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Weak convergence of iterative methods for solving quasimonotone variational inequalities

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Abstract
In this work, we introduce self-adaptive methods for solving variational inequalities with Lipschitz continuous and quasimonotone mapping(or Lipschitz continuous mapping without monotonicity) in real Hilbert space. Under suitable assumptions, the convergence of algorithms are established without the knowledge of the Lipschitz constant of the mapping. The results obtained in this paper extend some recent results in the literature. Some preliminary numerical experiments and comparisons are reported.

Keywords Variational inequalities · Projection · Gradient method · Quasimonotone mapping · Convex set

Mathematics Subject Classification 47J20 · 90C25 · 90C30 · 90C52

1 Introduction

Let C be a nonempty closed and convex set in a real Hilbert space H and $F : H \longrightarrow H$ is a continuous mapping, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the induced norm in H, respectively. The variational inequality ($VI(C, F)$) is

$$\text{find} \quad x^* \in C \quad \text{such that} \quad \langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1}$$

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Weak converge of the sequence $\{x_n\}$ to a point x is denoted by $x_n \rightharpoonup x$ while $x_n \to x$ to denote the sequence $\{x_n\}$ converges strongly to x. Let S be the solution set of (1) and S_D be the solution set of the following problem,

$$\text{find} \quad x^* \in C \quad \text{such that} \quad \langle F(y), y - x^* \rangle \geqslant 0, \quad \forall y \in C. \tag{2}$$

It is obvious that S_D is a closed convex set (possibly empty). As F is continuous and C is convex, we get

$$S_D \subseteq S. \tag{3}$$

If F is a pseudomonotone and continuous mapping, then $S = S_D$(see Lemma 2.1 in [1]). The inclusion $S \subset S_D$ is false, if F is a quasimonotone and continuous mapping [2]. For solving quasimonotone variational inequalities, the convergence of interior proximal algorithm [3, 4] was obtained under more assumptions than $S_D \neq \emptyset$. Under the assumption of $S_D \neq \emptyset$, Ye and He [2] proposed a double projection algorithm for solving quasimonotone (or without monotonicity) variational inequalities in R^n. For a nonempty closed and convex set $C \subseteq H$, P_C is called the projection from H onto C, that is, for every element $x \in H$ such that $\|x - P_C(x)\| = min\{\| y - x \| | y \in C\}$. It is easy to check that problem (1) is equivalent to the following fixed point problem:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad x^* = P_C(x^* - \lambda F(x^*)) \tag{4}$$

for any $\lambda > 0$. Various projection algorithms have been proposed and analyzed for solving variational inequalities [2, 5–27]. Among them, the extragradientmethod was proposed by Korpelevich [5] and Antipin [18], that is

$$y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = P_C(x_n - \lambda F(y_n)), \tag{5}$$

where $\lambda \in (0, \frac{1}{L})$ and L is the Lipschitz constant of F. Tseng [8] modified the extragradient method with the following method

$$y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = y_n + \lambda(F(x_n) - F(y_n)), \tag{6}$$

where $\lambda \in (0, \frac{1}{L})$. Recently, Censor et al.[7] introduced the following subgradient extragradient algorithm

$$y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)), \tag{7}$$

where $T_n = \{x \in H | \langle x_n - \lambda F(x_n) - y_n, x - y_n \rangle \leq 0\}$ and $\lambda \in (0, \frac{1}{L})$. In the original papers, the methods (5), (6) and (7) were applied for solving monotone variational inequalities. It is known that these methods can be applied for solving pseudomonotone variational inequalities in infinite dimensional Hilbert spaces [22]. Very recently, Yang et al. [19–21] proposed modifications of gradient methods for solving monotone variational inequalities with the new step size rules. But the step sizes are non-increasing and the algorithms [19–21] may depend on the choice of initial step size. The natural question is whether the gradient algorithms still hold using non-monotonic step sizes for solving quasimonotone variational inequalities (or without monotonicity). The goal of this paper is to give an answer to this question.

The paper is organized as follows. In Sect. 2 , we first give some preliminaries that will be needed in the sequel. In Sect. 3 , we present algorithms and analyze their convergence. Finally, in Sect. 4 we provides numerical examples and comparisons.

2 Preliminaries

In this section, we give some concepts and results for further use.

Definition 2.1 A mapping $F : H \longrightarrow H$ is said to be as follows:

(a) *γ-strongly monotone* on H if there exists a constant $\gamma > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \gamma \parallel x - y \parallel^2, \quad \forall x, y \in H. \tag{8}$$

(b) *monotone* on H if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in H. \tag{9}$$

(c) *pseudomonotone* on H, if

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in H. \tag{10}$$

(d) *quasimonotone* on H, if

$$\langle F(y), x - y\rangle > 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \forall x, y \in H. \tag{11}$$

(e) *Lipschitz-continuous* on H, if there exists $L > 0$ such that

$$\parallel F(x) - F(y) \parallel \leq L \parallel x - y \parallel, \quad \forall x, y \in H. \tag{12}$$

From the above definitions, we see that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. But the converses are not true.

In this paper, we assume that the following conditions hold

$(A1)$ $S_D \neq \emptyset$.
$(A2)$ The mapping F is Lipschitz-continuous with constant $L > 0$.
$(A3)$ The mapping F is sequentially weakly continuous, i.e., for each sequence $\{x_n\}$: $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $F(x)$.
$(A4)$ The mapping F is quasimonotone on H.
$(A4')$ If $x_n \rightharpoonup \bar{x}$ and $\limsup_{n\to\infty}\langle F(x_n), x_n\rangle \leq \langle F(\bar{x}), \bar{x}\rangle$, then $\lim\limits_{n\to\infty}\langle F(x_n), x_n\rangle = \langle F(\bar{x}), \bar{x}\rangle$.
$(A5)$ The set $A = \{z \in C : F(z) = 0\} \setminus S_D$ is a finite set.
$(A5')$ The set $B = S \setminus S_D$ is a finite set.

