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INITIAL-VALUE TECHNIQUE FOR SELF-ADJOINT SINGULAR PERTURBATION BOUNDARY VALUE PROBLEMS

H. K. Mishra,[1] **M. Kumar,**[2] **and P. Singh**[3]

We have developed an initial-value technique for self-adjoint singularly perturbed two-point boundary value problems. The original problem is reduced to its normal form, and the reduced problem is converted into first-order initial-value problems. These initial-value problems are solved by the cubic spline method. Numerical illustrations are given at the end to demonstrate the efficiency of our method. Graphs are also depicted in support of the results.

1. Introduction

We consider the following class of self-adjoint singularly perturbed two-point boundary value problems

$$Ly \equiv -\varepsilon(a(x)y'(x))' + b(x)y(x) = f(x), \tag{1}$$

where $0 \le x \le 1$, subject to

$$y(0) = \alpha, \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where $\alpha, \beta \in R$, ε is a small positive parameter and $a(x)$, $b(x)$, and $f(x)$ are smooth functions and satisfy

$$a(x) \ge a^* > 0, \quad a'(x) \ge 0, \quad b(x) \ge b^* > 0.$$

Under these conditions the operator L admits a maximum principle [9]. Earlier these types of problems have been solved by numerous researchers. Boglave [1] and Schatz and Wahlbin [11] used finite element techniques to solve such problems. Niijima [7, 8] gave uniformly second-order accurate difference schemes, whereas Miller [6] gave sufficient conditions for the uniform first-order convergence of a general three-point difference scheme. These classes of problems arise in various fields of science and engineering, for instance, fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics etc. Surla and Jerkovic [12] considered the singularly perturbed boundary value problem using the spline collocation method. Sakai and Usmani [10] gave a new concept of B-splines in terms of hyperbolic and trigonometric splines which are different from earlier ones. It is proved that the hyperbolic and trigonometric B-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval [0, 1].

In order to solve the singular perturbation problem, first we reduce Eq. (1) to its normal form and then the reduced problem is solved by the initial-value method using cubic splines.

[1] Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad, India; e-mail: hkm@mnnit.ac.in.

[2] Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad, India; e-mail: manoj@mnnit.ac.in.

[3] Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad, India; e-mail: psingh@mnnit.ac.in.

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In general, finding the numerical solution of a second-order boundary value problem with y' term is more difficult as compare to a second-order boundary value problem without y' term. Therefore it is better to convert the second-order boundary value problem without y' term, i.e., to its normal form.

In Section 2, we give a description of the method. In section 3, we give three numerical examples to demonstrate the efficiency of the method. In Section 4, we present the conclusion.

2. Description of the Method

Consider the singularly perturbed two-point boundary value problem

$$-\varepsilon(a(x)y'(x))' + b(x)y = f(x) \tag{4}$$

for $0 \le x \le 1$ with the boundary conditions

$$y(0) = \alpha, \tag{5}$$

$$y(1) = \beta, \tag{6}$$

where ε is a small parameter $(0 < \varepsilon << 1)$, α, β are given constants, and the functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth in $[0, 1]$. We also suppose that $a(x)$, $b(x)$ satisfy

$$a(x) \ge a^* > 0, \quad a'(x) \ge 0, \quad b(x) \ge b^* > 0. \tag{7}$$

Equation (4) can be written as

$$-\varepsilon a(x)y''(x) - \varepsilon a'(x)y'(x) + b(x)y(x) = f(x)$$

or

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x), \tag{8}$$

where

$$P(x) = \frac{a'(x)}{a(x)}, \qquad Q(x) = \frac{-b(x)}{\varepsilon a(x)}, \quad \text{and} \quad R(x) = \frac{-f(x)}{\varepsilon a(x)}.$$

Consider the transformation

$$y(x) = U(x)V(x). \tag{9}$$

Then Eq. (8) can be written as its normal form as

$$V''(x) + A(x)V(x) = H(x), \tag{10}$$

where

$$A(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2, \tag{11}$$

$$H(x) = R(x)\exp\left(\frac{1}{2}\int_0^x P(s)\,ds\right), \tag{12}$$

$$U(x) = \exp\left(-\frac{1}{2}\int_0^x P(s)\,ds\right), \tag{13}$$

with $V(0) = \frac{y(0)}{U(0)} = \gamma$, $V(1) = \frac{y(1)}{U(1)} = \delta$, $x \in (0, 1)$, and $\gamma, \delta \in R$. Multiplying Eq. (10) by $-\varepsilon$, we get

$$L_1 V(x) = -\varepsilon V''(x) + W(x)V(x) = Z(x), \tag{14}$$

with boundary conditions

$$V(0) = \gamma, \qquad V(1) = \delta, \tag{15}$$

where $W(x) = -\varepsilon A(x)$, $Z(x) = -\varepsilon H(x)$, and $W(x) \geq W^* > 0$.

