Comput Optim Appl (2016) 64:177–214
DOI 10.1007/s10589-015-9804-y

Exact MIP-based approaches for finding maximum quasi-cliques and dense subgraphs

Alexander Veremyev[1] · Oleg A. Prokopyev[2] · Sergiy Butenko[3] · Eduardo L. Pasiliao[4]

Received: 23 December 2014 / Published online: 21 November 2015

Abstract Given a simple graph and a constant $\gamma \in (0, 1]$, a γ-quasi-clique is defined as a subset of vertices that induces a subgraph with an edge density of at least γ. This well-known clique relaxation model arises in a variety of application domains. The maximum γ-quasi-clique problem is to find a γ-quasi-clique of maximum cardinality in the graph and is known to be NP-hard. This paper proposes new mixed integer programming (MIP) formulations for solving the maximum γ-quasi-clique problem. The corresponding linear programming (LP) relaxations are analyzed and shown to be tighter than the LP relaxations of the MIP models available in the literature on sparse graphs. The developed methodology is naturally generalized for solving the maximum $f(\cdot)$-dense subgraph problem, which, for a given function $f(\cdot)$, seeks for the largest k such that there is a subgraph induced by k vertices with at least $f(k)$

✉ Sergiy Butenko
butenko@tamu.edu

Alexander Veremyev
averemyev@ufl.edu

Oleg A. Prokopyev
droleg@pitt.edu

Eduardo L. Pasiliao
eduardo.pasiliao@eglin.af.mil

1 Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611-6595, USA

2 Department of Industrial Engineering, University of Pittsburgh, Pittsburgh, PA 15261, USA

3 Department of Industrial and Systems Engineering, Texas A&M University, College Station, TX 77843, USA

4 Munitions Directorate, Air Force Research Laboratory, Eglin AFB, FL 32542, USA

edges. The performance of the proposed exact approaches is illustrated on real-life network instances with up to 10,000 vertices.

Keywords Quasi-clique · s-Defective clique · Average s-plex · Dense subgraph · Clique relaxation · Mixed integer programming

1 Introduction

Let $G = (V, E)$ be a simple graph with the sets of vertices (nodes) and edges denoted by V and E, respectively. Graph G is *complete* if it has all possible edges, i.e., $(i, j) \in E$ for any $i, j \in V$ $(i \neq j)$. For any subset $S \subseteq V$, $G[S] = (S, (S \times S) \cap E)$ denotes the subgraph induced by S in G. A *clique* C is a subset of V such that $G[C]$ is a complete graph [30]. The *maximum clique problem* is to find a clique of maximum cardinality in G [12,35]. This problem is known to be NP-hard [20]. The size of the maximum clique in G is denoted by $\omega(G)$ and is referred to as the *clique number* of G.

The concept of a clique is used in a number of application areas due to its elegance and inherent ability to logically represent cohesive subgroups (also, referred to as *clusters*), of "tightly knit" elements (i.e., nodes) of complex systems modeled as a graph [12,14]. Indeed, cliques have a number of ideal cohesiveness properties [37]. For example, each vertex is connected to all other vertices in a clique, the distance between any pair of vertices in a clique is one, a clique has maximum possible edge density, etc. However, in many real-life applications, using cliques for discovering *large* cohesive clusters is impractical due to the fact that the definition of a clique is rather idealistic and, thus, can be too limiting. Consequently, a number of *clique relaxation* definitions has appeared in the literature in recent years, see, e.g., [6,7], a unifying taxonomic framework in [37], and references therein.

In particular, one of the most popular and widely applied clique relaxation models is the γ-*quasi-clique*, an *edge-based* clique relaxation defined as a subset $S \subseteq V$ such that the subgraph $G[S]$ induced by S in G has the edge density of at least γ, that is, $\rho(G[S]) = |(S \times S) \cap E|/\binom{|S|}{2} \geq \gamma$, where $\gamma \in (0, 1]$ is a fixed constant parameter [2]. The problem of finding a maximum γ-quasi-clique is known to be NP-hard for any fixed $\gamma \in (0, 1]$, see [36]. The cardinality of a maximum γ-quasi-clique in G is denoted by $\omega_\gamma(G)$. Clearly, $\gamma = 1$ corresponds to the clique and $\omega_1(G) = \omega(G)$.

The popularity of the γ-quasi-clique can be attributed to the fact that the underlying concept is rather intuitive and relatively simple. Moreover, missing edges (i.e., links in the networked system under consideration) in *dense* clusters can often be justified by measurement errors in real-life data sets. For example, as discussed in [50], in some biological experiments there is "a higher absolute degree of confidence when an interaction is observed, but a much lower degree when no interaction is detected."

The concept of the γ-quasi-clique is further generalized using the notion of the $f(k)$-*dense-subgraph*, defined as a k-vertex subset $S \subseteq V$ such that subgraph $G[S]$ induced by S in G contains at least $f(k)$ edges, where $f : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is some fixed function. Observe that if $f(k) = \gamma k(k-1)/2$ then the $f(k)$-dense-subgraph is equivalent to the γ-quasi-clique. It is known that the problem of checking whether the graph contains a k-vertex $f(k)$-dense-subgraphs is NP-complete if

– $f(k) = \Theta(k^{1+\varepsilon})$, where $\varepsilon \in (0, 1)$ [4];
– $f(k) = k + \Omega(k^{\varepsilon})$, where $\varepsilon \in (0, 1)$ [24].

Moreover, the second result is "sharp" in the sense that the problem becomes polynomially solvable for $f(k) = k + c$ for any constant c [24].

The concept of $f(k)$-dense-subgraphs encompasses other edge-based clique relaxation models from the literature. For instance, if one sets $f(k) = k(k-1)/2 - s$, then the resulting subset of vertices is known as an *s-defective clique*, formally defined as a subset $S \subseteq V$ such that the induced subgraph $G[S]$ has at most s missing edges. Furthermore, $f(k) = k(k-s)/2$ corresponds to an *average s-plex*, which is a statistical clique relaxation defined as a subset $S \subseteq V$ such that the average degree of vertices in $G[S]$ is at least $|S| - s$. We refer the reader to [21,37] for a more detailed discussion of these clique relaxation models.

The problem of finding large dense subgraphs naturally arises in a number of application areas, including biology [5,9,13,23,25,31,38,42], social networks analysis [16,17,39,46,47], telecommunication [1,2] and finance [10,11,26,39,41]. Nevertheless, the literature on *exact* computational methods for this class of problems is extremely sparse, and most of the focus has been on the development and application of heuristic methods, see, e.g., [1,2]. In particular, to the best of our knowledge, there are no exact approaches for finding maximum $f(\cdot)$-dense subgraphs for a general function $f(\cdot)$. With respect to the maximum γ-quasi-clique problem, there are only two related papers dealing with exact methods. Namely,

(i) two mixed integer programming (MIP) models are proposed in [36] (we review them in Sect. 2.1), and
(ii) the work in [34] describes a combinatorial branch-and-bound (B&B) algorithm and compares its performance against an exact MIP solver with the models from [36] (we provide some additional discussion on the efficiency of the method in Sect. 4).

One can argue that relative scarcity of exact methods can be attributed to the fact that γ-quasi-cliques are not *hereditary* (i.e., a subset of a γ-quasi-clique is not necessarily a γ-quasi-clique), which "introduces interesting challenges in the development of exact algorithms" [34] and preprocessing techniques. In contrast, there exist very effective exact solvers for finding maximum cliques, e.g., [33], and maximum subgraphs that satisfy a given hereditary property [44], which heavily exploit the hereditary property in their algorithmic design. Additionally, one should mention about a considerable body of work on bounds for the clique number $\omega(G)$, in particular those that are computable in polynomial time; see a detailed discussion in [12]. Such bounds can be exploited within exact and heuristic solution approaches. A notable example includes the celebrated *Lovász number* $\vartheta(\bar{G})$ [29], which is computable in polynomial time, e.g., by using semidefinite programming methods [28].

In view of the discussion above, the contributions of this paper are as follows:

(i) We propose four new MIP models for solving the maximum γ-quasi-clique problem (Sect. 2.2). We provide theoretical analysis of the quality of their LP relaxations and demonstrate that the proposed models result in much stronger LP relaxations than previous MIPs from [36] for sufficiently sparse graphs (Sect. 2.3). Additionally, we describe two easily implementable iterative methods that solve a sequence of feasibility MIPs, obtained by fixing the values for some of the variables in the proposed MIP models (Sect. 2.4).

(ii) We demonstrate that our MIP models can be generalized (in a rather simple manner, see Sect. 3) to find maximum $f(\cdot)$-dense subgraphs for any nonnegative function $f(\cdot)$, including the two notable examples discussed above, namely, the s-defective clique and the average s-plex. Furthermore, our theoretical results regarding the quality of their LP relaxations established for γ-quasi-cliques can also be extended (admittedly, under some mild conditions).

(iii) Finally, our computational experiments (see Sect. 4) using real-life test instances (including social, biological and communication networks) demonstrate that the proposed solution methods for finding a maximum γ-quasi-clique outperform previous approaches (specifically, MIPs from [36]) for sufficiently sparse graphs, which provides an experimental illustration of our theoretical results. We should emphasize here that the vast majority of real-life graphs, including those used in this paper, are rather sparse.

2 MIP models

2.1 Known formulations from [36]

For each vertex $i \in V$, we define a binary variable x_i such that $x_i = 1$ iff $i \in S$, where $S \subseteq V$. Clearly, S is a γ-quasi-clique if the following condition holds:

$$\sum_{(i,j)\in E} x_i x_j \geq \gamma \sum_{i,j\in V,\ i<j} x_i x_j, \tag{1}$$

where the left- and right-hand sides of constraint (1) represent the number of edges in $G[S]$ and the γ-fraction of the maximum possible number of edges in $G[S]$, respectively.

Observe that constraint (1) is nonlinear. However, it can be linearized (see, e.g., discussion and references in [3]), which results in two linear MIP models proposed in [36]. Specifically, the first formulation is based on introducing a new variable $y_{ij} = x_i x_j$ for each i and $j \in V$, $i < j$, and is given by:

Model 1 (**F1**)

$$\omega_\gamma(G) = \max_{\mathbf{x},\mathbf{y}} \sum_{i\in V} x_i \tag{2a}$$

$$s.t. \quad \sum_{(i,j)\in E} y_{ij} \geq \gamma \sum_{i,j\in V,\ i<j} y_{ij}, \tag{2b}$$

$$y_{ij} \leq x_i, \quad y_{ij} \leq x_j, \quad y_{ij} \geq x_i + x_j - 1, \quad \forall i, j \in V,\ i < j, \tag{2c}$$

$$x_i \in \{0, 1\},\ y_{ij} \geq 0, \quad \forall i, j \in V,\ i < j, \tag{2d}$$

where constraints (2c) ensure that the linearization of the nonlinear term $y_{ij} = x_i x_j$ is valid. Model **F1** requires a number $|V|$ of binary and $\Theta(|V|^2)$ of continuous variables.

The second MIP is based on introducing a new variable of the form

$$u_i = x_i\Big(\gamma x_i + \sum_{j \in V}(\mathbb{1}_{(i,j)\in E} - \gamma)x_j\Big) \tag{3}$$

for each $i \in V$, and is given by:

Model 2 (**F2**)

$$\omega_\gamma(G) = \max_{\mathbf{u},\mathbf{x}} \quad \sum_{i \in V} x_i \tag{4a}$$

$$s.t. \quad \sum_{i \in V} u_i \geq 0, \tag{4b}$$

$$u_i \leq M x_i, \quad \forall i \in V, \tag{4c}$$

$$u_i \geq -M x_i, \quad \forall i \in V, \tag{4d}$$

$$u_i \geq \gamma x_i + \sum_{j \in V}(\mathbb{1}_{(i,j)\in E} - \gamma)x_j - M(1 - x_i), \quad \forall i \in V, \tag{4e}$$

$$u_i \leq \gamma x_i + \sum_{j \in V}(\mathbb{1}_{(i,j)\in E} - \gamma)x_j + M(1 - x_i), \quad \forall i \in V, \tag{4f}$$

$$x_i \in \{0, 1\}, \quad \forall i \in V, \tag{4g}$$

where M is a sufficiently large constant, e.g., $M \geq |V|$, symbol $\mathbb{1}$ denotes the standard indicator function (i.e., in (4e) and (4f) it returns 1 iff $(i, j) \in E$), and constraints (4c)–(4f) ensure that the linearization of the nonlinear term (3) is valid. Model **F2** requires $|V|$ binary and $|V|$ continuous variables. Note that model **F2** requires less variables than **F1**. However, the results of computational experiments reported in [36] indicate that, in general, neither of the models is dominated by the other one (both with respect to the quality of their LP relaxation bounds and the performance of commercial MIP solvers when solving the problem exactly).

