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A self-adaptive method for pseudomonotone equilibrium problems and variational inequalities

Jun Yang[1,2] **· Hongwei Liu**[1]

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Abstract
In this paper, we introduce and analyze a new algorithm for solving equilibrium problem involving pseudomonotone and Lipschitz-type bifunction in real Hilbert space. The algorithm requires only a strongly convex programming problem per iteration. A weak and a strong convergence theorem are established without the knowledge of the Lipschitz-type constants of the bifunction. As a special case of equilibrium problem, the variational inequality is also considered. Finally, numerical experiments are performed to illustrate the advantage of the proposed algorithm.

Keywords Equilibrium problem · Pseudomonotone bifunction · Gradient method · Variational inequality

Mathematics Subject Classification 65J15 · 90C33 · 90C25 · 90C52

1 Introduction

In this paper, we consider the equilibrium problem (EP) which is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C, \tag{1}$$

where C is a nonempty closed convex subset in a real Hilbert space H, $f : H \times H \longrightarrow \mathbb{R}$ is a bifunction. The solution set of (1) is denoted by $EP(f)$. Equilibrium problem is

✉ Jun Yang
xysyyangjun@163.com

Hongwei Liu
hwliu@mail.xidian.edu.cn

1 School of Mathematics and Statistics, Xidian University, Xi’an 710126, Shaanxi, China

2 School of Mathematics and Information Science, Xianyang Normal University, Xianyang 712000, Shaanxi, China

also called the Ky Fan inequality due to his contribution to this field [1]. The problem unifies many important mathematical models such as saddle point problem, fixed point problem, variational inequality and Nash equilibrium problem [2,3]. Recently, methods for solving equilibrium problem have been studied extensively [4–17]. One of the most popular methods is the proximal point method [4–6]. But the method cannot be applied to pseudomonotone equilibrium problem. Another method is the proximal-like method (the extragradient method) [7]. By using the idea of Korpelevich extragradient method in [8], this method was extended by Quoc et al. in [9]

$$\begin{cases} x_0 \in C, y_n = argmin\left\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\right\}, \\ x_{n+1} = argmin\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\right\}, \end{cases} \tag{2}$$

where λ is a suitable parameter. It was proved that the sequence $\{x_n\}$ generated by (2) converges to a solution of equilibrium problem under the assumptions of pseudomonotonicity and Lipschitz-type condition of f. But at each iteration, one needs to calculate two strongly convex programming problems. This method was improved by many authors; see,e.g., [10–15]. Based on the Malitsky's work in the variational inequality [18], Nguyen [15] proposed the following method

$$\begin{cases} x_0, y_1 \in C, x_n = \frac{(\varphi-1)y_n + x_{n-1}}{\varphi} \\ y_{n+1} = argmin\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\right\}, \end{cases} \tag{3}$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ and λ is a suitable parameter. It is easy to see this method need only one strongly convex programming problem per iteration. The main drawback of algorithms (2) and (3) is a requirement to know Lipschitz-type constants of equilibrium bifunction. In order to overcome this shortcoming, Dang [10,11,13,14] proposed the non-summable and diminishing step size sequence for solving strongly pseudomonotone equilibrium problem. In this work, we propose a new gradient method for solving pseudomonotone equilibrium problem. It is worth pointing out that the proposed algorithm uses a new step size and does not require the knowledge of the Lipschitz-type constants of the bifunction.

The remainder of this paper is organized as follows. In Sect. 2, we present some definitions and preliminaries that will be needed throughout the paper. In Sect. 3, we propose a new algorithm and analyze its convergence. In Sect. 4, we particularize our method to the variational inequality. Finally, preliminary numerical experiments are provided which demonstrate our algorithm performance.

2 Preliminaries

In this section, we recall some concepts and results for further use.

Definition 2.1 A bifunction $f : C \times C \longrightarrow \mathbb{R}$ is said to be as follows:

(i) *Monotone* on C if $f(x, y) + f(y, x) \leq 0,\ \forall x, y \in C$.
(ii) *Pseudomonotone* on C if $f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0,\ \forall x, y \in C$.

(iii) *Strong pseudomonotone* on C if $f(x, y) \geq 0 \implies f(y, x) \leq -\gamma\|x - y\|^2, \ \forall x, y \in C$. Where $\gamma > 0$.

Definition 2.2 A mapping $h : C \longrightarrow \mathbb{R}$ is called *subdifferentiable* at $x \in C$ if there exists a vector $w \in H$ such that $h(y) - h(x) \geq \langle w, y - x\rangle, \forall y \in C$.

Definition 2.3 A mapping $F : H \to H$ is said to be sequentially weakly continuous if the sequence $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $F(x)$.

For solving the equilibrium problem, we assume that the bifunction f satisfies the following conditions:

$(C1)$ f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.
$(C1')$ f is strong pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.
$(C2)$ f satisfies the Lipschitz-type condition on C. That is, there exist two positive constants c_1, c_2 such that $f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \ \forall x, y, z \in C$.
$(C3)$ $f(x, .)$ is convex and subdifferentiable on C for every fixed $x \in C$.
$(C4)$ $\limsup_{n\to\infty} f(x_n, y) \leq f(x, y)$ for every sequence
$\{x_n\}$ which converges weakly to x and for each $y \in C$.

For a proper, convex and lower semicontinuous function $g : C \to (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping of g associated with λ is defined by

$$prox_{\lambda g}(x) = argmin\left\{\lambda g(y) + \frac{1}{2}\|x - y\|^2 : y \in C\right\}, \ x \in H. \tag{4}$$

The following lemma is a property of the proximal mapping.

