Hitnarski and Ignazack [4]. The notation "hyperbolic" reflects the fact that thermal waves are modeled, avoiding the physical paradox of infinite propagation speed of the classical model. In the 1990's, Green and Naghdi [5–7] proposed three new thermoelastic theories based on an entropy equality rather than the usual entropy inequality. The constitutive assumptions for the heat flux vector are different in each theory. Thus, they obtained three theories that they called thermoelasticity of types I, II, and III. When the theory of type I is linearized we obtain the classical system of thermoelasticity. The theory of type II (a limiting case of type III) does not admit energy dissipation. In the context of the linearized version of this theory, theorems on uniqueness of solutions have been established by Hitnarski and Ignazack [4] and Green and Naghdi [7]. Boundary-initiated waves in a half-space and in an unbounded body with a cylindrical cavity have been studied by Green and Naghdi [5] and Chandrasekharaiah and Srinath [8], [9], and plane-wave thermal shock problems have been studied by Othman et al. [10] and Othman and Song [11–13].

The counterparts of our problem in the context of the uncoupled thermoelasticity theory, the coupled thermoelasticity theory, the Green–Lindsay theory (GL-theory) [14], and the Lord–Shulman theory (LS-theory) [15] have been considered by Othman [16], [17] and Othman and Song [18]. At appropriate stages of our analysis, we make a comparison of our results with those obtained in these works. This comparison reveals that, on the

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whole, the predictions of the GN-theory (as obtained here) are qualitatively similar to those of the LS-theory. More importantly, we notice that certain physically unrealistic features inherent in the conventional coupled thermo-elasticity theory and the GL-theory are not present in the GN-theory.

The exact solution of the governing equations of the generalized thermoelasticity theory for a coupled and nonlinear/linear system exists only for very special and simple initial and boundary problems. To calculate the solution of general problems, a numerical solution technique is used. For this reason the finite-element method is chosen. The method of weighted residuals offers the formulation of the finite-element equations and yields the best approximate solutions to linear and nonlinear ordinary and partial differential equations. Applying this method basically involves three steps. The first step is to assume that the general behavior of these approximating functions in the differential equations and boundary conditions results in some errors, called the residual. This residual has to vanish in an average sense over the solution domain. The second step is the time integration. The time derivatives of the unknown variables have to be determined by the former results. The third step is to solve the equations resulting from the first and the second step by using a finite-element algorithm program (see Zienkiewicz and Taylor [19]). Abbas [20], Youssef and Abbas [21], and Abbas and Abd-alla [22] applied the finite-element method in different problems.

In the present paper, we consider the thermal shock problem of generalized thermoelasticity of a homogeneous isotropic hollow cylinder based on the Green–Naghdi theory of types II and III. The problem has been solved numerically using a finite-element method (FEM). Numerical results for the temperature distribution, displacement, radial stress, and hoop stress are represented graphically.

2. Formulation of the Problem

In the context of generalized thermoelasticity theories, the system of equations that include the displacement, the stress, the strain, and the temperature for a linear, homogenous, and isotropic thermoelastic continuum take the following form (Abbas and Abd-alla [22]):

$$(\lambda+\mu)u_{j,ij} + \mu u_{i,jj} + F_i - \gamma T_{,i} = \rho \ddot{u}_i , \tag{1}$$

$$K^* T_{,ii} + K\dot{T}_{,ii} = \rho C_E \ddot{T} + g T_0 \ddot{u}_{i,i} , \tag{2}$$

$$\tau_{ij} = \lambda u_{i,i}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) - \gamma T \delta_{ij} , \tag{3}$$

where λ, μ are the Lamé constants, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, C_E is the specific heat at constant strain, T is the temperature above the reference temperature T_0, and K and K^* are respectively the thermal conductivity and material constant characteristic of the theory. When $K \to 0$, Eq. (2) reduces to the heat conduction equation of the GN II theory.

In a cylindrical coordinate system (r, θ, z) for the axially symmetric problem, $u_r = u_r(r, z, t)$, $u_\theta = 0$, $u_z = u_z(r, z, t)$. Furthermore, if only the axisymmetric plane strain problem is considered, we have $u_r = u(r, t)$ and $u_\theta = u_z = 0$ The strain-displacement relations are

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = e_{rz} = e_{r\theta} = e_{\theta z} = 0 . \tag{4}$$

The stress-strain relations are

$$\tau_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma T \,, \tag{5}$$

$$\tau_{\theta\theta} = 2\mu \frac{u}{r} + \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma T \tag{6}$$

$$\tau_{zz} = \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma T, \quad \tau_{rz} = \tau_{r\theta} = \tau_{\theta z} = 0 \,. \tag{7}$$

If it is assumed that there are no body forces and heat sources in the medium, the equations of motion and energy equation have the form

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2} \,, \tag{8}$$

$$K^* \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + K \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^2 T}{\partial t \partial r} \right) = \rho C_E \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2}{\partial t^2} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right). \tag{9}$$

It is convenient to cast the preceding equations into their dimensionless forms. To do this, the dimensionless parameters are introduced as

$$(r^\circ, u^\circ) = \frac{(r,u)}{c_1 \omega_1}, \quad t^\circ = \frac{t}{\omega_1}, \quad \left(\tau^0_{rr}, \tau^0_{\theta\theta}, \tau^0_{zz} \right) = \frac{1}{\mu} \left(\tau_{rr}, \tau_{\theta\theta}, \tau_{zz} \right), \quad \theta^\circ = \frac{\gamma T}{\rho c_1^2} \,, \tag{10}$$

where, $c_1^2 = \dfrac{\lambda + 2\mu}{\rho}$, $\omega_1 = \dfrac{K}{\rho C_E c_1^2}$.

