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Tensor *Z*-eigenvalue complementarity problems

Meilan Zeng[1]

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Abstract
This paper studies tensor Z-eigenvalue complementarity problems. We formulate the tensor Z-eigenvalue complementarity problem as constrained polynomial optimization, and propose a semidefinite relaxation algorithm for solving the complementarity Z-eigenvalues of tensors. For every tensor that has finitely many complementarity Z-eigenvalues, we can compute all of them and show that our algorithm has the asymptotic and finite convergence. Numerical experiments indicate the efficiency of the proposed method.

Keywords Complementarity Z-eigenvalue · Semidefinite relaxation · Asymptotic convergence · Finite convergence

Mathematics Subject Classification 15A18 · 65K10 · 90C22

1 Introduction

Let $\mathbb{R}, \mathbb{C}$ respectively be the sets of real and complex numbers. Let $T^m(\mathbb{R}^n)$ denote the space of all real m-order n-dimensional tensors, $\mathbb{R}^{n\times n}$ be the space of all real n-by-n matrices. A tensor $\mathcal{A} \in T^m(\mathbb{R}^n)$ is a multi-array indexed as

$$\mathcal{A} = (a_{i_1,\ldots,i_m})_{1\le i_1,\ldots,i_m\le n}.$$

The tensor $\mathcal{A}$ is called symmetric if the value of $a_{i_1,\ldots,i_m}$ is invariant under any permutation of its index $\{i_1, \ldots, i_m\}$. Let $S^m(\mathbb{R}^n)$ be the space of all symmetric tensors in $T^m(\mathbb{R}^n)$.

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✉ Meilan Zeng
zml571@126.com

[1] School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, Hubei 432000, China

$\mathcal{A}x^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}x^m := x^T(\mathcal{A}x^{m-1}) = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m},$$

where $x \in \mathbb{R}^n$, and $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$, which is defined by

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\le i\le n}.$$

The matrix eigenvalue complementarity problem (MEiCP) is that: for given two matrices $A, B \in \mathbb{R}^{n\times n}$, we find a number $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$0 \le x \perp (\lambda Bx - Ax) \ge 0. \tag{1.1}$$

In the above, $a \perp b$ means that the two vectors a, b are perpendicular to each other. For (λ, x) satisfying (1.1), λ is called a complementary eigenvalue of (A, B) and x is called the associated complementary eigenvector. MEiCPs have wide applications, such as static equilibrium states of mechanical systems with unilateral friction [24], the dynamic analysis of structural mechanical systems [17] and the contact problem in mechanics [18].

Tensor eigenvalue complementarity problems (TEiCPs) have received much attention lately. It is a generalization of the matrix eigenvalue complementarity problem, which has a broad range of interesting applications. A tensor eigenvalue complementarity problem can be formulated: for two nonzero tensors $\mathcal{A}\Leftrightarrow\mathcal{B} \in T^m(\mathbb{R}^n)$, if a pair $(\lambda, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ satisfies the equation

$$0 \le x \perp (\lambda \mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}) \ge 0, \tag{1.2}$$

then λ is called a complementarity eigenvalue of $(\mathcal{A}, \mathcal{B})$ and x is the associated complementarity eigenvector. Such (λ, x) is called a C-eigenpair. Chen et al. [3] have further work on tensor eigenvalue complementarity problems. When the tensors are symmetric, they reformulated the problem as nonlinear optimization and proposed a shifted projected power method. Chen and Qi [2] reformulated the TEiCP as a system of nonlinear equations and proposed a damped semi-smooth Newton method for solving it. Fan, Nie and Zhou [6] proposed a semidefinite relaxation method for computing all the complementarity eigenvalues. In this paper, we study tensor Z-eigenvalue complementarity problems.

Lim [14] and Qi [25] introduced the definition of tensor eigenvalues. There is more than one possible definition for tensor eigenvalue in [25, 28]. In this paper, we specifically use the following definitions.

Definition 1.1 Let $\mathcal{A} \in T^m(\mathbb{R}^n)$. The pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ is called E-eigenpair, and λ is called E-eigenvalue and x is the corresponding E-eigenvector of $\mathcal{A}$ if they satisfy the equations

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^T x = 1. \tag{1.3}$$

We call (λ, x) a Z-eigenpair if they are both real.

In the following, we define complementarity Z-eigenvalues of tensors.

Definition 1.2 For $\mathcal{A} \in T^m(\mathbb{R}^n)$, if a pair $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ satisfies the equations

$$0 \le x \perp (\lambda x - \mathcal{A}x^{m-1}) \ge 0 \quad \text{and} \quad x^T x = 1, \tag{1.4}$$

then λ is called a complementarity Z-eigenvalue of $\mathcal{A}$ and x is the associated complementarity Z-eigenvector. Such (λ, x) is called a complementarity Z-eigenpair. For convenience, complementary Z-eigenvalues and Z-eigenvectors are respectively called CZ-eigenvalues and CZ-eigenvectors. The above (λ, x) is called a CZ-eigenpair.