Lemma 2.1 [19] *For all $x \in H$, $y \in C$ and $\lambda > 0$, the following inequality holds:*

$$\lambda\{g(y) - g(prox_{\lambda g}(x))\} \geq \langle x - prox_{\lambda g}(x), y - prox_{\lambda g}(x)\rangle. \tag{5}$$

Remark 2.1 From Lemma 2.1, we note that if $x = prox_{\lambda g}(x)$, then

$$x \in Argmin\{g(y) : y \in C\} := \{x \in C : g(x) = \min_{y\in C} g(y)\}. \tag{6}$$

Lemma 2.2 *Let $\delta \in (0, +\infty)$ and $x, \ y \in H$. Then*

$$\|(\delta + 1)x - \delta y\|^2 = (\delta + 1)\|x\|^2 - \delta\|y\|^2 + \delta(\delta + 1)\|x - y\|^2.$$

Lemma 2.3 *Let $\{a_n\}$, $\{b_n\}$ be two nonnegative real sequences such that $\exists N > 0, \ \forall n > N$, $a_{n+1} \leq a_n - b_n$. Then $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$.*

Lemma 2.4 *Let $\{x_n\}$ be a sequence in H such that $x_n \rightharpoonup x$. Then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \ \forall y \neq x.$$

For a closed and convex $K \subseteq H$, the (metric) projection $P_K : H \longrightarrow C$ is defined, for all $x \in H$ by $P_K(x) = argmin\{\| y - x \| | y \in K\}$.

Lemma 2.5 *Let C be a nonempty, closed and convex set in H and $x \in H$. Then*

$$\langle P_C x - x, y - P_C x \rangle \geq 0, \quad \forall y \in C.$$

3 Algorithm and its convergence

In this section, we propose an iterative algorithm for solving the equilibrium problem (1). The algorithm is designed as follows:

Algorithm 3.1

(Step 0) Choose $\lambda_1 > 0$, $x_0, y_0, y_1 \in C$, $\mu \in (0, 1)$, $\alpha \in (0, 1)$, $\theta \in (0, 1]$, $\delta \in (\frac{\sqrt{1+4(\frac{\alpha}{2-\theta}+1-\alpha)}-1}{2}, 1)$.

(Step 1) Given the current iterate x_{n-1}, y_{n-1}, y_n, compute

$$x_n = (1-\delta)y_n + \delta x_{n-1}. \tag{7}$$

$$y_{n+1} = argmin\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\right\} = prox_{\lambda_n f(y_n,.)}(x_n). \tag{8}$$

If $y_{n+1} = x_n = y_n$, then stop: y_n is a solution. Otherwise, go to step 2.

(Step 2) Compute

$$\lambda_{n+1} = \begin{cases} min\left\{\frac{\alpha\mu\theta(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))}, \lambda_n\right\}, \\ \qquad if \quad f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}) > 0, \\ \lambda_n, \qquad otherwise. \end{cases} \tag{9}$$

Set $n := n + 1$ and return to step 1.

Remark 3.1 Under hypotheses $(C1)$ and $(C3)$, from Lemma 2.1 and Remark 2.1, we obtain that if Algorithm 3.1 terminates at some iterate, i.e., $y_{n+1} = x_n = y_n$, then $y_n \in EP(f)$.

Lemma 3.1 *The sequence $\{\lambda_n\}$ generated by* Algorithm 3.1 *is a monotonically decreasing sequence with lower bound* $\min\{\frac{\alpha\mu\theta}{4\delta \max\{c_1,c_2\}}, \lambda_1\}$.

Proof It is easily checked that $\{\lambda_n\}$ is a monotonically decreasing sequence. Since f is a Lipschitz-type bifunction with constants c_1 and c_2, in the case of $f(y_{n-1}, y_{n+1}) -$

$f(y_{n-1}, y_n) - f(y_n, y_{n+1}) > 0$, we have

$$\begin{aligned}
&\frac{\alpha\mu\theta(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))} \\
&\geq \frac{\alpha\mu\theta(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta(c_1\|y_{n-1} - y_n\|^2 + c_2\|y_{n+1} - y_n\|^2)} \\
&\geq \frac{\alpha\mu\theta(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta \max\{c_1, c_2\}(\|y_{n-1} - y_n\|^2 + \|y_{n+1} - y_n\|^2)} \\
&= \frac{\alpha\mu\theta}{4\delta \max\{c_1, c_2\}}. \qquad (10)
\end{aligned}$$

It is clear that the sequence $\{\lambda_n\}$ has the lower bound $\min\{\frac{\alpha\mu\theta}{4\delta\max\{c_1,c_2\}}, \lambda_1\}$. □

Remark 3.2 The limit of $\{\lambda_n\}$ exists and we denote $\lambda = \lim_{n\to\infty} \lambda_n$. It is obvious that $\lambda > 0$. If $\lambda_1 \leq \frac{\alpha\mu\theta}{4\delta\max\{c_1,c_2\}}$, then $\{\lambda_n\}$ is a constant sequence. The following lemma plays a crucial role in the Proof of the Theorem 3.1.

Lemma 3.2 *Under the conditions* $(C1)$, $(C2)$ *and* $(C3)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* Algorithm 3.1 *and* $EP(f) \neq \emptyset$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *are bounded.*

Proof Since $y_{n+1} = prox_{\lambda_n f(y_n,.)}(x_n)$. By Lemma 2.1, we get

$$\lambda_n(f(y_n, y) - f(y_n, y_{n+1})) \geq \langle x_n - y_{n+1}, y - y_{n+1}\rangle, \ \forall y \in C. \qquad (11)$$

Let $u \in EP(f)$. Substituting $y = u$ into the last inequality, we have

$$\lambda_n(f(y_n, u) - f(y_n, y_{n+1})) \geq \langle x_n - y_{n+1}, u - y_{n+1}\rangle. \qquad (12)$$

