Computational Mathematics and Modeling, Vol. 21, No. 1, 2010

SIMULATION OF A NONLINEAR STEKLOV EIGENVALUE PROBLEM USING FINITE-ELEMENT APPROXIMATION

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Elliptic problems with parameters in the boundary conditions are called Steklov problems. With the tool of computational approximation (finite-element method), we estimate the solution of a nonlinear Steklov eigenvalue problem for a second-order, self-adjoint, elliptic differential problem. We discussed the behavior of the nonlinear problem with the help of computational results using Matlab.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in N$, with Lipschitz-continuous boundary, and let the boundary $\Gamma = \partial\Omega$, i.e., the set $\overline{\Omega} = \Omega \bigcup \Gamma$. We define the operator

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + a.u\,,$$

where $a_{ij}(x)$ and $a(x)$ belong to $C^\infty(\mathbb{R}^n)$, $a_{ij} = a_{ji}$, $i, j = 1, 2, \ldots, n$, and $a(x) \geq a_0 > 0 \quad \forall x \in \Omega$. Let us assume that L is uniformly elliptic, i.e., there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \;\geq\; \alpha\sum_{i=1}^{n}\xi_i^2 \qquad \forall \xi, x \in \mathbb{R}^n.$$

Let Γ_0 and Γ_1 be two complementary parts of Γ:

$$\Gamma = \Gamma_0 \bigcup \Gamma_1, \qquad \Gamma_0 \bigcap \Gamma_1 = \phi, \qquad \text{meas}(\Gamma_1) > 0.$$

We study the nonlinear Steklov eigenvalue problem for a nontrivial function $u(x)$ of the given form:

$$Lu = f(x, u) \qquad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \qquad \text{on} \quad \Gamma_0, \qquad \frac{\partial u}{\partial n} = \lambda u \qquad \text{on} \quad \Gamma_1, \tag{2}$$

where

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1046–283X/10/2101–0109

$$\frac{\partial u}{\partial n} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i$$

is the co-normal derivative and n_i is the i th component of the outward unit normal on Γ_1.

Elliptic problems of the above form with parameters in the boundary conditions are called Steklov problems from their first appearance in [14]. Linear problems with an eigenvalue parameter on the boundary appear in many physical situations [1]. For example, we find them in the separation of the variables of parabolic or hyperbolic equations with dynamical boundary conditions [2], or in the dynamics of liquids in moving containers in sloshing problems. There are many interesting problems for the one-dimensional case, like those of vibrations of a pendulum [3], those of eigenoscillations of mechanical systems with boundary conditions containing the frequency (Hinton & Shaw, 1990), and many others. In [13] Andreev discussed the isoparametric finite-element approximation of a linear Steklov eigenvalue problem. Bramble & Osborn in [7] studied the Galerkin method for the approximation of the Steklov problem (1), (2) of a non-self-adjoint second-order differential operator L. Several model eigenvalue problems emerging in physics and engineering, as well as their approximations, are presented in Babuska & Osborn in [9]. In the case of the biharmonic operator, these problems were first considered by Kuttler and Sigillito [4] and Payne [5] who studied the isoperimetric properties of the first eigenvalue λ_1. More recently, the whole spectrum of the biharmonic Steklov problem was studied in [6], where one can also find a physical interpretation of λ_1 and of the Steklov boundary conditions.

In recent years, numerical approximations of spectral problems arising in fluid mechanics have received increasing attention [10, 11]. Some of these spectral problems lead to a Steklov eigenvalue problem similar to the one considered here, for instance, in the study of surface waves, in the analysis of stability of mechanical oscillators immersed in a viscous fluid, and in the study of the vibration modes of a structure in contact with an incompressible fluid.

1. Steklov Eigenvalue Problem

Given the Sobolev space $W^{m,p}(\Omega)$ with norm

$$\| v \|_{m,p,\Omega} = \left(\sum_{|\alpha| \le m} \left\| \partial^{\alpha} v \right\|_{L_p(\Omega)}^{p} \right)^{1/p}$$

and seminorm

$$| v |_{m,p,\Omega} = \left(\sum_{|\alpha| = m} \left\| \partial^{\alpha} v \right\|_{L_p(\Omega)}^{p} \right)^{1/p}.$$

We make the usual changes when $p = \infty$. As usual, we drop the index p when it is equal to 2, and we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$.

Let V be a closed subspace of $H^1(\Omega)$ defined by

$$V = \left\{ v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_0 \right\}. \tag{3}$$

Obviously, $H_0^1(\Omega) \subset V \subseteq H^1(\Omega)$.