For $x \ge 0$ and $\lambda x - \mathcal{A}x^{m-1} \ge 0$, $x \perp (\lambda x - \mathcal{A}x^{m-1})$ holds if and only if

$$x \circ (\lambda x - \mathcal{A}x^{m-1}) = 0, \tag{1.5}$$

where $\circ$ denotes the Hadamard product defined as in Sect. 2.

Z-eigenvalues have important applications in numerical multilinear algebra [23], image processing [27, 29], higher order Markov chains [15, 16], and spectral hypergraph theory [9], etc. For symmetric tensors, some methods for computing Z-eigenvalues are proposed. Kolda and Mayo [10] proposed a shifted power method. Cui, Dai and Nie [4] proposed a semidefinite relaxation approach for computing all the real eigenvalues. For nonsymmetric tensors, Nie and Zhang [22] proposed a semidefinite relaxation method for computing all the real eigenvalues. A tensor may not have the Z-eigenvalues, but has the CZ-eigenvalues (see Example 1.1). Under some generic conditions, $\mathcal{A}$ has finitely many CZ-eigenvalues. Tensor Z-eigenvalue complementarity problems make some practical problems have more natural and precise mathematical descriptions. Its applications need further research.

Example 1.1 Consider the tensor $\mathcal{A} \in T^4(\mathbb{R}^2)$ such that $a_{ijkl} = 0$ except

$$a_{1112} = a_{1222} = 1, a_{2111} = a_{2122} = -1.$$

By the above definitions and analysis, (λ, x) is a Z-eigenpair if and only if

$$\begin{cases} (x_1^2 + x_2^2)x_2 = \lambda x_1, \\ -(x_1^2 + x_2^2)x_1 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

(λ, x) is a CZ-eigenpair if and only if

$$\begin{cases} (x_1^2 + x_2^2)x_2x_1 = \lambda x_1^2, \\ -(x_1^2 + x_2^2)x_1x_2 = \lambda x_2^2, \\ x_1^2 + x_2^2 = 1, x \geq 0, \\ \lambda x_1 - (x_1^2 + x_2^2)x_2 \geq 0, \\ \lambda x_2 + (x_1^2 + x_2^2)x_1 \geq 0. \end{cases}$$

$\mathcal{A}$ has no Z-eigenvalues and neither E-eigenvalues [22], but $\mathcal{A}$ has one CZ-eigenvalue $\lambda = 0$ with the associated CZ-eigenvector $(1, 0)$.

It is possible that a tensor has infinitely many Z-eigenvalues and CZ-eigenvalues.

Example 1.2 Consider the tensor $\mathcal{A} \in T^3(\mathbb{R}^2)$ such that $a_{ijk} = 0$ except

$$a_{111} = a_{221} = 1.$$

By the equations

$$\begin{cases} x_1^2 = \lambda x_1, \\ x_1x_2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1, \end{cases}$$

one can check that every real number $\lambda \in [0, 1]$ is a Z-eigenvalue of $\mathcal{A}$, associated with Z-eigenvectors $(\lambda, \pm\sqrt{1 - \lambda^2})$ [26]. By the equations

$$\begin{cases} \lambda x_1^2 = x_1^3, \\ \lambda x_2^2 = x_1x_2^2, \\ x_1^2 + x_2^2 = 1, \\ x \geq 0, \lambda x_1 - x_1^2 \geq 0, \\ \lambda x_2 - x_1x_2 \geq 0, \end{cases}$$

one can check that every real number $\lambda \in [0, 1]$ is a CZ-eigenvalue of $\mathcal{A}$, associated with the CZ-eigenvector $(\lambda, \sqrt{1 - \lambda^2})$.

In this paper, we study to how to solve all the CZ-eigenvalues of $\mathcal{A}$, when $\mathcal{A}$ has finitely many CZ-eigenvalues.

The organization of this paper is as follows. Section 2 reviews some basics in polynomial optimization. We propose the semidefinite relaxation algorithm for computing all the CZ-eigenvalues for every tensor that has finitely many CZ-eigenvalues, and prove its asymptotic and finite convergence in Sect. 3. Section 4 demonstrates the numerical experiments. Conclusions are drawn in Sect. 5.

2 Preliminaries

This section reviews some basics in polynomial optimization. We refer to [11–13, 21] for surveys in the area.