As $u \in EP(f)$, we obtain $f(u, y_n) \geq 0$. Thus $f(y_n, u) \leq 0$ because of the pseudomonotonicity of f. Hence, from (12) and $\lambda_n > 0$, we obtain

$$-\lambda_n f(y_n, y_{n+1}) \geq \langle x_n - y_{n+1}, u - y_{n+1}\rangle. \qquad (13)$$

Since $y_n = prox_{\lambda_{n-1} f(y_{n-1},.)}(x_{n-1})$, we get

$$\lambda_{n-1}(f(y_{n-1}, y) - f(y_{n-1}, y_n)) \geq \langle x_{n-1} - y_n, y - y_n\rangle, \ \forall y \in C. \qquad (14)$$

In particular, substituting $y = y_{n+1}$ into the last inequality, we have

$$\lambda_{n-1}(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geq \langle x_{n-1} - y_n, y_{n+1} - y_n\rangle. \qquad (15)$$

Since $x_n = (1-\delta)y_n + \delta x_{n-1}$, we obtain $y_n = \frac{1}{1-\delta}x_n - \frac{\delta}{1-\delta}x_{n-1}$. Hence,

$$y_n - x_n = \frac{\delta}{1-\delta}(x_n - x_{n-1}) = \delta(y_n - x_{n-1}). \qquad (16)$$

Combining (15), (16) and $\lambda_n > 0$, we have

$$\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geq \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}\langle x_n - y_n, y_{n+1} - y_n\rangle. \tag{17}$$

Adding (13) and (17), we get

$$\begin{aligned}
&2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))\\
&\quad \geq 2\langle x_n - y_{n+1}, u - y_{n+1}\rangle + 2\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}\langle x_n - y_n, y_{n+1} - y_n\rangle\\
&\quad = \|x_n - y_{n+1}\|^2 + \|y_{n+1} - u\|^2 - \|x_n - u\|^2\\
&\qquad + \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2 - \|x_n - y_{n+1}\|^2).
\end{aligned} \tag{18}$$

That is

$$\begin{aligned}
\|y_{n+1} - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_{n+1}\|^2 - \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1}\\
&\quad - y_n\|^2 - \|x_n - y_{n+1}\|^2) + 2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))\\
&= \|x_n - u\|^2 + \left(\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta} - 1\right)\|x_n - y_{n+1}\|^2 - \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2)\\
&\quad + 2\lambda_{n+1}\frac{\lambda_n}{\lambda_{n+1}}(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1})).
\end{aligned} \tag{19}$$

By definition of λ_n and (19), we obtain

$$\begin{aligned}
\|y_{n+1} - u\|^2 &\leq \|x_n - u\|^2 + \left(\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta} - 1\right)\|x_n - y_{n+1}\|^2\\
&\quad - \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2)\\
&\quad + \frac{1}{2}\mu\frac{\lambda_n}{\lambda_{n+1}}\frac{1}{\delta}\alpha\theta(\|y_n - y_{n-1}\|^2 + \|y_n - y_{n+1}\|^2).
\end{aligned} \tag{20}$$

In the last inequality, in the case of $f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}) \leq 0$, it is obvious that

$$\begin{aligned}
&2\lambda_{n+1}\frac{\lambda_n}{\lambda_{n+1}}(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))\\
&\quad \leq 0 \leq \frac{1}{2}\mu\frac{\lambda_n}{\lambda_{n+1}}\frac{1}{\delta}\alpha\theta(\|y_n - y_{n-1}\|^2 + \|y_n - y_{n+1}\|^2).
\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n-1}}=1>\alpha,\quad \lim_{n\to\infty}\lambda_n\frac{\mu}{\lambda_{n+1}}=\mu,\ \ 0<\mu<1. \tag{21}$$

we have that $\exists N\geq 0$, such that $\forall n\geq N$, $\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}-1>0$, $0<\lambda_n\frac{\mu}{\lambda_{n+1}}<1$ and $\alpha<\frac{\lambda_n}{\lambda_{n-1}}$.

From the relation $y_{n+1}=\frac{1}{1-\delta}x_{n+1}-\frac{\delta}{1-\delta}x_n$, by Lemma 2.2 and (16), we have

$$\begin{aligned}\|y_{n+1}-u\|^2&=\|\frac{1}{1-\delta}(x_{n+1}-u)-\frac{\delta}{1-\delta}(x_n-u)\|^2\\&=\frac{1}{1-\delta}\|x_{n+1}-u\|^2-\frac{\delta}{1-\delta}\|x_n-u\|^2+\frac{1}{1-\delta}\frac{\delta}{1-\delta}\|x_{n+1}-x_n\|^2\\&=\frac{1}{1-\delta}\|x_{n+1}-u\|^2-\frac{\delta}{1-\delta}\|x_n-u\|^2+\delta\|y_{n+1}-x_n\|^2.\end{aligned} \tag{22}$$

Also, from $\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}-1\leq\frac{\lambda_{n-1}}{\lambda_{n-1}}\frac{1}{\delta}-1=\frac{1}{\delta}-1$, (20), (21) and (22), it implies that $\forall n\geq N$,