We introduce the $L_2(\Gamma)$ scalar product $\langle \bullet, \bullet \rangle$ by

$$\langle u, v \rangle = \int_\Gamma uv\,dl .$$

We define the bilinear form $a(\bullet, \bullet)$ on $V \times V$ by

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_\Omega a(x) uv\,dx - \int_\Omega f(x, u)\, v\,dx .$$

Let $(\lambda, u) \in (R, V)$ be an exact Steklov eigenpair of the problem (1), (2) in the weak formulation:

$$a(u, v) = \lambda(u, v) \quad \forall\, v \in V . \tag{4}$$

The seminorm $|v|_{1,\Omega}$ is a norm on V, which is equivalent to the norm $\|v\|_{1,\Omega}$. We can infer that the bilinear form $a(\bullet, \bullet)$ is V-elliptic. Then there exists a constant $\beta_1 > 0$ such that $\forall\, v \in V$

$$\beta_1 \|v\|_{1,\Omega}^2 \le a(v, v) .$$

Since the coefficients in L are bounded, it is clear that there is another constant $\beta_2 > 0$ such that

$$|a(u, v)| \le \beta_2 \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \forall u, v \in H^1(\Omega) .$$

Then from the classical theory of abstract elliptic eigenvalue problems given in [8], we now infer the following:

(i) The problem (4) has a countable infinite set of eigenvalues, all having finite multiplicity and being strictly positive, without finite accumulation point arranged as

$$0 < \lambda_1 \le \lambda_2 \le \ldots \lambda_m \le \ldots \to \infty .$$

(ii) There is a Hilbert basis of $L_2(\overline{\Omega})$ formed by orthonormal eigenfunctions u_m, $m \ge 1$.

In Section 2, we discuss the finite-element approximation using the Newton method for a second-order nonlinear Steklov eigenvalue problem. In Section 3, we discuss the numerical results using the finite-element method. Figure 1 depicts the results and verifies the theoretical arguments. In the last section we discuss the conclusion.

2. Numerical Method

In this section we present our numerical method, which is one of the discretization methods. Here the purpose of the discretization method is to reduce a continuous system to a simple discrete system that is equivalent with it. In this section, initially we give an overview of weak formulations of nonlinear elliptic equations, as required for the use of finite-element methods. Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $\partial\Omega$ denote the boundary, which can be thought of as a set in $\mathbb{R}$. Consider now the following nonlinear scalar second-order elliptic equation on Ω, a class of elliptic equations of the form

$$-\nabla.(a(x)\nabla u(x)) + a(x)u(x) = f(x, u(x)) \quad \text{in} \quad \Omega,$$

$$u(x) = 0 \quad \text{on} \quad \Gamma_0, \qquad \frac{\partial u(x)}{\partial n} = \lambda u(x) \quad \text{on} \quad \Gamma_1, \tag{5}$$

where $a : \Omega \to \mathbb{R}^{2\times 2}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

The above form of the equation is sometimes referred to as the strong form, in that the solution is required to be twice differentiable for the equation to hold in the classical sense. To produce a weak formulation, which is more suitable for finite-element methods in that it will require that derivatives of the solution exist in the classical sense, we first choose trial and test spaces of functions. For second-order scalar elliptic equations of this form, the appropriate trial and test space can be seen to be $H^1(\Omega)$, the Sobolev space of functions that are differentiable one time under the integral. This function space is simply the set of all scalar-valued functions over the domain for which the following integral (the energy norm, or H^1-norm) is always finite:

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 + |u|^2 \, dx \right)^{1/2}.$$

In other words, the set (or space) of functions $H^1(\Omega)$ is defined as

$$H^1(\Omega) = \left\{ u : \|u\|_{H^1(\Omega)} < \infty \right\}.$$

A closely related function space is that of square-integrable functions:

$$L^2(\Omega) = \left\{ u : \|u\|_{L^2(\Omega)} < \infty \right\},$$

where

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 \, dx \right)^{1/2}.$$

(To be precise, all integrals here and below must be interpreted in the Lebesgue rather than Riemann sense.) It is important that the trial and test spaces satisfy a zero boundary condition on the boundary $\partial\Omega$ on which the Dirichlet boundary condition (3) holds, so that in fact we choose the following subspace of $H^1(\Omega)$:

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : \; u = 0 \;\; \text{on } \partial\Omega \right\}.$$

Now, multiplying our elliptic equation by a test function $v \in H_0^1(\Omega)$ produces

$$\int_\Omega \left(-\nabla.(a\nabla u) + a.u - f(x,u)\right) v dx = 0,$$

which becomes, after applying a generalized form of Green's integral identities,

$$\int_\Omega (a\nabla u).\nabla v \, dx - \int_{\partial\Omega} v\,(a\nabla u).n \, ds + \int_\Omega a.u.v dx - \int_\Omega f(x,u)\, v dx = 0. \tag{6}$$

Note that the boundary integral above vanishes due to the fact that the test function v vanishes on the boundary.