Remark 2.1

(i) If $x_n \rightharpoonup \overline{x}$ and the function $g(x) = \langle F(x), x\rangle$ is weak lower semicontinuous(i.e., $\liminf_{n\to\infty}\langle F(x_n), x_n\rangle \geq \langle F(\overline{x}), \overline{x}\rangle$ for every sequence $\{x_n\}$ converges weakly to $\overline{x}$), then F satisfies $(A4')$.

(ii) If $x_n \rightharpoonup \overline{x}$ and F is sequentially weakly-strongly continuous(i.e., $x_n \rightharpoonup \overline{x} \Rightarrow F(x_n) \to F(\overline{x})$), then F satisfies $(A4')$. Indeed, since $x_n \rightharpoonup \overline{x}$ and F is sequentially weakly-strongly continuous, we obtain $\lim_{n\to\infty}\langle F(\overline{x}), x_n\rangle = \langle F(\overline{x}), \overline{x}\rangle$ and $\lim_{n\to\infty}\|F(x_n) - F(\overline{x})\| = 0$. Thus, we have

$$\lim_{n\to\infty}(\langle F(x_n), x_n\rangle - \langle F(\overline{x}), \overline{x}\rangle) = \lim_{n\to\infty}\langle F(x_n) - F(\overline{x}), x_n\rangle + \lim_{n\to\infty}(\langle F(\overline{x}), x_n\rangle - \langle F(\overline{x}), \overline{x}\rangle) = 0.$$

That is, $\limsup_{n\to\infty}\langle F(x_n), x_n\rangle = \lim_{n\to\infty}\langle F(x_n), x_n\rangle = \langle F(\overline{x}), \overline{x}\rangle$.

(iii) If $x_n \rightharpoonup \overline{x}$ and F is sequentially weakly continuous and monotone mapping, we get

$$\langle F(x_n) - F(\overline{x}), x_n - \overline{x}\rangle \geq 0.$$

Which means that

$$\langle F(x_n), x_n\rangle \geq \langle F(x_n), \overline{x}\rangle + \langle F(\overline{x}), x_n\rangle - \langle F(\overline{x}), \overline{x}\rangle.$$

Let $n \to \infty$ in the last inequality, we have $\limsup_{n\to\infty}\langle F(x_n), x_n\rangle \geq \langle F(\overline{x}), \overline{x}\rangle$. We have F satisfies $(A4')$.

Remark 2.2 If $intC \neq \emptyset$, F is a continuous and quasimonotone mapping, then the condition $(A5)$ is equivalent to $(A5')$. It is easy to see that $(A5') \Rightarrow (A5)$. On the other hand, from $intC \neq \emptyset$, we obtain $\{z \in C : F(z) = 0\} = \{z \in C : \langle F(z), y - z\rangle = 0, \forall y \in C\}$ (see Theorem 2.1 in [2]). Since F is a continuous and quasimonotone mapping, we have $S \setminus \{z \in C : \langle F(z), y - z\rangle = 0, \forall y \in C\} \subset S_D$ (see Lemma 2.7 in [2]). Hence $S \setminus \{z \in C : F(z) = 0\} \subset S_D \subset S$. It follow that $S \setminus S_D \subset \{z \in C : F(z) = 0\} \setminus S_D$. By $(A5)$, we get $\{z \in C : F(z) = 0\} \setminus S_D$ is a finite set. This implies that (A5′) holds.

Lemma 2.1 [2] *If either*

(i) *F is pseudomonotone on C and $S \neq \emptyset$;*

(ii) *F is the gradient of G, where G is a differentiable quasiconvex function on anopen set $K \supset C$ and attains its global minimum on C;*

(iii) *F is quasimonotone on C, $F \neq 0$ on C and C is bounded;*

(iv) *F is quasimonotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle F(x), y - x\rangle \leq 0$;*

(v) *F is quasimonotone on C, $intC$ is nonempty and there exists $x^* \in S$ such that $F(x^*) \neq 0$. Then S_D is nonempty.*

Proof See Proposition 2.1 in [2] and Proposition 1 in [28]. □

Lemma 2.2 *Let C be a nonempty, closed and convex set in H and* $x \in H$. *Then*

$$\langle P_C x - x, y - P_C x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 (Opial) *For any sequence* $\{x_n\}$ *in H such that* $x_n \rightharpoonup x$, *then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \neq x. \tag{13}$$

3 Main results

First, we give a iterative algorithm for solving variational inequality.

Algorithm 3.1 (Step 0) Take $\lambda_0 > 0$, $x_0 \in H, 0 < \mu < 1$. Choose a nonnegative real sequence $\{p_n\}$ such that $\sum_{n=0}^{\infty} p_n < +\infty$.

(Step 1) Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n F(x_n)). \tag{14}$$

If $x_n = y_n$ (or $F(y_n) = 0$), then stop: y_n is a solution. Otherwise,

(Step 2) Compute

$$x_{n+1} = y_n + \lambda_n(F(x_n) - F(y_n)) \tag{15}$$

and

$$\lambda_{n+1} = \begin{cases} min\left\{\frac{\mu\|x_n - y_n\|}{\|F(x_n) - F(y_n)\|}, \lambda_n + p_n\right\}, & if \quad F(x_n) - F(y_n) \neq 0, \\ \lambda_n + p_n, & otherwise. \end{cases} \tag{16}$$

Set $n := n + 1$ and return to step 1.

Remark 3.1 If Algorithm 3.1 stops in a finite step of iterations, then y_n is a solution of the variational inequality. So in the rest of this section, we assume that the Algorithm 3.1 does not stop in any finite iterations, and hence generates an infinite sequence.

Remark 3.2 In numerical experiments, the condition $A5$(or $A5'$) does not need to be considered. Indeed, if $\|y_n - x_n\| < \epsilon$, Algorithm 3.1 terminates in a finite step of iterations. In the process of proving the conclusion $\lim_{n\to\infty} \|x_n - y_n\| = 0$, the condition $A5$(or $A5'$) is not used (see(26) in Lemma 3.2).