$$\begin{aligned}&\frac{1}{1-\delta}\|x_{n+1}-u\|^2-\frac{\delta}{1-\delta}\|x_n-u\|^2+\delta\|y_{n+1}-x_n\|^2\\&\leq\|x_n-u\|^2+\left(\frac{1}{\delta}-1\right)\|x_n-y_{n+1}\|^2-\frac{\alpha}{\delta}(\|x_n-y_n\|^2+\|y_{n+1}-y_n\|^2)\\&+\frac{\alpha}{2\delta}\theta(\|y_n-y_{n-1}\|^2+\|y_n-y_{n+1}\|^2).\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}&\frac{1}{1-\delta}\|x_{n+1}-u\|^2+\frac{\alpha\theta}{2\delta}\|y_{n+1}-y_n\|^2\\&\leq\frac{1}{1-\delta}\|x_n-u\|^2+\frac{\alpha\theta}{2\delta}\|y_n-y_{n-1}\|^2+\left(\frac{\theta\alpha}{\delta}-\frac{\alpha}{\delta}\right)\|y_{n+1}-y_n\|^2\\&\quad+\left(\frac{1}{\delta}-1-\delta\right)\|y_{n+1}-x_n\|^2-\frac{\alpha}{\delta}\|y_n-x_n\|^2\\&=\frac{1}{1-\delta}\|x_n-u\|^2+\frac{\alpha\theta}{2\delta}\|y_n-y_{n-1}\|^2\\&\quad+\frac{(\theta-1)\alpha}{\delta}\|y_{n+1}-y_n\|^2+\left(\frac{1}{\delta}-1-\delta\right)\|y_{n+1}-x_n\|^2\\&\quad-\frac{\alpha}{\delta}(\|x_n-y_{n+1}\|^2+\|y_n-y_{n+1}\|^2+2\langle x_n-y_{n+1},y_{n+1}-y_n\rangle).\end{aligned} \tag{24}$$

For $n \geq N$, let

$$a_n = \frac{1}{1-\delta}\|x_n - u\|^2 + \frac{\alpha\theta}{2\delta}\|y_n - y_{n-1}\|^2,$$
$$\eta = \frac{1}{2-\theta}. \tag{25}$$

Combining (24), (25) and $-2\langle x_n - y_{n+1}, y_{n+1} - y_n\rangle \leq \eta\|x_n - y_{n+1}\|^2 + \frac{1}{\eta}\|y_n - y_{n+1}\|^2$, we have

$$\begin{aligned} a_{n+1} &\leq a_n + \left(\frac{(\theta-1)\alpha}{\delta} - \frac{\alpha}{\delta} + \frac{\alpha}{\delta}\frac{1}{\eta}\right)\|y_{n+1} - y_n\|^2 + \left(\frac{1}{\delta} - 1 - \delta - \frac{\alpha}{\delta}\right. \\ &\quad \left. + \frac{\alpha\eta}{\delta}\right)\|y_{n+1} - x_n\|^2 \\ &= a_n + \left(\frac{1}{\delta} - 1 - \delta - \frac{\alpha}{\delta} + \frac{\alpha\eta}{\delta}\right)\|y_{n+1} - x_n\|^2. \end{aligned} \tag{26}$$

Since $\delta \in (\frac{\sqrt{1+4(\frac{\alpha}{2-\theta}+1-\alpha)}-1}{2}, 1)$, we obtain $\frac{1}{\delta} - 1 - \delta - \frac{\alpha}{\delta} + \frac{\alpha\eta}{\delta} < 0$.

For $n \geq N$, let

$$b_n = -\left(\frac{1}{\delta} - 1 - \delta - \frac{\alpha}{\delta} + \frac{\alpha\eta}{\delta}\right)\|y_{n+1} - x_n\|^2. \tag{27}$$

Then (26) can be written as $a_{n+1} \leq a_n - b_n, \forall n \geq N$. From Lemma 2.3, we can conclude that $\{a_n\}$ is bounded, $\lim_{n\to\infty} b_n = 0$ and the limit of $\{a_n\}$ exists. By definition of b_n, we can show that $\lim_{n\to\infty}\|y_{n+1} - x_n\| = 0$. From the relation (16), $\|y_n - y_{n-1}\| \leq \|y_n - x_n\| + \|x_n - y_{n-1}\|$ and $\|x_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - y_{n-1}\|$, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|x_n - x_{n-1}\| = \lim_{n\to\infty}\|y_n - y_{n-1}\| = \lim_{n\to\infty}\|y_{n+1} - x_n\| = 0. \tag{28}$$

Also, we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{1-\delta}\|x_n - u\|^2$. This implies that the sequence $\{x_n\}$ is bounded and so $\{y_n\}$ is bounded. That is the desired result. □

Theorem 3.1 *Assume that* $(C1)-(C4)$ *and* $EP(f) \neq \emptyset$ *hold. Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by* Algorithm 3.1 *converge weakly to a solution of the equilibrium problem.*

Proof By Lemma 3.2, the sequence $\{x_n\}$ is bounded and there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $x^* \in H$. Then $y_{n_k} \rightharpoonup x^*$, $y_{n_k+1} \rightharpoonup x^*$ and $x^* \in C$. From the relation (11), we have

$$\lambda_{n_k}(f(y_{n_k}, y) - f(y_{n_k}, y_{n_k+1})) \geq \langle x_{n_k} - y_{n_k+1}, y - y_{n_k+1}\rangle, \ \forall y \in C. \tag{29}$$

Since f satisfies the Lipschitz-type condition on C, we have

$$\lambda_{n_k}(f(y_{n_k}, y_{n_k+1})) \geq \lambda_{n_k}(f(y_{n_k-1}, y_{n_k+1}) - f(y_{n_k-1}, y_{n_k})) - \lambda_{n_k} c_1 \|y_{n_k} - y_{n_k-1}\|^2 - \lambda_{n_k} c_2 \|y_{n_k} - y_{n_k+1}\|^2. \tag{30}$$

From the relations (17) and (30), it follows that

$$\lambda_{n_k}(f(y_{n_k}, y_{n_k+1})) \geq \frac{\lambda_{n_k}}{\lambda_{n_k-1}} \frac{1}{\delta} \langle x_{n_k} - y_{n_k}, y_{n_k+1} - y_{n_k}\rangle - \lambda_{n_k} c_1 \|y_{n_k} - y_{n_k-1}\|^2 - \lambda_{n_k} c_2 \|y_{n_k} - y_{n_k+1}\|^2. \tag{31}$$