Let $\mathbb{N}$ be the set of nonnegative integer numbers. For two vectors $a, b \in \mathbb{R}^n$, $a \circ b$ denotes the Hadamard product of a and b, i.e. the product is defined componentwise. The symbol $\mathbb{R}[x] := \mathbb{R}[x_1, x_2, \dots, x_n]$ denotes the polynomial ring in $x = (x_1, x_2, \dots, x_n)$ with real coefficients. For the vector $\alpha = (\alpha_1, \dots, \alpha_n)$, denote $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n \,|\, |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \le d\}$. The symbol $deg(p)$ denotes the degree of polynomial p. For $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer $\ge t$. For $x = (x_1, x_2, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, denote

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad [x]_d := [1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_1x_n, \dots, x_1^d, \dots, x_n^d]^T.$$

The superscript T denotes the transpose of a matrix/vector. By writing $X \succeq 0$ (resp., $X \succ 0$), we mean that X is a symmetric positive semidefinite (resp., positive definite) matrix. For matrices $X_1, \dots, X_r$, $diag(X_1, \dots, X_r)$ denotes the block diagonal matrix whose diagonal blocks are $X_1, \dots, X_r$. For a vector x, $\|x\|$ denotes its standard Euclidean norm.

An ideal I in $\mathbb{R}[x]$ is a subset such that $I \cdot \mathbb{R}[x] \subseteq I, I + I \subseteq I$. For a tuple $h = (h_1, \dots, h_m)$ in $\mathbb{R}[x]$, $I(h)$ denotes the smallest ideal containing all h_i, i.e. $I(h) := h_1 \cdot \mathbb{R}[x] + \cdots + h_m \cdot \mathbb{R}[x]$. The kth truncation of the ideal $I(h)$ is denoted as $I_k(h)$, which is the set

$$I_k(h) := h_1 \cdot \mathbb{R}[x]_{k-deg(h_1)} + \cdots + h_m \cdot \mathbb{R}[x]_{k-deg(h_m)}.$$

Clearly, $I(h) = \cup_{k \in \mathbb{N}} I_k(h)$.

A polynomial σ is called a sum of squares (SOS) if $\sigma = p_1^2 + \cdots + p_k^2$ for some polynomials $p_1, \dots, p_k \in \mathbb{R}[x]$. $\Sigma[x]$ denotes the set of all SOS polynomials in x. For a degree m, $\Sigma[x]_m$ denotes the truncation $\Sigma[x] \cap R[x]_m$. For a tuple $g = (g_1, \dots, g_t)$, its quadratic module is the set

$$Q(g) := \Sigma[x] + g_1 \cdot \Sigma[x] + \cdots + g_t \cdot \Sigma[x].$$

The kth truncation of $Q(g)$ is the set

$$Q_k(g) := \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{2k-deg(g_1)} + \cdots + g_t \cdot \Sigma[x]_{2k-deg(g_t)}.$$

Note that $Q(g) = \cup_{k \in \mathbb{N}} Q_k(g)$. If the tuple g is empty, then $Q(g) = \Sigma[x], Q_k(g) = \Sigma[x]_{2k}$
.

Let $Pr(g)$ be the quadratic module generated by the set of all possible cross products:

$$g_1, \dots, g_t, g_1g_2, \dots, g_{t-1}g_t, \dots, g_1g_2 \cdots g_t.$$

The set $Pr_k(g)$ is the kth truncated preordering generated by $g = (g_1, \dots, g_t)$.

The set $I(h) + Q(g)$ is called archimedean if there exists some real number $R > 0$ such that $R - \|x\|^2 \in I(h) + Q(g)$. If there exists $p \in I(h) + Q(g)$ such that

$p(x) \geq 0$ defines a compact set in $\mathbb{R}^n$, then $I(h) + Q(g)$ is archimedean. For the tuples h and g as above, denote

$$K = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \geq 0\}.$$

Clearly, if $I(h) + Q(g)$ is archimedean, then K must be a compact set.

Let $\mathbb{R}^{\mathbb{N}_d^n}$ be the space of real sequences indexed by $\alpha \in \mathbb{N}_d^n$. A vector y in $\mathbb{R}^{\mathbb{N}_d^n}$ is called a truncated moment sequences (tms) of degree d, i.e.

$$y := (y_\alpha)_{\alpha \in \mathbb{N}_d^n}.$$

A tms $y \in \mathbb{R}^{\mathbb{N}_d^n}$ defines a Riesz function $\mathscr{L}$ on $\mathbb{R}[x]_d$ as

$$\mathscr{L}_y(\sum_{\alpha \in \mathbb{N}_d^n} p_\alpha x^\alpha) := \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha y_\alpha.$$

For convenience, we denote $\langle p, y \rangle := \mathscr{L}_y(p)$. For an integer $t \leq d$ and $y \in \mathbb{R}^{\mathbb{N}_d^n}$, denote the tth truncation of y as

$$y|_t := (y_\alpha)_{\alpha \in \mathbb{N}_t^n}.$$

Let $q \in \mathbb{R}[x]_{2k}$. The kth localizing matrix of q, generated by $y \in \mathbb{R}^{\mathbb{N}_{2k}^n}$, is the symmetric matrix $L_q^{(k)}(y)$ such that