Lemma 3.1 *Let* $\{\lambda_n\}$ *be the sequence generated by Algorithm* 3.1. Then we get $\lim_{n\to\infty} \lambda_n = \lambda$ *and* $\lambda \in [\min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + P]$. *Where* $P = \sum_{n=0}^{\infty} p_n$.

Proof Since F is Lipschitz-continuous with constant $L > 0$, in the case of $F(y_n) - F(x_n) \neq 0$, we get

$$\frac{\mu\|x_n - y_n\|}{\|F(x_n) - F(y_n)\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}. \tag{17}$$

By the definition of λ_{n+1} and mathematical induction, then the sequence $\{\lambda_n\}$ has upper bound $\lambda_0 + P$ and lower bound $\min\{\frac{\mu}{L}, \lambda_0\}$. Let $(\lambda_{n+1} - \lambda_n)^+ = \max\{0, \lambda_{n+1} - \lambda_n\}$ and $(\lambda_{n+1} - \lambda_n)^- = \max\{0, -(\lambda_{n+1} - \lambda_n)\}$. From the definition of $\{\lambda_n\}$, we have

$$\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+ \leq \sum_{n=0}^{\infty} p_n < +\infty. \tag{18}$$

That is, the series $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+$ is convergence. Next we prove the convergence of the series $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^-$. Assume that $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^- = +\infty$. From the fact that

$$\lambda_{n+1} - \lambda_n = (\lambda_{n+1} - \lambda_n)^+ - (\lambda_{n+1} - \lambda_n)^-, \tag{19}$$

we get

$$\lambda_{k+1} - \lambda_0 = \sum_{n=0}^{k}(\lambda_{n+1} - \lambda_n) = \sum_{n=0}^{k}(\lambda_{n+1} - \lambda_n)^+ - \sum_{n=0}^{k}(\lambda_{n+1} - \lambda_n)^-. \tag{20}$$

Taking $k \to +\infty$ in (20), we have $\lambda_k \to -\infty (k \to \infty)$. That is a contradiction. From the convergence of the series $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+$ and $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^-$, taking $k \to +\infty$ in (20), we obtain $\lim_{n\to\infty} \lambda_n = \lambda$. Since $\{\lambda_n\}$ has the lower bound $\min\{\frac{\mu}{L}, \lambda_0\}$ and the upper bound $\lambda_0 + P$, we have $\lambda \in [\min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + P]$. □

Remark 3.3 The step size in Algorithm 3.1 is allowed to increase from iteration to iteration and so Algorithm 3.1 reduces the dependence on the initial step size λ_0. Since the sequence $\{p_n\}$ is summable, we get $\lim_{n\to\infty} p_n = 0$. So the step size λ_n may non-increasing when n is large. If $p_n \equiv 0$, then the step size in Algorithm 3.1 is similar to the methods in [19, 21].

Lemma 3.2 *Under the conditions* (A1) *and* (A2). *Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *is bounded and* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof Let $u \in S_D$, since $x_{n+1} = y_n + \lambda_n(F(x_n) - F(y_n))$, we obtain

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|y_n-\lambda_n(F(y_n)-F(x_n))-u\|^2\\ &= \|y_n-u\|^2+\lambda_n^2\|F(y_n)-F(x_n)\|^2-2\lambda_n\langle F(y_n)-F(x_n),y_n-u\rangle\\ &= \|x_n-u\|^2+\|y_n-x_n\|^2+2\langle y_n-x_n,x_n-u\rangle\\ &\quad+\lambda_n^2\|F(y_n)-F(x_n)\|^2-2\lambda_n\langle F(y_n)-F(x_n),y_n-u\rangle\\ &= \|x_n-u\|^2+\|y_n-x_n\|^2-2\langle y_n-x_n,y_n-x_n\rangle+2\langle y_n-x_n,y_n-u\rangle\\ &\quad+\lambda_n^2\|F(y_n)-F(x_n)\|^2-2\lambda_n\langle F(y_n)-F(x_n),y_n-u\rangle.\end{aligned} \tag{21}$$

Note that $y_n=P_C(x_n-\lambda_nF(x_n))$ and $u\in S_D\subseteq S\subseteq C$, by Lemma 2.2, we obtain $\langle y_n-x_n+\lambda_nF(x_n),y_n-u\rangle\le 0$. It follows that

$$\langle y_n-x_n,y_n-u\rangle\le-\lambda_n\langle F(x_n),y_n-u\rangle. \tag{22}$$

Using $y_n\in C$ and $u\in S_D$, we get $\langle F(y_n),y_n-u\rangle\ge 0,\ \forall n\ge 0$. From (16), (21) and (22), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 \le&\|x_n-u\|^2-\|y_n-x_n\|^2-2\lambda_n\langle F(x_n),y_n-u\rangle\\ &+\lambda_n^2\|F(y_n)-F(x_n)\|^2-2\lambda_n\langle F(y_n)-F(x_n),y_n-u\rangle\\ =&\|x_n-u\|^2-\|y_n-x_n\|^2+\lambda_n^2\|F(y_n)-F(x_n)\|^2-2\lambda_n\langle y_n-u,F(y_n)\rangle\\ \le&\|x_n-u\|^2-\|y_n-x_n\|^2+\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n-x_n\|^2.\end{aligned} \tag{23}$$

Since

$$\lim_{n\to\infty}\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)=1-\mu^2>0, \tag{24}$$

we obtain $\exists N\ge 0,\ \forall n\ge N$, such that $1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}>0$.