2.2 New formulations

2.2.1 Quasi-clique size decomposition

Let ω^u and ω^ℓ be some upper and lower bounds on the size of a maximum γ-quasi-clique in G, respectively. The lower bound can be set to 1 if there is no information available about the sizes of γ-quasi-cliques in G. On the other hand, the value of ω^ℓ can be increased using the size of some heuristically identified quasi-clique, e.g., it can be set to be the size of any known (possibly, maximum) clique in G. The upper bound ω^u can be simply set to $|V|$, or we can use the result from [36] given by:

$$\omega^u = \left\lfloor \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\frac{|E|}{\gamma}} \right\rfloor. \tag{5}$$

It is easy to verify that if $\omega^u \geq |V|$ in (5), then $|E| \geq \gamma \frac{|V|(|V|-1)}{2}$, i.e., graph G is a γ-quasi-clique. We should also note that in case of very large graphs, in order to derive some non-trivial lower and upper bounds, one can exploit asymptotic results regarding the size of the maximum γ-quasi-clique available for some classes of graphs, see, e.g., recent work in [46].

Next, we redefine y_{ij} for each $(i, j) \in E$ to be a binary variable such that $y_{ij} = 1$ iff $(i, j) \in E \cap (S \times S)$, i.e., an edge (i, j) is in a subgraph $G[S]$. Also, define z_k, $k = 1, \ldots, |V|$, to be a set of binary variables that determine the size of S, namely, $z_k = 1$ iff $|S| = k$. Using this notation, we propose the following formulation for finding a maximum γ-quasi-clique based on the classical value-disjunction idea (see, e.g., [32]), which is applied to the size of a maximum γ-quasi-clique:

Model 3 (**F3**)

$$\omega_\gamma(G) = \max_{\mathbf{x},\mathbf{y},\mathbf{z}} \sum_{i \in V} x_i \tag{6a}$$

$$s.t. \quad \sum_{(i,j)\in E} y_{ij} \geq \gamma \sum_{k=\omega^\ell}^{\omega^u} \frac{k(k-1)}{2} z_k, \tag{6b}$$

$$y_{ij} \leq x_i, \quad y_{ij} \leq x_j, \quad \forall (i, j) \in E, \tag{6c}$$

$$\sum_{i \in V} x_i = \sum_{k=\omega^\ell}^{\omega^u} k z_k, \quad \sum_{k=\omega^\ell}^{\omega^u} z_k = 1, \tag{6d}$$

$$x_i \in \{0, 1\}, \quad y_{ij} \geq 0, \quad \forall i, j \in V, \ i < j, \tag{6e}$$

$$z_k \geq 0, \quad \forall k \in \{\omega^\ell, \ldots, \omega^u\}, \tag{6f}$$

where constraints (6c) ensure that y_{ij} can be set to 1 only if both vertices i and j are in S, i.e., $x_i = x_j = 1$. Constraint (6b) represents the edge density requirements for the induced subgraph $G[S]$, while constraints (6d) enforce the proper value in the right hand-side of (6b). Clearly, variables y_{ij} can be relaxed to be continuous due to the structure of (6b) and (6c) and the fact that (6a) involves maximization of a linear function of x_i, $i \in V$, with positive coefficients. Note that the binary restrictions for z_k's are replaced by nonnegativity in constraint (6f). The following proposition shows that this relaxation is valid.

Proposition 1 *There exists an optimal solution* $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ *of MIP* ***F3*** *such that* $\mathbf{z}^*$ *is a binary vector.*

Proof Consider **F3** without integrality restrictions for z_k, $k \in \{\omega^\ell, \ldots, \omega^u\}$. Suppose its optimal solution is given by $(\mathbf{x}^*, \mathbf{y}^*, \bar{\mathbf{z}})$, where $\bar{\mathbf{z}}$ is not a 0–1 vector. Define $\mathbf{z}^* = (z^*_{\omega^\ell}, \ldots, z^*_{\omega^u})$ as follows:

$$z_k^* = \begin{cases} 1, & \text{if } \sum_{i \in V} x_i^* = k, \\ 0, & \text{if } \sum_{i \in V} x_i^* \neq k. \end{cases} \tag{7}$$

Clearly, $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ satisfies constraints (6c) and (6d). In particular,

$$\sum_{k=\omega^\ell}^{\omega^u} k z_k^* = \sum_{k=\omega^\ell}^{\omega^u} k \bar{z}_k = \sum_{i \in V} x_i^* \quad \text{and} \quad \sum_{k=\omega^\ell}^{\omega^u} z_k^* = 1 \tag{8}$$

by our construction. Then

$$\begin{aligned} \sum_{(i,j) \in E} y_{ij}^* &\geq \frac{\gamma}{2} \sum_{k=\omega^\ell}^{\omega^u} k(k-1)\bar{z}_k \geq \frac{\gamma}{2} \left(\sum_{k=\omega^\ell}^{\omega^u} k\bar{z}_k \right) \left(\sum_{k=\omega^\ell}^{\omega^u} k\bar{z}_k - 1 \right) \\ &= \frac{\gamma}{2} \left(\sum_{k=\omega^\ell}^{\omega^u} k z_k^* \right) \left(\sum_{k=\omega^\ell}^{\omega^u} k z_k^* - 1 \right) = \frac{\gamma}{2} \sum_{k=\omega^\ell}^{\omega^u} k(k-1) z_k^*, \end{aligned}$$

where the first inequality holds by the definition of $(\mathbf{x}^*, \mathbf{y}^*, \bar{\mathbf{z}})$, the second inequality holds by Jensen's inequality (see, e.g., [22], and note that function $f(k) = k(k-1)$ is convex) and the last two equalities follow from (8) and the fact that $\mathbf{z}^*$ is a binary vector by construction. Therefore, $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ satisfies (6b) and is an optimal solution of **F3**. □

To derive the next MIP model, we use an idea similar to the one behind **F2**. Note that $y_{ij} = x_i x_j$ for all $(i, j) \in E$. Thus, the left-hand side of constraint (6b) can be rewritten as

$$\sum_{(i,j) \in E} y_{ij} = \frac{1}{2} \sum_{i \in V} \sum_{j:\,(i,j) \in E} x_i x_j = \frac{1}{2} \sum_{i \in V} x_i \sum_{j:\,(i,j) \in E} x_j.$$

Defining a new set of variables v_i, $i \in V$, such that

$$v_i = x_i \sum_{j:\,(i,j) \in E} x_j, \tag{9}$$

we obtain:

Model 4 (**F4**)

$$\omega_\gamma(G) = \max_{\mathbf{x}, \mathbf{v}, \mathbf{z}} \sum_{i \in V} x_i \tag{10a}$$

$$s.t. \quad \sum_{i \in V} v_i \geq \gamma \sum_{k=\omega^\ell}^{\omega^u} k(k-1) z_k, \tag{10b}$$

$$v_i \le \mu_i x_i, \quad v_i \le \sum_{j:\ (i,j)\in E} x_j, \quad \forall i \in V, \tag{10c}$$

$$\sum_{i\in V} x_i = \sum_{k=\omega^\ell}^{\omega^u} k z_k, \quad \sum_{k=\omega^\ell}^{\omega^u} z_k = 1, \tag{10d}$$

$$x_i \in \{0,1\}, \ v_i \ge 0, \quad \forall i \in V, \tag{10e}$$

$$z_k \ge 0, \quad \forall k \in \{\omega^\ell, \ldots, \omega^u\}, \tag{10f}$$

where μ_i is a sufficiently large constant parameter. In particular, we let $\mu_i = deg_G(i)$, where $deg_G(i)$ denotes the degree of vertex i, which is its number of neighbors in G. Constraints (10c) ensure that the linearization of (9) is valid and v_i can be set to $\sum_{j:\ (i,j)\in E} x_j$ only if $i \in S$, i.e., $x_i = 1$. Just like in formulation **F3**, the integrality restrictions for variables z_k are relaxed in (10f). We can show that there is always an optimal solution with $z_k \in \{0,1\}$ for all k similarly to how it was done in Proposition 1.

Note that one could easily generalize both models **F3** and **F4** to consider positive vertex weights in their objectives. In terms of the number of variables, **F3** needs $|V|$ binary and $O(|V| + |E|)$ continuous variables, while **F4** requires $|V|$ binary and $O(|V|)$ continuous variables.

2.2.2 Logarithmic reduction

Observe that MIPs **F3** and **F4** include ω^u variables $z_i, i \in \{1, \ldots, \omega^u\}$ in the worst case (if a nontrivial lower bound is not known, i.e., ω^ℓ is set to 1). However, this number can be reduced to $\lfloor \log_2 \omega^u + 1 \rfloor + \lfloor \log_2 \omega^u + 1 \rfloor^2$ by using the standard logarithmic reformulation technique [27,32]. Specifically, let $t_k, k \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}$, be binary variables such that

$$\sum_{i\in V} x_i = \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k, \tag{11}$$

which implies that

$$\left(\sum_{i\in V} x_i\right)^2 = \left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right)^2 = \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} t_k t_\ell$$
$$= \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell},$$

where $s_{k\ell} = t_k t_\ell$ for $k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}$. Then the right-hand side of constraint (6b) can be rewritten as

$$\gamma \frac{\sum_{i\in V} x_i \left(\sum_{i\in V} x_i - 1\right)}{2} = \frac{\gamma}{2}\left(\left(\sum_{i\in V} x_i\right)^2 - \sum_{i\in V} x_i\right)$$
$$= \frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right). \quad (12)$$

Using (11) and (12), model **F3** can be modified as follows:

Model 5 (**F3log**)

$$\omega_\gamma(G) = \max_{\mathbf{s},\mathbf{t},\mathbf{x},\mathbf{y}} \sum_{i\in V} x_i \quad (13a)$$
$$s.t. \quad \sum_{(i,j)\in E} y_{ij} \geq \frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right), \quad (13b)$$
$$y_{ij} \leq x_i, \quad y_{ij} \leq x_j, \quad \forall (i,j) \in E, \quad (13c)$$
$$\sum_{i\in V} x_i = \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k, \quad (13d)$$
$$s_{k\ell} \leq t_k, \quad s_{k\ell} \leq t_\ell, \quad s_{k\ell} \geq t_k + t_\ell - 1, \quad \forall k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}, \quad (13e)$$
$$s_{k\ell}, \ t_k, \ x_i \in \{0,1\}, \quad \forall k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}, \ i \in V, \quad (13f)$$
$$y_{ij} \geq 0, \quad i, j \in V, \ i < j, \quad (13g)$$

where constraints (13e) ensure that the linearization of the nonlinear term $s_{k\ell} = t_k t_\ell$ is valid. In fact, due to the structure of (13e) variables $s_{k\ell}$ can be relaxed to be nonnegative continuous.

Finally, we note that, using equalities (11) and (12), formulation **F4** can be modified in a similar manner. The resulting formulation is omitted for brevity. However, in the remainder in the paper we refer to the corresponding MIP model as **F4log**.

2.3 LP relaxation bounds

Performance of standard MIP solvers based on the branch-and-bound framework is heavily dependent on the quality of the LP relaxations in the considered MIP models. Next, we provide theoretical analysis of such relaxations in the context of the MIP formulations described in Sects. 2.1 and 2.2.

Formally, let $\omega^1_{LP}(\gamma)$, $\omega^2_{LP}(\gamma)$, $\omega^3_{LP}(\gamma)$, $\omega^4_{LP}(\gamma)$, $\omega^{3\ell og}_{LP}(\gamma)$, $\omega^{4\ell og}_{LP}(\gamma)$ be the optimal objective function values of the LP relaxations of **F1**, **F2**, **F3**, **F4**, **F3log** and **F4log**, respectively.

Theorem 1 *The following inequalities hold:*

$$\text{(i)} \quad \omega^\alpha_{LP}(\gamma) \geq |V|/2, \quad \alpha \in \{1,2\}; \quad (14)$$

$$\text{(ii)} \quad \omega_{LP}^{\alpha}(\gamma) \le \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\frac{|E|}{\gamma}}, \quad \alpha \in \{3, 4\}; \tag{15}$$

$$\text{(iii)} \quad \omega_{LP}^{\alpha}(\gamma) \le \frac{1}{2} + \frac{1}{2}\sqrt{1 + 16(\omega^u)^2 + 8\frac{|E|}{\gamma}}, \quad \alpha \in \{3\ell og, 4\ell og\}; \tag{16}$$

$$\text{(iv)} \quad \omega_{LP}^{3}(\gamma) \le \omega_{LP}^{3\ell og}(\gamma) \quad and \quad \omega_{LP}^{4}(\gamma) \le \omega_{LP}^{4\ell og}(\gamma); \tag{17}$$

$$\text{(v)} \quad \omega_{LP}^{3}(\gamma) \le \omega_{LP}^{4}(\gamma) \quad and \quad \omega_{LP}^{3\ell og}(\gamma) \le \omega_{LP}^{4\ell og}(\gamma); \tag{18}$$

$$\text{(vi)} \quad \omega_{LP}^{\alpha}(\gamma) \ge \frac{2|E|}{\gamma|V|} + 1, \quad \alpha \in \{3, 4\}. \tag{19}$$

Proof (i) Let $x_i = \frac{1}{2}$ for all $i \in V$, $y_{ij} = 0$ for all $i, j \in V$, $i < j$, and $u_i = 0$ for all $i \in V$. Then one can verify that $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{u})$ are feasible solutions of the LP relaxations of **F1** and **F2**, respectively. Clearly, inequality (14) holds by construction.

