Comput Optim Appl (2012) 52:609–627
DOI 10.1007/s10589-011-9431-1

Comparing SOS and SDP relaxations of sensor network localization

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Received: 18 January 2010 / Published online: 13 September 2011
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Abstract We investigate the relationships between various sum of squares (SOS) and semidefinite programming (SDP) relaxations for the sensor network localization problem. In particular, we show that Biswas and Ye's SDP relaxation is equivalent to the degree one SOS relaxation of Kim et al. We also show that Nie's sparse-SOS relaxation is stronger than the edge-based semidefinite programming (ESDP) relaxation, and that the trace test for accuracy, which is very useful for SDP and ESDP relaxations, can be extended to the sparse-SOS relaxation.

Keywords Sensor network localization · Semidefinite programming relaxation · Sum of squares relaxation · Individual trace

1 Introduction

In its basic form, the sensor network localization problem is that of finding the coordinates of some sensors $x_i = (x_i^1, x_i^2)^T \in \mathbb{R}^2, i = 1, \dots, m$, given the Cartesian coordinates of $n - m$ points $x_{m+1}, \dots, x_n$ (called *anchors*) in $\mathbb{R}^2$ and the Euclidean distances $\|x_i - x_j\|$ for all $(i, j) \in \mathcal{A}$, where $\mathcal{A} \subset \{(i, j) \in \mathbb{N}^2 : 1 \le i < j \le n\}$ is the set of *edges*. We say that the two points x_i and x_j are *neighbors* if $(i, j) \in \mathcal{A}$. In practice, measurements may be inexact so that we only know some estimated d_{ij},

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where

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A},$$

$\delta = (\delta_{ij})_{(i,j)\in\mathcal{A}}$ is the noise vector and x_i^{true} is the true position of the ith point. When $\delta = 0$, we call this problem the *noiseless* sensor network localization problem.

The sensor network localization problem is NP-hard in general, and thus efforts have been directed at solving this problem approximately. In particular, various convex relaxations have been proposed to approximate the problem. Examples include second-order cone programming (SOCP) relaxations [7, 18], semidefinite programming (SDP) relaxations [3–5, 8–12, 16], edge-based semidefinite programming (ESDP) relaxations [15, 19] and sum of squares (SOS) relaxations [13]. We will describe known relationships between these relaxations in Sect. 3. In this paper, we will add to the list a comparison between SOS relaxations and some SDP type relaxations, and in particular, show that SOS relaxations are stronger.

The paper is organized as follows. We introduce notations in Sect. 2. In Sect. 3, we briefly describe some existing convex relaxation approaches and their known relationships. We compare SOS relaxations with some SDP type relaxations in Sect. 4. A simple condition for testing solution accuracy is given in Sect. 5. We present some numerical examples and simulations in Sect. 6 to illustrate the strength of SOS type relaxations.

2 Notation

Throughout this paper, sensor positions x_i are 2×1 vectors, $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and ^{T} denotes transpose. For a vector $v \in \mathbb{R}^p$, $\|v\|$ denotes the Euclidean norm of v. For $A \in \mathbb{R}^{p\times q}$, a_{ij} denotes the (i,j)th entry of A. For $A, B \in \mathcal{S}^p$, $A \succeq B$ means $A - B$ is positive semidefinite. For $A \in \mathcal{S}^p$ and an index set $\mathcal{I}$, $A_{\mathcal{I}} = \left(a_{ij}\right)_{i,j\in\mathcal{I}}$ denotes the principal submatrix of A comprising the rows and columns of A indexed by $\mathcal{I}$.

Any instance of the sensor network localization problem has an associated graph structure, namely the graph $\mathcal{G} = (\{1,\ldots,n\},\mathcal{A})$. We will work under the standard assumptions that every connected component of $\mathcal{G}$ has at least one index corresponding to an anchor and that each sensor connects to at least one other sensor. The first assumption is justified since if a connected component has no anchors, all associated sensors are clearly not localizable, i.e. their positions are not uniquely determined from the known distances; while the second assumption is reasonable since if a sensor is only connected to anchors, determining its location can be treated as a separate problem. We partition the set $\mathcal{A}$ of edges into the sets $\mathcal{A}^s = \{(i,j) \in \mathcal{A} : i < j \le m\}$ (edges from a sensor to a sensor) and $\mathcal{A}^a = \{(i,j) \in \mathcal{A} : i \le m < j\}$ (edges from a sensor to an anchor). The set β^k will be the set of all monomials in variables $\{x_i^1, x_i^2 : i = 1,\ldots,m\}$ with degree up to k, while for $(i,j) \in \mathcal{A}$, the set β_{ij}^k will denote the set of all monomials of degree up to k in variables $\{x_i^1, x_i^2, x_j^1, x_j^2\}$ if $(i,j) \in \mathcal{A}^s$, or in variables $\{x_i^1, x_i^2\}$ if $(i,j) \in \mathcal{A}^a$. Let β be any set of monomials. We define ξ_β to be the column vector indexed by β with polynomial entries such that for each $s \in \beta$,

$[\xi_\beta]_s = s(x)$. Let Γ be the set of monomials obtained by taking all possible pairwise products of the elements of β. Then

$$\xi_\beta \xi_\beta^T = \sum_{s\in\Gamma} s(x) A_s, \tag{1}$$

for some $|\beta| \times |\beta|$ real symmetric matrices A_s. Given a real vector y indexed by a set of monomials containing Γ, we define the moment matrix of y with respect to β as

$$M_\beta(y) = \sum_{s\in\Gamma} y_s A_s,$$

a linearization of (1).

3 Convex relaxations for sensor network localization

In this section, we discuss briefly some existing convex relaxations for the sensor network localization problem. We remark on algorithms for solving these relaxations at the end of this section.

3.1 SDP type relaxations

The sensor network localization can be formulated as the following optimization problem:

$$\min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|. \tag{2}$$

In the SDP approach of Biswas and Ye [3, 4], letting $X = \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix} \in \mathbb{R}^{2\times m}$ and I_2 denote the 2×2 identity matrix, they considered the following SDP relaxation of (2):

$$\begin{aligned} \upsilon_{\text{sdp}} := \min_Z & \sum_{(i,j)\in\mathcal{A}} \left| \ell_{ij}(Z) - d_{ij}^2 \right| \\ \text{s.t.} \quad & Z = \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix} \succeq 0, \end{aligned} \tag{3}$$

where $U = \left(u_{ij}\right)_{1\le i,j\le m}$ and

$$\ell_{ij}(Z) := \begin{cases} u_{ii} - 2u_{ij} + u_{jj} & \text{if } i < j \le m; \\ u_{ii} - 2x_i^T x_j + \|x_j\|^2 & \text{if } i \le m < j. \end{cases}$$

When m is large, SDP relaxation (3) can be hard to solve. In [19], Wang et al. further relaxed (3) to the so-called ESDP relaxation, by requiring only the principal 4×4 submatrices of Z associated with $\mathcal{A}^s$ to be positive semidefinite. Specifically, the ESDP relaxation takes the form

$$\begin{aligned} \upsilon_{\text{esdp}} := \min_{Z} & \sum_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z) - d_{ij}^2\right| \\ \text{s.t.} & \begin{pmatrix} u_{ii} & u_{ij} & x_i^T \\ u_{ij} & u_{jj} & x_j^T \\ x_i & x_j & I_2 \end{pmatrix} \succeq 0 \; \forall (i,j) \in \mathcal{A}^s, \end{aligned} \tag{4}$$

where Z stands for the matrix $\left(\begin{smallmatrix} U & X^T \\ X & I_2 \end{smallmatrix}\right)$.