Inserting (10) into Eqs. (5)–(9) one obtains (after dropping the superscript $^\circ$ for convenience)

$$\tau_{rr} = (2 + a_1) \frac{\partial u}{\partial r} + a_1 \frac{u}{r} - (2 + a_1)\theta \,, \tag{11}$$

$$\tau_{\theta\theta} = a_1 \frac{\partial u}{\partial r} + (2 + a_1) \frac{u}{r} - (2 + a_1)\theta \,, \tag{12}$$

$$\tau_{zz} = a_1 \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - (2 + a_1)\theta \,, \tag{13}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{\partial \theta}{\partial r} = \frac{\partial^2 u}{\partial t^2} \,, \tag{14}$$

$$\left(\varepsilon_2 + \varepsilon_3 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right) - \frac{\partial^2 \theta}{\partial t^2} = \varepsilon_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right), \tag{15}$$

$$a_1 = \frac{\lambda}{\mu}, \quad \varepsilon_1 = \frac{\gamma^2 T_0}{\rho^2 c_1^2 C_E}, \quad \varepsilon_2 = \frac{K^*}{\rho c_1^2 C_E}, \quad \varepsilon_3 = \frac{K}{\rho c_1^2 C_E \omega_1}.$$

From the preceding description, the initial and boundary conditions may be expressed as

$$u(r,0) = \frac{\partial u(r,0)}{\partial t} = 0, \quad \theta(r,0) = \frac{\partial \theta(r,0)}{\partial t} = 0, \tag{16}$$

$$\tau_{rr}(a,t) = 0, \quad \tau_{rr}(b,t) = 0, \quad \theta(a,t) = H(t), \quad \frac{\partial \theta(b,t)}{\partial r} = 0, \tag{17}$$

where a and b are the inner and outer radii of the hollow cylinder and H is the Heaviside unit step function.

3. Finite-Element Method

In order to investigate the numerical solution of the thermal shock problem of generalized thermoelasticity of a homogeneous isotropic hollow cylinder, we use the finite-element method (FEM) (Reddy [23] and Cook et al. [24]) due to its flexibility in modeling layered structures and its ability to yield the full-field numerical solution. The governing equations (14) and (15) are coupled with initial and boundary conditions (16) and (17). The numerical values of the dependent variables such as the displacement u and the temperature θ are obtained at interesting points called the degrees of freedom. The weak formulations of the nondimensional governing equations are derived. The set of independent test functions consisting of the displacement δu and the temperature $\delta\theta$ is prescribed. The governing equations are multiplied by independent weighting functions and are then integrated over the spatial domain with the boundary. Integrating by parts and making use of the divergence theorem, we reduce the order of the spatial derivatives, which allows for the application of the boundary conditions. The same shape functions are defined piecewise on the elements. Using the Galerkin procedure, we approximate the unknown fields u and θ and the corresponding weighting functions by the same shape functions. The last step towards the finite-element discretization is to choose the element type and the associated shape functions. Three nodes of quadrilateral elements are used. The shape function is usually denoted by the letter N and is usually the coefficient that appears in the interpolation polynomial. A shape function is written for each individual node of a finite element and has the property that its magnitude is 1 at that node and 0 for all other nodes in that element. We assume that the master element has its local coordinates in the range $[-1,1]$. In our case, the one-dimensional quadratic elements are used, which are given by:

Linear shape functions

$$N_1 = \frac{1}{2}(1-\xi), \quad N_2 = \frac{1}{2}(1+\xi). \tag{18}$$

Quadratic shape functions

$$N_1 = \frac{1}{2}\left(\xi^2 - \xi\right), \quad N_2 = 1 - \xi^2, \quad N_3 = \frac{1}{2}\left(\xi^2 + \xi\right). \tag{19}$$

On the other hand, the time derivatives of the unknown variables have to be determined by the Newmark time integration method (Cook et al. [24]).

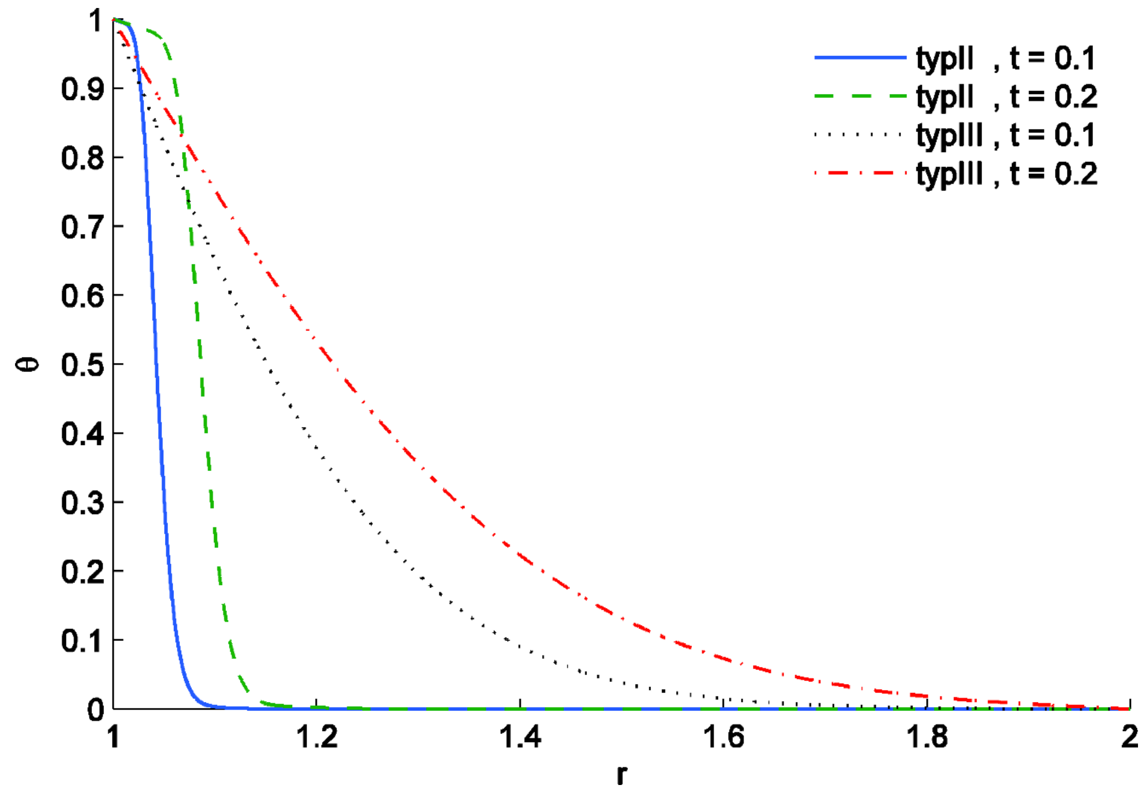

Fig. 1. The temperature distribution in different values of time.