Combining the relations (29) and (31), it follows that, for all $y \in C$,

$$f(y_{n_k}, y) \geq \frac{1}{\lambda_{n_k-1}} \frac{1}{\delta} \langle x_{n_k} - y_{n_k}, y_{n_k+1} - y_{n_k}\rangle + \frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k+1}, y - y_{n_k+1}\rangle - c_1 \|y_{n_k} - y_{n_k-1}\|^2 - c_2 \|y_{n_k} - y_{n_k+1}\|^2. \tag{32}$$

Let $k \to \infty$, using the facts $\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - y_{n_k+1}\| = \lim_{k\to\infty} \|y_{n_k} - y_{n_k-1}\| = 0$, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and the hypothesis $(C4)$, we obtain $f(x^*, y) \geq 0, \quad \forall y \in C$. That is $x^* \in EP(f)$. Next we prove that $x_n \rightharpoonup x^*$. Assume that $\{x_n\}$ has at least two weak cluster points $x^* \in EP(f)$ and $\bar{x} \in EP(f)$ such that $x^* \neq \bar{x}$. Let $\{x_{n_i}\}$ be a sequence such that $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$, noting the fact that $\forall u \in EP(f)$,

$$\lim_{n\to\infty} \|x_n - u\| = \lim_{n\to\infty} \sqrt{(1-\delta)a_n}. \tag{33}$$

By Lemma 2.4, we get

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| = \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| < \liminf_{i\to\infty} \|x_{n_i} - x^*\| \\
&= \lim_{n\to\infty} \|x_n - x^*\| = \lim_{k\to\infty} \|x_{n_k} - x^*\| = \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|.
\end{aligned} \tag{34}$$

Which is impossible. Hence we deduce that $x_n \rightharpoonup x^*$. Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have $y_n \rightharpoonup x^*$. That is the desired result. □

Next, we prove Algorithm 3.1 converges strongly to the solution of (1) under a strong pseudomonotonicity assumption of the bifunction f.

Theorem 3.2 *Assume that* $(C1')$, $(C2)$, $(C3)$ *and* $EP(f) \neq \emptyset$ *hold. Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by* Algorithm 3.1 *converge strongly to the unique solution* u *of the equilibrium problem.*

Proof The strong pseudomonotonicity assumption of the bifunction f implies that (1) has a unique solution, which we denote by u. Since $y_n \in C$, we have $f(u, y_n) \geq 0$. As f is strong pseudomonotone, we get $f(y_n, u) \leq -\gamma \|y_n - u\|^2$. Hence, from (12) and $\lambda_n > 0$, we have

$$-\lambda_n f(y_n, y_{n+1}) \geq \langle x_n - y_{n+1}, u - y_{n+1}\rangle + \lambda_n \gamma \|y_n - u\|^2. \tag{35}$$

Adding (35) and (17), we obtain

$$\begin{aligned} &2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1})) \\ &\geq 2\langle x_n - y_{n+1}, u - y_{n+1}\rangle + 2\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}\langle x_n - y_n, y_{n+1} - y_n\rangle + 2\lambda_n\gamma\|y_n - u\|^2 \\ &= \|x_n - y_{n+1}\|^2 + \|y_{n+1} - u\|^2 - \|x_n - u\|^2 + 2\lambda_n\gamma\|y_n - u\|^2 \\ &+ \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2 - \|x_n - y_{n+1}\|^2). \end{aligned} \tag{36}$$

Moreover, by Lemma 3.1, Remark 3.2 and (36), we also have

$$\begin{aligned} &\|y_{n+1} - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_{n+1}\|^2 \\ &- \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2 - \|x_n - y_{n+1}\|^2) \\ &+ 2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1})) - 2\lambda_n\gamma\|y_n - u\|^2 \\ &\leq \|x_n - u\|^2 + \left(\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta} - 1\right)\|x_n - y_{n+1}\|^2 - \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2) \\ &+ 2\lambda_{n+1}\frac{\lambda_n}{\lambda_{n+1}}(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1})) - 2\lambda\gamma\|y_n - u\|^2. \end{aligned} \tag{37}$$

By definition of λ_n and (37), we obtain

$$\begin{aligned} \|y_{n+1} - u\|^2 &\leq \|x_n - u\|^2 + \left(\frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta} - 1\right)\|x_n - y_{n+1}\|^2 \\ &- \frac{\lambda_n}{\lambda_{n-1}}\frac{1}{\delta}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2) \\ &+ \frac{1}{2}\mu\frac{\lambda_n}{\lambda_{n+1}}\frac{1}{\delta}\alpha\theta(\|y_n - y_{n-1}\|^2 + \|y_n - y_{n+1}\|^2) - 2\lambda\gamma\|y_n - u\|^2. \end{aligned} \tag{38}$$

Using (38) and the same techniques as in the proof of (21)–(24), we have $\exists N \geq 0$, such that $\forall n \geq N$,

$$\begin{aligned}
&\frac{1}{1-\delta}\|x_{n+1}-u\|^2 + \frac{\alpha\theta}{2\delta}\|y_{n+1}-y_n\|^2 \\
&\quad \leq \frac{1}{1-\delta}\|x_n-u\|^2 + \frac{\alpha\theta}{2\delta}\|y_n-y_{n-1}\|^2 \\
&\quad + \frac{(\theta-1)\alpha}{\delta}\|y_{n+1}-y_n\|^2 + \left(\frac{1}{\delta}-1-\delta\right)\|y_{n+1}-x_n\|^2 \\
&\quad - \frac{\alpha}{\delta}(\|x_n-y_{n+1}\|^2 + \|y_n-y_{n+1}\|^2 + 2\langle x_n-y_{n+1}, y_{n+1}-y_n\rangle) - 2\lambda\gamma\|y_n-u\|^2.
\end{aligned} \tag{39}$$