Let $\mathcal{P} \subseteq \mathbb{R}^{2\times m}$ be the set of minimizers of problem (2). Let $\mathcal{S}_{\text{sdp}}$ and $\mathcal{S}_{\text{esdp}}$ denote the solution set of (3) and (4) respectively and let $\mathcal{P}_{\text{sdp}}$ and $\mathcal{P}_{\text{esdp}}$ be the set of corresponding recovered sensor positions X. Then it is obvious from the definition of ESDP relaxation that $\mathcal{P}_{\text{sdp}} \subseteq \mathcal{P}_{\text{esdp}}$ in the noiseless case. Furthermore, it is suggested by [19, Example 1] that the inclusion is strict in general. In other words, in the noiseless case, SDP relaxation can be strictly stronger than ESDP relaxation. In the noisy case, one cannot expect inclusion of solution sets but it is not hard to see that $\upsilon_{\text{sdp}} \geq \upsilon_{\text{esdp}}$.

We next present a simple example to illustrate the above relaxations.

Example 1 Let $n = 4$ and $m = 2$. The anchors are $x_3 = (-1, 0)^T$ and $x_4 = (1, 0)^T$, and the true positions of the sensors are $x_1 = (0, 0)^T$ and $x_2 = (0, 1)^T$. We set $\mathcal{A} = \{(1, 2), (1, 3), (1, 4)\}$. Since there are only two sensors, ESDP and SDP relaxations coincide.

According to (2), the sensor network localization problem is formulated as

$$\min_{x_1,x_2} \; |\|x_1 - x_2\|^2 - 1| + |\|x_1 - x_3\|^2 - 1| + |\|x_1 - x_4\|^2 - 1|. \tag{5}$$

By definition, $\ell_{12}(Z) = u_{11} - 2u_{12} + u_{22}$, $\ell_{13}(Z) = u_{11} + 2x_1^1 + 1$ and $\ell_{14}(Z) = u_{11} - 2x_1^1 + 1$. Hence, the ESDP relaxation of (5) is given by

$$\begin{aligned} \min \quad & |u_{11} - 2u_{12} + u_{22} - 1| + |u_{11} + 2x_1^1| + |u_{11} - 2x_1^1| \\ \text{s.t.} \quad & \begin{pmatrix} u_{11} & u_{12} & x_1^T \\ u_{12} & u_{22} & x_2^T \\ x_1 & x_2 & I_2 \end{pmatrix} \succeq 0. \end{aligned}$$

Before ending this subsection, we would like to mention that an SDP relaxation other than (3) can be derived by considering an Euclidean distance matrix (EDM) completion problem; see for example [1, 2, 9–12]. The derivation starts by treating anchors the same as sensors, and the resulting relaxation is equivalent to the above Biswas-Ye SDP relaxation in the sense that they give the same set of Z. In other words, the Biswas-Ye SDP relaxation is essentially equivalent to the classical EDM relaxations used in the literature. We refer the readers to [12, Sect. 5] for a detailed discussion.

3.2 SOCP relaxation

SOCP relaxation was proposed in [7] and extensively studied in [18]. In this approach, one replaces the absolute value in (2) by the function $(\cdot)_+ := \max\{\cdot, 0\}$, yielding a convex optimization problem:

$$\upsilon_{\rm socp} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left(\|x_i - x_j\|^2 - d_{ij}^2\right)_+ . \tag{6}$$

It is shown in [18] that (6) can be reformulated into a standard SOCP.

Let $\mathcal{S}_{\rm socp}$ denote the solution set of (6) and let $\mathcal{P}_{\rm socp}$ denote the set of corresponding recovered sensor positions X. In [18, Proposition 3.1], it is shown that in the noiseless case, $\mathcal{P}_{\rm sdp} \subseteq \mathcal{P}_{\rm socp}$. Following essentially the same line of arguments, it is shown in [19, Theorem 4.5] that $\mathcal{P}_{\rm esdp} \subseteq \mathcal{P}_{\rm socp}$ in the noiseless case. This last inclusion is in general strict. To see this, recall from [18, Proposition 6.2] that, in the noiseless case, if some sensor is not in the convex hull of other points, $\mathcal{P}_{\rm socp}$ is not a singleton set. However, using the same proof as [20, Theorem 2.2], we see that $\mathcal{P}_{\rm esdp}$ is a singleton set for any d-trilateration graphs such that $\{x_1^{\rm true}, \ldots, x_n^{\rm true}\}$ are generic[1] and that the first $d+1$ points in the trilateration ordering are anchors, when there is no noise in the distance measurements. Thus, there are cases where sensor positions can be recovered by solving ESDP relaxation but not SOCP relaxation. Combining the discussion in this and the previous subsection, we have in the noiseless case that

$$\mathcal{P} \subseteq \mathcal{P}_{\rm sdp} \subseteq \mathcal{P}_{\rm esdp} \subseteq \mathcal{P}_{\rm socp}, \tag{7}$$

where the third inclusion can be strict in general from the above discussion, and [19, Example 1] suggests that the second inclusion can also be strict in general.

In the noisy case, it is proved in [18, Proposition 6.2] that $\upsilon_{\rm sdp} \geq \upsilon_{\rm socp}$ and in [19, Theorem 4.5] that $\upsilon_{\rm esdp} \geq \upsilon_{\rm socp}$. Hence, combining with the discussion in the previous subsection, we have in the noisy case that

$$\upsilon_{\rm sdp} \geq \upsilon_{\rm esdp} \geq \upsilon_{\rm socp}. \tag{8}$$

3.3 SOS type relaxations

There are two different SOS relaxations, based on different formulations of the sensor network localization problem.