$$\mathscr{L}(qp_1p_2) = vec(p_1)^T(L_q^{(k)}(y))vec(p_2)$$

for all $p_1, p_2 \in R[x]_{k-\lceil deg(q)/2 \rceil}$. In the above, $vec(p_i)$ denotes the coefficient vector of the polynomial p_i. For example, $n = 2, k = 2, q = 1 - x_1^2 - x_2^2$, it follows

$$L_{1-x_1^2-x_2^2}^{(2)}(y) = \begin{pmatrix} 1 - y_{20} - y_{02} & y_{10} - y_{30} - y_{12} & y_{01} - y_{21} - y_{03} \\ y_{10} - y_{30} - y_{12} & y_{20} - y_{40} - y_{22} & y_{11} - y_{31} - y_{13} \\ y_{01} - y_{21} - y_{03} & y_{11} - y_{31} - y_{13} & y_{02} - y_{22} - y_{04} \end{pmatrix}.$$

When $q = (q_1, \ldots, q_r)$ is a tuple of polynomials, we define

$$L_q^{(k)}(y) := diag(L_{q_1}^{(k)}(y), \ldots, L_{q_r}^{(k)}(y)),$$

a block diagonal matrix. When $q = 1, L_1^{(k)}(y)$ is called the kth moment matrix generated by y, denoted as $M_k(y)$. For instance, $n = 2, k = 2$,

$$M_2(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix}.$$

An important question is whether or not a tms $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ admits a representing measure whose support is contained in K. For this to be true, a necessary condition [5, 7] is that

$$L_h^{(k)}(y) = 0, \quad M_k(y) \succeq 0, \quad L_g^{(k)}(y) \succeq 0. \tag{2.1}$$

However, the above is typically not sufficient. Let $d' = \max\{1, \lceil deg(h)/2 \rceil, \lceil deg(g)/2 \rceil\}$. y admits a measure supported in K if y also satisfies the rank condition [5]

$$rankM_{k-d'}(y) = rankM_k(y). \tag{2.2}$$

In such case, y admits a unique finitely atomic measure on K. We call that y is flat with respect to $h = 0$ and $g \geq 0$ if both Problems (2.1) and (2.2) are satisfied.

3 Computing all the *CZ*-eigenvalues

Suppose that the tensor $\mathcal{A}$ has finite CZ-eigenvalues, we discuss how to compute all of them.

Recall that (λ, x) is a CZ-eigenpair of $\mathcal{A} \in T^m(\mathbb{R}^n)$, if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ satisfies

$$0 \leq x \perp (\lambda x - \mathcal{A}x^{m-1}) \geq 0 \quad \text{and} \quad x^T x = 1. \tag{3.1}$$

Then,

$$0 = x^T(\lambda x - \mathcal{A}x^{m-1}) = \lambda x^T x - \mathcal{A}x^m = \lambda - \mathcal{A}x^m.$$

Thus $\lambda = \mathcal{A}x^m$.

x is a CZ-eigenvector of $\mathcal{A}$ if and only if

$$\begin{cases} h = (x \circ ((\mathcal{A}x^m)x - \mathcal{A}x^{m-1}), x^T x - 1) = 0, \\ g = (x, (\mathcal{A}x^m)x - \mathcal{A}x^{m-1}) \geq 0, \end{cases} \tag{3.2}$$

where $\circ$ denotes the Hadamard product of two vectors, and the associated CZ-eigenvalue is $\mathcal{A}x^m$. Since the tensor $\mathcal{A}$ has finite CZ-eigenvalues, we suppose that the CZ-eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_L$, where L is the total number of distinct CZ-eigenvalues. They can be ordered monotonically as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_L.$$

In the following subsections, we give the semidefinite relaxation method for computing all the CZ-eigenvalues of $\mathcal{A}$.

3.1 The smallest *CZ*-eigenvalue

Let $f(x) := \mathcal{A}x^m$. The smallest CZ-eigenvalue λ_1 equals the optimal value of the optimization problem

$$\begin{aligned} &\min f(x) \\ &s.t.\ h(x) = 0,\ \ g(x) \geq 0, \end{aligned} \tag{3.3}$$

where h, g are as in (3.2). Let K be its feasible set. For a tms $y \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ with degree $2k \geq m$, denote

$$k_0 = \lceil m/2 \rceil.$$

We apply Lasserre's semidefinite relaxations [11] to solve (3.3). For the orders $k = k_0, k_0 + 1, \ldots$, the kth Lasserre relaxation is

$$\begin{cases} \rho_k^{(1)} := \min\ \langle f, y \rangle \\ \qquad\qquad \text{s.t.}\ \ \langle 1, y \rangle = 1, L_h^{(k)}(y) = 0, \\ \qquad\qquad\quad M_k(y) \succeq 0, L_g^{(k)}(y) \succeq 0. \end{cases} \tag{3.4}$$

The dual problem of (3.4) is

$$\eta_k^{(1)} := \max \gamma \quad \text{ s.t. } \quad f - \gamma \in I_{2k}(h) + Pr_k(g). \tag{3.5}$$

Under the weak duality, we have $\eta_k^{(1)} \leq \rho_k^{(1)} \leq \lambda_1$ for all k and the sequences $\{\rho_k^{(1)}\}$ and $\{\eta_k^{(1)}\}$ are monotonically increasing (cf. [11]).