For $n \geq N$, let

$$\begin{aligned}
a_n &= \frac{1}{1-\delta}\|x_n-u\|^2 + \frac{\alpha\theta}{2\delta}\|y_n-y_{n-1}\|^2, \qquad \eta = \frac{1}{2-\theta}, \\
b_n &= -\left(\frac{1}{\delta}-1-\delta-\frac{\alpha}{\delta}+\frac{\alpha\eta}{\delta}\right)\|y_{n+1}-x_n\|^2 + 2\lambda\gamma\|y_n-u\|^2.
\end{aligned} \tag{40}$$

Using (40) and the same argument as in the proof of (26), we obtain $a_{n+1} \leq a_n - b_n$, $\forall n \geq N$. From Lemma 2.3 and the definition of b_n, we can conclude that $\{a_n\}$ is bounded, $\lim_{n\to\infty} b_n = 0$. Using $\frac{1}{\delta} - 1 - \delta - \frac{\alpha}{\delta} + \frac{\alpha\eta}{\delta} < 0$ and (28), we have

$$\lim_{n\to\infty}\|x_n-u\| = \lim_{n\to\infty}\|y_n-u\| = \lim_{n\to\infty} b_n = 0. \tag{41}$$

The proof is complete. □

4 The case of variational inequalities

Let $f(x, y) = \langle F(x), y-x\rangle$, $\forall x, y \in C$, where $F : C \to H$ is a mapping. Then the equilibrium problem becomes the variational inequality. That is, find $x^* \in C$ such that

$$\langle F(x^*), y-x^*\rangle \geq 0, \quad \forall y \in C. \tag{42}$$

Moreover, we have $prox_{\lambda_n f(y_n,.)}(x_n) = P_C(x_n - \lambda_n F(y_n))$. The solution set of (42) is denoted by $VI(F, C)$. It is well known that $x^* \in VI(F, C)$ if and only if it satisfies the following projection equation

$$x^* = P_C(x^* - \lambda F(x^*)), \tag{43}$$

where λ is any positive real number. For solving pseudomonotone variational inequality, we propose the following method.

Algorithm 4.1 (Step 0) Choose $\lambda_1 > 0$, $x_0, y_0, y_1 \in C$, $\mu \in (0,1)$, $\alpha \in (0,1)$, $\theta \in (0,1]$, $\delta \in (\frac{\sqrt{1+4(\frac{\alpha}{2-\theta}+1-\alpha)}-1}{2}, 1)$.

(Step 1) Given the current iterate x_{n-1}, y_{n-1}, y_n, compute

$$x_n = (1-\delta)y_n + \delta x_{n-1}.$$
$$y_{n+1} = P_C(x_n - \lambda_n F(y_n)).$$

If $y_{n+1} = x_n = y_n$(or $F(y_n) = 0$), then stop: y_n is a solution. Otherwise, go to step 2.

(Step 2) Compute

$$\lambda_{n+1} = \begin{cases} min\{\frac{\alpha\mu\theta(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta\langle F(y_{n-1}) - F(y_n), y_{n+1} - y_n\rangle}, \lambda_n\}, & if \ \langle F(y_{n-1}) - F(y_n), y_{n+1} - y_n\rangle > 0, \\ \lambda_n, & otherwise. \end{cases}$$

Set $n := n+1$ and return to step 1.

Remark 4.1 If $F(y_n) = 0$, we have $y_n = P_C(y_n - \lambda F(y_n))$. Thus $y_n \in VI(F, C)$ follows directly from (43).

Recall that the mapping F is Lipschitz-continuous with constant $L > 0$, if there exists $L > 0$ such that

$$\| F(x) - F(y) \| \leq L \| x - y \|, \quad \forall x, y \in C. \tag{44}$$

If F is Lipschitz-continuous and pseudomonotone, then the conditions $(C1)-(C3)$ hold for f with $c_2 = c_1 = \frac{L}{2}$. Then the following conclusion follows from Lemma 3.2.

Lemma 4.1 *Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by* Algorithm 4.1 *and $VI(F, C) \neq \emptyset$. Then $\{x_n\}$ and $\{y_n\}$ are bounded.*

The next statement is classical.

Lemma 4.2 [20] *Assume that $F : C \to \mathcal{H}$ is a continuous and pseudomonotone mapping. Then $x^* \in VI(F, C)$ if and only if x^* is a solution of the following problem*

$$find \quad x \in C \quad s.t. \ \langle F(y), y - x\rangle \geqslant 0, \ \ \forall y \in C.$$

We analyze the finite and infinite dimensions separately.

Theorem 4.1 *Let H be a finite dimensional real Hilbert space. Assume that F is a pseudomonotone Lipschitz continuous mapping on C and $VI(F, C)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by* Algorithm 4.1. *Then $\{x_n\}$ and $\{y_n\}$ converge to the same point $x^* \in VI(F, C)$.*

Proof Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ that converges to some $x^* \in H$. From the relation (28), we have $y_{n_k} \to x^*$, $y_{n_k+1} \to x^*$ and $x^* \in C$. Noting the fact that

$$y_{n_k+1} = P_C(x_{n_k} - \lambda_{n_k} F(y_{n_k})). \tag{45}$$

By the continuity of F and the projection, we get

$$x^* = \lim_{k\to\infty} y_{n_k+1} = \lim_{k\to\infty} P_C(x_{n_k} - \lambda_{n_k} F(y_{n_k})) = P_C(x^* - \lambda F(x^*)). \tag{46}$$

We deduce from (43) that $x^* \in VI(F, C)$. By using (33), we obtain $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Combining $\lim_{k\to\infty} x_{n_k} = x^*$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. That is the desired result. □

Inspired by [21], we give the proof of the following theorem.