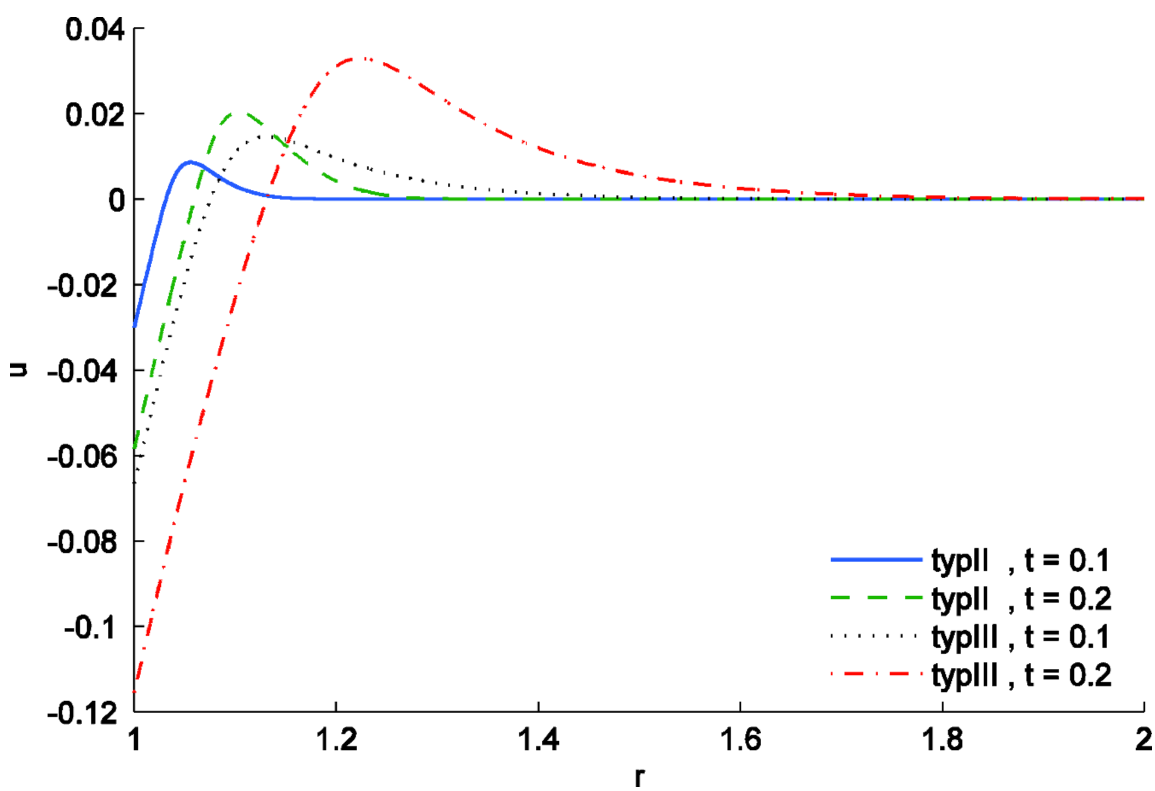

Fig. 2. The displacement distribution in different values of time.

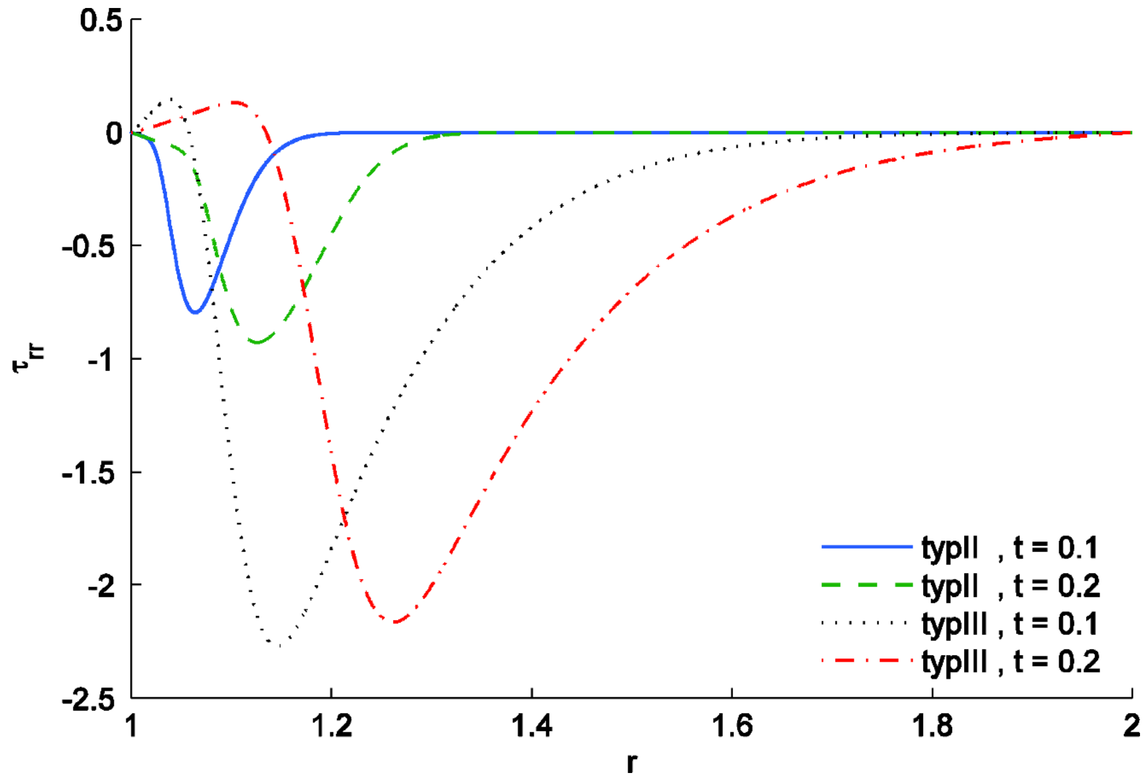

Fig. 3. The radial stress distribution in different values of time.