(ii) Consider $\alpha = 3$. Denote by $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ an optimal solution of the LP relaxation of **F3**. Then

$$|E| \ge \sum_{(i,j)\in E} y_{ij} \ge \gamma \sum_{k=\omega^\ell}^{\omega^u} \frac{k(k-1)}{2} z_k \ge \gamma \frac{\omega_{LP}^3(\gamma)(\omega_{LP}^3(\gamma) - 1)}{2}, \tag{20}$$

where the last relation holds by Jensen's inequality taking into account that $\omega_{LP}^3(\gamma) = \sum_{i\in V} x_i = \sum_{k=\omega^\ell}^{\omega^u} k z_k$. Then the upper bound (15) follows by solving the quadratic inequality with respect to $\omega_{LP}^3(\gamma)$ obtained by considering the left- and right-hand sides of (20). For $\alpha = 4$, the result follows from a similar observation that any feasible solution $(\mathbf{x}, \mathbf{v}, \mathbf{z})$ of the LP relaxation of **F4** has to satisfy:

$$2|E| \ge \sum_{i\in V} v_i \ge \gamma \sum_{k=\omega^\ell}^{\omega^u} k(k-1) z_k \ge \gamma \omega_{LP}^4(\gamma)(\omega_{LP}^4(\gamma) - 1), \tag{21}$$

where the first inequality is due to constraints (10c) and our choice of μ_i for all $i \in V$ in **F4**.

(iii) Consider $\alpha = 3\ell og$. For any optimal solution $(\mathbf{s}, \mathbf{t}, \mathbf{x}, \mathbf{y})$ of the LP relaxation of **F3log** we observe that

$$\begin{aligned} |E| \ge \sum_{(i,j)\in E} y_{ij} &\ge \frac{\gamma}{2} \left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k \right) \\ &= \frac{\gamma}{2} \left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \omega_{LP}^{3\ell og}(\gamma) \right), \end{aligned} \tag{22}$$

where the last equality follows from (13d). Note that $s_{k\ell} \geq t_k + t_\ell - 1$ for all $k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}$ by (13e), and $t_k + t_\ell \geq t_k t_\ell$ since $t_k, t_\ell \in [0, 1]$. Hence:

$$\begin{aligned}
\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} &\geq \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell}(t_k + t_\ell - 1) \\
&\geq \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell}(t_k t_\ell) - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} \sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} \\
&\geq \left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k \right)^2 - \left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k \right)^2 \\
&\geq \left(\omega_{LP}^{3\ell og}(\gamma) \right)^2 - 4(\omega^u)^2, \qquad (23)
\end{aligned}$$

where we use (13d) and the fact that $2\omega^u \geq \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k$. Combining (22) and (23) we obtain:

$$|E| + 2\gamma(\omega^u)^2 \geq \gamma \frac{\omega_{LP}^{3\ell og}(\gamma)(\omega_{LP}^{3\ell og}(\gamma) - 1)}{2},$$

which is a quadratic inequality with respect to $\omega_{LP}^{3\ell og}(\gamma)$. It can be easily solved to derive (16). The case of $\alpha = 4\ell og$ can be proved similarly.

(iv) We provide the proof only for the first inequality in (17), namely, $\omega_{LP}^{3}(\gamma) \leq \omega_{LP}^{3\ell og}(\gamma)$. The proof of the second inequality in (17) can be constructed in a similar manner.

Specifically, to establish the result we show that for any feasible solution of the LP relaxation of **F3** there exists a feasible solution of the LP relaxation of **F3log** with the same objective function value. Let $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be a feasible solution of the LP relaxation of **F3**. Hence, it satisfies (6b)–(6d). Thus, using Jensen's inequality, we can conclude that:

$$\begin{aligned}
\sum_{(i,j)\in E} y_{ij} \geq \gamma \sum_{k=\omega^\ell}^{\omega^u} \frac{k(k-1)}{2} z_k &\geq \frac{\gamma \left(\sum_{i\in V} x_i \left(\sum_{i\in V} x_i - 1 \right) \right)}{2} \\
&= \frac{\gamma}{2} \left(\left(\sum_{i\in V} x_i \right)^2 - \sum_{i\in V} x_i \right).
\end{aligned} \qquad (24)$$

Observe that $g(\mathbf{t}) = \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k$ is a continuous function for $\mathbf{t} \in [0, 1]^{\lfloor \log_2 \omega^u \rfloor + 1} \subset \mathbb{R}_+^{\lfloor \log_2 \omega^u \rfloor + 1}$, which takes all possible values in $[0, \omega^u]$. Note that $\sum_{i\in V} x_i \leq \omega^u$. Thus, there exists $\mathbf{t} \in [0, 1]^{\lfloor \log_2 \omega^u \rfloor + 1} \subset \mathbb{R}_+^{\lfloor \log_2 \omega^u \rfloor + 1}$ such that

$$g(\mathbf{t}) = \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k = \sum_{i \in V} x_i,$$

and (13d) in **F3log** holds for $\mathbf{x}$ and $\mathbf{t}$. Then, the right-hand side of (24) can be rewritten as:

$$\frac{\gamma}{2}\left(\left(\sum_{i \in V} x_i\right)^2 - \sum_{i \in V} x_i\right) = \frac{\gamma}{2}\left(\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right)^2 - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right)$$
$$= \frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor}\sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} t_k t_\ell - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right). \quad (25)$$

Define $\mathbf{s}$ as $s_{k\ell} = \max\{0, t_k + t_\ell - 1\}$ for all $k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\}$, which implies that constraints (13e) in **F3log** hold. Moreover, $(1-t_k)(1-t_\ell) = 1-t_k-t_\ell+t_k t_\ell \geq 0$. Thus,

$$s_{k\ell} \leq t_k t_\ell, \quad \forall k, \ell \in \{0, \ldots, \lfloor \log_2 \omega^u \rfloor\},$$

and

$$\frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor}\sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} t_k t_\ell - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right)$$
$$\geq \frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor}\sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right) \quad (26)$$

Combining (24), (25) and (26) we obtain:

$$\sum_{(i,j) \in E} y_{ij} \geq \frac{\gamma}{2}\left(\sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor}\sum_{\ell=0}^{\lfloor \log_2 \omega^u \rfloor} 2^{k+\ell} s_{k\ell} - \sum_{k=0}^{\lfloor \log_2 \omega^u \rfloor} 2^k t_k\right),$$

which implies that constraint (13b) is satisfied for $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{y}$. Summarizing all the above observations, we conclude that $(\mathbf{s}, \mathbf{t}, \mathbf{x}, \mathbf{y})$ is a feasible solution of the LP relaxation of **F3log** with the same objective function value as the LP relaxation of **F3**.

(v) We provide the proof only for the first inequality in (18), namely, $\omega^3_{LP}(\gamma) \leq \omega^4_{LP}(\gamma)$. The proof of the second inequality in (18) can be derived similarly. In particular, to establish the result we show that for any feasible solution of the LP relaxation of **F3** there exists a feasible solution of the LP relaxation of **F4** with the same objective function value.

Let $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be a feasible solution of the LP relaxation of **F3**. Without loss of generality, we assume that $y_{ij} = \min\{x_i, x_j\}$ for any $(i, j) \in E$. Then for any $i \in V$, we define $v_i = \min\{deg_G(i)x_i, \sum_{j:\,(i,j)\in E} x_j\}$. Clearly, $\mathbf{v} = (v_1, \ldots, v_n)^T$ satisfies constraint (10c). Moreover,

$$2\sum_{(i,j)\in E} y_{ij} = 2\sum_{(i,j)\in E}\min\{x_i, x_j\} = \sum_{i\in V}\sum_{j:\,(i,j)\in E}\min\{x_i, x_j\}$$
$$\leq \sum_{i\in V}\min\{deg_G(i)x_i, \sum_{j:\,(i,j)\in E} x_j\} = \sum_{i\in V} v_i.$$

Therefore, $\mathbf{v}$ satisfies constraint (10b), and $(\mathbf{x}, \mathbf{v}, \mathbf{z})$ is a feasible solution of the LP relaxation of **F4** with the same objective function value as the LP relaxation of **F3**.

(vi) Consider $\alpha = 3$. Let $x_i = \delta$ for all $i \in V$, and $y_{ij} = \delta$ for all $(i, j) \in E$ for some $\delta \in \left[0, \frac{\omega^u}{|V|}\right] \subset \mathbb{R}_+$. Note that there exists $\mathbf{z} = (z_{\omega^\ell}, \ldots, z_{\omega^u})^\top$, such that

$$\delta|V| = \sum_{i\in V} x_i = \sum_{k=\omega^\ell}^{\omega^u} k z_k \quad \text{and} \quad \sum_{k=\omega^\ell}^{\omega^u} z_k = 1.$$

Therefore, constraints (6c) and (6d) hold, and $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a feasible solution of the LP relaxation of **F3** if constraint (6b) is also satisfied, i.e.,

$$\delta|E| = \sum_{(i,j)\in E} y_{ij} \geq \gamma \sum_{k=\omega^\ell}^{\omega^u} \frac{k(k-1)}{2} z_k \geq \frac{\gamma}{2}\left(\left(\sum_{i\in V} x_i\right)^2 - \sum_{i\in V} x_i\right)$$
$$= \frac{\gamma}{2}\delta|V| \cdot (\delta|V| - 1),$$

or, equivalently,

$$\delta|V| \leq \frac{2|E|}{\gamma|V|} + 1.$$

Hence, if one sets $\delta = \frac{1}{|V|}\left(\frac{2|E|}{\gamma|V|} + 1\right)$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ provides a feasible solution of the LP relaxation of **F3** and

$$\omega^\alpha_{LP}(\gamma) \geq \delta|V| = \frac{2|E|}{\gamma|V|} + 1.$$

Finally, it is easy to verify that $\delta \leq \frac{\omega^u}{|V|}$ and $\frac{2|E|}{\gamma|V|} + 1 \leq \omega^u$ as long as G is not a γ-quasi-clique. The case of $\alpha = 4$ can be shown similarly. □

Remark 1 Note that in proving inequalities (15) and (17)–(19) we do not use any non-trivial lower or upper bound for the size of the maximum γ-quasi-clique (see (5) and the related discussion at the beginning of Sect. 2.2.1). Thus, (15)–(19) hold even if ω^ℓ and ω^u are set to 1 and $|V|$, respectively, in MIPs **F3**, **F4**, **F3log** and **F4log**.

Remark 2 The upper bound given by (15) is sharp. Specifically, for any $\gamma \in (0, 1]$ there exists G such that

$$\omega_\gamma(G) = \omega^\alpha_{LP}(\gamma) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\frac{|E|}{\gamma}},$$

where $\alpha \in \{3, 4\}$.

More importantly, the LP relaxation bounds established in Theorem 1 allow us to conclude that it may be preferable to use formulations **F3** and **F4** instead of **F1** and **F2** if G is sufficiently sparse. Formally:

Corollary 1 *Let $\alpha \in \{1, 2\}$ and $\beta \in \{3, 4\}$. If $|E| \leq \gamma \frac{|V|^2 - 2|V|}{8}$, then $\omega^\alpha_{LP}(\gamma) \geq \omega^\beta_{LP}(\gamma)$. Moreover:*

$$\omega^\alpha_{LP}(\gamma) - \omega^\beta_{LP}(\gamma) \geq \frac{1}{2}\left(|V| - 1 - \sqrt{1 + 8\frac{|E|}{\gamma}}\right).$$

In particular, note that if $|E| = O(|V|)$, then the difference between $\omega^\alpha_{LP}(\gamma)$ and $\omega^\beta_{LP}(\gamma)$ for the aforementioned α and β becomes quite significant, namely, $\Theta(|V|)$. On the other hand, if $|E| = \Theta(|V|^2)$, then the bounds established in (14) and (19) imply that both $\omega^\alpha_{LP}(\gamma)$ and $\omega^\beta_{LP}(\gamma)$ behave as $\Theta(|V|)$.