An SOS relaxation is proposed in [8], where the original problem is formulated as in (2) and a degree one SOS relaxation is used. More specifically, let

$$\|x_i - x_j\|^2 - d_{ij}^2 =: \sum_{s\in\beta^2} p_s^{ij} s(x) \quad \forall (i,j) \in \mathcal{A}.$$

The relaxation is given by

$$\begin{aligned} \upsilon_{\rm mom}^1 := \min_y \ & \sum_{(i,j)\in\mathcal{A}} \left| \sum_{s\in\beta^2} p_s^{ij} y_s \right| \\ \text{s.t.} \quad & M_{\beta^1}(y) \succeq 0, \\ & y_1 = 1, \end{aligned} \tag{9}$$

[1] We say that the points $\{x_1^{\rm true}, \ldots, x_n^{\rm true}\}$ are generic if there does not exist a non-zero polynomial $h : \mathbb{R}^{2\times n} \to \mathbb{R}$ with integer coefficients such that $h(x_1^{\rm true}, \ldots, x_n^{\rm true}) = 0$.

where $M_{\beta^1}(y)$ is the moment matrix generated by moment vector $y=(y_s)_{s\in\beta^2}$.

Instead of (2), the sensor network localization can also be formulated as the following polynomial optimization problem:

$$\min_{x_1,\ldots,x_m} p(x) := \sum_{(i,j)\in\mathcal{A}} (\|x_i - x_j\|^2 - d_{ij}^2)^2. \tag{10}$$

This is an unconstrained polynomial optimization problem and can be relaxed using sum of squares, as proposed in [13] by Nie:

$$\begin{aligned} \upsilon_{\text{sos}} &:= \max_{q_i,\gamma} \gamma \\ \text{s.t.} &\quad p(x)-\gamma = \textstyle\sum_{i=1}^r q_i(x)^2, \end{aligned} \tag{11}$$

where $q_i(x)$ are arbitrary polynomials. It is well known that this problem can be reformulated as the following SDP

$$\begin{aligned} \upsilon_{\text{sos}} &:= \max_{W,\gamma} \gamma \\ \text{s.t.} &\quad p(x)-\gamma = \xi_{\beta^2}^T W \xi_{\beta^2}, \quad W \succeq 0. \end{aligned} \tag{12}$$

Write $p(x)=\sum_{s\in\beta^4} p_s s(x)$, the dual of (12) can then be written as

$$\begin{aligned} \upsilon_{\text{mom}} &:= \min_y \sum_{s\in\beta^4} p_s y_s \\ \text{s.t.} &\quad M_{\beta^2}(y) \succeq 0, \\ &\quad y_1 = 1, \end{aligned} \tag{13}$$

where y is a real vector indexed by β^4. The solution set of (13) is denoted by $\mathcal{S}_{\text{mom}}$ and the sensor x_i is recovered from a solution $y\in\mathcal{S}_{\text{mom}}$ by setting $x_i=(y_{x_i^1}, y_{x_i^2})^T$. The set of all sensor positions (each sensor position is denoted by a 2×1 vector) obtained this way is denoted by $\mathcal{P}_{\text{mom}} \subseteq \mathbb{R}^{2\times m}$.

In view of the special structure of (10), Nie proposed the following sparse-SOS relaxation:

$$\begin{aligned} \upsilon_{\text{spsos}} &:= \max_{W_{ij},\gamma} \gamma \\ \text{s.t.} &\quad p(x)-\gamma = \sum_{(i,j)\in\mathcal{A}^s} \xi_{\beta_{ij}^2}^T W_{ij} \xi_{\beta_{ij}^2}, \\ &\quad W_{ij} \succeq 0 \quad \forall (i,j)\in\mathcal{A}^s. \end{aligned} \tag{14}$$

This corresponds to demanding not only that $p(x)-\gamma$ is a sum of squares, but also that each of its summands is the square of a polynomial depending only on x_i and x_j, for some $(i,j)\in\mathcal{A}^s$. The dual of (14) is

$$\upsilon_{\text{spmom}} := \min_y \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\beta_{ij}^4} p_s^{ij} y_s$$

$$\begin{aligned} \text{s.t.} \quad & M_{\beta_{ij}^2}(y) \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s, \\ & y_1 = 1, \end{aligned} \tag{15}$$

where

$$(\|x_i - x_j\|^2 - d_{ij}^2)^2 =: \sum_{s \in \beta_{ij}^4} p_s^{ij} s(x) \quad \forall (i,j) \in \mathcal{A}.$$

Note that $M_{\beta_{ij}^2}(y) \succeq 0$ for $(i,j) \in \mathcal{A}^s$ implies $M_{\beta_{ij}^2}(y) \succeq 0$ for $(i,j) \in \mathcal{A}^a$, since each sensor is connected to at least one other sensor. The solution set of (15) is denoted by $\mathcal{S}_{\text{spmom}}$. The sensor x_i is recovered from a solution y of (15) by setting $x_i = (y_{x_i^1}, y_{x_i^2})^T$. The set of all sensor positions obtained this way is denoted by $\mathcal{P}_{\text{spmom}} \subseteq \mathbb{R}^{2\times m}$. It was shown in [13, Theorem 3.4] that $\upsilon_{\text{spsos}} = \upsilon_{\text{spmom}}$ and it is easy to see that, in the noiseless case, $\mathcal{P} \subseteq \mathcal{P}_{\text{mom}} \subseteq \mathcal{P}_{\text{spmom}}$ and that $\mathcal{S}_{\text{mom}} \subseteq \mathcal{S}_{\text{spmom}}$. A general study of these sparse SOS relaxations can be found in [14].

We next present an example to illustrate Nie's sparse-SOS relaxation.

Example 2 Consider the network in Example 1. Since there are only two sensors, sparse-SOS and SOS relaxations coincide.

According to (10), the sensor network localization problem is formulated as

$$\min_{x_1,x_2} (\|x_1 - x_2\|^2 - 1)^2 + (\|x_1 - x_3\|^2 - 1)^2 + (\|x_1 - x_4\|^2 - 1)^2. \tag{16}$$

The corresponding objective function in sparse-SOS relaxation (15) is obtained by expanding the objective function in (16) and replacing polynomials s by variables y_s. Hence, sparse-SOS relaxation of (16) is

$$\begin{aligned} \upsilon_1 := & \min_y f(y) \\ \text{s.t.} \quad & M_{\beta^2}(y) \succeq 0, \\ & y_1 = 1, \end{aligned} \tag{17}$$

where

$$\begin{aligned} f(y) = 1 &- 2y_{(x_2^2)^2} + y_{(x_2^2)^4} - 2y_{(x_2^1)^2} + 2y_{(x_2^1 x_2^2)^2} + y_{(x_2^1)^4} + 4y_{x_1^2 x_2^2} - 4y_{x_1^2 (x_2^2)^3} \\ &- 4y_{x_1^2 (x_2^1)^2 x_2^2} - 2y_{(x_1^2)^2} + 6y_{(x_1^2 x_2^2)^2} + 2y_{(x_1^2 x_2^1)^2} - 4y_{(x_1^2)^3 x_2^2} + 3y_{(x_1^2)^4} \\ &+ 4y_{x_1^1 x_2^1} - 4y_{x_1^1 x_2^1 (x_2^2)^2} - 4y_{x_1^1 (x_2^1)^3} + 8y_{x_1^1 x_1^2 x_2^1 x_2^2} - 4y_{x_1^1 (x_1^2)^2 x_2^1} + 6y_{(x_1^1)^2} \\ &+ 2y_{(x_1^1 x_2^2)^2} + 6y_{(x_1^1 x_2^1)^2} - 4y_{(x_1^1)^2 x_1^2 x_2^2} + 6y_{(x_1^1 x_1^2)^2} - 4y_{(x_1^1)^3 x_2^1} + 3y_{(x_1^1)^4}. \end{aligned}$$