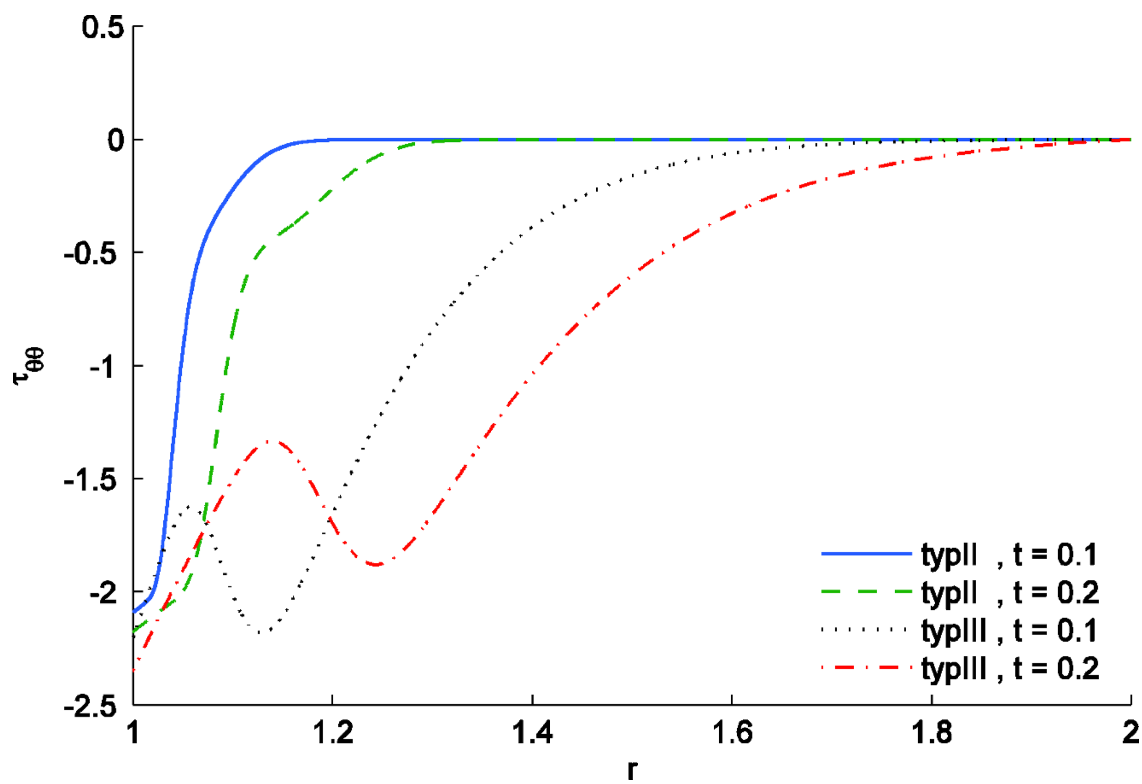

Fig. 4. The hoop stress distribution in different values of time.

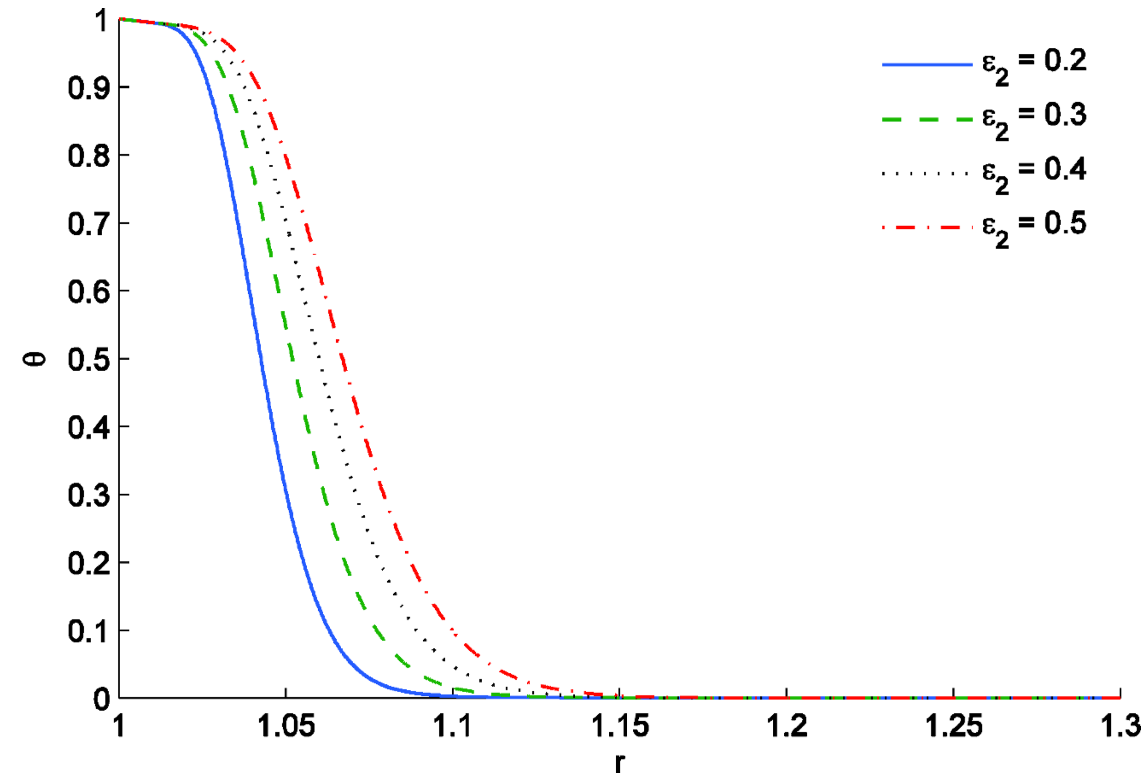

Fig. 5. The temperature distribution in different values of ε_2 .