With respect to formulations **F3log** and **F4log** and their comparison to **F1** and **F2**, in addition to sparsity of G, we should also require availability of some non-trivial upper bound. Specifically:

Corollary 2 *Let $\alpha \in \{1, 2\}$ and $\beta \in \{3\ell og, 4\ell og\}$. If $|E| \leq \gamma \frac{|V|^2 - 2|V| - 16(\omega^u)^2}{8}$, then $\omega^\alpha_{LP}(\gamma) \geq \omega^\beta_{LP}(\gamma)$. Moreover:*

$$\omega^\alpha_{LP}(\gamma) - \omega^\beta_{LP}(\gamma) \geq \frac{1}{2}\left(|V| - 1 - \sqrt{1 + 16(\omega^u)^2 + 8\frac{|E|}{\gamma}}\right).$$

2.4 Feasibility MIPs and exact iterative algorithms

Next, we focus on feasibility versions of the proposed MIPs. In particular, we describe two simple and easily implementable exact methods for finding maximum γ-quasi-cliques that iteratively solve multiple feasibility versions of models **F3** and **F4**. Formally, we define:

Model 6 (**F3**(k))

$$\sum_{(i,j)\in E} y_{ij} \geq \gamma \frac{k(k-1)}{2}, \tag{27a}$$

$$y_{ij} \leq x_i, \quad y_{ij} \leq x_j \quad \forall (i, j) \in E, \tag{27b}$$

$$\sum_{i \in V} x_i = k, \tag{27c}$$

$$x_i \in \{0, 1\}, \ y_{ij} \geq 0, \quad \forall i, j \in V, \ i < j, \tag{27d}$$

which is obtained from **F3** by fixing $z_k = 1$ in (6b) and (6d). Then by solving mixed integer feasibility problem **F3**(k) it can be verified whether G contains a γ-quasi-clique of size k. Thus, in order to find the size of a maximum γ-quasi-clique, one can simply re-solve **F3**(k) for different values of k starting from $k = \omega^\ell + 1$. (Recall one of our initial assumptions in Sect. 2.2.1 that ω^ℓ is obtained by applying a heuristic approach; thus, a γ-quasi-clique of size ω^ℓ is known.) Clearly, if for some k model **F3**(k) is infeasible then G does not contain a γ-quasi-clique of size $k' \geq k$ (due to the quasi-hereditary property of γ-quasi-cliques [37]). Hence, we can apply the linear search with respect to the value of k by solving **F3**(k) for $k = 1, 2, \ldots$ and stopping when **F3**(k) becomes infeasible. The largest value of k for which the problem is feasible is output as $\omega_\gamma(G)$. We refer to this algorithmic approach as **AlgF3**.

Similar to **F3**(k), by fixing $z_k = 1$ in (10b) and (10d) we construct a feasibility version of **F4** and denote it by **F4**(k). Consequently, by replacing **F3**(k) by **F4**(k) in algorithm **AlgF3** we obtain another exact iterative algorithm, which we refer to as **AlgF4** in the remainder of the paper. Note that formal descriptions of **F4**(k) and **AlgF4** are omitted here for brevity.

One should mention that the iterative methods described above are parallelizable. Naturally, MIPs **F3**(k) and **F4**(k) can be solved concurrently for different values of k.

Finally, one could also use the binary search on k instead of the linear search. However, our computational experiments showed that the linear search performs better. This can be explained by observing that models **F3**(k) and **F4**(k) appear to be much easier to solve when they are feasible (see an additional discussion in Sect. 4). Thus, applying the linear search rather than binary one allows to avoid solving multiple infeasible instances of **F3**(k) and **F4**(k).

3 Finding maximum $f(\cdot)$-dense subgraphs

As mentioned in Sect. 1, the maximum γ-quasi-clique problem can be viewed as a special case of the maximum $f(\cdot)$-dense subgraph problem with $f(k) = \gamma k(k-1)/2$. Next, we demonstrate that models **F3** and **F4** can be extended to handle these more general settings. Specifically, given some fixed nonnegative function $f(\cdot)$ consider the following MIP:

Model 7 (**GF3**(f))

$$\omega(f) = \max_{\mathbf{x},\mathbf{y},\mathbf{z}} \ \sum_{i \in V} x_i \tag{28a}$$

$$\text{s.t.} \quad \sum_{(i,j) \in E} y_{ij} \geq \sum_{k=\omega^\ell}^{\omega^u} f(k) z_k, \tag{28b}$$

$$y_{ij} \leq x_i, \quad y_{ij} \leq x_j, \quad \forall (i, j) \in E, \tag{28c}$$

$$\sum_{i\in V} x_i = \sum_{k=\omega^\ell}^{\omega^u} k z_k, \quad \sum_{k=\omega^\ell}^{\omega^u} z_k = 1, \tag{28d}$$

$$x_i \in \{0,1\},\ y_{ij} \geq 0, \quad \forall i, j \in V,\ i < j, \tag{28e}$$

$$z_k \in \{0,1\}, \quad \forall k \in \{\omega^\ell, \ldots, \omega^u\}, \tag{28f}$$

where constraint (28b) is a generalization of (6b) in **F3** for an arbitrary nonnegative function $f(\cdot)$. In this section, we assume that ω^ℓ and ω^u correspond to the lower and upper bounds, respectively, on the size of a maximum $f(\cdot)$-dense subgraph in G. If non-trivial bounds, e.g., ones similar to (5) for γ-quasi-cliques, are not available for the given function $f(\cdot)$, then we set $\omega^\ell = 1$ and $\omega^u = |V|$.

Furthermore, we observe that $f(k) = \gamma k(k-1)/2$ is convex. Therefore, it is not surprising that Proposition 1 can also be extended to an arbitrary nonnegative convex function. Thus, in case of convex $f(\cdot)$, binary restrictions in (28f) can be replaced by nonnegativity ones, i.e., $z_k \geq 0$ for all k. To the best of our knowledge, in most of the related work in the literature, function $f(\cdot)$ is convex, see [4,24], including the special cases considered below.

Naturally, MIP **F4** can be generalized in a similar fashion by replacing constraint (10b) with:

$$\sum_{i\in V} v_i \geq \sum_{k=\omega^\ell}^{\omega^u} f(k) z_k,$$

and we refer to the modified model as **GF4**(f).

Next, denote by $\omega^3_{LP}(f)$ and $\omega^4_{LP}(f)$ the optimal objective function values of the LP relaxations of MIPs **GF3**(f) and **GF4**(f), respectively. Then the following result is a direct generalization of the statement (ii) in Theorem 1.

Proposition 2 *If $f(\cdot)$ is a strictly increasing, nonnegative, convex function, then the following inequalities hold:*

$$\omega^3_{LP}(f) \leq f^{-1}(|E|), \tag{29}$$

$$\omega^4_{LP}(f) \leq f^{-1}(|E|). \tag{30}$$

Proof Denote by $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ an optimal solution of the LP relaxation of **GF3**(f). Then using an approach similar to the one applied in the Proof of Theorem 1, by convexity of $f(\cdot)$ we conclude that

$$|E| \geq \sum_{(i,j)\in E} y_{ij} \geq \sum_{k=\omega^\ell}^{\omega^u} f(k) z_k \geq f\left(\sum_{k=\omega^\ell}^{\omega^u} k z_k\right) = f\left(\omega^3_{LP}(f)\right), \tag{31}$$

where (31) is analogous to (20). Note that $f(\cdot)$ is strictly increasing. Thus, its inverse function exists, and inequality (29) follows from (31). Inequality (30) can be shown similarly. □

It is worth noting that algorithms **AlgF3** and **AlgF4** can also be extended to handle the general functional case of $f(\cdot)$ using the feasibility versions of **GF3**(f) and **GF4**(f), respectively.

Sample special cases: s-defective clique and average s-plex. Recall that for an s-defective clique, $f(k) = k(k-1)/2 - s$, which implies that constraint (28b) in **GF3**(f) reduces to:

$$\sum_{(i,j)\in E} y_{ij} \geq \sum_{k=\omega^\ell}^{\omega^u} \frac{k(k-1)}{2} z_k - s. \tag{32}$$

For an average s-plex we have $f(k) = k(k-s)/2$, and constraint (28b) is replaced by:

$$2\sum_{(i,j)\in E} y_{ij} \geq \sum_{k=\omega^\ell}^{\omega^u} k(k-1)z_k - (s-1)\sum_{i\in V} x_i. \tag{33}$$

Finally, we note that the term $\sum_{k=\omega^\ell}^{\omega^u} k(k-1)z_k$ in the right-hand sides of (32) and (33) also appears in **F3** and **F4**. Therefore, it is clear that one can derive “logarithmic” versions of (32) and (33) in a similar fashion as described in Sect. 2.2.2 for models **F3log** and **F4log**.

4 Computational experiments

The focus of the computational study presented in this section is on the following issues. First, in Tables 1 and 2, we compare the proposed approaches for finding maximum γ-quasi-cliques (namely, MIP models **F3**, **F4**, **F3log** and **F4log** as well as algorithms **AlgF3** and **AlgF4**) with MIPs **F1** and **F2** from [36] for small and medium-sized network instances (both real-life and synthetic). Note that all of these solution methods can be implemented using off-the-shelf MIP solvers. We do not provide comparisons of our methods with the approach from [34], which is a tailored combinatorial B&B algorithm and is not publicly available. However, we should mention that the computational results reported in [34] indicate that the developed B&B algorithm does not significantly outperform the MIP solver with **F2** for a substantial subset of the considered test instances. Moreover, for the majority of the large-scale graphs in [34] the B&B algorithm fails to converge within the time limit (however, it was able to obtain the optimality gaps that were significantly better than those provided by the MIP solver with **F2**). On the other hand, the computational experiments reported here indicate that for sufficiently sparse graphs MIPs **F1** and **F2** are dominated by the proposed MIP models, which is not surprising given the results regarding the quality of their LP relaxations derived in Sect. 2.3.

Second, in Tables 3 and 4, for several large-scale graph instances with 5000–10,000 vertices (which cannot be tackled by **F1** and **F2** within a reasonable time limit),

Table 1 Finding maximum γ-quasi-cliques in real-life and synthetic networks for $\gamma = 0.9$ and $\gamma = 0.5$

Network parameters				$\omega_\gamma(G)$	Running time (s)							
Instance	Type	$\|V\|$	$\|E\|$		**F1**	**F2**	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
$\gamma = 0.9$												
USAir97	Transport	332	2126	35	TL	TL	5.9	**4.5**	**4.5**	22.6	20.4	6.8
Harvard500	Internet	500	2043	23	TL	TL	1	2	**0.3**	0.4	13.6	4.4
Email	Internet	1133	5451	13	TL	TL	1443.4	TL	2854	TL	TL	**531.4**
Homer	Book	561	1628	14	TL	TL	**3.4**	8.6	7.1	140.4	9	8.1
SmallW	Collaboration	396	994	11	TL	2929.5	**1.6**	7.4	3.5	6.8	7.3	2.9
Erdos971	Collaboration	429	1312	8	TL	TL	5.4	8.3	54.5	1061.4	20	**3.8**
Netscience	Collaboration	1589	2742	21	TL	TL	0.5	0.5	**0.3**	0.7	9.2	6.5
C.Elegans	Biological	453	2025	12	TL	TL	**3.4**	15.5	5.6	26.9	13.6	6.9
H.Pylori	Biological	1570	1399	3	TL	TL	13.1	88.6	304.9	323.2	24.8	**4**
S.Cerevisae	Biological	2112	2203	7	ML	TL	**0.5**	1.1	1.7	1.3	2.1	5.1
p500-1	Power-law	500	1500	7	TL	TL	103.6	1251.5	919.5	TL	177.4	**5.8**
p500-2	Power-law	500	1750	9	TL	TL	248.7	264.1	3417.4	TL	755	**13.5**
p500-3	Power-law	500	2000	8	TL	TL	3382	TL	TL	TL	TL	**37.9**
u100-1	$G(n, M)$	100	300	3	190.3	11.2	2.1	4.3	4.4	14.8	1.9	**0.5**
u100-2	$G(n, M)$	100	400	3	356.2	12.6	8.2	5.7	190.7	61.7	10.3	**0.5**
u100-3	$G(n, M)$	100	500	4	211.6	11.2	23.9	64.3	TL	TL	54.1	**0.7**

Table 1 continued

Network parameters				$\omega_\gamma(G)$	Running time (s)							
Instance	Type	$\|V\|$	$\|E\|$		**F1**	**F2**	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
$\gamma = 0.5$												
USAir97	Transport	332	2126	67	TL	TL	**0.2**	0.7	0.1	0.9	16	2.9
Harvard500	Internet	500	2043	37	TL	TL	15.9	21	**7.7**	13.6	86.3	17.7
Email	Internet	1133	5451	25	TL	TL	3002.9	3543.1	TL	TL	**1245**	TL
Homer	Book	561	1628	33	TL	TL	1.5	1.4	**0.7**	2.7	9.8	17.1
SmallW	Collaboration	396	994	28	TL	TL	1.2	1	**0.5**	1.1	5.7	2.7
Erdos971	Collaboration	429	1312	23	TL	TL	**1.7**	2.2	2.2	11	6.1	736.3
Netscience	Collaboration	1589	2742	31	TL	TL	0.4	1.8	**0.3**	1.5	12	9.7
C.Elegans	Biological	453	2025	30	TL	TL	1.2	1.7	**0.7**	1.5	10.3	2.9
H.Pylori	Biological	1570	1399	9	TL	TL	15.8	8.8	61.5	640.6	**15.5**	TL
S.Cerevisae	Biological	2112	2203	10	ML	TL	**2.2**	3.9	4.9	13.1	10.8	10.8
p500-1	Power-law	500	1500	21	TL	TL	**41.2**	462.3	562	TL	695.8	TL
p500-2	Power-law	500	1750	30	TL	TL	**5.4**	6.8	17.9	1539.3	12.6	TL
p500-3	Power-law	500	2000	31	TL	TL	46.9	3464	1766.6	TL	**41.9**	TL
u100-1	$G(n, M)$	100	300	8	TL	TL	2.6	4.7	36	TL	**1.4**	34.1
u100-2	$G(n, M)$	100	400	9	TL	TL	**16.5**	62.2	88.7	2885.2	27.8	TL
u100-3	$G(n, M)$	100	500	11	TL	TL	356.6	433.3	TL	TL	**353.8**	TL