In order to obtain the solution $V(x)$ of the problem, let $V_R(x)$ be the solution of the reduced problem obtained by putting $\varepsilon = 0$ in (14) and by neglecting both boundary conditions (15),

$$W(x)V_R(x) = Z(x), \quad x \in [0, 1]. \tag{16}$$

It is known [3] that the solution of (14)–(15) is given by

$$V(x) = V_R(x) + \left\{(\gamma - V_R(0))(W(0)/W(x))^{1/4} V_1(x)\right\} + \left\{(\delta - V_R(1))(W(1)/W(x))^{1/4} V_2(x)\right\} + O(\sqrt{\varepsilon}), \tag{17}$$

where $V_1(x)$ and $V_2(x)$ are defined on $[0, 1]$ by

$$V_1(x) = \exp\left(-\int_0^x \sqrt{-W(s)/\varepsilon}\,ds\right), \tag{18}$$

$$V_2(x) = \exp\left(-\int_x^1 \sqrt{-W(s)/\varepsilon}\,ds\right). \tag{19}$$

Clearly, $V_1(x)$ can be found as the solution of the following initial-value problem:

$$\sqrt{\varepsilon}\,V_1'(x) + \sqrt{-W(x)/\varepsilon}\,V_1(x) = 0, \quad x \in [0, 1], \tag{20}$$

$$V_1(0) = 1, \tag{21}$$

while $V_2(x)$ is the solution of

$$\sqrt{\varepsilon}\,V_2'(x) - \sqrt{-W(x)/\varepsilon}\,V_2(x) = 0, \quad x \in [0, 1], \tag{22}$$

$$V_2(0) = 1. \tag{23}$$

Therefore, we can approximate the solution of problem (14) by combining the function $V_R(x)$ given by (16) with the solutions of the initial-value problems (20) and (22). Hence the solution of Eq. (4) is given by

$$y(x) = U(x)V(x). \tag{24}$$

The whole method is extremely simple to implement. To this end, it is enough to use suitable codes for initial-value problems taking into account that the solution of Eqs. (20) and (22) may change character in the interval of integration. In fact, these problems are generally nonstiff near the starting point, but we must expect stiffness as the integration goes to the end of the interval. So, the numerical solution of these problems requires a scheme automatically determining whether the problem can be solved more efficiently using a class of methods designed for nonstiff problems or a class of methods suitable for stiff problems. We use the cubic spline method [4, 5] for the solution of Eqs. (20) and (22). In fact, any standard analytical or numerical method can be used.

3. Numerical Examples

To demonstrate the applicability of the method, we will discuss three examples: a nonhomogeneous SPP and an SPP with variable coefficients. These examples have been chosen because either analytical or approximate solutions are available for comparison.

Example 3.1. Consider the following equation:

$$-\varepsilon\big(y'(x)\big)' + y(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x)$$

having boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 0,$$

which has the exact solution [2]

$$y(x) = \frac{\left[\exp\left(\frac{-1(1-x)}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\right]}{\left[1+\exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\right] - \cos^2(\pi x)}.$$

Here, $a(x) = 1$, $b(x) = 1$ and $f(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x)$,

$$P(x) = 0, \qquad Q(x) = \frac{-1}{\varepsilon}, \qquad R(x) = \frac{[\cos^2(\pi x) + 2\varepsilon\pi^2\cos(2\pi x)]}{\varepsilon}.$$

Therefore

$$A(x) = \frac{-1}{\varepsilon}, \qquad H(x) = \frac{[\cos^2(\pi x) + 2\varepsilon\pi^2\cos(2\pi x)]}{\varepsilon},$$

$$U(x) = 1, \qquad V(0) = 0, \qquad V(1) = 0,$$

$$-\varepsilon V''(x) + V(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), \tag{25}$$

$$V(0) = 0, \quad V(1) = 0. \tag{26}$$

Putting $\varepsilon = 0$ in Eq. (25), we have

$$V_R(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), \tag{27}$$

$$V_R(0) = -1 - 2\varepsilon\pi^2, \tag{28}$$

$$V_R(1) = -\cos^2(\pi) - 2\varepsilon\pi^2 \cos(2\pi), \tag{29}$$

and

$$V_1'(x) = \frac{-1}{\sqrt{\varepsilon}} V_1(x), \quad V_1(0) = 1, \tag{30}$$

$$V_2'(x) = \frac{1}{\sqrt{\varepsilon}} V_2(x), \quad V_2(1) = 1. \tag{31}$$

Putting the values of Eqs. (27)–(31) in Eq. (17), we get $V(x)$ and hence $y(x)$.