It implies that $\forall n\ge N, \|x_{n+1}-u\|\le\|x_n-u\|$. This implies that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-u\|$ exists. Returning to (23), we have

$$\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-x_n\|^2\le\|x_n-u\|^2-\|x_{n+1}-u\|^2. \tag{25}$$

This implies that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{26}$$

Observe that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|y_n+\lambda_n(F(x_n)-F(y_n))-x_n\|\\ &\le\|y_n-x_n\|+(\lambda_0+P)L\|y_n-x_n\|,\end{aligned}$$

we obtain $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. □

Remark 3.4 We note that the quasimonotonicity of the mapping is not used in the proof of the Lemma 3.2. In finite dimensional Hilbert space, under the conditions ($A1$) and ($A2$), then all the accumulation points of $\{x_n\}$ belong to S. Indeed, the existence of the accumulation points can be obtained by boundedness of the $\{x_n\}$. If x^* is an accumulation point of $\{x_n\}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some $x^* \in C$. In view of the fact that $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k} F(x_{n_k}))$ and the continuity of F, we get

$$x^* = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} P_C(x_{n_k} - \lambda_{n_k} F(x_{n_k})) = P_C(x^* - \lambda F(x^*)).$$

We deduce from (4) that $x^* \in S$.

Lemma 3.3 *Assume that (A1)–(A4) hold. Let* $\{x_n\}$ *be the sequence generated by Algorithm 3.1. Then at least one of the following must hold* : $x^* \in S_D$ or $F(x^*) = 0$. Where x^* is one of the weak cluster point of $\{x_{\rm n}\}$.

Proof By Lemma 3.2, we have the sequence $\{x_n\}$ is bounded. Moreover, there exist a subsequence $\{x_{n_k}\}$ that converges weakly to $x^* \in H$. Using (26), we obtain $y_{n_k} \rightharpoonup x^*$ and $x^* \in C$. We divide the following proof into two cases.

Case 1 If $\limsup_{k\to\infty} \|F(y_{n_k})\| = 0$, we have $\lim\limits_{k\to\infty} \|F(y_{n_k})\| = \liminf\limits_{k\to\infty} \|F(y_{n_k})\| = 0$. Since $\{y_{n_k}\}$ converges weakly to $x^* \in C$ and F is sequentially weakly continuous on C, we have $\{F(y_{n_k})\}$ converges weakly to $F(x^*)$. Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 \le \|F(x^*)\| \le \liminf_{k\to\infty} \|F(y_{n_k})\| = 0.$$

We obtain $F(x^*) = 0$.

Case 2 If $\limsup_{k\to\infty} \|F(y_{n_k})\| > 0$, without loss of generality, we take $\lim\limits_{k\to\infty} \|F(y_{n_k})\| = M > 0$(Otherwise, we take a subsequence of $\{F(y_{n_k})\}$). That is, there exists a $K \in \mathbb{N}$ such that $\|F(y_{n_k})\| > \frac{M}{2}$ for all $k \ge K$. Since $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k} F(x_{n_k}))$ and Lemma 2.2, we have

$$\langle y_{n_k} - x_{n_k} + \lambda_{n_k} F(x_{n_k}), z - y_{n_k} \rangle \ge 0, \quad \forall z \in C.$$

That is

$$\langle x_{n_k} - y_{n_k}, z - y_{n_k} \rangle \le \lambda_{n_k} \langle F(x_{n_k}), z - y_{n_k} \rangle, \quad \forall z \in C.$$

Therefore, we get

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, z - y_{n_k} \rangle - \langle F(x_{n_k}) - F(y_{n_k}), z - y_{n_k} \rangle \le \langle F(y_{n_k}), z - y_{n_k} \rangle, \quad \forall z \in C. \tag{27}$$

Fixing $z \in C$, let $k \to \infty$, using the facts $\lim\limits_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, $\{y_{n_k}\}$ is bounded and $\lim\limits_{k\to\infty} \lambda_{n_k} = \lambda > 0$, we obtain

$$0 \leq \liminf_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle \leq \limsup_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle < +\infty. \tag{28}$$

If $\limsup_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle > 0$, there exists a subsequence $\{y_{n_{k_i}}\}$ such that $\lim_{i\to\infty}\langle F(y_{n_{k_i}}), z - y_{n_{k_i}}\rangle > 0$. Then there exists an $i_0 \in \mathbb{N}$ such that $\langle F(y_{n_{k_i}}), z - y_{n_{k_i}}\rangle > 0$ for all $i \geq i_0$. By the definition of quasidomonotone, we obtain $\forall i \geq i_0$,

$$\langle F(z), z - y_{n_{k_i}}\rangle \geq 0.$$

Let $i \to \infty$, we get $x^* \in S_D$.

If $\limsup_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle = 0$, we deduce from (28) that

$$\lim_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle = \limsup_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle = \liminf_{k\to\infty}\langle F(y_{n_k}), z - y_{n_k}\rangle = 0.$$

Let $\varepsilon_k = |\langle F(y_{n_k}), z - y_{n_k}\rangle| + \frac{1}{k+1}$. This implies that

$$\langle F(y_{n_k}), z - y_{n_k}\rangle + \varepsilon_k > 0. \tag{29}$$

Let $z_{n_k} = \frac{F(y_{n_k})}{\|F(y_{n_k})\|^2}$ $(\forall k \geq K)$, we get $\langle F(y_{n_k}), z_{n_k}\rangle = 1$. Moreover, from (29), we have $\forall k > K$,

$$\langle F(y_{n_k}), z + \varepsilon_k z_{n_k} - y_{n_k}\rangle > 0.$$

By the definition of quasidomonotone, we obtain $\forall k > K$,

$$\langle F(z + \varepsilon_k z_{n_k}), z + \varepsilon_k z_{n_k} - y_{n_k}\rangle \geq 0. \tag{30}$$