The time limit (denoted by "TL") is set to 3600 s. The symbol "ML" indicates instances, where the solver returned out-of-memory status. The best results are in bold

Table 2 Comparisons of the LP relaxations (at the root node) for problems in Table 1

Network instance	$\gamma = 0.9$							$\gamma = 0.5$						
	$\omega_\gamma(G)$	ω^1_{LP}	ω^2_{LP}	ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}	$\omega_\gamma(G)$	ω^1_{LP}	ω^2_{LP}	ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}
USAir97	35	166.6	181	**38.8**	52.9	40.7	55.8	67	170.5	229.6	**67.5**	83.8	68.7	84.5
Harvard500	23	250.3	267	**24**	33	24.4	33	37	252.5	338.5	**41.2**	56.4	41.3	56.9
Email	13	ML	600.6	**18.2**	23.6	23.2	32.4	25	ML	761.3	**32**	41.7	40.1	54.7
Homer	14	280.6	298.2	**19.6**	26.3	23	32.1	33	282.1	377.8	**34.5**	45.2	38.8	50.8
SmallW	11	197.7	210.6	**16.9**	21.6	19	24	28	199.1	266.8	**29.6**	38.8	31.9	40.6
Erdos971	8	214.6	228.6	**15.2**	19.4	18.2	23.2	23	216.1	289.8	**26.6**	35.6	30.9	39.4
Netscience	21	ML	837.9	**21.3**	26	**21.3**	26	31	ML	1061.5	**31.6**	38.5	**31.6**	38.5
C.Elegans	12	227	242.3	**17.7**	22.9	21.4	29.4	30	229.5	307.4	**31.2**	40.6	36	46.1
H.Pylori	3	ML	827.4	**8.5**	10.4	11.1	15.5	9	ML	1048	**14.6**	18.7	18.9	24.4
S.Cerevisae	7	ML	1112.7	**7**	8.5	8.1	9.8	10	ML	1409.4	**11.3**	15.6	13.4	17.3
p500-1	7	250.19	266.23	**17.42**	22.34	20.71	28.19	21	251.66	337.37	**30.57**	39.9	35.59	45.93
p500-2	9	250.22	266.72	**20.16**	27.51	24.29	33.31	30	251.97	338.03	**35.49**	46.79	41.78	56.44
p500-3	8	250.25	267.19	**22.56**	32.19	26.38	35.22	31	252.27	338.65	**39.82**	55.16	45.57	64.34
u100-1	3	50.19	55	**7.98**	9.66	8.49	10.34	8	51.74	70.13	**13.55**	17.66	14.44	18.51
u100-2	3	50.27	55.67	**10.08**	13.1	10.43	13.97	9	52.4	71.17	**17.34**	22.22	18	23.2
u100-3	4	50.36	56.38	**12.25**	16.49	12.64	16.82	11	53.08	72.31	**21.26**	30.33	21.92	31.66

The symbol "ML" indicates instances, where the solver returned out-of-memory status. The best results are in bold we study how the performance of the proposed methods depends on values of the parameter γ.

Third, in Tables 5 and 6 for two graph instances and two distinct values of $\gamma \in \{0.5, 0.9\}$ we further explore the performance of the proposed methods with respect to the quality of the available upper and lower bounds on the size of the maximum γ-quasi-clique. Specifically, in Table 5 we report the computational results (including the total running time and the number of B&B nodes explored) for the proposed MIPs **F3**, **F4**, **F3log** and **F4log** with respect to the value of the upper bound ω_γ^u. Additionally, in Table 6 we report the performance of **AlgF3** and **AlgF4** for each iteration of the methods, i.e., we provide the solution times for feasibility MIPs **F3**(k) and **F4**(k) for each value of k.

Finally, in Tables 7 and 8, we illustrate the applicability of our models for finding $f(\cdot)$-dense subgraphs. We want to emphasize that one can apply the proposed solution methodology for any arbitrary type of edge density function $f(\cdot)$ using an off-the-shelf MIP solver.

4.1 Hardware and software

The computational experiments were performed on a Dell laptop equipped with Windows 7 × 64 operating system, an Intel Core i7 940XM processor (CPU 2.13 GHz, L2 8 MB) and RAM 8 GB. All MIPs were solved using FICO Xpress-Optimizer [49] with the time limit of 3600 s (1 h). The corresponding CPU times in tables below are presented in seconds.

4.2 Test instances

In our computational experiments, we use real-life instances obtained from the University of Florida Sparse Matrix Collection [18] and Pajek datasets [8], as well as graph coloring instances from [43] and biological networks from [7]. More specifically, we consider:

- Social (book, collaboration, corporate inter-relationships and citation) networks:
  - Matrix group from **SNAP** in [18] (loops in graphs were removed): **ca-GRQC** ($|V| = 5242$, $|E| = 14484$) and **ca-HEPTh** ($|V| = 9877$, $|E| = 25973$);
  - Matrix group **Pajek** in [18]: **SmallW** ($|V| = 396$, $|E| = 994$), **Erdos971** ($|V| = 472$, $|E| = 1314$), **Geom** ($|V| = 7343$, $|E| = 11898$) and **EVA** ($|V| = 8497$, $|E| = 6726$);
  - Matrix group **Newman** in [18]: **netscience** ($|V| = 1589$, $|E| = 2742$);
  - **Erdos02** ($|V| = 6927$, $|E| = 8472$) [8];
  - **Homer** ($|V| = 561$, $|E| = 1629$) [43].
- Internet and communication networks:
  - Matrix group **Pajek** in [18]: **California** ($|V| = 9664$, $|E| = 15969$);
  - Matrix group **SNAP** in [18]: **AS -735** ($|V| = 7716$, $|E| = 13895$);
  - Matrix group **Arenas** in [18]: **email** ($|V| = 1, 133$, $|E| = 5451$) and **PGP-giantcompo** ($|V| = 10680$, $|E| = 24316$);

Table 3 Performance of the proposed approaches for $\gamma \in \{0.1, 0.2, \ldots, 0.9\}$

γ	$\omega_\gamma(G)$	LP relaxation objective				Running time (s)					
		ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
Erdos02 ($\|V\| = 6927, \|E\| = 8472$)											
0.9	10	**18.37**	23.9	30.77	38.63	**753**	1722.9	TL	TL	2227.8	TL
0.8	13	**20.54**	28.43	33.88	42.31	**512.7**	771.9	TL	TL	718.9	TL
0.7	16	**23.33**	32.8	37.77	47.81	**92.3**	757.8	TL	TL	580.2	TL
0.6	21	**27.07**	36.11	42.87	56.72	**35.7**	74.9	651	TL	285.6	TL
0.5	29	**32.28**	42.04	49.74	66.5	**22.6**	97.8	49.4	606.5	270.7	TL
0.4	38	**40.1**	55.8	59.62	75.58	155.5	**16.1**	22.2	489.8	221.3	TL
0.3	52	**53.13**	72.1	75.2	94.93	11.9	9.9	**8.6**	TL	223.4	TL
0.2	79	**79.16**	107.5	103.8	136.09	**3.4**	258.1	6.8	45.2	246.2	TL
0.1	149	**150.04**	194.8	177.58	234.1	8.4	**5.8**	11.3	25.4	333.6	TL
Geom ($\|V\| = 7343, \|E\| = 11898$)											
0.9	23	**25.19**	34.37	31.92	40.88	**16.6**	19.7	20.7	115.8	176.4	722.9
0.8	26	**28.22**	37.27	35.56	45.63	**10.1**	11.4	18.4	63.5	146.4	TL
0.7	30	**32.11**	41.82	40.13	53.12	16.8	**16.1**	135.5	361.6	167.2	TL
0.6	36	**37.29**	49.81	46	64.31	**7.4**	23	8.4	30	184	TL
0.5	44	**44.55**	64.2	53.95	70.9	**4.1**	10.5	5.1	5.1	255.8	152.1
0.4	54	**55.05**	73.28	65.43	82.66	**5.2**	5.9	7.5	10.6	274.8	TL
0.3	72	**72.18**	94.83	83.16	108.33	6.3	**4.2**	7.4	7.4	254.2	TL
0.2	104	**104.92**	138.96	114.97	146.57	6	**5.4**	6.5	5.5	288.1	TL
0.1	184	**184.66**	248.41	196.12	260.57	**4.2**	4.5	**4.2**	5.3	394.3	2170.1

Table 3 continued

γ	$\omega_\gamma(G)$	LP relaxation objective				Running time (s)					
		ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
ca-HEPTh ($\lvert V\rvert = 9877$, $\lvert E\rvert = 25998$)											
0.9	33	**34.74**	43.21	34.77	43.25	**228.6**	2785.4	474.8	TL	994.6	TL
0.8	36	**38.14**	48.28	38.17	48.33	TL	TL	TL	TL	**2974.6**	TL
0.7	39	**42.46**	56.33	42.5	56.41	**766.8**	TL	TL	TL	TL	TL
0.6	43	**48.18**	65.17	48.23	65.21	**1067.8**	2482.4	TL	TL	TL	TL
0.5	56	**56.11**	71.79	56.17	71.84	78	**50.2**	118.4	94.4	505.4	TL
0.4	64	**66.85**	83.45	66.92	83.83	**1067.4**	1860.7	TL	TL	3268.5	TL
0.3	82	**83.95**	106.7	84.37	111.02	TL	TL	TL	**2892.4**	TL	TL
0.2	111	**112.91**	142.25	117.18	150.17	2111.1	**1302**	3288.7	TL	TL	TL
0.1	186	**187.76**	256.8	207.04	274.47	180.5	**180.3**	358.4	TL	1618.7	TL
ca-GRQC ($\lvert V\rvert = 5242$, $\lvert E\rvert = 14496$)											
0.9	49	**50.09**	67.61	50.2	67.71	**18.9**	21	16.3	45.3	266.8	248.8
0.8	52	**55.42**	72.58	55.55	72.72	61.8	128.3	**41**	107.9	327.4	783.4
0.7	57	**62.25**	80.02	62.4	80.2	342.4	323.8	**175.2**	475.2	1893.2	400.5
0.6	63	**71.32**	92.05	71.5	92.27	TL	TL	TL	**2082.3**	TL	TL
0.5	81	**83.97**	113.38	84.19	113.69	182.1	**45.6**	119.8	57.6	364.7	676.8
0.4	98	**101.85**	135.42	102.05	135.88	**33.2**	62.7	45.4	107.3	294.5	560.6
0.3	129	**129.45**	159.24	130.11	159.98	16.9	18.2	**8.6**	60.9	301.6	578.3
0.2	173	**173.48**	220.05	174.34	222.49	**4.5**	6.7	7.4	9	353.1	734.1
0.1	280	**280.51**	340.4	283.23	343.29	**4.3**	17	8.7	9.1	482.6	1193

The time limit (denoted by "TL") is set to 3600 s. The best results are in bold

Table 4 Performance of the proposed approaches for $\gamma \in \{0.1, 0.2, \ldots, 0.9\}$

γ	$\omega_\gamma(G)$	LP relaxation objective				Running time (s)					
		ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
AS-735 ($\lvert V\rvert = 7716, \lvert E\rvert = 13895$)											
0.9	15	**20.71**	28.86	31.26	39.01	**19.5**	102.1	228.4	2231.5	126	731
0.8	19	**23.18**	32.67	34.36	42.69	**15.8**	38.4	151.6	479.8	111.5	TL
0.7	24	**26.35**	35.43	38.23	48.1	15.6	**10.5**	13.4	61.7	150.4	TL
0.6	29	**30.58**	39.91	43.22	56.84	11.6	**6.4**	7.3	16.5	157.4	TL
0.5	36	**36.49**	47.53	49.94	66.25	4.8	**4.2**	7.1	6.7	168.4	100.8
0.4	44	**44.9**	64.05	59.57	74.93	11.8	4.3	6.6	**3.3**	163.6	TL
0.3	58	**58.33**	76.44	74.43	92.64	6.5	5.2	6.1	**5**	187.6	TL
0.2	84	**84.93**	114.29	101.27	132.72	5.6	**4.8**	8	5.8	248.3	TL
0.1	150	**150.63**	188.5	168.61	213.47	3.9	4.5	5.1	**3.4**	416.8	341.6
PGPgiantcompo ($\lvert V\rvert = 10680, \lvert E\rvert = 24316$)											
0.9	42	**43.37**	62.93	45.92	65.47	133.8	144.6	17.1	**15.2**	1304.1	367.3
0.8	48	**48.54**	67.4	51.43	70.03	178.4	151.6	**20.1**	49.3	1209.7	343.2
0.7	52	**55.03**	73.41	58.49	76.79	**444.4**	758.9	3088.9	TL	TL	TL
0.6	61	**63.68**	83.24	67.72	87.93	257.2	489.8	**218.8**	340	2913.4	TL
0.5	73	**75.79**	102.23	80.17	109.7	230.2	413	**154.5**	277.3	TL	TL
0.4	93	**93.94**	133.34	98.8	136.1	172.7	229.3	**10.6**	88.1	1485.3	TL
0.3	122	**124.15**	161.12	128.91	164.53	272.1	394.1	**86.1**	1098	2070.6	TL
0.2	176	**176.77**	226.92	180.39	235.12	134.1	150.8	**8.3**	12.1	1890.6	872.6
0.1	289	**289.39**	355.66	298.77	370.19	217.9	159.3	**24.7**	1160.3	2509.7	980.7