While the relationships for SOCP, ESDP and SDP relaxations in the noiseless case (as described in (7)) are well-known, up to now there were no such results relating them to SOS relaxations. In the next section we fill in this gap, by establishing relationships between SOS and SDP relaxations.

3.4 Algorithms

All the aforementioned relaxations can be recast into standard SDPs, which can then be solved by standard interior point solvers such as SeDuMi [17]. Besides, there are methods specialized to solve some of these relaxations.

For SOCP relaxation (6), in [18], a fast distributed algorithm (SCGD algorithm) has been proposed, exploiting the partial separable structure of the problem. Distributed algorithms have also been proposed in [4, 5] for SDP relaxation (3). On the other hand, in [15], a fast distributed algorithm (LPCGD algorithm) has been proposed to solve a variant of ESDP relaxation (the ρ-ESDP), which exploits its partial separable structure. Specialized algorithms exploiting the clique structure of the network have been proposed in [10, 11] for SDP relaxation (3),[2] and in [8] for SDP relaxation (3) and degree one SOS relaxation (9). In contrast to the many specialized algorithms developed for these relaxations, for the SOS relaxations proposed by Nie, there is currently no specialized algorithm exploiting the structure of the problems.

4 Relationship between SOS and SDP relaxations

In this section, we study the relationship between SOS and SDP relaxations. Our first result shows that SDP relaxation (3) is equivalent to degree one SOS relaxation (9), regardless of distance measurement noise. The proof presented is a simplification of the original argument by the second author and is due to Paul Tseng. We also note that part (b) of the theorem was already proved in [8].

Theorem 1

(a) *Let Z be a feasible solution of* (3), *then there is a vector y indexed by β^2 that is feasible for* (9) *and has the same objective value.*
(b) *If y is a feasible solution of* (9), *then there exists Z that is feasible for* (3) *and has the same objective value.*

Proof (a) Let $Z = \left(\begin{smallmatrix} U & X^T \\ X & I_2 \end{smallmatrix}\right)$, where $X = (x_1 \ \dots \ x_m)$. Define y by setting $y_1 = 1$, $y_{x_i^k} = x_i^k$, $y_{x_i^1 x_j^k} = x_i^1 x_j^k$ and $y_{x_i^2 x_j^2} = u_{ij} - x_i^1 x_j^1$ for all $i, j = 1, \dots, m$ and $k = 1, 2$. Let v_k be the vector $(x_1^k \ \dots \ x_m^k)$, for $k = 1, 2$, then we have

$$M_{\beta^1}(y) = \begin{pmatrix} 1 \\ v_1^T \\ v_2^T \end{pmatrix} (1 \ v_1 \ v_2) + \begin{pmatrix} 0 & 0 \\ 0 & U - X^T X \end{pmatrix}.$$

The first matrix is positive semidefinite of rank 1 while the second matrix is positive semidefinite since, by (3) and a basic property of Schur complement, $U - X^T X \succeq 0$.

[2]As remarked earlier, these works consider an equivalent SDP relaxation obtained via an Euclidean distance matrix completion problem.

Thus y is a feasible solution of (9) and it is easy to check that y gives the same objective value as Z.

(b) Consider the submatrices U_1 and U_2 of $M_{\beta^1}(y)$ indexed by $\{1, x_1^1, \ldots, x_m^1\}$ and $\{1, x_1^2, \ldots, x_m^2\}$ respectively. Let $w_k = (y_{x_1^k} \ \ldots \ y_{x_m^k})$, $k = 1, 2$. By the same property of Schur complement as above, we have $U_k \succeq w_k^T w_k$ for $k = 1, 2$. Let $U = U_1 + U_2$ and $X = \binom{w_1}{w_2}$, then

$$U = U_1 + U_2 \succeq w_1^T w_1 + w_2^T w_2 = X^T X,$$

hence $Z = \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$ is a feasible solution of (3). Again, it is easy to check that Z gives the same objective value as y. □

Our next result shows that, in the noiseless case, sparse-SOS relaxation is stronger than ESDP relaxation.

Theorem 2 *In the noiseless case*, $\mathcal{P}_{\text{spmom}} \subseteq \mathcal{P}_{\text{esdp}}$.

Proof Take any $X = (x_1 \ \ldots \ x_m) \in \mathcal{P}_{\text{spmom}}$ and the corresponding $y \in \mathcal{S}_{\text{spmom}}$. Then for each $(i, j) \in \mathcal{A}^s$, it holds that $M_{\beta_{ij}^2}(y) \succeq 0$. Hence, both $M_{\{1, x_i^2, x_j^2\}}(y)$ and $M_{\{1, x_i^1, x_j^1\}}(y)$, being principal submatrices of $M_{\beta_{ij}^2}(y)$, are positive semidefinite. For $(i, j) \in \mathcal{A}^s$, define $u_{kl} := y_{x_k^1 x_l^1} + y_{x_k^2 x_l^2}$ for $k, l \in \{i, j\}$. We claim that

$$\begin{pmatrix} u_{ii} & u_{ij} & x_i^T \\ u_{ij} & u_{jj} & x_j^T \\ x_i & x_j & I_2 \end{pmatrix} \succeq 0.$$

To see this, it suffices to show that the Schur complement of I_2,

$$\begin{pmatrix} u_{ii} & u_{ij} \\ u_{ij} & u_{jj} \end{pmatrix} - \begin{pmatrix} \|x_i\|^2 & x_i^T x_j \\ x_i^T x_j & \|x_j\|^2 \end{pmatrix},$$

is positive semidefinite. But this matrix is the sum of the Schur complement of 1 in $M_{\{1, x_i^1, x_j^1\}}(y)$, which is

$$\begin{pmatrix} y_{(x_i^1)^2} - (x_i^1)^2 & y_{x_i^1 x_j^1} - x_i^1 x_j^1 \\ y_{(x_j^1)^2} - x_i^1 x_j^1 & y_{x_i^1 x_j^1} - (x_j^1)^2 \end{pmatrix},$$

and the Schur complement of 1 in $M_{\{1, x_i^2, x_j^2\}}(y)$, which is

$$\begin{pmatrix} y_{(x_i^2)^2} - (x_i^2)^2 & y_{x_i^2 x_j^2} - x_i^2 x_j^2 \\ y_{(x_j^2)^2} - x_i^2 x_j^2 & y_{x_i^2 x_j^2} - (x_j^2)^2 \end{pmatrix}.$$

Both matrices are positive semidefinite since $M_{\{1, x_i^2, x_j^2\}}(y)$ and $M_{\{1, x_i^1, x_j^1\}}(y)$ are. Hence, the claim follows.