3.2 The other *CZ*-eigenvalues

We discuss how to compute λ_i for $i = 2, \cdots, L$. Suppose λ_{i-1} is already computed, we need to determine the next *CZ*-eigenvalue λ_i. Consider the optimization problem

$$\begin{aligned} &\min f(x) \\ &s.t.\ h(x) = 0,\ \ g(x) \geq 0, f(x) \geq \lambda_{i-1} + \delta. \end{aligned} \tag{3.6}$$

The optimal value of (3.6) is equal to λ_i if

$$0 < \delta < \lambda_i - \lambda_{i-1}. \tag{3.7}$$

Similarly, Lasserre's semidefinite relaxations can be applied to solve (3.6). For the orders $k = k_0, k_0 + 1, \cdots$, the k-Lasserre relaxation is

$$\begin{cases} \rho_k^{(i)} := \min\ \langle f, y \rangle \\ \qquad\qquad \text{s.t.}\ \ \langle 1, y \rangle = 1, L_h^{(k)}(y) = 0, \\ \qquad\qquad\quad M_k(y) \succeq 0, L_g^{(k)}(y) \succeq 0, L_{f-\lambda_{i-1}-\delta}^{(k)}(y) \succeq 0. \end{cases} \tag{3.8}$$

The dual problem of (3.8) is

$$\eta_k^{(i)} := \max \gamma \quad \text{ s.t. } \quad f - \gamma \in I_{2k}(h) + Pr_k(g, f - \lambda_{i-1} - \delta). \tag{3.9}$$

In practice, we usually do not know whether λ_i exists or not. If it exists, how to choose δ to satisfy (3.7). The existence of λ_i and the relation (3.7) can be checked by solving the optimization problem

$$\chi_i := \max f(x) \quad s.t.\ h(x)=0, g(x)\ge 0, f(x)\le \lambda_{i-1}+\delta. \tag{3.10}$$

Its optimal value can be computed by semidefinite relaxations like (3.8, 3.9).

Proposition 3.1 *Let $\mathcal{A}\in T^m(\mathbb{R}^n)$. Let λ_{i-1} be the $(i-1)$-th smallest CZ-eigenvalue of $\mathcal{A}$. For all $\delta>0$, we have the following properties*:

(i) *The relaxation (3.8) is infeasible for some k if and only if $\lambda_{i-1}+\delta>\lambda_{max}$.*
(ii) *If $\chi_i=\lambda_{i-1}$ and λ_i exists, then δ satisfies (3.7).*
(iii) *If $\chi_i=\lambda_{i-1}$ and (3.8) is infeasible for some k, then λ_i does not exist and $\lambda_{max}=\lambda_{i-1}$.*

Proof

(i) Necessity: Note that every *CZ*-eigenpair (λ,μ) of $\mathcal{A}$ with $\lambda\ge\lambda_{i-1}+\delta$, the tms $[u]_{2k}$ is always feasible for (3.8). If the relaxation (3.8) is infeasible for some k, then $\mathcal{A}$ clearly has no *CZ*-eigenvalues $\ge\lambda_{i-1}+\delta$. Therefore, $\lambda_{i-1}+\delta>\lambda_{max}$.
Sufficiency is obvious.
(ii) If $\chi_i=\lambda_{i-1}$ and λ_i exists, then $\lambda_{i-1}+\delta<\lambda_i$. Therefore, δ satisfies (3.7).
(iii) From (i), we know $\lambda_{i-1}\le\lambda_{max}<\lambda_{i-1}+\delta$. Owing to $\chi_i=\lambda_{i-1}$, we get $\lambda_{max}=\lambda_{i-1}$, otherwise, $\chi_i=\lambda_{max}>\lambda_{i-1}$. This is a contradiction to $\chi_i=\lambda_{i-1}$.

□

3.3 The semidefinite relaxation algorithm

Let $Z(\mathcal{A})$ be the set of all the *CZ*-eigenvalues of $\mathcal{A}$. If $Z(\mathcal{A})$ is nonempty and finite, we can compute all the *CZ*-eigenvalues sequentially as follows. First, we compute the smallest one λ_1 by solving the hierarchy of semidefinite relaxations (3.4, 3.5). As shown in Theorem 3.1, this hierarchy converges in finitely many steps. After getting λ_1, we solve the hierarchy of (3.8–3.10) for $i=2$. If $\chi_2=\lambda_1$ and (3.8) is infeasible for some order k, then λ_1 is also the largest eigenvalue. If $\chi_2=\lambda_1$ and (3.8) is feasible for k big enough, then λ_2 is the 2-th smallest *CZ*-eigenvalue of $\mathcal{A}$. Otherwise, decrease the value δ as $\delta:=\frac{\delta}{5}$ and solve (3.6 and 3.10) again. After repeating this process for several times, we can always get $\chi_2=\lambda_1$, and the resulting δ satisfies (3.7). Repeating this procedure, we can get $\lambda_3,\lambda_4,\cdots$, if they exist, or we get the largest eigenvalue and stop.