This implies that $\forall k > K$,

$$\begin{aligned}
\langle F(z), z + \varepsilon_k z_{n_k} - y_{n_k}\rangle =& \langle F(z) - F(z + \varepsilon_k z_{n_k}), z + \varepsilon_k z_{n_k} \\
&- y_{n_k}\rangle + \langle F(z + \varepsilon_k z_{n_k}), z + \varepsilon_k z_{n_k} - y_{n_k}\rangle \\
\geq& \langle F(z) - F(z + \varepsilon_k z_{n_k}), z + \varepsilon_k z_{n_k} - y_{n_k}\rangle \\
\geq& -\|F(z) - F(z + \varepsilon_k z_{n_k})\|\|z + \varepsilon_k z_{n_k} - y_{n_k}\| \\
\geq& -\varepsilon_k L\|z_{n_k}\|\|z + \varepsilon_k z_{n_k} - y_{n_k}\| \\
=& -\varepsilon_k \frac{L}{\|F(y_{n_k})\|}\|z + \varepsilon_k z_{n_k} - y_{n_k}\| \\
\geq& -\varepsilon_k \frac{2L}{M}\|z + \varepsilon_k z_{n_k} - y_{n_k}\|.
\end{aligned} \tag{31}$$

Let $k \to \infty$ in (31), using the fact $\lim_{k\to\infty}\varepsilon_k = 0$ and boundedness of $\{\|z + \varepsilon_k z_{n_k} - y_{n_k}\|\}$, we get

$$\langle F(z), z - x^*\rangle \geq 0. \quad \forall z \in C.$$

This implies that $x^* \in S_D$. □

Lemma 3.4 *Assume that* $(A1)-(A5)$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 has finite weak cluster points in* S.

Proof First we prove $\{x_n\}$ has at most one weak cluster point in S_D.

Assume that $\{x_n\}$ has at least two weak cluster points $x^* \in S_D$ and $\bar{x} \in S_D$ such that $x^* \neq \bar{x}$. Let $\{x_{n_i}\}$ be a sequence such that $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$, noting the fact that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in S_D$, by Lemma 2.3, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - \bar{x}\| \\
&\quad = \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| = \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| \\
&\quad < \liminf_{i\to\infty} \|x_{n_i} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\| = \lim_{k\to\infty} \|x_{n_k} - x^*\| = \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&\quad < \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned} \tag{32}$$

which is impossible. Since $A = \{z \in C : F(z) = 0\} \setminus S_D$ is a finite set, using Lemma 3.3, then $\{x_n\}$ has finite weak cluster points in S. □

Lemma 3.5 *Assume that* $(A1)$–$(A5)$ *hold and* $\{x_n\}$ *has finite weak cluster points* $z^1, z^2, \dots z^m$. *Then There exists* $N_1 > N$, $\forall n \geq N_1$, *such that* $x_n \in B$. *Where* $B = \bigcup_{j=1}^m B^j$, $B^l = \bigcap_{j=1, j\neq l}^m \{x : \langle x, \frac{z^l - z^j}{\|z^l - z^j\|}\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|}\}$ *and* $\epsilon_0 = min\{\frac{\|z^l - z^j\|}{4} : l, j \in \{1, 2, \dots, m\}, l \neq j\}$.

Proof Let $\{x_{n_i}^l\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i}^l \rightharpoonup z^l$ as $\to \infty$, we get $\forall\, j \neq l$

$$\lim_{i\to\infty} \langle x_{n_i}^l, z^l - z^j\rangle = \langle z^l, z^l - z^j\rangle. \tag{33}$$

For $j \neq l$, one has

$$\begin{aligned}
\langle z^l, z^l - z^j\rangle =& \|z^l\|^2 - \langle z^l, z^j\rangle = \frac{\|z^l - z^j\|^2}{2} + \frac{\|z^l\|^2}{2} - \frac{\|z^j\|^2}{2} \\
&> \frac{\|z^l - z^j\|^2}{4} + \frac{\|z^l\|^2}{2} - \frac{\|z^j\|^2}{2}.
\end{aligned} \tag{34}$$

This implies that $\forall\, j \neq l$

$$\left\langle z^l, \frac{z^l - z^j}{\|z^l - z^j\|} \right\rangle > \frac{\|z^l - z^j\|}{4} + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|}. \tag{35}$$

From (33) and (35), when i is sufficiently large, we have $x_{n_i}^l \in \{x : \langle x, \frac{z^l - z^j}{\|z^l - z^j\|}\rangle > \frac{\|z^l - z^j\|}{4} + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|}\}$. Therefore, when i is sufficiently large, we have

$$x_{n_i}^l \in B^l,$$

where

$$B^l = \bigcap_{j=1, j\neq l}^{m} \left\{ x : \left\langle x, \frac{z^l - z^j}{\|z^l - z^j\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|} \right\} \tag{36}$$

and $\epsilon_0 = min\left\{ \frac{\|z^l - z^j\|}{4} : l, j \in \{1, 2, \dots, m\}, l \neq j \right\}$. It is obvious that

$$B^l = \bigcap_{j=1, j\neq l}^{m} \left\{ x : \left\langle -x, \frac{z^l - z^j}{\|z^j - z^l\|} \right\rangle < -\epsilon_0 + \frac{\|z^j\|^2 - \|z^l\|^2}{2\|z^l - z^j\|} \right\}. \tag{37}$$

Set $B = \bigcup_{j=1}^{m} B^j$. Next we prove $x_n \in B$ for sufficiently large n. Assume that there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin B$ $(\forall k)$. Using the boundedness of the $\{x_{n_k}\}$, there exists a subsequence of x_{n_k} convergent weakly to $\bar{z} \in C$. Without loss of generality, we still denote the subsequence as $\{x_{n_k}\}$, that is, $x_{n_k} \rightharpoonup \bar{z}$. Since $x_{n_k} \notin B$, for $\forall l \in \{1, 2, \dots, m\}$, we have

$$x_{n_k} \notin B^l = \bigcap_{j=1, j\neq l}^{m} \left\{ x : \left\langle x, \frac{z^l - z^j}{\|z^l - z^j\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|} \right\}. \tag{38}$$

Using the principle of drawer, there exists subsequence $\left\{x_{n_{k_i}}\right\}$ of $\{x_{n_k}\}$ and $l_0 \in \{1, 2, \dots, m\} \setminus \{l\}$ such that $\forall i \geq 0$

$$x_{n_{k_i}} \notin \left\{ x : \left\langle x, \frac{z^l - z^{l_0}}{\|z^l - z^{l_0}\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^{l_0}\|^2}{2\|z^l - z^{l_0}\|} \right\}, \tag{39}$$