The maximum absolute errors for Example 3.1 for $\varepsilon = 10^{-4}$, $h = 10^{-3}$ are presented in Table 1.

Example 3.2. Consider the following equation:

$$-\varepsilon\left(y'(x)\right)' + (1+x)y(x) = -40\left[x(x^2-1) - 2\varepsilon\right]$$

having boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 0,$$

which has the exact solution [7] $y(x) = 40x(1-x)$. Here,

$$a(x) = 1, \quad b(x) = (1+x), \quad f(x) = -40\left[x(x^2-1) - 2\varepsilon\right],$$

$$P(x) = 0, \quad Q(x) = \frac{-(1+x)}{\varepsilon}, \quad R(x) = \frac{40\left[x(x^2-1) - 2\varepsilon\right]}{\varepsilon}.$$

Therefore

$$A(x) = \frac{-(1+x)}{\varepsilon}, \quad H(x) = \frac{40\left[x(x^2-1) - 2\varepsilon\right]}{\varepsilon},$$

Table 1

x	$y(x)$	Exact solution	Max. absolute error
0.000	0.0000000	0.0000000	0.0000000
0.001	−0.0953418	−0.0951527	0.0001891
0.010	−0.6323828	−0.6311339	0.0012489
0.020	−0.8624169	−0.8607221	0.0016948
0.030	−0.9431993	−0.9413565	0.0018428
0.040	−0.9678527	−0.9659759	0.0018768
0.050	−0.9706548	−0.9687903	0.0018645
0.100	−0.9060599	−0.9044631	0.0015968
0.300	−0.3448815	−0.3454915	0.0006100
0.500	0.0019739	0.0000000	0.0019739
0.700	−0.3448814	−0.3454914	0.0006100
0.900	−0.9060598	−0.9044631	0.0015967
1.000	0.0000000	0.0000000	0.0000000

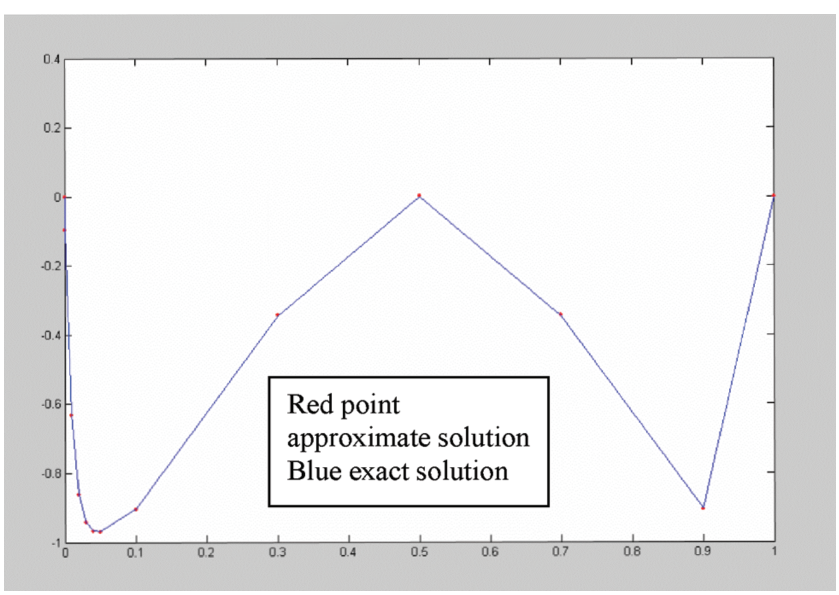

Fig. 1

$$U(x) = 1, \quad V(0) = 0, \quad V(1) = 0,$$

$$-\varepsilon V''(x) + V(x) = -40\left[x(x^2-1) - 2\varepsilon\right], \tag{32}$$

$$V(0) = 0, \quad V(1) = 0. \tag{33}$$

Putting $\varepsilon = 0$ in Eq. (32), we have

$$V_R(x) = \frac{-40\left[x(x^2-1) - 2\varepsilon\right]}{(1+x)}, \tag{34}$$

$$V_R(0) = 80\varepsilon, \tag{35}$$

$$V_R(1) = 40\varepsilon, \tag{36}$$

and

$$V_1'(x) = \frac{-1}{\sqrt{\varepsilon}}\sqrt{(1+x)}\,V_1(x), \quad V_1(0) = 1, \tag{37}$$

$$V_2'(x) = \frac{1}{\sqrt{\varepsilon}}\sqrt{(1+x)}\,V_2(x), \quad V_2(1) = 1. \tag{38}$$

Putting the values of Eqs. (34)–(38) in Eq. (17), we get $V(x)$ and hence $y(x)$. The maximum absolute errors for example 3.2 for $\varepsilon = 10^{-4}$, $h = 10^{-3}$ are presented in Table 2.