Table 4 continued

γ	$\omega_\gamma(G)$	LP relaxation objective				Running time (s)					
		ω^3_{LP}	ω^{3log}_{LP}	ω^4_{LP}	ω^{4log}_{LP}	**F3**	**F3log**	**F4**	**F4log**	**AlgF3**	**AlgF4**
EVA ($\lvert V\rvert = 8497$, $\lvert E\rvert = 6726$)											
0.9	4	**5.82**	7.63	8.86	10.56	**4.5**	9.6	24.6	78.7	27.7	30.1
0.8	5	**6.42**	8.12	9.71	11.88	**3.6**	5.1	5.4	36.9	19.2	37.6
0.7	5	**7.21**	8.84	10.8	14.04	12.8	**8.9**	14.3	20.3	28.2	60.3
0.6	7	**8.25**	10.03	12.24	16.15	4.3	7.8	**3.9**	20.7	20.2	35.7
0.5	9	**9.71**	12.21	14.24	17.89	**3.8**	4.7	5.9	4.7	17.3	27.5
0.4	11	**11.83**	15.98	17.11	21.09	5.2	4.2	**3**	3.4	21.6	34
0.3	15	**15.2**	19.18	21.64	28.66	4.1	4.3	**3.3**	5.2	29.9	54.8
0.2	21	**21.91**	31.41	30.17	37.83	5.2	4	4.3	**3.3**	31.1	79.3
0.1	41	**41.9**	57.3	52.5	68.95	4.5	4.2	**3.1**	5	68.2	138.6
California ($\lvert V\rvert = 9664$, $\lvert E\rvert = 15969$)											
0.9	12	**19.96**	27.05	23.82	33.01	**1540.1**	TL	TL	TL	TL	2110
0.8	13	**22.33**	32.01	26.58	35.44	**3501.2**	TL	TL	TL	TL	TL
0.7	$14^\dagger$	**25.37**	34.53	30.14	39.11	TL	TL	TL	TL	TL	TL
0.6	$14^\dagger$	**29.44**	38.59	34.84	45.32	TL	TL	TL	TL	TL	TL
0.5	26	**35.13**	46.2	41.4	57.85	**901.7**	TL	TL	TL	TL	TL
0.4	40	**43.67**	63.72	51.16	69.65	**56.5**	167	365.5	TL	621	TL
0.3	57	**57.89**	74.87	67.13	87.22	30.9	23.4	**20.6**	17	470.8	TL
0.2	80	**82.18**	110.4	97.85	135.09	**397.6**	574.1	909.7	TL	850.7	TL
0.1	145	**152.29**	203.05	183.41	258.42	TL	TL	TL	TL	**2103.7**	TL

The time limit (denoted by "TL") is set to 3600 s. The symbol "†" indicates the objective function value of the best feasible solution obtained by one of the algorithms at the time limit. The best results are in bold

Table 5 The performance of MIPs for different values of the upper bound $\omega_\gamma^u(G)$

ω_γ^u	**F3**			**F3log**			**F4**			**F4log**		
	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes
Harvard500 ($\|V\| = 500$, $\|E\| = 2043$)												
$\gamma = 0.9$, $\omega_\gamma(G) = 23$												
500	1.5	**1.66**	1	2.12	2.33	1	1.95	2.1	49	1.84	1.85	38
33	1.38	**1.39**	1	2.2	2.51	1	2.73	2.75	85	1.74	1.76	10
32	1.55	**1.75**	1	2.79	2.88	9	2.94	2.97	23	1.17	2.45	1
31	1.62	**1.7**	1	2.64	2.65	1	1.92	1.94	81	2.17	2.19	30
30	1.53	1.69	1	2.34	3.85	3	1.77	1.8	107	1.57	**1.59**	29
29	2.04	2.42	1	3.55	4.68	5	1.77	**1.78**	43	1.81	1.82	14
28	1.42	1.55	1	1.64	1.65	15	2	2.04	195	1.51	**1.53**	58
27	1.39	1.54	1	1.86	1.88	5	1.42	1.95	19	1.49	**1.51**	12
26	1.4	**1.54**	1	2.64	4.59	3	2.42	2.45	43	2.02	2.04	10
25	1.47	1.63	1	3.14	3.16	1	2.59	2.61	53	1.08	**1.09**	9
24	1.1	**1.56**	5	2.29	2.31	9	2.42	2.45	90	2.21	2.23	9
$23^\dagger$	0.34	**0.35**	1	3.6	3.62	13	0.13	0.13	1	1.67	1.69	7
$\gamma = 0.5$, $\omega_\gamma(G) = 37$												
500	1.51	16.2	5389	5.08	21.26	4961	2.7	19.22	8041	2.88	**15.98**	10309
47	3.5	27.62	5521	8.32	44.09	6839	52.61	58.1	17827	2.69	**21.35**	10199
46	3.28	**19.88**	5411	3	37.77	6305	35.4	42.29	18303	6.31	22.86	7587
45	3.53	26.02	5585	7.39	18.2	3575	47.73	55.77	18897	4.44	**16.3**	6613
44	3.38	**24.63**	5935	5.16	24.58	4693	20.58	55.57	22703	5.46	25.6	12063
43	3.64	**22.03**	5589	5.8	23.8	4261	5.8	56.01	22347	4.78	23.93	12979

Table 5 continued

ω_γ^u	F3			F3log			F4			F4log		
	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes
42	3.63	24.18	5493	2.82	47.99	9189	43.11	43.48	17029	4.23	**22.94**	9371
41	8.19	**13.09**	2561	3.45	52.1	7985	37.22	98.42	33803	5.47	15.97	5795
40	5.2	**10.17**	1577	5.23	31.98	6739	39.47	84.53	31867	1.53	13.75	7631
39	2.26	14.94	2813	6.44	29.94	5793	5.76	33.99	14441	2.58	**3.55**	911
38	1.09	26.48	5165	5.25	49.1	8133	48.62	48.67	15546	4.04	**15.29**	6429
37†	0.93	0.94	1	7.67	23.07	4129	0.85	**0.86**	1	4.06	23.01	9613
ca-GRQC ($\lvert V\rvert = 5242$, $\lvert E\rvert = 14496$)												
$\gamma = 0.9, \omega_\gamma(G) = 49$												
5242	26.3	**26.42**	187	29.61	29.68	106	46.74	46.83	590	33.01	33.08	270
59	21.43	**21.53**	151	29.59	29.66	128	26.34	26.39	149	95.66	95.73	1511
58	22.81	**22.9**	171	34.77	34.84	141	32.89	32.95	284	103.9	103.97	1574
57	24.53	24.63	157	19.6	**19.7**	101	42.66	42.71	345	59.46	59.53	1173
56	22.42	**22.52**	163	27.32	27.43	70	44.12	44.18	392	50.04	50.12	453
55	21.34	**21.43**	121	34.76	34.83	52	50.82	50.88	392	29.07	29.12	91
54	22.18	**22.27**	149	36.21	36.28	38	41.58	41.64	224	115.65	115.72	2107
53	25.17	**25.28**	169	41.69	41.76	138	47.03	47.1	443	34.41	34.47	293
52	24.09	24.19	137	30.32	30.39	31	40.2	40.26	262	15.55	**15.57**	1
51	25.79	**25.89**	171	30.33	30.4	33	43.84	43.9	353	35.5	35.57	325
50	47.95	**48.05**	57	30.34	30.42	90	174.39	174.45	2035	48.79	48.85	381
49†	52.68	52.78	34	27.96	28.14	67	10.77	**10.79**	1	47.29	47.37	579

Table 5 continued

ω^u_γ	F3			F3log			F4			F4log		
	Opt found	Totall time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes	Opt found	Total time	B&B nodes
$\gamma = 0.5, \omega_\gamma(G) = 81$												
5242	42.41	59.73	259	16.25	**43.51**	193	122.3	123.25	5775	49.33	56.39	1997
91	40.5	66.37	499	20.33	**42.68**	177	53.85	53.96	1148	17.29	49.82	559
90	1560.08	1564.96	12145	35.46	**49.34**	189	100.31	100.6	1280	101.79	101.94	3376
89	398.04	399.22	4367	67.2	79.18	509	127.33	130.56	1131	21.1	**44.73**	227
88	154.89	155.55	2783	21.53	42.75	189	70.82	82.06	1135	19.61	**24.07**	227
87	734.31	738.75	9291	27.58	**37.96**	167	81.51	86.58	1103	23.33	47.84	259
86	29.72	49.5	415	22.28	**41.73**	225	62.49	68.64	1821	19.44	43.94	371
85	45.18	62.75	381	18.08	36.09	267	57.74	62.16	1203	15.82	**24.31**	259
84	34.27	50.01	281	25.15	41.77	185	70.14	74.99	2591	14.89	**28.06**	265
83	73.5	86.23	427	32.87	53.1	185	53.51	57.61	1161	19.4	**33.96**	467
82	72.24	94.46	635	77.39	79.97	979	68.6	71.94	913	61.51	**62.05**	2469
$81^\dagger$	42.47	42.7	42	25.72	40.01	57	28.73	**28.88**	88	23.28	50.19	607

The lower bound $\omega^\ell_\gamma(G)$ is set to 2 except for the rows marked by "$\dagger$," where we set $\omega^u_\gamma(G) = \omega^\ell_\gamma(G) = \omega_\gamma(G)$. The table reports the total computational time, the time when the optimal solution $\omega_\gamma(G)$ is first obtained (both in seconds) and the total number of branch-and-bound nodes explored by the solver. The best results with respect to the total running time are in bold

Table 6 The performance of **AlgF3** and **AlgF4** for each iteration k, i.e., we report the solution times for feasibility MIPs **F3**(k) and **F4**(k) for each value of k

k	**AlgF3**		**AlgF4**		k	**AlgF3**		**AlgF4**	
	Time	B&B Nodes	Time	B&B Nodes		Time	B& B Nodes	Time	B&B Nodes
Harvard500 ($\|V\| = 500$, $\|E\| = 2043$)									
$\gamma = 0.9$, $\omega_\gamma(G) = 23$					$\gamma = 0.5$, $\omega_\gamma(G) = 37$				
3–14	0.54	1	0.32	19.9	3–28	0.43	1	0.15	1
15	0.46	1	0.15	1	29	0.71	1	0.14	1
16	0.46	1	0.15	1	30	0.59	1	0.76	1
17	0.4	1	0.14	1	31	0.43	1	0.34	1
18	0.49	1	0.13	1	32	0.63	1	0.14	1
19	0.45	1	0.15	1	33	1.69	1	1.43	334
20	0.44	1	0.16	1	34	5.06	1	0.46	1
21	0.46	1	0.15	1	35	0.54	1	0.16	1
22	0.33	1	0.11	1	36	1.45	1	0.3	1
23	0.33	1	0.15	1	37	0.62	1	0.35	1
24	3.63	1	0.32	1	38	33.42	4561	9.76	4383
Total	14.01	22	15.51	249		56.35	4596	17.81	4747
ca-GRQC ($\|V\| = 5242$, $\|E\| = 14496$)									
$\gamma = 0.9$, $\omega_\gamma(G) = 49$					$\gamma = 0.5$, $\omega_\gamma(G) = 81$				
3–40	3.49	1	6.18	1	3–72	3.7	1	6.26	1
41	5.5	1	6.87	1	73	2.91	1	6.8	1
42	4.35	1	6.29	1	74	3.94	1	22.52	48
43	4.24	1	6.18	1	75	3.44	1	7.82	1
44	4.35	1	6.07	1	76	3.5	1	7.14	1
45	4.73	1	6.3	1	77	4.05	1	7.07	1
46	2	1	6.7	1	78	5.69	1	7.08	1
47	2.04	1	6.39	1	79	38.23	1	7.44	1
48	3.93	1	7.11	1	80	83.32	428	72.73	2444
49	59.13	21	23.36	69	81	27.81	12	22.12	100
50	86.12	3	8.89	1	82	72.58	49	25.54	153
Total	309.39	70	319.14	116		504.98	566	624.8	2821

When the value k is given by a range (e.g., "3–14" in the first row), then the average running time (in seconds) and the number of B&B nodes explored per iteration are reported

  – Matrix group **MathWorks** in [18]: undirected version of **Harvard500** ($|V| = 500$, $|E| = 2043$);
- Biological networks [7]: **C.Elegans** ($|V| = 453$, $|E| = 2025$), **H.Pylori** ($|V| = 1570$, $|E| = 1399$), and **S.Cerevisae** ($|V| = 2112$, $|E| = 2203$);
- Transportation network from matrix group **Pajek** in [18]: **USAir97** ($|V| = 332$, $|E| = 2126$).