We have

$$\bar{z} \notin \left\{ x : \left\langle x, \frac{z^l - z^{l_0}}{\|z^l - z^{l_0}\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^{l_0}\|^2}{2\|z^l - z^{l_0}\|} \right\}. \tag{40}$$

Combining (34), (40) and $\left\langle z^l, \frac{z^l - z^{l_0}}{\|z^l - z^{l_0}\|} \right\rangle > \frac{\|z^l - z^{l_0}\|}{4} + \frac{\|z^l\|^2 - \|z^{l_0}\|^2}{2\|z^l - z^{l_0}\|} \geq \epsilon_0 + \frac{\|z^l\|^2 - \|z^{l_0}\|^2}{2\|z^l - z^{l_0}\|}$, we get $\bar{z} \neq z^l$. As l is arbitrary, we have $\bar{z} \notin \{z^1, z^2, \dots z^m\}$, which is impossible. So we have $x_n \in B$ for sufficiently large n. That is, $\exists N_1 > N,\ \forall n \geq N_1$, such that $x_n \in B$.

□

Theorem 3.1 *Assume that* (A1) – (A5) *hold. Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.1. *Then* $\{x_n\}$ *converges weakly to a point* $x^* \in S$.

Proof By Lemma 3.2, we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. It follows that $\exists N_2 > N_1 > N,\ \forall n \geq N_2$, such that $\|x_{n+1} - x_n\| < \epsilon_0$. Assume that $\{x_n\}$ has more than one weak cluster point, from Lemma 3.5, then $\exists N_3 \geq N_2 > N_1 > N$, we have $x_{N_3} \in B^l$, $x_{N_3+1} \in B^j$. Where $l \neq j$, $l, j \in \{1, 2, \dots, m\}$ and $m \geq 2$. In particular, we have

$$\|x_{N_3+1} - x_{N_3}\| < \epsilon_0. \tag{41}$$

Using (36) and (37), we get

$$x_{N_3} \in B^l = \bigcap_{j=1, j\neq l}^{m} \left\{ x : \left\langle x, \frac{z^l - z^j}{\|z^l - z^j\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|} \right\} \tag{42}$$

and

$$x_{N_3+1} \in B^j = \bigcap_{l=1, l\neq j}^{m} \left\{ x : \left\langle -x, \frac{z^j - z^l}{\|z^j - z^l\|} \right\rangle < -\epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^j - z^l\|} \right\}. \tag{43}$$

Moreover, we obtain

$$\left\langle x_{N_3}, \frac{z^l - z^j}{\|z^l - z^j\|} \right\rangle > \epsilon_0 + \frac{\|z^l\|^2 - \|z^j\|^2}{2\|z^l - z^j\|} \tag{44}$$

and

$$\left\langle -x_{N_3+1}, \frac{z^l - z^j}{\|z^j - z^l\|} \right\rangle > \epsilon_0 + \frac{\|z^j\|^2 - \|z^l\|^2}{2\|z^l - z^j\|}. \tag{45}$$

Combining (44) and (45), we get

$$2\epsilon_0 < \left\langle x_{N_3} - x_{N_3+1}, \frac{z^l - z^j}{\|z^j - z^l\|} \right\rangle \le \|x_{N_3+1} - x_{N_3}\| < \epsilon_0. \tag{46}$$

Which is impossible. This implies that $\{x_n\}$ has only a weak cluster point in S. Hence we deduce that $x_n \rightharpoonup x^*$. □

Now we prove the convergence of the Algorithm 3.1 without monotonicity.

Theorem 3.2 *Assume that* $(A1)-(A3)$, $(A4')$ *and* $(A5')$ *hold. Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.1. *Then* $\{x_n\}$ *converges weakly to a point* $x^* \in S$.

Proof According to (28) and the proof in Lemma 3.3, fixing $z \in C$, we have $y_n \rightharpoonup x^*$, $x^* \in C$ and

$$\liminf_{k\to\infty} \langle F(y_{n_k}), z - y_{n_k} \rangle \ge 0.$$

We choose a positive sequence $\{\varepsilon_k\}$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and

$$\langle F(y_{n_k}), z - y_{n_k} \rangle + \varepsilon_k > 0, \quad \forall k \ge 0.$$

It follows that

$$\langle F(y_{n_k}), z \rangle + \varepsilon_k > \langle F(y_{n_k}), y_{n_k} \rangle, \quad \forall k \ge 0. \tag{47}$$

In particular, set $z = x^*$ in (47), we have

$$\langle F(y_{n_k}), x^* \rangle + \varepsilon_k > \langle F(y_{n_k}), y_{n_k} \rangle, \quad \forall k \ge 0.$$

Let $k \to \infty$ in the last inequality, from $(A3)$ and $y_{n_k} \rightharpoonup x^*$, we obtain

$$\langle F(x^*), x^* \rangle \geq \limsup_{k\to\infty} \langle F(y_{n_k}), y_{n_k} \rangle.$$

From $(A4')$, we get $\lim_{n\to\infty} \langle F(y_{n_k}), y_{n_k} \rangle = \langle F(x^*), x^* \rangle$.

At the same time, from (47), we obtain

$$\begin{aligned}\langle F(x^*), z \rangle &= \lim_{k\to\infty} (\langle F(y_{n_k}), z \rangle + \varepsilon_k) \\ &\geq \liminf_{k\to\infty} \langle F(y_{n_k}), y_{n_k} \rangle \\ &= \lim_{n\to\infty} \langle F(y_{n_k}), y_{n_k} \rangle = \langle F(x^*), x^* \rangle.\end{aligned}$$

which implies that

$$\langle F(x^*), z - x^* \rangle \geq 0. \quad \forall z \in C.$$

We have $x^* \in S$. By the condition $(A5')$, similar to the proof of Lemmas 3.4, 3.5 and Theorem 3.1, we get $\{x_n\}$ converges weakly to a point $x^* \in S$. □

For the extragradient method and the subgradient extragradient, we introduce the following algorithms.