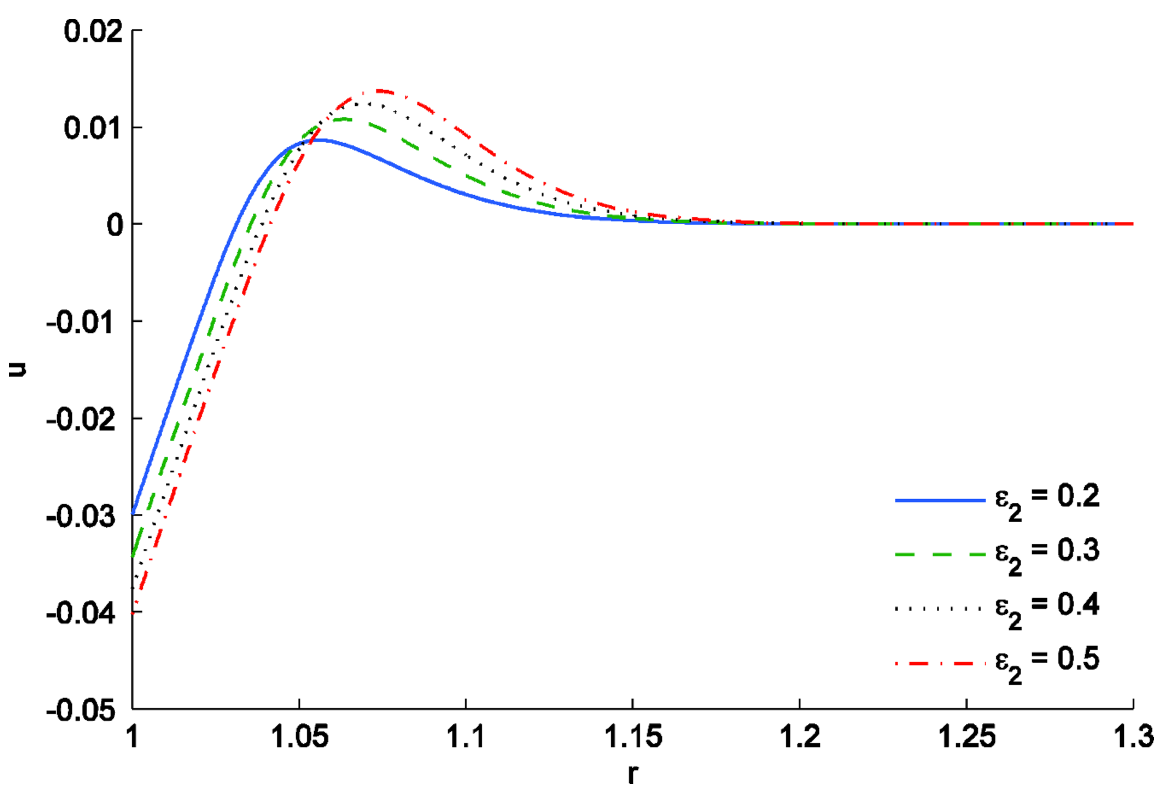

Fig. 6. The displacement distribution in different values of ε_2 .

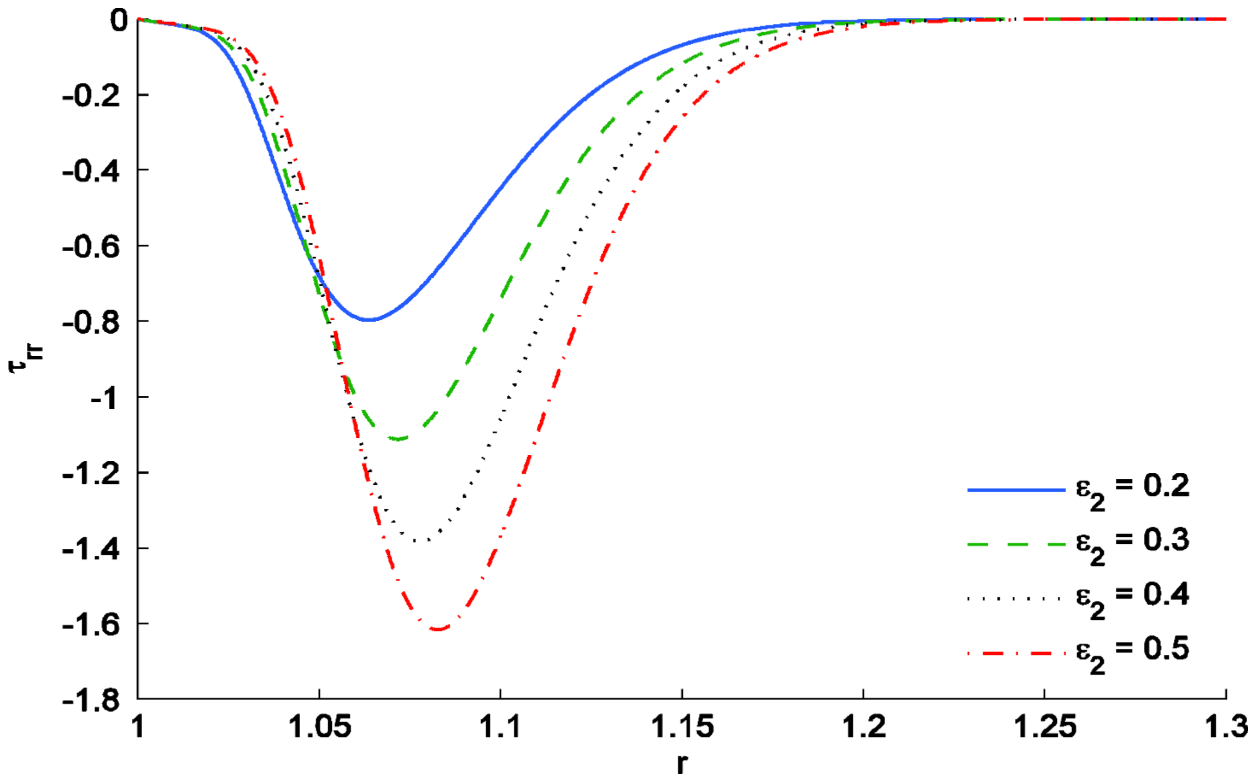

Fig. 7. The radial stress distribution in different values of ε_2 .