Example 3.3. Consider the following equation:

$$-\varepsilon\left(y'(x)\right)' + (1 + x(1-x))\,y(x) = 1 + x(1-x)$$
$$+ \left[2\sqrt{\varepsilon} - x^2(1-x)\right]\exp\left(-\frac{1-x}{\sqrt{\varepsilon}}\right) + \left[2\sqrt{\varepsilon} - x(1-x)^2\right]\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)$$

having boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 0,$$

which has the exact solution [13]

$$y(x) = 1 + (x-1)\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right] - x\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right].$$

Here, $a(x) = 1$, $b(x) = 1 + x(1-x)$ and

$$f(x) = 1 + x(1-x) + \left[2\sqrt{\varepsilon} - x^2(1-x)\right]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right] + \left[2\sqrt{\varepsilon} - x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right],$$

Table 2

x	$y(x)$	Exact solution	Max. absolute error
0.000	0.0000000	0.0000000	0.0000000
0.001	0.0407962	0.0399600	0.0008362
0.010	0.4011096	0.3960000	0.0051096
0.020	0.7909375	0.7840000	0.0069375
0.030	1.1715688	1.1640000	0.0075688
0.040	1.5437557	1.5360000	0.0077557
0.050	1.9077790	1.9000000	0.0077790
0.100	3.6074901	3.6000001	0.0074900
0.300	8.4063807	8.4000006	0.0063801
0.500	10.0055685	10.0000000	0.0055685
0.700	8.4049492	8.4000000	0.0049492
0.900	3.6044607	3.6000009	0.0044598
1.000	0.0000000	0.0000000	0.0000000

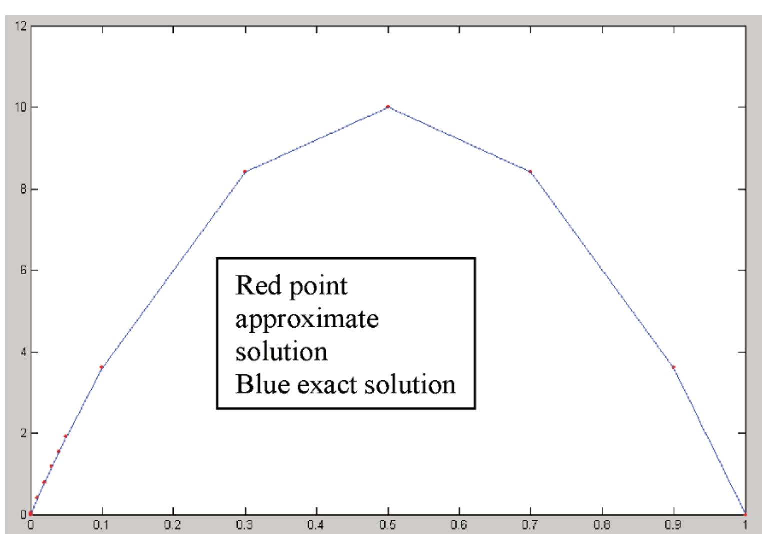

Fig. 2

$$P(x) = 0, \qquad Q(x) = \frac{-[1+x(1-x)]}{\varepsilon},$$

$$R(x) = -\frac{(1+x(1-x)+\left[2\sqrt{\varepsilon}-x^2(1-x)\right]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right]+\left[2\sqrt{\varepsilon}-x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{\varepsilon}.$$