If the boundary function is smooth enough, there is a mathematical result called the trace theorem [12] that guarantees the existence of a function $\bar{u} \in H^1(\Omega)$ such that

$$u(x) = 0 \quad \text{on} \quad \Gamma_0$$

and

$$\frac{\partial u(x)}{\partial n} = \lambda u(x) \quad \text{on} \quad \Gamma_1.$$

We will not be able to construct such a trace function $\bar{u}$ in practice, but we will be able to approximate it as accurately as necessary in order to use this weak formulation in finite-element methods. Employing such a function $\bar{u} \in H^1(\Omega)$, we can easily verify that the solution u to the problem (5), if it exists, lies in the space of functions $\bar{u} + H_0^1(\Omega)$.

We have therefore shown that the solution to the original problem (5) also solves the following problem:

$$\text{Find} \quad u \in \bar{u} + H_0^1(\Omega) \quad \text{such that} \quad \langle F(u), v\rangle = 0 \quad \forall\ v \in H_0^1(\Omega), \tag{7}$$

where from Eq. (6) the scalar-valued function of u and v, nonlinear in u but linear in v, is defined as

$$\langle F(u), v\rangle = \int_\Omega \left(a\nabla u.\nabla v + a.u.v - f(x,u)v\right) dx - \int_{\partial\Omega} \lambda.u.v \, dl.$$

Clearly, the weak formulation of the problem given by Eq. (7) imposes only one order of differentiability on the solution u, and only in the weak sense (under an integral). Under suitable growth restrictions on the nonlinearities of function f, it can be shown that this weak formulation makes sense, in that the form $\langle F(\cdot), \cdot\rangle$ is finite for all arguments. Moreover, it can be shown under somewhat stronger assumptions that the weak formulation is well posed, in that there exists a unique solution depending continuously on the problem data.

To apply a Newton iteration to solve this nonlinear problem numerically, we will need a linearization of some sort. Rather than discretize and then linearize the discretized equations, we will exploit the fact that with projection-type discretizations such as the finite-element method, these operations actually commute; we can first linearize the differential equation, and then discretize the linearization in order to employ a Newton iteration. To linearize the weak form operator $\langle F(u), v\rangle$, a bilinear linearization form $\langle DF(u)w, v\rangle$ is produced as its directional (variational, Gâteaux) derivative as follows:

$$\langle DF(u)w, v\rangle = \frac{d}{dt}\langle F(u + tw), v\rangle\Big|_{t=0}$$

$$= \frac{d}{dt}\int_{\Omega}\left(a\nabla(u+tw).\nabla v + a.u(u+t.w) - f(x,(u+tw))v\right)dx\Big|_{t=0} - \frac{d}{dt}\int_{\partial\Omega}\lambda u.v\,dl\Big|_{t=0}$$

$$= \int_{\Omega}\left(a\nabla w.\nabla v + a.w.v - \frac{\partial f(x,u)}{\partial u}wv\right)dx - \int_{\partial\Omega}\lambda uv\,dl. \tag{8}$$

This scalar-valued function of the three arguments u, v, and w is linear in w and v but possibly nonlinear in u. For fixed u, it is referred to as a bilinear form (linear in each of the remaining arguments w and v). We will see shortly that the nonlinear weak form $\langle F(u), v\rangle$ and the associated bilinear linearization form $\langle DF(u)w, v\rangle$, together with a continuous piecewise polynomial subspace of the solution space $\bar{u} + H_0^1(\Omega)$, are all that are required to employ the finite-element method for numerical solution of the original elliptic equation.

3. Numerical Result

To illustrate our numerical scheme, we consider a second-order nonlinear Steklov eigenvalue problem

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a.u = u^3 \quad \text{in} \quad \Omega. \tag{9}$$