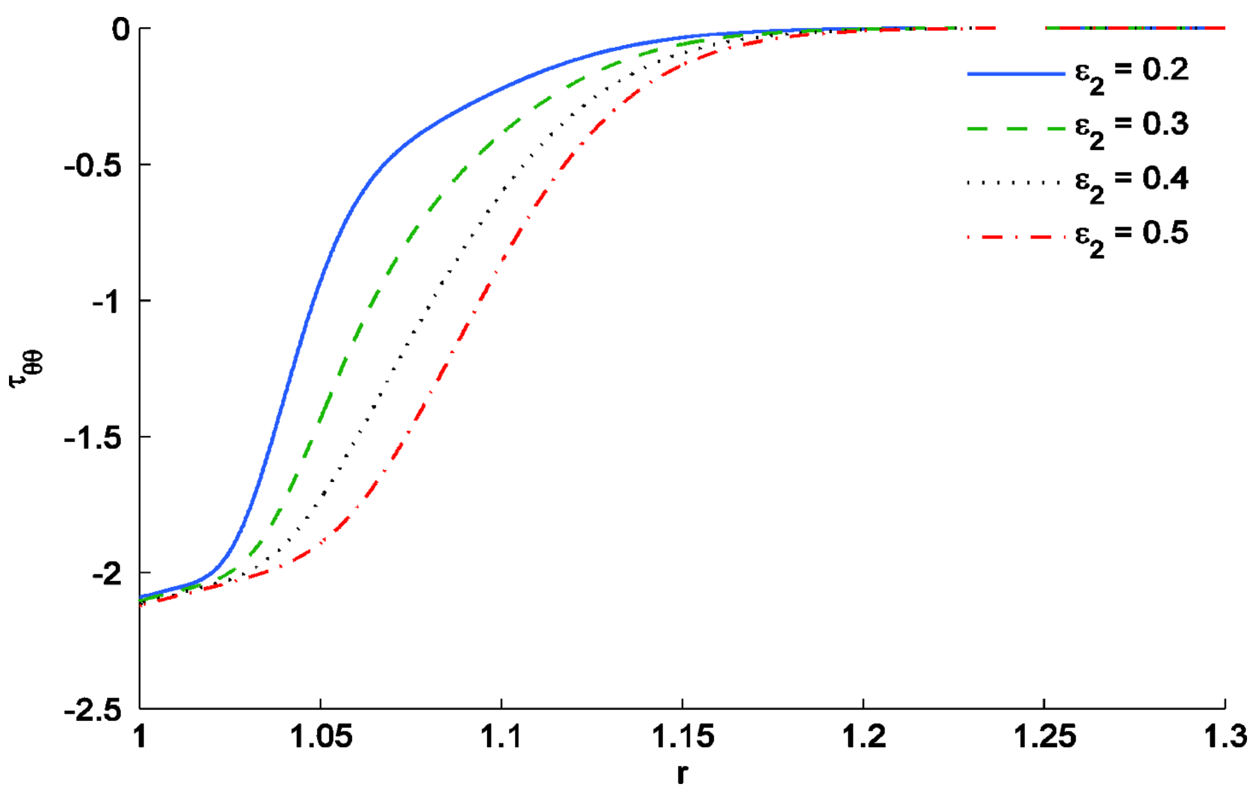

Fig. 8. The hoop stress distribution in different values of ε_2 .

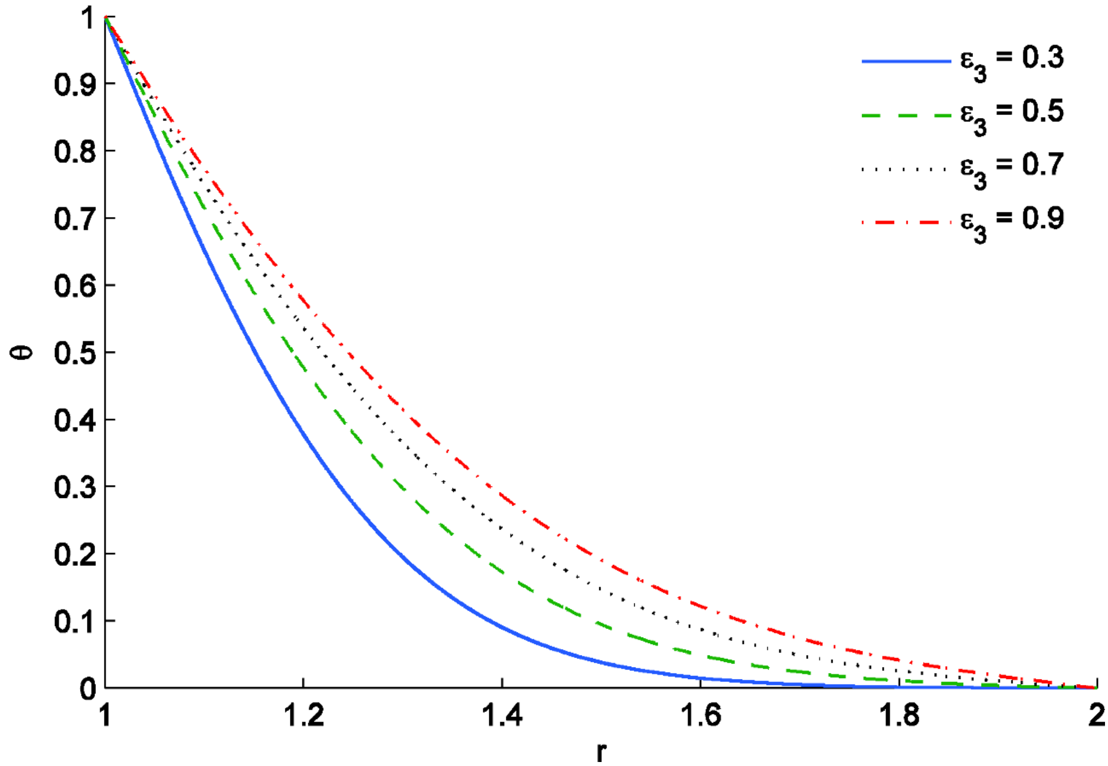

Fig. 9. The temperature distribution in different values of ε_3 .