Algorithm 3.2 (Step 0) Take $\lambda_0 > 0$, $x_0 \in H$, $\mu \in (0,1)$. Choose a nonnegative real sequence $\{p_n\}$ such that $\sum_{n=0}^{\infty} p_n < +\infty$.

(Step 1) Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n F(x_n)).$$

If $x_n = y_n$(or $F(y_n) = 0$), then stop: y_n is a solution. Otherwise, go to Step 2.

(Step 2) Compute

$$x_{n+1} = P_C(x_n - \lambda_n F(y_n)).$$

(Step 3) Compute

$$\lambda_{n+1} = \begin{cases} min\left\{ \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\langle F(x_n) - F(y_n), x_{n+1} - y_n \rangle}, \lambda_n + p_n \right\}, & if \;\; \langle F(x_n) - F(y_n), x_{n+1} - y_n \rangle > 0, \\ \lambda_n + p_n, & otherwise. \end{cases}$$

Set $n := n + 1$ and return to step 1.

Algorithm 3.3 (Step 0) Take $\lambda_0 > 0$, $x_0 \in H$, $\mu \in (0,1)$. Choose a nonnegative real sequence $\{p_n\}$ such that $\sum_{n=0}^{\infty} p_n < +\infty$.

(Step 1) Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n F(x_n)).$$

If $x_n = y_n$(or $F(y_n) = 0$), then stop: y_n is a solution. Otherwise, go to Step 2.

(Step 2) Construct $T_n = \{x \in H | \langle x_n - \lambda_n F(x_n) - y_n, x - y_n \rangle \leq 0\}$ and compute

$$x_{n+1} = P_{T_n}(x_n - \lambda_n F(y_n)).$$

(Step 3) Compute

$$\lambda_{n+1} = \begin{cases} min\left\{ \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\langle F(x_n) - F(y_n), x_{n+1} - y_n \rangle}, \lambda_n + p_n \right\}, & if \quad \langle F(x_n) - F(y_n), x_{n+1} - y_n \rangle > 0, \\ \lambda_n + p_n, & otherwise. \end{cases}$$

Set $n := n + 1$ and return to step 1.

Remark 3.5 If $p_n \equiv 0$, then the step sizes in Algorithms 3.2 and 3.3 are similar to the method in [20]. The conclusion in Theorems 3.1 and 3.2 still hold for Algorithms 3.2 and 3.3.

4 Numerical experiments

In this section, we compare the proposed methods with the Algorithm 2.1 in [2]. We choose $\mu = 0.5$, $p_n = \frac{100}{(n+1)^{1.1}}$ and $\lambda_0 = 1$ for our algorithms. We choose $\gamma = 0.4$ and $\sigma = 0.99$ for Algorithm 2.1 in [2]. The stopping criterion are the following

Algorithms 3.1, 3.2, 3.3 $\|y_n - x_n\| / \min\{\lambda_n, 1\} \le \varepsilon$ and $\|F(y_n)\| \le \varepsilon$. (45)
Algorithm 2.1 in [2] $\|x_n - z_n\| = \|x_n - P_C(x_n - F(x_n))\| \le \varepsilon$. (46)

We do not use the criterion $\|x_n - P_C(x_n - F(x_n))\| \le \varepsilon$ and $\|F(y_n)\| \le \varepsilon$ for Algorithms 3.1, 3.2 and 3.3 because we do not want to compute $\|x_n - P_C(x_n - F(x_n))\|$ extra. If $\lambda_n < 1$, we have

$$\|x_n - P_C(x_n - F(x_n))\| \le \|x_n - P_C(x_n - \lambda_n F(x_n))\| / \lambda_n = \|y_n - x_n\| / \min\{\lambda_n, 1\}. \tag{47}$$

If $\lambda_n \ge 1$, we have

$$\|x_n - P_C(x_n - F(x_n))\| \le \|x_n - P_C(x_n - \lambda_n F(x_n))\| = \|y_n - x_n\| / \min\{\lambda_n, 1\}. \tag{48}$$

Moreover, we notice that termination criteria (45) is stronger than (46). We denoted by x_0 the starting point of the experiment and by x the solution of the variational inequality. We also added the total number (nf) of all values F that is evaluated. For the test problems, we also have generated random samples with different choice of x_0 in C. For all algorithms, we take $\varepsilon = 10^{-6}$.

Problem 1 Let $C = [-1, 1]$ and

$$F(x) = \begin{cases} 2x - 1, & x > 1, \\ x^2, & x \in [-1, 1], \\ -2x - 1, & x < -1. \end{cases}$$

Table 1 Problem 1

x_0	Algorithm 3.1		Algorithm 3.2		Algorithm 3.3		Algorithm 2.1 in [2]		x
	Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)	
0.8	53(108)	1×10^{-4}	50(102)	7×10^{-5}	50(102)	7×10^{-5}	990(991)	6×10^{-5}	0
0.5	48(98)	9×10^{-5}	45(92)	7×10^{-5}	45(92)	6×10^{-5}	992(993)	6×10^{-5}	0
0.1	35(72)	7×10^{-5}	33(68)	4×10^{-5}	33(68)	6×10^{-5}	986(987)	9×10^{-5}	0
−0.8	21(44)	3×10^{-5}	1(4)	7×10^{-6}	1(4)	7×10^{-6}	1(2)	7×10^{-6}	− 1
random	43(88)	7×10^{-5}	40(82)	4×10^{-5}	40(82)	7×10^{-5}	991(992)	4×10^{-5}	0
random	34(70)	6×10^{-5}	33(68)	4×10^{-5}	33(68)	6×10^{-5}	985(986)	6×10^{-5}	0

Then F is a quasimonotone and Lipschitz continuous mapping. We have $S_D = \{-1\}$ and $S = \{-1, 0\}$. The results are presented in Table 1. As we can see from Table 1, the number of iterations of our Algorithms is much smaller than Algorithm 2.1 in [2].