Table 7 Performance of **GF3**(f) for the maximum s-defective clique and the maximum average s-plex problems, where $f(|S|) = |S|(|S|-1)/2 - s$ and $f(|S|) = |S|(|S|-s)/2$, respectively

Network instance	Maximum s-defective clique						Maximum average s-plex					
	$\omega(f)$		$\omega^3_{LP}(f)$		Runtime		$\omega(f)$		$\omega^3_{LP}(f)$		Runtime	
	$s = 5$	$s = 10$	$s = 5$	$s = 10$	$s = 5$	$s = 10$	$s = 1$	$s = 2$	$s = 1$	$s = 2$	$s = 1$	$s = 2$
USAir97	25	27	35.32	35.6	487.1	284.6	28	31	36.04	37.04	65.7	22.4
Harvard500	22	22	22.32	22.73	0.6	0.5	22	23	22.84	23.78	0.6	0.6
Email	13	14	17.11	17.65	1302.5	169	13	14	17.51	18.51	194.8	444.5
Homer	13	14	18.3	18.82	6.8	6.8	13	15	18.76	19.76	7.3	5.3
SmallW	11	12	15.97	16.54	4	3.5	11	13	16.34	17.34	4.1	3.1
Erdos971	9	10	14.49	15.13	5.7	5.6	9	11	14.81	15.81	9.4	4.9
Netscience	20	20	20.32	20.65	0.9	0.9	20	21	20.67	21.35	0.8	0.9
C.Elegans	11	12	16.7	17.26	5.2	6.5	12	13	17.11	18.11	4.6	4.5
H.Pylori	6	7	8.93	9.83	28	16.7	5	6	8.8	9.8	16.3	17.4
S.Cerevisae	7	8	7.65	8.61	0.9	0.9	7	8	7.37	8.3	0.9	2.7
p500-1	8	9	16.38	16.97	120.1	128.3	8	9	16.78	17.78	115	160.8
p500-2	9	11	18.77	19.28	166.4	112.1	9	12	19.24	20.24	368.8	130
p500-3	9	11	20.89	21.34	2047.9	1583.5	$9^{\dagger}$	11	21.4	22.4	TL	2441.2
u100-1	6	7	8.44	9.39	3.2	2.4	4	6	8.26	9.26	6.2	2.9
u100-2	6	7	10.15	10.99	7.2	10.8	5	6	10.16	11.16	10.8	13
u100-3	6	7	11.97	12.69	29.7	56.8	5	7	12.13	13.13	35.1	55.3
Erdosh02	10	12	17.21	17.76	225.9	222.8	10	12	17.63	18.63	487.1	212
Geom	22	22	23.2	23.61	599.8	85.5	22	23	23.77	24.77	153.6	817.3
ca-HEPTh	32	32	32.24	32.49	327.5	294.1	32	33	32.8	33.61	361.5	283.1
ca-GRQC	46	46	46	46.18	4.9	5.8	46	47	46.67	47.54	6.3	25.3

Table 7 continued

Network instance	Maximum s-defective clique						Maximum average s-plex					
	$\omega(f)$		$\omega^3_{LP}(f)$		Runtime		$\omega(f)$		$\omega^3_{LP}(f)$		Runtime	
	$s=5$	$s=10$	$s=5$	$s=10$	$s=5$	$s=10$	$s=1$	$s=2$	$s=1$	$s=2$	$s=1$	$s=2$
AS-735	13	15	19.26	19.75	37.1	29.4	14	16	19.74	20.74	312.9	177.4
PGPgiantcompo	$28^{\dagger}$	$31^{\dagger}$	39.38	39.63	TL	TL	$33^{\dagger}$	36	40.13	41.13	TL	1437.5
EVA	6	7	6.81	7.89	6.3	5.5	5	6	6.33	7.33	7.9	6
California	12	13	18.59	19.11	2862.8	3401.7	12	$11^{\dagger}$	19.07	20.07	3409.2	TL

The time limit (denoted by "TL") is set to 3600 s. The symbol "$\dagger$" indicates the objective function value of the best feasible solution obtained by one of the algorithms at the time limit. The running times are in seconds

Table 8 Performance of **GF3**(f) for finding maximum f-dense subgraphs, where $f(|S|) = \gamma|S|^{3/2}$ and $f(|S|) = (2|E|/|V| + s)|S|$

Network instance	$f(\lvert S\rvert) = \gamma\lvert S\rvert^{3/2}$						$f(\lvert S\rvert) = \left(\frac{2\lvert E\rvert}{\lvert V\rvert} + s\right)\lvert S\rvert$					
	$\omega(f)$		$\omega^3_{LP}(f)$		Runtime		$\omega(f)$		$\omega^3_{LP}(f)$		Runtime	
	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 1$	$s = 1$	$s = 2$	$s = 1$	$s = 2$	$s = 1$	$s = 2$
USAir97	255	142	255.04	142.38	0.7	0.4	303	280	303.96	280.23	0.8	0.3
Harvard500	193	80	194.07	81.57	3	4.1	430	364	430.42	364.71	0.3	1.1
Email	238	47	241.09	60.2	27.9	746.7	1001	877	1001.71	877.56	0.8	1.3
Homer	164	67	164.66	67.67	0.4	0.5	448	371	448.59	371.3	1.1	0.4
SmallW	133	50	133	50	0.2	0.3	330	283	330.39	283.32	0.2	0.2
Erdos971	119	40	119.18	41.03	0.4	1.8	387	313	387.55	314.01	0.4	5.1
Netscience	106	44	106.89	45.09	2.7	3.2	1137	772	1137.8	772.67	15	2.9
C.Elegans	163	55	163.71	55.69	0.6	0.5	371	294	371.7	294.89	0.3	0.5
H.Pylori	34	–	46.39	–	13.1	–	1006	740	1006.05	740.05	0.3	0.3
S.Cerevisae	24	7	24.4	7	3.7	0.7	916	364	916.06	366.17	2.3	7.9
p500-1	160	51	160.36	54.64	0.4	6.1	428	363	428.57	363.67	0.2	0.5
p500-2	192	74	192.08	74.22	0.5	0.5	437	381	437.5	381.14	0.2	0.3
p500-3	218	93	218.67	93	0.5	0.5	444	393	444.44	393	0.3	0.3
u100-1	24	–	39.31	–	2.5	–	–	–	–	–	–	–
u100-2	55	–	66.42	–	4.3	–	–	–	–	–	–	–
u100-3	99	–	99.75	–	0.1	–	–	–	–	–	–	–
Erdosh02	203	60	203.14	61.01	4	19	4066	2878	4066.66	2878.07	16.8	7.5
Geom	267	107	267.74	107.31	5.5	5	5442	3905	5443.07	3905.71	99.5	15.5
ca-HEPTh	281	116	282.305	119.499	427	797.6	7697	5865	7697.76	5865.39	405.9	318.7
ca-GRQC	462	232	462.93	232.79	32.3	6.3	4182	3383	4182.6	3383.9	17.8	17.6

Table 8 continued

Network instance	$f(\|S\|) = \gamma \|S\|^{3/2}$						$f(\|S\|) = \left(\frac{2\|E\|}{\|V\|} + s\right) \|S\|$					
	$\omega(f)$		$\omega^3_{LP}(f)$		Runtime		$\omega(f)$		$\omega^3_{LP}(f)$		Runtime	
	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 1$	$s = 1$	$s = 2$	$s = 1$	$s = 2$	$s = 1$	$s = 2$
AS-735	192	72	192.82	72	5.5	12.3	5400	3509	5400.05	3509.22	20.5	5.8
PGPgiantcompo	536	241	536.77	241.2	680.2	638.1	7674	5989	7674.84	5989.38	823.8	651.3
EVA	17	–	17	–	4.2	–	612	141	612.47	141.46	12.7	4.8
California	207	69	217.13	69.45	820.9	95.2	7419	5989	7419.32	5989.38	143.9	651.3

The symbol "–" indicates that the problem turned out to be infeasible. The running times are in seconds

While the main focus of our experiments is on real-life networks, for illustrative purposes, we also test our approaches on several randomly generated synthetic graph instances. In particular, uniform random graphs are constructed according to a classical $G(n, M)$ model [19]. These instances are denoted by $u100$-1, $u100$-2 and $u100$-3. Finally, power-law random graphs (i.e., the probability that a vertex has degree k is $\Theta(k^{-\beta})$ for some constant β) are generated as in [15,36] and denoted by $p500$-1, $p500$-2 and $p500$-3.

4.3 Results and discussion

Tables 1 and 2: In the first table, we report running times for the considered solution approaches, while the second one contains the objective function values for the corresponding LP relaxations. As one would expected, due to sparsity of the graphs in our study, MIP models **F3** and **F4** have the best LP relaxations, with **F3log** and **F4log** lagging not too far behind. The quality of the LP relaxations of **F1** and **F2** is very poor. These experimental observations are verified by the theoretical results established in Theorem 1 and Corollary 1. Therefore, it is not surprising that MIPs **F1** and **F2** are dominated by the other approaches with respect to their running times.

Models **F3** and **F4** show consistently good performance for all test instances in Table 1. We should note that, while **F3** has better LP relaxations than **F4**, the latter model requires less variables. **AlgF4** provides the best results for most of the instances with $\gamma = 0.9$. We attribute this to the rather small values of $\omega_\gamma(G)$ for sufficiently large γ, which results in a small number of feasibility MIPs of the form **F4**(k) required to be solved during the execution of **AlgF4**. Also, recall that **F4**(k) has less variables than **F3**(k), which may explain the somewhat worse performance of **AlgF3**. On the other hand, **AlgF3** outperforms **AlgF4** for smaller values of γ, which is, perhaps, due to a better quality of the LP relaxations of the underlying feasibility MIPs.

Tables 3 and 4: In this set of experiments, we solve the maximum γ-quasi-clique problem for large graph instances with 5000–10,000 vertices and different values of $\gamma \in \{0.1, 0.2, \ldots, 0.9\}$. The models **F1** and **F2** cannot handle large real-life graphs (see also the results for small- and medium-size instances in Table 1 for another confirmation of this observation), hence, we focus on the newly proposed approaches.

First, we note that for most of the instances **AlgF3** and **AlgF4** are not competitive, which is due to large values of $\omega_\gamma(G)$ in the optimal solutions. Therefore, both algorithms are forced to solve a large number of feasibility MIPs, which results in their ineffectiveness for these instances (recall our previous discussion for instances with $\gamma = 0.9$ in Table 1).

Second, similar to the experiments discussed above, models **F3** and **F4** yield the best (or close to the best) results for the majority of the considered graphs. We should point out again the extremely high quality of their LP relaxations. However, in contrast to the previous experiments, there exist instances, where **F3log** and **F4log** outperform the other MIPs. One should recall that **F3log** and **F4log** have a less number of variables than **F3** and **F4**; however, as we show in Theorem 1, the quality of their LP relaxations is somewhat worse. This trade-off is clearly evident in the results of our experiments for the considered large real-life graphs, e.g., networks **ca-HEPTh** and **ca-GRQC** in Table 3 and **AS-735** in Table 4.

Table 5: In this set of experiment we explore how the performance of the proposed MIPs **F3**, **F4**, **F3log** and **F4log** depends on the value of the upper bound ω_γ^u. We report the total running time, the running time required for the solver to identify an optimal solution (thus, the difference in these values indicates the time required by the solver to prove optimality of the obtained solution) and the total number of B&B nodes explored by the solver.

First, we observe that the total running of the MIP solver does not usually improve significantly (or may even increase) for the tighter values of ω_γ^u in the MIP models. In particular, for the easier instance (**Harvard500**) most of the total running time is spent for proving optimality of the obtained optimal solution (except for the model **F4**), which is typically identified rather early by the MIP solver (due to a good quality of the MIP-based heuristics). Naturally, if $\omega_\gamma^u = \omega_\gamma^\ell = \omega_\gamma$ (see the row marked by "†"), then the solution is obtained almost immediately for **F3** and **F4**.

On the other hand, for the more difficult instance (**ca-GRQC**) a considerable portion of the total running time is spent for identifying an optimal solution. Consequently, if $\omega_\gamma^u = \omega_\gamma^\ell = \omega_\gamma$ (see the row marked by "†"), then the total running times for the MIP models do not improve significantly (except for the model **F4**) in comparison to the MIPs, where ω_γ^u is set to $|V|$. Therefore, the variability of the total running times required to obtain optimal solutions in our experiments may be attributed to the effect of the MIP solver parameters (recall that we use the default settings) that guide dynamically the search in the underlying B&B approach.