Define $Z := \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$. Then Z is feasible for (4). We shall show that $Z \in \mathcal{S}_{\rm esdp}$. To this end, let q^{ij} be a column vector such that

$$\|x_i - x_j\|^2 - d_{ij}^2 =: \sum_{s\in\beta_{ij}^2} q_s^{ij} s(x) \quad \forall (i,j)\in\mathcal{A}.$$

Since $M_{\beta_{ij}^2}(y) \succeq 0$ for all $(i,j)\in\mathcal{A}$, it follows that

$$\begin{pmatrix} 1 & \sum_{s\in\beta_{ij}^2} q_s^{ij} y_s \\ \sum_{s\in\beta_{ij}^2} q_s^{ij} y_s & \sum_{s\in\beta_{ij}^4} p_s^{ij} y_s \end{pmatrix} = \begin{pmatrix} e_1^T \\ q^T \end{pmatrix} M_{\beta_{ij}^2}(y)(e_1\ q) \succeq 0$$

for all $(i,j)\in\mathcal{A}$, where e_1 is the vector that is one in the first entry and zero elsewhere. This last relation implies

$$\sum_{s\in\beta_{ij}^4} p_s^{ij} y_s \ge \left(\sum_{s\in\beta_{ij}^2} q_s^{ij} y_s\right)^2 = \left(\ell_{ij}(Z) - d_{ij}^2\right)^2 \quad \forall (i,j)\in\mathcal{A}.$$

Since $y\in\mathcal{S}_{\rm spmom}$, the noiseless assumption implies that $\sum_{(i,j)\in\mathcal{A}}(\ell_{ij}(Z) - d_{ij}^2)^2 = 0$, and hence Z solves (4). This proves that $\mathcal{P}_{\rm spmom} \subseteq \mathcal{P}_{\rm esdp}$. □

From the above proof, we obtain the following corollary.

Corollary 1 *Consider the noiseless case. Let* $y\in\mathcal{S}_{\rm spmom}$. *For* $(i,j)\in\mathcal{A}^s$, *define* $x_k^s := y_{x_k^s}$, $u_{kl} := y_{x_k^1 x_l^1} + y_{x_k^2 x_l^2}$, *for* $s = 1,2$ *and* $k,l\in\{i,j\}$. *Then* $Z := \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix} \in \mathcal{S}_{\rm esdp}$.

Remark 1 Similarly one can show that, in the noiseless case, $\mathcal{P}_{\rm mom} \subseteq \mathcal{P}_{\rm sdp}$.

We shall illustrate in Sect. 6 by numerical examples that the inclusions $\mathcal{P}_{\rm mom} \subseteq \mathcal{P}_{\rm sdp}$ and $\mathcal{P}_{\rm spmom} \subseteq \mathcal{P}_{\rm esdp}$ are likely strict in general. However, we do not have explicit proofs for them. The same examples also suggest that $\mathcal{P}_{\rm sdp}$ and $\mathcal{P}_{\rm spmom}$ are not comparable.

5 Testing accuracy of individual sensors

As in [15, 18] and [19], one is interested in identifying sensors whose recovered locations remain the same for all solutions since, in the noiseless case, these sensors will turn out to be in their true position. Hence, we are interested in the following set

$$\mathcal{I}_{\rm spmom} := \left\{ i\in\{1,\ldots,m\} \mid (y_{x_i^1}, y_{x_i^2}) \text{ is invariant over } \mathcal{S}_{\rm spmom} \right\}.$$

By invariance over $\mathcal{S}_{\text{spmom}}$, we mean $(y_{x_i^1}, y_{x_i^2})$ is the same for any $y \in \mathcal{S}_{\text{spmom}}$. In order to identify elements in $\mathcal{I}_{\text{spmom}}$, we consider a version of individual trace for SOS relaxations.

Definition 1 For any $y \in \mathcal{S}_{\text{spmom}}$, the ith individual trace of y is defined as

$$\text{Tr}_i(y) := y_{(x_i^1)^2} + y_{(x_i^2)^2} - (y_{x_i^1})^2 - (y_{x_i^2})^2.$$

Note that the trace is always nonnegative since $y_{(x_i^k)^2} - (y_{x_i^k})^2$ is the determinant of a principal submatrix of $M_{\beta_{ij}^2}(y)$, for $k = 1, 2$. We have the following simple result, generalizing the zero trace test to the setting of SOS relaxations. The proof parallels that of [18, Proposition 4.1]. We include it for completeness.

Theorem 3 *If* $\text{Tr}_i(y) = 0$ *for some* y *in the relative interior of* $\mathcal{S}_{\text{spmom}}$, *then* $i \in \mathcal{I}_{\text{spmom}}$.

Proof We shall show that $y_{x_i^1}$ is invariant over $\mathcal{S}_{\text{spmom}}$. The proof for $y_{x_i^2}$ is similar. Note that $\text{Tr}_i(y) = 0$ implies $y_{(x_i^1)^2} = (y_{x_i^1})^2$. Take any $w \in \mathcal{S}_{\text{spmom}}$. Since y is in the relative interior of $\mathcal{S}_{\text{spmom}}$, there exists $\epsilon > 0$ so that both

$$\eta := y + \epsilon(w - y) \quad \text{and} \quad \zeta := y - \epsilon(w - y)$$

belong to $\mathcal{S}_{\text{spmom}}$. Thus, $y = \frac{\eta+\zeta}{2}$ and hence

$$\begin{aligned} 0 = y_{(x_i^1)^2} - (y_{x_i^1})^2 &= \frac{1}{2}[\eta_{(x_i^1)^2} - (\eta_{x_i^1})^2] + \frac{1}{2}[\zeta_{(x_i^1)^2} - (\zeta_{x_i^1})^2] + \frac{1}{4}(\eta_{x_i^1} - \zeta_{x_i^1})^2 \\ &\geq \frac{1}{4}(\eta_{x_i^1} - \zeta_{x_i^1})^2 = \epsilon^2(w_{x_i^1} - y_{x_i^1})^2. \end{aligned}$$

This shows that $w_{x_i^1} = y_{x_i^1}$ and the proof is complete. □

Hence, after getting a solution y in the relative interior of $\mathcal{S}_{\text{spmom}}$ (say, by solving the sparse-SOS relaxation using path-following interior point methods), we look at the individual traces. If $\text{Tr}_i(y) = 0$, then the true position of the ith sensor is given by $(y_{x_i^1}, y_{x_i^2})^T$.