Algorithm 3.1

Step 0. Choose a small positive value δ (e.g., 0.05). Set $i:=1$.

Step 1. Solve the hierarchy (3.4) and get the smallest *CZ*-eigenvalue λ_1.

Step 2. Set $i:=i+1$ and solve the hierarchy of (3.10). If $\chi_i=\lambda_{i-1}$, then go to Step 3; If $\chi_i>\lambda_{i-1}$, let $\delta:=\frac{\delta}{5}$ and compute (3.10). Repeat this process until (3.7) holds.

Step 3. Solve the hierarchy (3.8). If (3.8) is infeasible for some k, then λ_{i-1} is the largest eigenvalue and stop. Otherwise, we can get the next smallest eigenvalue λ_i.

Step 4. Go to Step 2.

In the following, we show the asymptotic and finite convergence of Algorithm 3.1.

Theorem 3.1 *Let $\mathcal{A} \in T^m(\mathbb{R}^n)$ and $Z(\mathcal{A})$ be the set of its CZ-eigenvalues. Then,*

(i) *The set $Z(\mathcal{A}) = \emptyset$ if and only if the semidefinite relaxation (3.4) is infeasible for some k.*

(ii) *If the set $Z(\mathcal{A}) \neq \emptyset$, then the smallest CZ-eigenvalue λ_1 always exists and*

$$\lim_{k\to\infty} \eta_k^{(1)} = \lim_{k\to\infty} \rho_k^{(1)} = \lambda_1. \tag{3.11}$$

In addition, if $Z(\mathcal{A})$ is finite, then for all k sufficiently large,

$$\eta_k^{(1)} = \rho_k^{(1)} = \lambda_1. \tag{3.12}$$

Suppose y^* is an optimal solution of (3.4). If there exists $t \in [k_0, k]$, such that

$$rankM_{t-k_0}(y^*) = rankM_t(y^*), \tag{3.13}$$

then there are $r := rankM_t(y^*)$ distinct CZ-eigenvectors associated with λ_1.

(iii) *For $i \geq 2$, suppose that λ_i exists and $0 < \delta < \lambda_i - \lambda_{i-1}$, then*

$$\lim_{k\to\infty} \eta_k^{(i)} = \lim_{k\to\infty} \rho_k^{(i)} = \lambda_i. \tag{3.14}$$

If the set $Z(\mathcal{A})$ is finite, then for all k sufficiently large,

$$\eta_k^{(i)} = \rho_k^{(i)} = \lambda_i.$$

Suppose y^* is an optimal solution of (3.8). If there exists $t \in [k_0, k]$, such that (3.13) holds, then there are $r := rankM_t(y^*)$ distinct CZ-eigenvectors associated with λ_i.

Proof

(i) Necessity: If $Z(\mathcal{A}) = \emptyset$, then the feasible set K is empty. By Positivstellensatz [1], $-1 \in I(h) + Pr(g)$. So, when k is big enough, $-1 \in I_{2k}(h) + Pr_k(g)$, and then the optimization (3.5) is unbounded from above. By duality theory, (3.4) must be infeasible, for all k big enough.

Sufficiency: Assume (3.4) is infeasible for some k. Then $\mathcal{A}$ has no CZ-eigenpairs. Otherwise, suppose (λ, u) is such one CZ-eigenpair. Then the tms $[u]_{2k}$ [see the notation in Sect. 2] is always feasible for (3.4), which is a contradiction. So $Z(\mathcal{A}) = \emptyset$.

(ii) Firstly, we prove the asymptotic convergence. Since $Z(\mathcal{A})$ is nonempty, then $\mathcal{A}$ has at least one CZ-eigenvalue. So λ_1 always exists. Note that $x^T x - 1$ is a polynomial in the tuple h, then $-(x^Tx-1)^2 \in I(h)$ and the set $-(x^Tx-1)^2 \geq 0$ is compact. The ideal $I(h)$ is archimedean, which implies that $I(h)+Q(g)$ is also archimedean. So K is compact, then $\{\eta_k^{(1)}\}$ asymptotically converges to λ_1 (cf. [11]). Therefore, the asymptotic convergence (3.11) is obtained.