Theorem 4.2 *Assume that F is pseudomonotone on a infinite dimensional H, sequentially weakly continuous and Lipschitz continuous on C and $VI(F, C)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by* Algorithm 4.1. *Then $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $x^* \in VI(F, C)$.*

Proof From Lemma 4.1, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $x^* \in H$. Then $y_{n_k} \rightharpoonup x^*$ and $x^* \in C$. Next we prove $x^* \in VI(F, C)$. Since $y_{n_k+1} = P_C(x_{n_k} - \lambda_{n_k} F(y_{n_k}))$, by Lemma 2.5, we have

$$\langle y_{n_k+1} - x_{n_k} + \lambda_{n_k} F(y_{n_k}), z - y_{n_k+1}\rangle \geq 0, \quad \forall z \in C. \tag{47}$$

That is

$$\langle x_{n_k} - y_{n_k+1}, z - y_{n_k+1}\rangle \leq \lambda_{n_k} \langle F(y_{n_k}), z - y_{n_k+1}\rangle, \quad \forall z \in C. \tag{48}$$

Therefore, we get

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k+1}, z - y_{n_k+1}\rangle + \langle F(y_{n_k}), y_{n_k+1} - y_{n_k}\rangle \leq \langle F(y_{n_k}), z - y_{n_k}\rangle, \quad \forall z \in C. \tag{49}$$

Fixing $z \in C$, let $k \to \infty$, using the facts (28), $\{y_n\}$ is bounded and $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$, we obtain

$$\liminf_{k\to\infty} \langle F(y_{n_k}), z - y_{n_k}\rangle \geq 0. \tag{50}$$

We choose a decreasing positive sequence $\{\varepsilon_k\}$ such that $\lim_{k\to\infty} \varepsilon_k = 0$. By definition of the lower limit, for each ε_k , we denote by m_k the smallest positive integer such that

$$\langle F(y_{n_i}), z - y_{n_i}\rangle + \varepsilon_k \geq 0, \ \forall i \geq m_k. \tag{51}$$

As $\{\varepsilon_k\}$ is decreasing, it is easy to see that the sequence $\{m_k\}$ is increasing. From Remark 4.1, for each k, $F(y_{n_{m_k}}) \neq 0$. Let $z_{n_{m_k}} = \frac{F(y_{n_{m_k}})}{\|F(y_{n_{m_k}})\|^2}$. Then we get $\langle F(y_{n_{m_k}}), z_{n_{m_k}}\rangle = 1$ for each k. Moreover, from (51), we have

$$\langle F(y_{n_{m_k}}), z + \varepsilon_k z_{n_{m_k}} - y_{n_{m_k}}\rangle \geq 0. \tag{52}$$

By definition of pseudomonotone, we obtain

$$\langle F(z+\varepsilon_k z_{n_{m_k}}), z+\varepsilon_k z_{n_{m_k}} - y_{n_{m_k}}\rangle \geq 0. \tag{53}$$

Since $\{y_{n_k}\}$ converges weakly to $x^* \in C$ and F is sequentially weakly continuous on C, we have $\{F(y_{n_k})\}$ converges weakly to $F(x^*)$. We can suppose that $F(x^*) \neq 0$ (otherwise, x^* is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$\|F(x^*)\| \leq \liminf_{k\to\infty} \|F(y_{n_k})\|. \tag{54}$$

As $\{y_{n_{m_k}}\} \subset \{y_{n_k}\}$ and $\lim_{k\to\infty} \varepsilon_k = 0$, we have

$$0 \leq \lim_{k\to\infty} \|\varepsilon_k z_{n_{m_k}}\| = \lim_{k\to\infty} \frac{\varepsilon_k}{\|F(y_{n_{m_k}})\|} \leq \frac{0}{\|F(x^*)\|} = 0. \tag{55}$$

Let $k \to \infty$ in (53), we get

$$\langle F(z), z - x^*\rangle \geq 0. \ \forall z \in C. \tag{56}$$

By Lemma 4.2, we obtain $x^* \in VI(F, C)$ and as in the proof of Theorem 3.1, we have $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup x^*$. That is the desired result. □

Remark 4.2 When F is monotone, as in [22,23], it is not necessary to impose the sequential weak continuity of F.

Now applying Theorem 3.2 with variational inequalities, we have the following result.

Theorem 4.3 *Assume that F is strong pseudomonotone on a infinite dimensional H, Lipschitz continuous on C and $VI(F, C)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by* Algorithm 4.1. *Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique solution $u \in VI(F, C)$.*

5 Numerical experiments

In this section, we provide numerical experiments to illustrate our algorithm and compare them with other existing algorithms in [15,23,24]. First, we compare Algorithm 3.1 with the Algorithm (3) (Algorithm 3.1 in [15]). Then we compare Algorithm 4.1 with Algorithm A in [23], Algorithm 2.1 in [24] and Algorithm 3.2 in [24]. We report the number of iterations (iter.) and computing time (time) measured in seconds for all the tests. The termination criteria are the following

Alg. 3.1, Alg. 4.1, Alg. (3).	$max\{\|y_{n+1} - x_n\|, \|y_n - x_n\|\} \leq \varepsilon.$
Alg. A in [23].	$max\{\|x_{n+1} - y_n\|, \|x_{n+1} - x_n\|\} \leq \varepsilon.$
Alg. 2.1 in [24], Alg. 3.2 in [24].	$\|x_n - P_C(x_n - F(x_n))\| \leq \varepsilon.$

For Algorithm 3.1, we take $\alpha = \mu = 0.98$, $\theta = 1(\delta = 0.62)$ and $\theta = 0.75(\delta = 0.53)$. For Algorithm A in [21], we use $\alpha = \lambda_0 = 0.4$ and $\delta = 1.001$. For Algorithm 3.2 in [24], we choose $P = I$, $\alpha_{-1} = 1$, $\theta = 1.5$, $\rho = 0.1$ and $\beta = 0.3$. We take $\varepsilon = 10^{-6}$ for all algorithms.