It is not known whether the converse of Theorem 3 is true. Nonetheless, we are able to establish a partial converse to the theorem in the noiseless case. This follows from the fact that if $y \in \mathcal{S}_{\text{spmom}}$, and $Z \in \mathcal{S}_{\text{esdp}}$ is obtained from y according to Corollary 1, then the trace $\text{Tr}_i(y)$ equals the ESDP trace $\text{tr}_i(Z)$ defined in [19]. Then the proofs of [15, Lemmas 2,3] follow through and we get the following result.

Lemma 1 *In the noiseless case, let* $i \leq m$ *and* $y \in \mathcal{S}_{\text{spmom}}$ *be such that the corresponding recovered sensor positions verify* $\|x_i - x_j\| = d_{ij}$. *Then if* $j > m$, *we have* $\text{Tr}_i(y) = 0$, *and if* $j \leq m$, *we have* $\text{Tr}_i(y) = \text{Tr}_j(y)$.

The next theorem follows from Lemma 1 by a simple induction argument.

Theorem 4 *In the noiseless case, let $i \in \mathcal{I}_{\text{spmom}}$ be such that there exists a path with nodes in $\mathcal{I}_{\text{spmom}}$ connecting x_i to an anchor. Then* $\text{Tr}_i(y) = 0$ *for all* $y \in \mathcal{S}_{\text{spmom}}$.

We next illustrate Theorems 3 and 4 by an example.

Example 3 Consider sparse-SOS relaxation (17) in Example 2. Since $p(x)$ is a sum of squares, we have $\upsilon_1 \geq 0$. By letting $\bar{y}_1 = 1$ and $\bar{y}_s = s(x_1^{\text{true}}, x_2^{\text{true}})$ for all $s \in \beta^2$, we obtain that $\bar{y}$ is feasible for (17) and $f(\bar{y}) = 0$. Thus, $\upsilon_1 = 0$ and $\bar{y} \in \mathcal{S}_{\text{spmom}}$.

Let $y^\diamond \in \mathcal{S}_{\text{spmom}}$ and let $f_1(y)$ and $f_2(y)$ be the linearization of $(\|x_1 - x_2\|^2 - 1)^2$ and $(\|x_1 - x_3\|^2 - 1)^2 + (\|x_1 - x_4\|^2 - 1)^2$ respectively, obtained by replacing s with y_s. Since these polynomials are sums of squares, it follows easily that $f_1(y) \geq 0$ and $f_2(y) \geq 0$ for any y feasible for (17). Using these facts, $\upsilon_1 = 0$, the expression of $f_2(y)$ and the optimality of $y^\diamond$, we obtain in particular that

$$f_2(y^\diamond) = 2y^\diamond_{(x_1^2)^4} + 8y^\diamond_{(x_1^2)^2} + 4y^\diamond_{(x_1^1 x_1^2)^2} + 2y^\diamond_{(x_1^1)^4} = 0. \tag{18}$$

It follows from (18) and $M(y^\diamond) \succeq 0$ that $y^\diamond_{x_1^1} = y^\diamond_{x_1^2} = y^\diamond_{(x_1^1)^2} = y^\diamond_{(x_1^2)^2} = 0$. Hence, $(y^\diamond_{x_1^1}, y^\diamond_{x_1^2})^T$ has to equal $(0,0)^T$, the true position of the first sensor.

On the other hand, consider $\hat{y}_1 = 1$ and $\hat{y}_s = s(x_1^{\text{true}}, -x_2^{\text{true}})$ for all $s \in \beta^2$. Obviously, this is also a solution to (17). Thus, $y^* := \frac{\bar{y}+\hat{y}}{2}$ also solves (17). However, $(y^*_{x_2^1}, y^*_{x_2^2})^T = (0,0)^T$, which is not the true position of the sensor. Computing the trace for this sensor, since $y^*_{(x_2^1)^2} = 0$ and $y^*_{(x_2^2)^2} = 1$, we get $\text{Tr}_2(y^*) = 1 > 0$, and so Theorem 4 would already guarantee us that this sensor position is not uniquely determined.

6 Numerical comparison

6.1 Numerical examples

Theorem 2 says that in the noiseless case, sparse-SOS relaxation is at least as strong as ESDP relaxation. We illustrate this fact in Example 4, which is taken from [19, Example 1]. In [19], the same example was used to illustrate that SDP relaxation is stronger than ESDP relaxation. All computations presented were done with SeDuMi1.1R3 [17] interfaced in Matlab7.8.0 (2009a), on Dell POWEREDGE 1950 equipped with Debian 5.0.6 (Linux). The Matlab codes that construct the problem data in SeDuMi format are written based on the SDP and ESDP codes sent to Paul Tseng by Yinyu Ye in a private communication in 2006. We used the default tolerance of SeDuMi.

Example 4 Let $n = 6$ and $m = 3$. The anchors are $x_4 = (-0.4, 0)^T$, $x_5 = (0.4, 0)^T$ and $x_6 = (0, 0.4)^T$, and the true positions of the sensors are $x_1 = (-0.05, 0.3)^T$, $x_2 = (-0.08, 0.2)^T$ and $x_3 = (0.2, 0.3)^T$. We set $\mathcal{A} = \{(1,2), (1,3), (1,4), (1,6), (2,3), (2,4), (2,6), (3,5), (3,6)\}$.