Next, we prove the finite convergence. Since $Z(\mathcal{A})$ is finite, let $Z(\mathcal{A}) = \{\lambda_1, \lambda_2, \cdots, \lambda_L\}$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_L$. Let $\varphi_1, \varphi_2, \cdots, \varphi_L$ be the univariate real polynomials in t such that $\varphi_i(\lambda_j) = 0$ when $i \neq j$ and $\varphi_i(\lambda_j) = 1$ when $i = j$. For $i = 1, 2, \cdots, L$, let

$$a_i := (\lambda_i - \lambda_1)(\varphi_i(f(x)))^2.$$

Let $a := a_1 + \cdots + a_L$, then $a \in \Sigma[x]$. The polynomial

$$\hat{f} := f - \lambda_1 - a$$

vanishes identically on K. By Positivstellensatz (cf. [1], Corollary 4.4.3), there exists integers $\ell > 0$ and $N_1 > 0$ such that

$$q \in Pr_{N_1}(g), \hat{f}^{2\ell} + q \in I_{N_1}(h),$$

where $Pr_{N_1}(g)$ denotes the N_1-th truncated preordering generated by the tuple g (cf. [1]). For all $\varepsilon > 0$ and $c > 0$, we can write $\hat{f} + \varepsilon = \phi_\varepsilon + \theta_\varepsilon$, where

$$\phi_\varepsilon = -c\varepsilon^{1-2\ell}(\hat{f}^{2\ell} + q),$$
$$\theta_\varepsilon = \varepsilon(1 + \hat{f}/\varepsilon + c(\hat{f}/\varepsilon)^{2\ell}) + c\varepsilon^{1-2\ell}q.$$

By Lemma 2.1 of [19], when $c \geq \frac{1}{2\ell}$, there exists $N \geq N_1$ such that for all $\varepsilon > 0$,

$$\phi_\varepsilon \in I_{2N}(h), \theta_\varepsilon \in Pr_N(g).$$

Therefore, we have

$$f - (\lambda_1 - \varepsilon) = \phi_\varepsilon + \sigma_\varepsilon,$$

where $\sigma_\varepsilon = \theta_\varepsilon + a \in Pr_N(g)$ for all $\varepsilon > 0$. This implies that for all $\varepsilon > 0$, $\gamma = \lambda_1 - \varepsilon$ is feasible in (3.5) for the order N. Thus, we get $\eta_N^{(1)} \geq \lambda_1$. Note that $\eta_k^{(1)} \leq \rho_k^{(1)} \leq \lambda_1$ for all k and the sequence $\{\eta_k^{(1)}\}$ is monotonically increasing. So, (3.12) must be true for all $k \geq N$.

Note that $L_h^{(t)}(y^*) = 0, M_t(y^*) \succeq 0, L_g^{(t)}(y^*) \succeq 0$ $(t \leq k)$. When (3.13) is satisfied, there exist $r := rankM_t(y^*)$ distinct vectors $u_1, \cdots, u_r \in K$ and scalars $c_1, \cdots, c_r$ [20] such that

$$y^*|_{2t} = c_1[u_1]_{2t} + \cdots + c_r[u_r]_{2t},$$
$$c_1 > 0, \cdots, c_r > 0.$$

The constraint $\langle 1, y^* \rangle = 1$ implies $c_1 + \cdots + c_r = 1$. We have

$$c_1 f(u_1) + \cdots + c_r f(u_r) = \langle f, y^*|_{2t} \rangle = \langle f, y^* \rangle = \rho_k^{(1)} \leq \lambda_1.$$

Since every $f(u_i) \geq \lambda_1$, then we have

$$f(u_i) = \lambda_1, i = 1, \cdots, r.$$

So each u_i is a CZ-eigenvector associated to $\rho_k^{(1)} = f(u_1) = \cdots = f(u_r) = \lambda_1$.

(iii) For $i \geq 2$, if $0 < \delta < \lambda_i - \lambda_{i-1}$ holds, the optimal value of (3.6) is equal to λ_i. The rest of the proof is the similar to that of Theorem 3.1(ii).

□

4 Numerical experiments

In this section, we present the numerical experiments to show how to compute all the CZ-eigenvalues of tensors. The Lasserre's semidefinite relaxations are solved by the software GloptiPoly 3 [8] and SeDuMi [30]. The program is coded in MATLAB (2016a). The experiments are implemented on a personal PC with 2.5 GHz and 2.7 GHz, 8.0 GB RAM, using Windows 10 operation system.

Example 4.1 ([31]). A tensor $\mathcal{A} \in T^3(\mathbb{R}^3)$ is defined by

$$\begin{array}{l} a_{111} = a_{333} = a_{121} = a_{231} = 1, a_{222} = 2, \\ a_{ijk} = 0, \mathit{elsewhere}. \end{array} \tag{4.1}$$

We apply Algorithm 3.1 and get four CZ-eigenvalues and the associated CZ-eigenvectors:

$$\begin{aligned} \lambda_1 &= 0.8944, u_1 = (0.0000, 0.4472, 0.8944), \\ \lambda_2 &= 1.0000, u_2 = (0.0000, 0.0001, 1.0000), \\ &\qquad\qquad\qquad (1.0000, 0.0000, 0.0000), \\ \lambda_3 &= 1.4142, u_3 = (0.7071, 0.7071, 0.0000), \\ \lambda_4 &= 2.0000, u_4 = (0.0000, 1.0000, 0.0000). \end{aligned}$$

The computation takes about 3 s.