Let

$$a = 1, \quad u = 0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial u}{\partial n} = \lambda u \quad \text{on} \quad \Gamma_1,$$

where

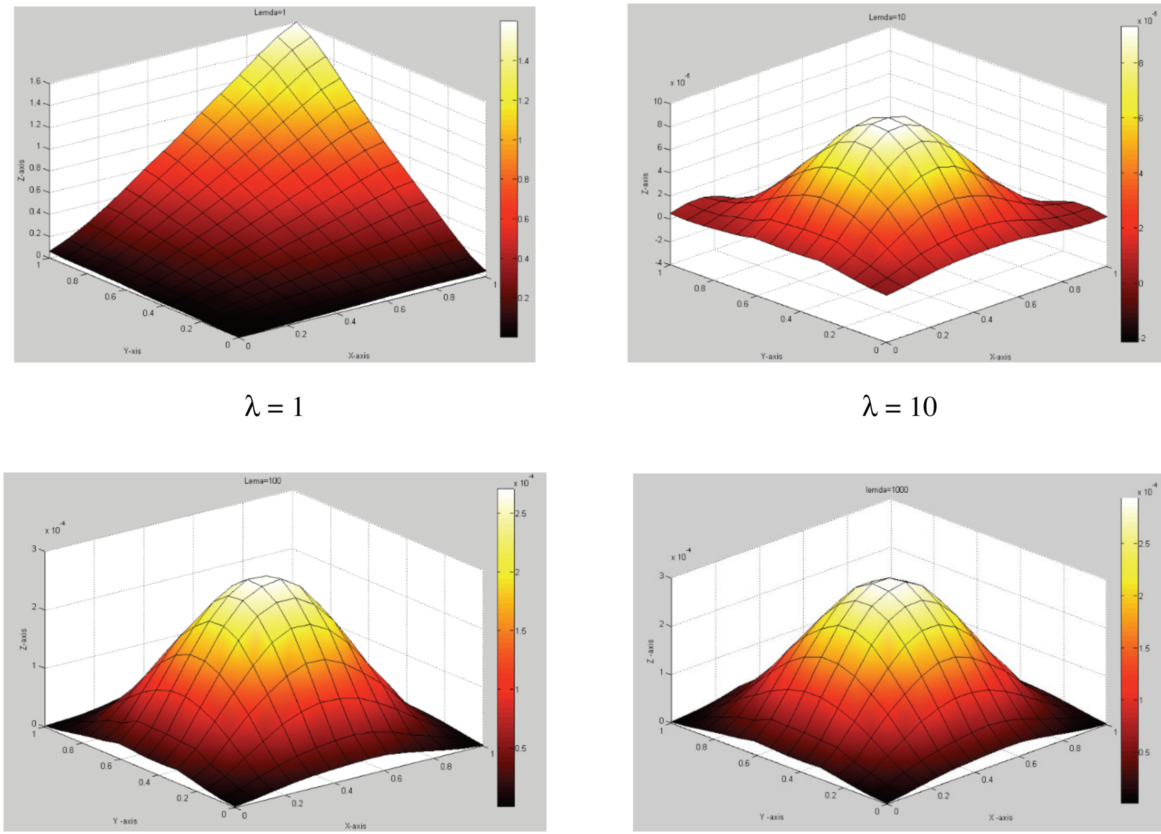
$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \phi, \quad \text{meas}(\Gamma_1) > 0.$$

In particular, let $a = 1$ in (7). The weak formulation of (7) becomes

$$J(u, v) := \int_{\Omega}\nabla u.\nabla v\,dx - \int_{\Gamma_1}\lambda u.v\,dx + \int_{\Omega}(u - u^3)v\,dx = 0, \quad v \in H_0^1(\Omega), \tag{10}$$

which can also be regarded as the necessary condition for the minimizer in the variational problem

$$\min\int_{\Omega}\left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2\right]dx.$$



$\lambda = 1$ $\lambda = 10$

$\lambda = 100$ $\lambda = 1000$

Fig. 1

We aim to solve (10) using the Newton method. Starting with some initial guess u^0, for each iteration, first we compute $u^n - u^{n+1}$ satisfying

$$DJ\left(u^n,\, v;\, u^n - u^{n+1}\right) = J(u^n, v), \qquad v \in H_0^1(\Omega)\,, \tag{11}$$

where

$$DJ\,(u, v;\, w) := \int_\Omega \nabla v.\nabla w\, dx - \int_{\Gamma_1} \lambda v.w\, dx + \int_\Omega (vw - 3vu^2 w)\, dx\,.$$

Here the integrals in $J(u,v)$ and $DJ(u,v;w)$ are calculated as a sum over all elements. This implementation of the Newton method gives more rapid results compared to other approximation schemes. The results for the nonlinear Steklov equation obtained by our scheme at different nodes are given in the following table, and the solution is properly depicted in Fig. 1.

4. Conclusion

Here we introduce a computational scheme, with the help of which researchers can easily handle the problem of solving a nonlinear Steklov problem. Nonlinear Steklov elliptic differential equations have many practical applications in science and engineering. Solution of this problem using the finite-element method shows the importance of eigenvalues in Steklov equations, which change the behavior of the solution as we increase the eigenvalue.

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