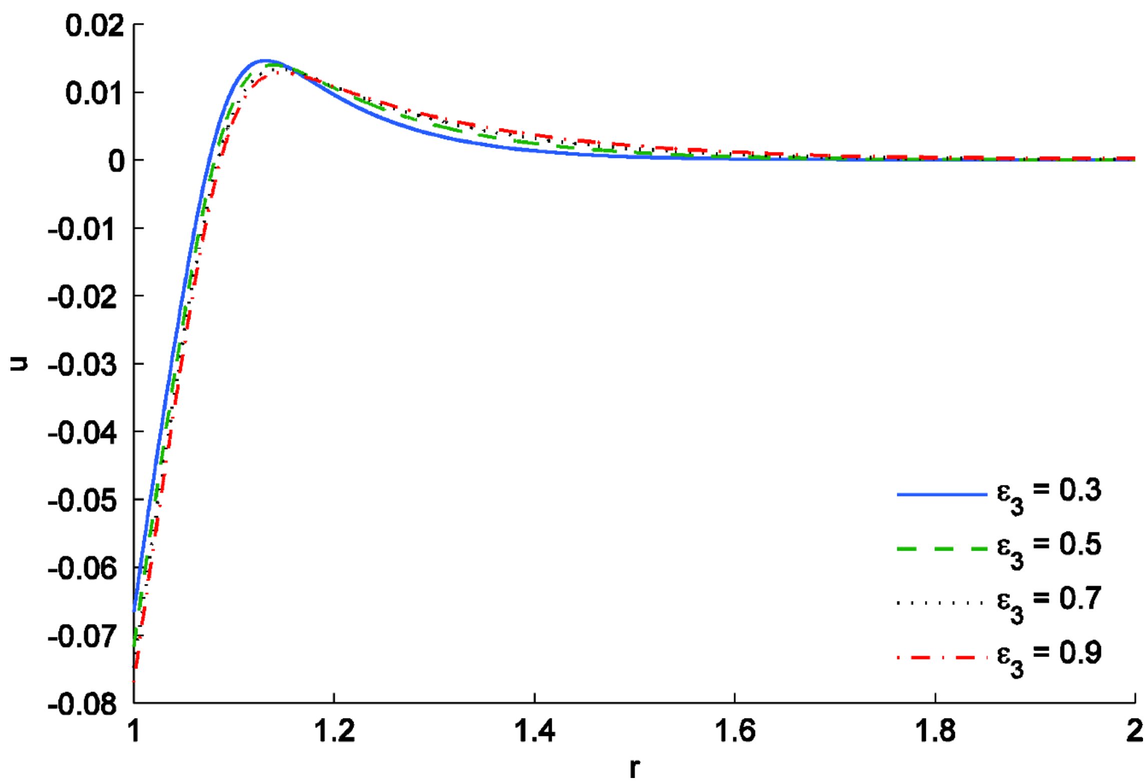

Fig. 10. The displacement distribution in different values of ε_3 .

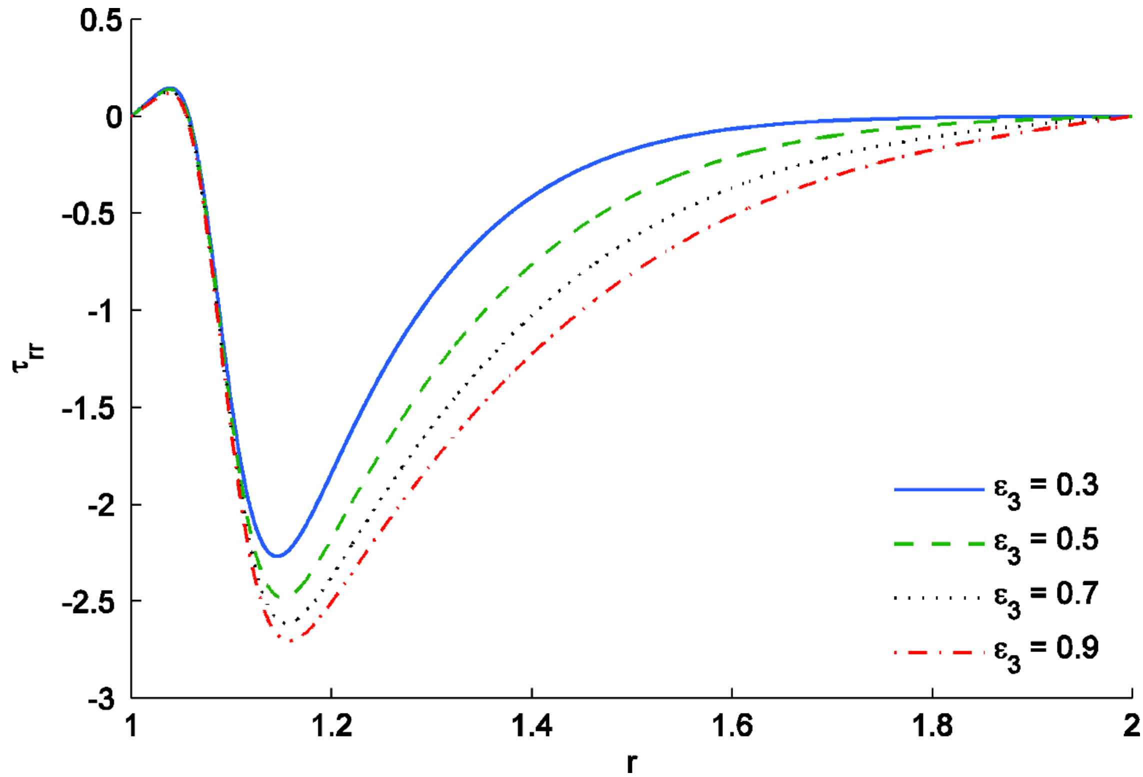

Fig. 11. The radial stress distribution in different values of ε_3 .