Problem 1 We consider the equilibrium problem for the following bifunction $f : H \times H \to \mathbb{R}$ which comes from the Nash-Cournot equilibrium model in [9–15].

$$f(x, y) = \langle Px + Qy + q, y - x\rangle, \tag{57}$$

where $q \in \mathbb{R}^m$ is chosen randomly with its elements in $[-m, m]$, and the matrices P and Q are two square matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is negative semidefinite. In this case, the bifunction f satisfies $(C1)-(C4)$ with the Lipschitz-type constants $c_1 = c_2 = \frac{\|P-Q\|}{2}$, see [9, Lemma 6.2]. For Algorithm 3.1, we take $\lambda_1 = \frac{1}{2c_1}$. For Algorithm (3), we take $\lambda = \frac{\varphi}{4c_1}$.

For numerical experiments: we suppose that the feasible set $C \subset \mathbb{R}^m$ has the form of

$$C = \{x \in \mathbb{R}^m : -2 \le x_i \le 5, i = 1, \ldots, m\}, \tag{58}$$

where $m = 10, 100, 500$. We take $y_1 = x_0 = y_0 = (1, \ldots, 1)$ for all algorithms. For every m, as shown in Table 1, we have generated two random samples with different choice of P, Q and q. The Table 1 shows that our algorithm may perform better, even if the Lipschitz constants are known.

Problem 2 The second problem is HpHard problem , we choose $F(x) = Mx + q$ with $q \in R^n$ and $M = NN^T + S + D$, where every entry of the $n \times n$ matrix N and of the $n \times n$ skew-symmetric matrix S is uniformly generated from $(-5, 5)$, and every diagonal entry of the $n \times n$ diagonal D is uniformly generated from $(0, 0.3)$ (so M is positive definite), with every entry of q uniformly generated from $(-500, 0)$. The feasible set is R_n^+. This problem was considered in [24]. For all tests, we take $y_1 = x_0 = y_0 = (1, 1, \ldots, 1)$. For Algorithm 4.1, we choose $\lambda_1 = 0.4$. For Algorithm 2.1 in [24], we take $P = (I + M^T)(I + M)$ and $\theta = 0.7$. For every n, as shown in Table 2, we have generated three random samples with different choice of M and q.

Table 1 Problem 1

m	Algorithm 3.1 ($\theta = 1$)		Algorithm 3.1 ($\theta = 0.75$)		Algorithm (3)	
	Iter.	Time	Iter.	Time	Iter.	Time
10	253	2.46	204	2.14	306	3.15
	361	3.69	202	1.94	302	2.88
100	430	8.13	398	6.99	594	8.39
	488	8.49	400	6.91	573	8.59
500	536	73.41	431	71.37	646	81.80
	577	77.83	440	54.27	617	78.09

Problem 3 The third problem was considered in [23,25], where

$$F(x) = (f_1(x), f_2(x), \dots, f_m(x)),$$
$$f_i(x) = x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_i x_{i+1} - 2x_{i-1} + 4x_i + x_{i+1} - 1,$$
$$i = 1, 2, \dots, m, \quad x_0 = x_{m+1} = 0.$$

The feasible set is $C = R_+^m$. We take $\lambda_1 = 0.4$ for Algorithm 4.1. For all tests, we take $y_1 = x_0 = y_0 = (0, 0, \dots, 0)$. The results are summarized in Table 3.

Problem 4 Kojima–Shindo Nonlinear Complementarity Problem (NCP) was considered in [23,25,26], where $n = 4$ and the mapping F is defined by

$$F(x_1, x_2, x_3, x_4) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}$$

The feasible set is $C = \{x \in R_4^+ | x_1 + x_2 + x_3 + x_4 = 4\}$. We choose as starting points: $y_1 = x_0 = y_0 = (1, 1, 1, 1)$ and $y_1 = x_0 = y_0 = (2, 0, 0, 2)$. We take $\lambda_1 = 0.8$ for Algorithm 4.1. The Tables 2, 3 and 4 illustrate that Algorithm 4.1 may work better. As in the previous experiments, Algorithms 3.1 and 4.1 may perform better when choosing $\theta = 0.75$.

Table 2 Problem 2

n	Algorithm 4.1 ($\theta = 1$)		Algorithm 4.1 ($\theta = 0.75$)		Algorithm 2.1 in [24]	
	Iter.	Time	Iter.	Time	Iter.	Time
30	2762	0.043	2470	0.034	5670	0.580
	3327	0.046	2949	0.040	6003	0.617
	4780	0.067	3932	0.054	4468	0.461
200	7214	0.998	5985	0.811	45887	94.2
	5591	0.748	4705	0.686	42884	87.9
	7214	1.060	6168	0.811	39370	80.4

Table 3 Problem 3

m	Algorithm 4.1 ($\theta = 1$)		Algorithm 4.1 ($\theta = 0.75$)		Algorithm A in [23]		Algorithm 3.2 in [24]	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
50	65	0.0017	53	0.0014	49	0.0015	295	0.013
500	70	0.005	58	0.003	53	0.004	313	0.039
5000	76	0.035	62	0.021	58	0.035	333	0.96
50000	81	0.499	66	0.421	62	0.468	353	4.99
500000	87	5.16	71	4.19	67	5.53	373	63.7

Table 4 Problem 4

x_0	Algorithm 4.1 ($\theta = 1$)		Algorithm 4.1 ($\theta = 0.75$)		Algorithm A in [23]	
	Iter.	Time	Iter.	Time	Iter.	Time
(2, 0, 0, 2)	52	0.154	47	0.149	72	0.200
(1, 1, 1, 1)	60	0.166	58	0.169	166	0.304

6 Conclusions

In this work, we consider a convergence result for equilibrium problem involving Lipschitz-type and pseudomonotone bifunctions but the Lipschitz-type constants are unknown. We modify the gradient method with a new step size. A weak and a strong convergence theorem are proved for sequences generated by the algorithm. The numerical experiments confirm the computational effectiveness of the proposed algorithm.

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