Example 4.2 Consider the tensor $\mathcal{A} \in T^3(\mathbb{R}^n)$ such that

$$a_{ijk} = \frac{(-1)^j}{i} + \frac{(-1)^k}{j} + \frac{(-1)^i}{k}.$$

For $n = 3$, we apply Algorithm 3.1 and get seven CZ-eigenvalues and the corresponding CZ-eigenvectors:

$$\lambda_1 = -6.0565,\ u_1 = (0.7996, 0.1172, 0.5889),$$
$$\lambda_2 = -5.8821,\ u_2 = (0.8090, 0.0000, 0.5878),$$
$$\lambda_3 = -3.0725,\ u_3 = (0.9955, 0.0952, 0.0000),$$
$$\lambda_4 = -3.0000,\ u_4 = (1.0000, 0.0000, 0.0000),$$
$$\lambda_5 = -1.1791,\ u_5 = (0.0000, 0.2203, 0.9754),$$
$$\lambda_6 = -1.0000,\ u_6 = (0.0000, 0.0000, 1.0000),$$
$$\lambda_7 = \ 1.9273,\ u_7 = (0.2554, 0.9668, 0.0000).$$

The computation takes about 5 s.

For $n = 4$, we apply Algorithm 3.1 and get eight CZ-eigenvalues and the corresponding CZ-eigenvectors:

$$\lambda_1 = -6.0670,\ u_1 = (0.7997, 0.1089, 0.5897, 0.0302),$$
$$\lambda_2 = -5.9269,\ u_2 = (0.8065, 0.0000, 0.5881, 0.0609),$$
$$\lambda_3 = -3.0732,\ u_3 = (0.9958, 0.0907, 0.0000, 0.0108),$$
$$\lambda_4 = -3.0186,\ u_4 = (0.9988, 0.0000, 0.0000, 0.0494),$$
$$\lambda_5 = -1.1802, u_5 = (0.0000, 0.2111, 0.9773, 0.0201),$$
$$\lambda_6 = -1.0523,\ u_6 = (0.0000, 0.0000, 0.9908, 0.1353),$$
$$\lambda_7 = -1.0000,\ u_7 = (0.0000, 0.0000, 1.0000, 0.0000),$$
$$\lambda_8 = \ 4.2185,\ u_8 = (0.2902, 0.7258, 0.0175, 0.6235).$$

The computation takes about 10 s.

Example 4.3 Consider the diagonal tensor $\mathcal{A} \in S^4(\mathbb{R}^3)$ such that

$$a_{1111} = 1, a_{2222} = 2, a_{3333} = 3,$$
$$a_{ijkl} = 0, elsewhere.$$

We apply Algorithm 3.1 and get seven CZ-eigenvalues and the associated CZ-eigenvectors:

$$\lambda_1 = 0.5454, u_1 = (0.7385, 0.5222, 0.4264),$$
$$\lambda_2 = 0.6667, u_2 = (0.8165, 0.5773, 0.0001),$$
$$\lambda_3 = 0.7500, u_3 = (0.8660, 0.0000, 0.5000),$$
$$\lambda_4 = 1.0000, u_4 = (1.0000, 0.0000, 0.0000),$$
$$\lambda_5 = 1.2000, u_5 = (0.0000, 0.7746, 0.6325),$$
$$\lambda_6 = 2.0000, u_6 = (0.0000, 1.0000, 0.0000),$$
$$\lambda_7 = 3.0000, u_7 = (0.0000, 0.0000, 1.0000).$$

The computation takes about 5 s.

Example 4.4 Consider the tensor $\mathcal{A} \in T^3(\mathbb{R}^3)$ such that

$$a_{ijk} = tan(i - \frac{j}{2} + \frac{k}{3}).$$

We apply Algorithm 3.1 and get one *CZ*-eigenvalue and the corresponding *CZ*-eigenvector:

$$\lambda = 1.1008, u = (1.0000, 0.0000, 0.0000).$$

The computation takes about 1 s.

Example 4.5 Consider the tensor $\mathcal{A} \in T^3(\mathbb{R}^3)$ such that

$$a_{ijk} = \frac{1}{10}(i + 2j + 3k - \sqrt{i^2 + j^2 + k^2}).$$

Using Algorithm 3.1, we get one *CZ*-eigenvalue and the corresponding *CZ*-eigenvector:

$$\lambda = 4.4033, u = (0.5445, 0.5810, 0.6050).$$

The computation takes about 1 s.

5 Conclusions

In this paper, we propose the semidefinite relaxation algorithm for computing all the *CZ*-eigenpairs of tensor that has finitely many *CZ*-eigenvalues, and prove its asymptotic and finite convergence. Numerical experiments demonstrate the efficiency of the proposed algorithm.

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