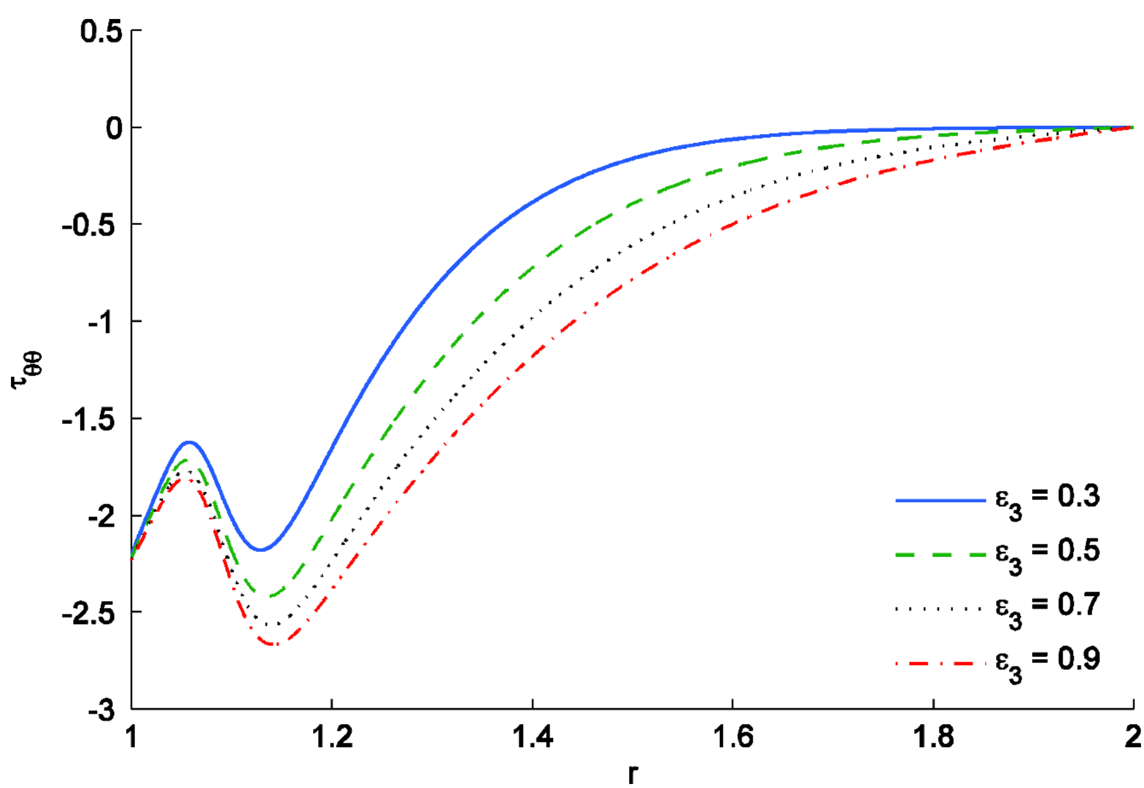

Fig. 12. The hoop stress distribution in different values of ε_3 .

4. Numerical Results

With an aim to illustrate the problem, we will present some numerical results. The copper material was chosen for purposes of numerical computation, the physical data for which are given as

$$\lambda = 7.76\times 10^{10}(\mathrm{kg})(\mathrm{m})^{-1}(\mathrm{sec})^{-2}, \quad \mu = 3.86\times 10^{10}(\mathrm{kg})(\mathrm{m})^{-1}(\mathrm{sec})^{-2}, \quad T_0 = 293(\mathrm{K}), \quad \varepsilon_1 = 0.0168,$$

$$\rho = 8.954\times 10^{3}(\mathrm{kg})(\mathrm{m})^{-3}, \quad C_E = 3.831\times 10^{2}(\mathrm{m})^{2}(\mathrm{K})^{-1}(\mathrm{sec})^{-2}, \quad \alpha_t = 17.8\times 10^{-6}(\mathrm{K})^{-1}.$$

The field quantities, temperature, displacement, and stresses depend not only on time t and space r, but also on the characteristic parameter of the Green–Naghdi theory of types II and III. Here all the variables are taken in nondimensional form. The results for the temperature, displacement, radial stress, and hoop stress have been obtained by taking $t = 0.2$ based on the Green–Naghdi theory of types II and III. Figures 1–4 exhibit the variation of the temperature, displacement, radial stress, and hoop stress with respect to r for the two types II, III of the Green–Naghdi theory and two different values of times $t = 0.1$ and $t = 0.2$. Figures 5–8 depict the variation of the temperature, displacement, radial stress, and hoop stress under the Green–Naghdi theory of type II (without energy dissipation) for four different values of the characteristic parameter $\varepsilon_2 = 0.2, 0.3, 0.4, 0.5$ and $\varepsilon_3 = 0$. It is seen from Fig. 5 that the characteristic parameter ε_2 has an increasing effect on the temperature for $0 < r < 1.15$. Figure 6 shows that the characteristic parameter ε_2 has a decreasing effect on the displacement component u for $0 < r < 1.05$ and an increasing effect for $1.05 < r < 1.2$. Figure 7 shows that the characteristic parameter ε_2 has an increasing effect on radial stress τ_{rr} for $0 < r < 1.05$ and a decreasing effect for $1.05 < r < 1.25$. Figure 8 shows that the characteristic parameter ε_2 has a decreasing effect on hoop stress $\tau_{\theta\theta}$ for $0 < r < 1.25$. Figures 9–12 show the variation of displacement, radial stress, and hoop stress under the Green–Naghdi theory of type III (with energy dissipation) for four different values of the characteristic parameter $\varepsilon_3 = 0.3, 0.5, 0.7, 0.9$ and $\varepsilon_2 = 0.2$. Figure 9 shows that the characteristic parameter ε_3 has an increasing effect on the temperature for $0 < r < 2$. Figure 10 shows that the characteristic parameter ε_3 has a decreasing effect on the displacement component u for $0 < r < 1.2$ and an increasing effect for $1.2 < r < 2$. Figures 11 and 12 show that the characteristic parameter ε_3 has a decreasing effect on the radial stress τ_{rr} and hoop stress $\tau_{\theta\theta}$.

5. Conclusion

In this paper we have investigated the solution of the thermal shock problem of generalized thermoelasticity of a homogeneous isotropic hollow cylinder based on the Green–Naghdi theory of types II and III) by using the finite-element method. The differences of the field quantities predicted by the GN theory of types II and III are remarkable. We conclude that the characteristic parameters ε_2 and ε_3 have a great effect on the field quantities. The results obtained in this paper can be used to design various homogeneous thermoelastic elements under thermal load to meet special engineering requirements.

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