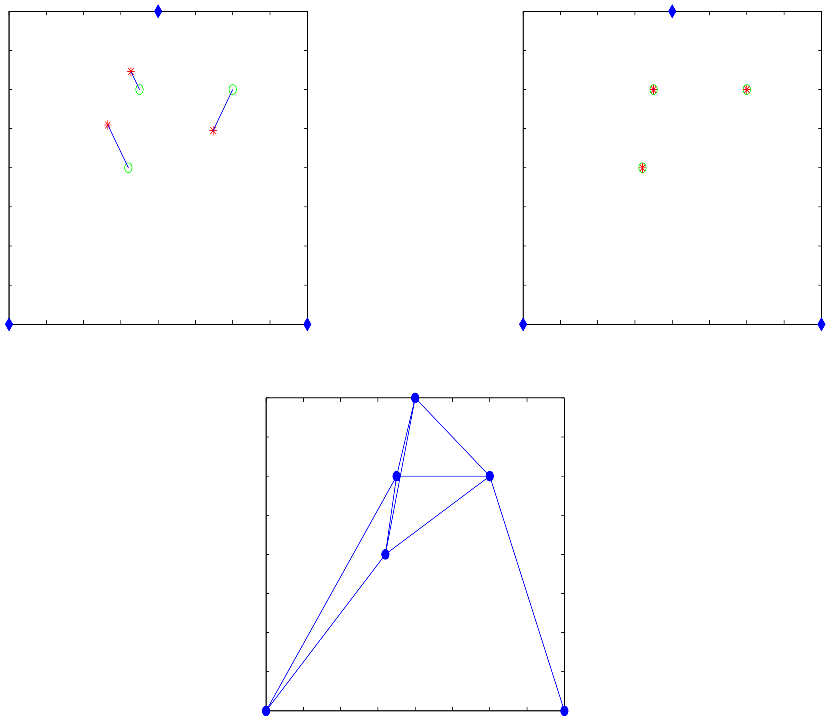

Fig. 1 The *top left figure* shows the anchor ("♦") and the solution found by solving ESDP relaxation (4). Each sensor position ("$*$") found is joined to its true position ("$\circ$") by a line. The *top right* figure shows the same information for the solution found by solving sparse-SOS relaxation (14). The *bottom figure* shows the location of the points ("$\bullet$") and the edges

First we solve ESDP relaxation (4); the result is inaccurate, as is shown in Fig. 1, with RMSD being 6e–2; where RMSD stands for Root Mean Square Deviance, defined by

$$\mathrm{RMSD} = \left(\frac{1}{m} \sum_{i=1}^{m} \|x_i - x_i^{\mathrm{true}}\|^2 \right)^{\frac{1}{2}}.$$

However, sparse-SOS relaxation seems to provide an accurate solution, as is shown in the figure, with RMSD 2e–4. This is also suggested by the small individual traces of the solution obtained by solving sparse-SOS relaxation: 1e–6, 4e–6 and 4e–6. By Theorem 3, the sensors are likely accurately positioned, since SeDuMi likely returns a relative interior solution. On the other hand, the individual traces of the solution obtained by solving ESDP relaxation turn out to be much larger: 1e–3, 7e–3 and 5e–3, so the solution is less likely to be accurate.

How does SOS relaxation compare with SDP relaxation? The next example shows a network that is not localizable by solving SDP relaxation, yet is likely localizable by solving SOS relaxation. This implies that the underlying graph has a unique realization in $\mathbb{R}^2$, but does not have a unique realization if we relax the dimension restriction. It is surprising that SOS relaxation is strong enough to restrict the dimensionality of the realization. Note that since this example only has two sensors, ESDP and SDP are equivalent, and so are SOS and sparse-SOS, thus it also provides numerical evidence for the relative strength of those methods.

Example 5 Let $n = 5$ and $m = 2$. The anchors are $x_3 = (0, 0)^T$, $x_4 = (0.5, 1)^T$ and $x_5 = (1, 0)^T$, and the true positions of the sensors are $x_1 = (0.4, 0.7)^T$ and $x_2 = (0.6, 0.7)^T$. We set $\mathcal{A} = \{(1, 2), (1, 4), (1, 5), (2, 3), (2, 4)\}$.

First we solve SDP relaxation (3); the result is inaccurate, as is shown in Fig. 2, with RMSD being 1e–1. It can also be shown manually that the solution to SDP relaxation is not unique. However, SOS relaxation seems to provide an accurate solution, as is shown in the figure, with RMSD 1e–4. This is also suggested by the individual traces of the solution obtained by solving SOS relaxation: both less than 2e–5.

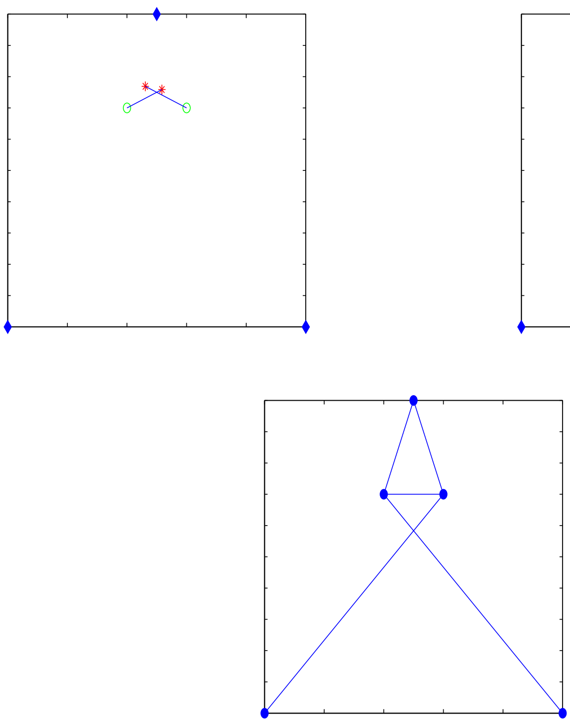

Fig. 2 The *top left figure* shows the anchor ("♦") and the solution found by solving SDP relaxation (3). Each sensor position ("∗") found is joined to its true position ("∘") by a line. The *top right figure* shows the same information for the solution found by solving SOS relaxation (12). The *bottom figure* shows the location of the points ("•") and the edges

6.2 Numerical simulations

In this subsection, we compare ESDP, SDP and sparse-SOS relaxations on randomly generated instances of sensor network localization problem.

As in [18], we randomly generate m points (sensors) uniformly in the unit square $[-0.5, 0.5]^2$ and fix four anchors at $(\pm 0.45, \pm 0.45)$. The measured distance is then set to be

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \sigma \epsilon_{ij}| \quad \forall (i, j) \in \mathcal{A}, \tag{19}$$

where $\sigma \in [0, 1]$ is the noise factor and $\epsilon_{ij} \sim N(0, 1)$ for all $(i, j) \in \mathcal{A}$.

We first consider the noiseless case, i.e., $\sigma = 0$ in (19). We consider three different methods of constructing $\mathcal{A}$:

Type I. Include (i, j) with probability p;
Type II. Include (i, j) if $\|x_i^{\text{true}} - x_j^{\text{true}}\| < \textit{radiorange}$;
Type III. Include (i, j) with probability 0.5 if $\|x_i^{\text{true}} - x_j^{\text{true}}\| < \textit{radiorange}$;

Type I was considered in [13], while Type II corresponds to the unit disc graph model widely used in the literature; see, for example, [3]. Type III models communication failure, where the probability of failure when both points are within the radio range is 50%. For each problem instance generated,[3] we solve the corresponding ESDP, SDP and sparse-SOS relaxation by calling SeDuMi. We use the default tolerance of SeDuMi, and declare the ith sensor to be accurately positioned if for the solution obtained, $\|x_i - x_i^{\text{true}}\| \le 5\text{e–}3$.