The above observations are not particularly surprising. However, from the practical perspective our results imply that if the decision-maker, who seeks maximum quasi-cliques, applies the MIP solver with the default settings (and is not interested in either tuning the solver parameters or using more advanced solver capabilities), then he/she could simply use $|V|$ as an upper bound in the proposed MIP models.

Table 6: In this table we report the performance of **AlgF3** and **AlgF4** for each iteration of the methods, i.e., we provide the solution times for feasibility MIPs **F3**(k) and **F4**(k) for each value of k. One can observe that the feasibility MIP models are usually more difficult to solve when the problem is either infeasible (i.e., $k = \omega_\gamma + 1$), or the value of k is close to ω_γ. In particular, the latter observation typically holds for more difficult instances; see the results for **ca-GRQC** with $k \in \{\omega_\gamma - 1, \omega_\gamma\}$. In view of the previous set of the experiments, these results are not too surprising. However, they also imply that the iterative scheme with feasibility MIPs may be favorable if there exist good quality lower and upper bounds for the size of the maximum quasi-clique. Furthermore, feasibility MIPs may be preferred if we are simply interested in verifying existence of a quasi-clique of a particular size, which may be the case in some practical applications.

Tables 7 and 8: Next, we consider the problem of finding maximum $f(\cdot)$-dense subgraphs for four types of the edge density function $f(\cdot)$. The first two correspond to the maximum s-defective clique and the maximum average s-plex problems, respectively (recall our discussion in Sects. 1 and 3). Note that for both of these functions we have $f(k) = \Theta(k^2)$, which implies a relatively strict edge density requirement for the obtained subgraphs. Thus, we also consider two other types of functions, namely, $f(k) = \gamma k^{3/2}$ and $f(k) = (2|E|/|V| + s)k$, which correspond to $f(k) = \Theta(k^{3/2})$ and $f(k) = \Theta(k)$, respectively. Recall that MIP **F3** turned out to be the most consistent model in our previous experiments. Therefore, we use **GF3**(f) in this set of experiments. The obtained results illustrate the applicability of our models for finding $f(\cdot)$-dense subgraphs, where $f(\cdot)$ can be an arbitrary nonnegative function. In particular, we want to emphasize a high quality of the LP relaxations, which allowed us to solve the problem for graphs with up to 10,000 vertices.

5 Concluding remarks

In this paper, we propose new MIP models for solving the maximum γ-quasi-clique problem. The key advantage of our MIPs is that for sparse graphs the corresponding LP relaxations are tighter than the LP relaxations of other MIP models available in the literature. We note that one can easily construct instances of dense graphs, for which this is not necessarily the case. However, we emphasize here that the vast majority of real-life graphs in the literature are sparse. Thus, it is not surprising that in our computational experiments the proposed exact solution approaches are capable of solving problems on large real-life instances with up to 10,000 vertices. Furthermore, we demonstrate that our methodology can be naturally generalized for solving the maximum $f(\cdot)$-dense subgraph problem, which seeks the largest k such that the graph has a k-vertex subgraph with at least $f(k)$ edges for a given nonnegative function $f(\cdot)$.

As a possible direction of future research, it is worth mentioning a somewhat related work in [45], which aims to enumerate all quasi-cliques in a graph. Therefore, it could be interesting to exploit our MIP models for developing effective methods that can solve the same problem (or a similar class of problems) and provide all (or almost all) maximal γ-quasi-cliques and general $f(\cdot)$-dense subgraphs in a given graph. Finally, a deeper investigation of the polyhedral relationships between the previous and the proposed MIP models could be interesting.

Acknowledgments The authors would like to thank the Associate Editor and the anonymous reviewers for their constructive comments that helped us to greatly improve the quality of the paper. This material is based upon work supported by the U.S. Air Force Research Laboratory (AFRL) Mathematical Modeling and Optimization Institute and the U.S. Air Force Office of Scientific Research (AFOSR). The research of the first author was performed while he held a National Research Council Research Associateship Award at AFRL Munitions Directorate. The research of the second author was also supported by the U.S. Air Force Summer Faculty Fellowship and by AFRL/RW under agreement number FA8651-14-2-0002. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of AFRL/RW or the U.S. Government.

References

1. Abello, J., Pardalos, P.M., Resende, M.G.C.: On maximum clique problems in very large graphs. In: Abello, J., Vitter, J. (eds.) External Memory Algorithms and Visualization, pp. 119–130. American Mathematical Society, Boston (1999)
2. Abello, J., Resende, M.G.C., Sudarsky, S.: Massive quasi-clique detection. In: Rajsbaum, S. (ed.) LATIN 2002: Theoretical Informatics, pp. 598–612. Springer, London (2002)

3. Adams, W.P., Forrester, R.J., Glover, F.W.: Comparisons and enhancement strategies for linearizing mixed 0–1 quadratic programs. Discret. Optim. **1**(2), 99–120 (2004)
4. Asahiro, Y., Hassin, R., Iwama, K.: Complexity of finding dense subgraphs. Discret. Appl. Math. **121**(1–3), 15–26 (2002)
5. Bader, G.D., Hogue, C.W.: An automated method for finding molecular complexes in large protein interaction networks. BMC Bioinf. **4**(1), 2 (2003)
6. Balasundaram, B., Butenko, S., Hicks, I.: Clique relaxations in social network analysis: the maximum k-plex problem. Oper. Res. **59**(1), 133–142 (2011)
7. Balasundaram, B., Butenko, S., Trukhanov, S.: Novel approaches for analyzing biological networks. J. Comb. Optim. **10**, 23–39 (2005)
8. Batagelj, V., Mrvar, A.: Pajek Datasets. http://vlado.fmf.uni-lj.si/pub/networks/data/ (2006). Accessed 26 Oct 2015
9. Bhattacharyya, M., Bandyopadhyay, S.: Mining the largest quasi-clique in human protein interactome. In: Proceedings of the 2009 International Conference on Adaptive and Intelligent Systems, ICAIS '09, pp. 194–199. IEEE Computer Society, Washington, DC, USA (2009)
10. Boginski, V., Butenko, S., Pardalos, P.: Mining market data: a network approach. Comput. Oper. Res. **33**(11), 3171–3184 (2006)
11. Boginski, V., Butenko, S., Pardalos, P.M.: Statistical analysis of financial networks. Comput. Stat. Data Anal. **48**(2), 431–443 (2005)
12. Bomze, I.M., Budinich, M., Pardalos, P.M., Pelillo, M.: The maximum clique problem. Handbook of Combinatorial Optimization, vol. 4, pp. 1–74. Kluwer Academic Publishers, Boston (1999)
13. Bu, D., Zhao, Y., Cai, L., Xue, H., Zhu, X., Lu, H., Zhang, J., Sun, S., Ling, L., Zhang, N., Li, G., Chen, R.: Topological structure analysis of the protein–protein interaction network in budding yeast. Nucleic Acids Res. **31**(9), 2443–2450 (2003)
14. Butenko, S., Wilhelm, W.: Clique-detection models in computational biochemistry and genomics. Eur. J. Oper. Res. **173**(1), 1–17 (2006)
15. Chung, F., Lu, L., Vu, V.: The spectra of random graphs with given expected degrees. Internet Math. **1**, 257–275 (2004)
16. Crenson, M.A.: Social networks and political processes in urban neighborhoods. Am. J. Polit. Sci. **22**(3), 578–594 (1978)
17. Davis, S., Trapman, P., Leirs, H., Begon, M., Heesterbeek, J.A.P.: The abundance threshold for plague as a critical percolation phenomenon. Nature **454**, 634–637 (2008)
18. Davis, T.A., Hu, Y.: The University of Florida sparse matrix collection. ACM Trans. Math. Softw. **38**(1), 1–25 (2011)
19. Erdős, P., Rényi, A.: On random graphs. Publicationes Mathematicae Debrecen **6**, 290–297 (1959)
20. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H Freeman, New York (1979)
21. Guo, J., Kanj, I.A., Komusiewicz, C., Uhlmann, J.: Editing graphs into disjoint unions of dense clusters. Algorithmica **61**(4), 949–970 (2011)
22. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1952)
23. Hartwell, L.H., Hopfield, J.J., Leibler, S., Murray, A.W.: From molecular to modular cell biology. Nature **402**(6761 Suppl), C47–C52 (1999)
24. Holzapfel, K., Kosub, S., Maaß, M.G., Täubig, H.: The complexity of detecting fixed-density clusters. Algorithms and Complexity, pp. 201–212. Springer, Berlin (2003)
25. Hu, H., Yan, X., Huang, Y., Han, J., Zhou, X.J.: Mining coherent dense subgraphs across massive biological networks for functional discovery. Bioinformatics **21**(suppl 1), i213–i221 (2005)
26. Huang, W.-Q., Zhuang, X.-T., Yao, S.: A network analysis of the Chinese stock market. Phys. A: Stat. Mech. Appl. **388**(14), 2956–2964 (2009)
27. Ibaraki, T.: Integer programming formulation of combinatorial optimization problems. Discret. Math. **16**(1), 39–52 (1976)
28. Kleinberg, J., Goemans, M.X.: The Lovász theta function and a semidefinite programming relaxation of vertex cover. SIAM J. Discret. Math. **11**(2), 196–204 (1998)
29. Lovász, L.: On the Shannon capacity of a graph. IEEE Trans. Inf. Theory **25**(1), 1–7 (1979)
30. Luce, R.D., Perry, A.D.: A method of matrix analysis of group structure. Psychometrika **14**(2), 95–116 (1949)
31. Matsuda, H., Ishihara, T., Hashimoto, A.: Classifying molecular sequences using a linkage graph with their pairwise similarities. Theor. Comput. Sci. **210**, 305–325 (1999)

32. Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization. Wiley, New York (1988)
33. Östergård, P.: A fast algorithm for the maximum clique problem. Discret. Appl. Math. **120**(1), 197–207 (2002)
34. Pajouh, F.M., Miao, Z., Balasundaram, B.: A branch-and-bound approach for maximum quasi-cliques. Ann. Oper. Re. **216**(1), 145–161 (2014)
35. Pardalos, P.M., Xue, J.: The maximum clique problem. J. Global. Optim. **4**(3), 301–328 (1994)
36. Pattillo, J., Veremyev, A., Butenko, S., Boginski, V.: On the maximum quasi-clique problem. Discret. Appl. Math. **161**(1–2), 244–257 (2013)
37. Pattillo, J., Youssef, N., Butenko, S.: On clique relaxation models in network analysis. Eur. J. Oper. Res. **226**(1), 9–18 (2013)
38. Ravasz, E., Somera, A.L., Mongru, D.A., Oltvai, Z.N., Barabási, A.L.: Hierarchical organization of modularity in metabolic networks. Science **297**(5586), 1551–1555 (2002)
39. Saban, D., Bonomo, F., Stier-Moses, N.E.: Analysis and models of bilateral investment treaties using a social networks approach. Phys. A: Stat. Mech. Appl. **389**(17), 3661–3673 (2010)
40. Scott, J.: Social Network Analysis: A Handbook, 2nd edn. Sage Publications, London (2000)
41. Sim, K., Li, J., Gopalkrishnan, V., Liu, G.: Mining maximal quasi-bicliques to co-cluster stocks and financial ratios for value investment. In: Proceedings of the Sixth International Conference on Data Mining, ICDM '06, pp. 1059–1063. IEEE Computer Society, Washington, DC, USA (2006)
42. Spirin, V., Mirny, L.A.: Protein complexes and functional modules in molecular networks. Proc. Natl. Acad. Sci. **100**(21), 12123–12128 (2003)
43. Trick, M.: COLOR02/03/04: Graph Coloring and Its Generalizations. http://mat.gsia.cmu.edu/COLOR03/ (2004). Accessed 26 Oct 2015
44. Trukhanov, S., Balasubramaniam, C., Balasundaram, B., Butenko, S.: Algorithms for detecting optimal hereditary structures in graphs, with application to clique relaxations. Comput. Optim. Appl. **56**(1), 113–130 (2013)
45. Uno, T.: An efficient algorithm for solving pseudo clique enumeration problem. Algorithmica **56**, 3–16 (2010)
46. Veremyev, A., Boginski, V., Krokhmal, P.A., Jeffcoat, D.E.: Dense percolation in large-scale mean-field random networks is provably "explosive". PLoS One **7**(12), e51883 (2012)
47. Vogiatzis, C., Veremyev, A., Pasiliao, E.L., Pardalos, P.M.: An integer programming approach for finding the most and the least central cliques. Optim. Lett. **9**(4), 615–633 (2014)
48. Wasserman, S., Faust, K.: Social Network Analysis. Cambridge University Press, New York (1994)
49. FICO$^{\text{TM}}$ Xpress Optimization Suite 7.6. http://www.fico.com/en/Products/DMTools/Pages/FICO-Xpress-Optimization-Suite.aspx (2014). Accessed 26 Oct 2015
50. Yu, H., Paccanaro, A., Trifonov, V., Gerstein, M.: Predicting interactions in protein networks by completing defective cliques. Bioinformatics **22**(7), 823–829 (2006)