Problem 2 Let $C = \{x \in R^2 : x_1^2 + x_2^2 \leq 1, 0 \leq x_1\}$ and $F(x_1, x_2) = (-x_1 e^{x_2}, x_2)$. It is not difficulty to check that F is not a quasimonotone mapping. Indeed, take $x = (0, \frac{1}{4})$ and $y = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, we have $\langle F(y), x - y\rangle = \frac{3}{4}e^{0.5} - \frac{1}{8} > 0$ and $\langle F(x), x - y\rangle = -\frac{1}{16} < 0$. It's easy to validate that $(1, 0) \in S_D$. By the KKT conditions to the $VI(C, F)$ and convexity of S_D, we have $S = \{(1, 0), (0, 0)\}$ and $(0, 0) \notin S_D$. This problem is tested in Table 2. Tables 2 shows that our Algorithms work better.

Problem 3 This problem was considered in [10, 29]. Let $C = [0, 1]^m$ and

$$\begin{aligned} F(x) =& (f_1(x), f_2(x), \dots, f_m(x)),\\ f_i(x) =& x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_i x_{i+1}\\ &- 2x_{i-1} + 4x_i + x_{i+1} - 1, i = 1, 2, \dots, m, x_0 = x_{m+1} = 0. \end{aligned}$$

When n greater than 1000, we aborted the evaluation of Algorithm 2.1 in [2](Since it involves the calculation of quadratic programming). The results are presented in Tables 3 and 4. In this example, our Algorithms are faster than Algorithm 2.1

Table 2 Problem 2

x_0	Algorithm 3.1		Algorithm 3.2		Algorithm 3.3		Algorithm2.1 in [2]		x
	Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)	
(0.5,0.5)	44(90)	0.002	21(44)	0.003	38(78)	0.002	1197(3536)	7.25	(1,0)
(0.1,0.2)	41(84)	0.003	22(46)	0.003	45(92)	0.005	1193(3539)	7.23	(1,0)
(0.25,0.25)	42(86)	0.002	21(44)	0.002	37(76)	0.002	1199(3581)	7.28	(1,0)
(0.75,0.25)	41(84)	0.002	20(42)	0.003	36(74)	0.002	1194(3537)	7.28	(1,0)
(0.5,-0.6)	44(90)	0.003	21(44)	0.003	39(80)	0.002	1743(4091)	11.9	(1,0)

Table 3 Problem 3

n	x_0	Algorithm 3.1		Algorithm 3.2		Algorithm 3.3		Algorithm 2.1 in [2]	
		Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)
500	$(0,0,\dots,0)$	71(144)	0.006	62(126)	0.006	72(146)	0.006	29(117)	7.44
500	Random	71(144)	0.005	63(128)	0.006	68(138)	0.006	41(170)	4.07
500	Random	70(142)	0.006	64(130)	0.008	68(138)	0.006	45(190)	4.43
1000	$(0,0,\dots,0)$	72(146)	0.009	63(128)	0.016	78(158)	0.012	29(177)	47.1
1000	Random	71(144)	0.009	65(132)	0.011	69(140)	0.011	44(184)	25.2
1000	Random	72(146)	0.009	65(130)	0.014	70(142)	0.010	43(179)	24.7

Table 4 Problem 3

n	x_0	Algorithm 3.1		Algorithm 3.2		Algorithm 3.3	
		Iter (nf)	Time (s)	Iter (nf)	Time (s)	Iter (nf)	Time (s)
10000	$(0,0,\dots,0)$	76(154)	0.115	68(138)	0.112	82(166)	0.131
50000	$(0,0,\dots,0)$	79(160)	0.546	70(142)	0.537	88(178)	0.639
100000	$(0,0,\dots,0)$	80(162)	1.09	72(146)	1.26	94(190)	1.662
1000000	$(0,0,\dots,0)$	84(170)	16.7	76(154)	18.9	92(186)	24.8

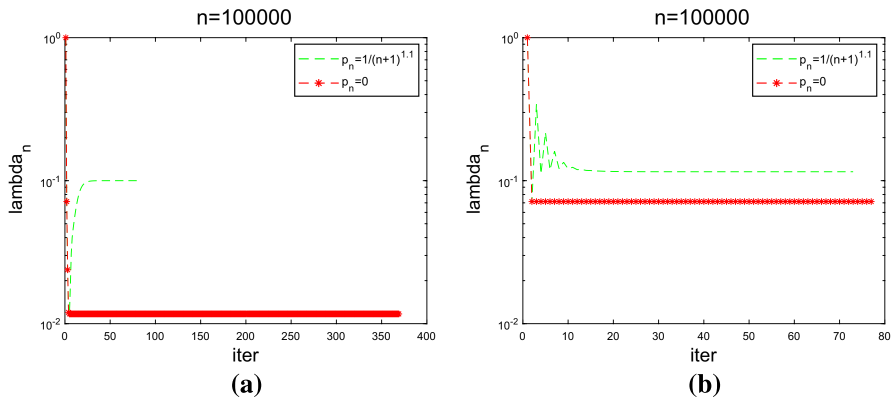

Fig. 1 Comparison of different p_n and $x_0 = (0,0,\dots,0)$ with $\epsilon = 10^{-6}$ for the Problem 3 with $n = 100000$, (a): λ_n for Algorithm 3.1; (b): λ_n for Algorithm 3.2

in [2]. In Fig. 1, we illustrate the behavior of the stepsisizes for this problem with $n = 100000$.

5 Conclusions

In this paper, we consider convergence results for variational inequalities involving Lipschitz continuous quasimonotone mapping (or without monotonicity) but the Lipschitz constant is unknown. We modify the gradient methods with the new step sizes. Weak convergence theorems are proved for sequences generated by the Algorithms. Numerical experiments confirm the effectiveness of the proposed Algorithms.

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