Therefore

$$A(x) = \frac{-[1+x(1-x)]}{\varepsilon},$$

$$H(x) = -\frac{1+x(1-x)+[2\sqrt{\varepsilon}-x^2(1-x)]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right]+\left[2\sqrt{\varepsilon}-x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{\varepsilon},$$

$$U(x) = 1, \qquad V(0) = 0, \qquad V(1) = 0,$$

$$-\varepsilon V''(x) + (1+x(1-x))V(x)$$

$$= \frac{(1+x(1-x)+\left[2\sqrt{\varepsilon}-x^2(1-x)\right]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right]+\left[2\sqrt{\varepsilon}-x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{\varepsilon}, \tag{37}$$

$$V(0) = 0, \qquad V(1) = 0. \tag{38}$$

Putting $\varepsilon = 0$ in Eq. (37), we have

$$V_R(x) = \frac{1+x(1-x)+\left[2\sqrt{\varepsilon}-x^2(1-x)\right]\exp\left[-\frac{1-x}{\sqrt{\varepsilon}}\right]+\left[2\sqrt{\varepsilon}-x(1-x)^2\right]\exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{1+x(1-x)}, \tag{39}$$

$$V_R(0) = \left[1+2\sqrt{\varepsilon}\exp\left[-\frac{1}{\sqrt{\varepsilon}}\right]+2\sqrt{\varepsilon}\right], \tag{40}$$

$$V_R(1) = \left[1+2\sqrt{\varepsilon}+2\sqrt{\varepsilon}\exp\left[-\frac{1}{\sqrt{\varepsilon}}\right]\right], \tag{41}$$

and

$$V_1'(x) = \frac{-1}{\sqrt{\varepsilon}}\sqrt{(1+x(1-x))}\,V_1(x), \qquad V_1(0) = 1, \tag{42}$$

$$V_2'(x) = \frac{1}{\sqrt{\varepsilon}}\sqrt{(1+x(1-x))}\,V_2(x), \qquad V_2(1) = 1. \tag{43}$$

Putting the values of Eqs. (39)–(43) in Eq. (17), we get $V(x)$ and hence $y(x)$.

The maximum absolute errors for Example 3.3 for $\varepsilon = 10^{-4}$, $h = 10^{-3}$ are presented in Table 3.

Table 3

x	$y(x)$	Exact solution	Max. absolute error
0.000	0.0000000	0.0000000	0.0000000
0.001	0.1117282	0.1020549	0.0096733
0.010	0.6522136	0.6602936	0.0080800
0.020	0.8915527	0.8853841	0.0061686
0.030	0.9812594	0.9615718	0.0196876
0.040	1.0147314	0.9871895	0.0275419
0.050	1.0271859	0.9957512	0.0314347
0.100	1.0346912	0.9999838	0.0347074
0.300	1.0356572	1.0000000	0.0356572
0.500	1.0359484	1.0000000	0.0359484
0.700	1.0356572	1.0000000	0.0356572
0.900	1.0346913	0.9999996	0.0346917
1.000	0.0000000	0.0000000	0.0000000

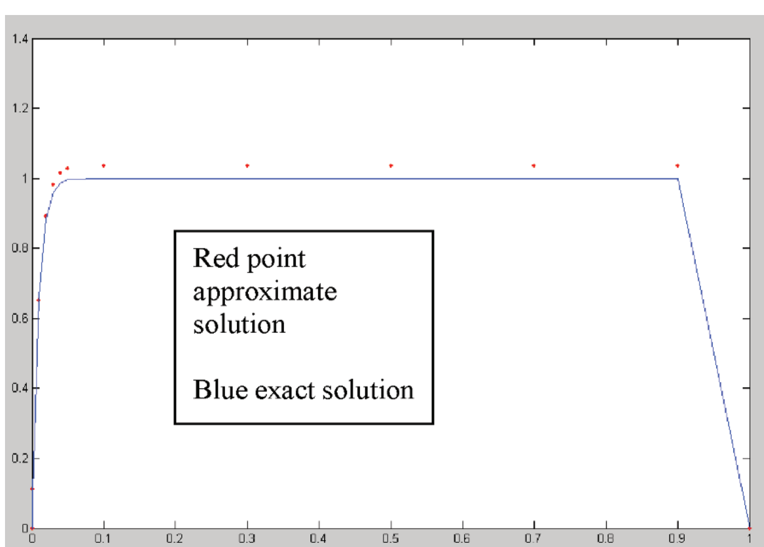

Fig. 3

4. Conclusion

We have described an initial-value method for solving self-adjoint singularly perturbed two-point boundary value problems using the cubic spline method. We have first transformed the original problem into the normal form and then converted it into two initial-value problems. It is a practical method and can be easily implemented on a computer to solve such problems. Three examples are given to demonstrate the efficiency of the proposed method. The maximum absolute errors $\max_i |y(x_i) - y_i|$ at different nodal points are tabulated in the table for $\varepsilon = 10^{-4}$. To further corroborate the applicability of the proposed method, graphs have been plotted in Figs. $1-3$ for problems $1-3$ for $x \in [0, 1]$ versus the computed solution obtained at different values of x for $\varepsilon = 10^{-4}$.

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