In Tables 1, 2 and 3, we report the number of accurately positioned sensors ($\#_{\text{ap}}$), the cpu time, to the nearest integer, and the RMSD of the solution obtained, to 2 significant figures, averaged over 50 problems. We observe in Table 1 that by solving sparse-SOS relaxation, we usually obtain a solution with the lowest RMSD and the

Table 1 Computational results for network Type I

p	m	SDP $\#_{\text{ap}}$/RMSD/cpu	sparse–SOS $\#_{\text{ap}}$/RMSD/cpu	ESDP $\#_{\text{ap}}$/RMSD/cpu
0.08	60	6/2.4e–1/1	25/1.6e–1/18	1/3.3e–1/0
0.08	80	66/6.8e–2/2	73/4.8e–2/75	10/2.4e–1/2
0.08	100	99/1.8e–2/4	97/1.6e–2/471	29/1.7e–1/4
0.10	60	42/7.9e–2/1	55/3.6e–2/29	8/2.5e–1/1
0.10	80	79/1.6e–2/2	79/1.4e–2/198	39/1.2e–1/3
0.10	100	100/2.9e–3/3	100/2.0e–3/1106	83/2.7e–2/8
0.12	60	59/3.2e–2/1	59/2.5e–2/44	32/1.3e–1/1
0.12	80	80/4.7e–3/2	80/3.1e–3/423	62/4.6e–2/4
0.12	100	100/5.8e–6/3	100/1.8e–4/2092	100/5.2e–4/13

[3]For each instance, we remove disconnected components and check that no sensors are connected only to anchors.

Table 2 Computational results for network Type II; $radiorange = \sqrt{\frac{\ln(n)}{n}}r$

r	m	SDP $\#_{ap}$/RMSD/cpu	sparse-SOS $\#_{ap}$/RMSD/cpu	ESDP $\#_{ap}$/RMSD/cpu
0.9	60	51/2.8e–2/1	13/8.0e–2/48	7/1.0e–1/1
0.9	80	77/1.7e–2/2	21/5.9e–2/84	19/7.7e–2/2
0.9	100	97/1.6e–2/4	18/5.4e–2/135	15/7.1e–2/3
1.0	60	58/1.1e–2/1	41/2.8e–2/64	33/4.7e–2/2
1.0	80	79/5.6e–3/2	49/2.2e–2/118	49/3.3e–2/4
1.0	100	99/4.1e–3/4	59/2.0e–2/215	72/2.4e–2/7
1.1	60	60/2.4e–3/1	56/5.7e–3/99	54/1.1e–2/3
1.1	80	79/2.2e–3/2	72/6.4e–3/209	71/1.0e–2/6
1.1	100	100/1.4e–3/4	89/5.8e–3/467	95/4.3e–3/10

Table 3 Computational results for network Type III; $radiorange = \sqrt{\frac{\ln(n)}{n}}r$

r	m	SDP $\#_{ap}$/RMSD/cpu	sparse-SOS $\#_{ap}$/RMSD/cpu	ESDP $\#_{ap}$/RMSD/cpu
1.2	60	48/4.3e–2/1	18/9.3e–2/28	6/1.3e–1/1
1.2	80	75/2.6e–2/2	24/6.4e–2/61	11/9.7e–2/2
1.2	100	95/2.3e–2/4	22/5.5e–2/107	15/8.7e–2/3
1.3	60	57/2.7e–2/1	37/4.0e–2/37	20/7.6e–2/1
1.3	80	75/2.1e–2/2	45/4.3e–2/92	29/7.2e–2/2
1.3	100	99/1.4e–2/4	58/2.5e–2/158	48/4.4e–2/4
1.4	60	58/2.0e–2/1	53/2.3e–2/46	43/4.4e–2/2
1.4	80	79/1.3e–2/2	72/1.8e–2/123	62/3.2e–2/3
1.4	100	99/1.1e–2/4	84/1.7e–2/311	83/2.5e–2/6

largest number of accurately positioned sensors. On the other hand, in Table 2 in which the usual disc graph model is considered, the solution obtained by solving SDP relaxation usually has the lowest RMSD and the largest number of accurately positioned sensors, while the solution quality achieved by solving sparse-SOS and ESDP relaxations is comparable. Finally, in Table 3, where the graph model is a hybrid of Type I and Type II, we see that the solution quality achieved by solving SDP relaxation is the best (lowest RMSD and largest number of accurately positioned sensors), followed by sparse-SOS relaxation and then by ESDP relaxation.

We next consider the case when distance measurements have noise. We consider only the most commonly used network Type II. The results are reported in Tables 4 and 5. In both tables, we see that sparse-SOS relaxation is comparable to SDP relaxation in terms of number of accurately positioned sensors but has a larger RMSD. The relative improvement in solution quality achieved by solving sparse-SOS relaxation in the noisy case is due to the fact that sparse-SOS relaxation is based on (10) rather

Table 4 Computational results for network Type II; $radiorange = \sqrt{\frac{\ln(n)}{n}}r$; $\sigma = 0.01$

r	m	SDP $\#_{ap}$/RMSD/cpu	sparse-SOS $\#_{ap}$/RMSD/cpu	ESDP $\#_{ap}$/RMSD/cpu
0.9	60	9/5.4e–2/1	7/8.3e–2/44	3/1.0e–1/1
0.9	80	16/3.5e–2/2	12/6.6e–2/78	8/8.4e–2/2
0.9	100	24/3.2e–2/4	16/6.0e–2/142	11/7.3e–2/4
1.0	60	17/2.9e–2/1	21/3.7e–2/64	10/6.2e–2/2
1.0	80	28/1.8e–2/2	34/2.6e–2/164	18/4.5e–2/3
1.0	100	37/1.4e–2/5	38/2.5e–2/233	24/3.9e–2/5
1.1	60	26/1.3e–2/1	39/1.6e–2/96	16/3.7e–2/2
1.1	80	34/9.8e–3/3	50/9.1e–3/210	22/2.3e–2/4
1.1	100	41/1.0e–2/6	53/1.3e–2/402	27/2.8e–2/7

Table 5 Computational results for network Type II; $radiorange = \sqrt{\frac{\ln(n)}{n}}r$; $\sigma = 0.05$

r	m	SDP $\#_{ap}$/RMSD/cpu	sparse-SOS $\#_{ap}$/RMSD/cpu	ESDP $\#_{ap}$/RMSD/cpu
0.9	60	1/7.6e–2/1	1/9.2e–2/41	1/1.0e–1/1
0.9	80	2/6.3e–2/2	3/7.9e–2/71	2/9.0e–2/2
0.9	100	3/5.3e–2/3	4/6.6e–2/124	2/7.7e–2/3
1.0	60	2/4.8e–2/1	3/5.6e–2/53	1/7.0e–2/2
1.0	80	3/4.7e–2/2	3/5.9e–2/108	2/7.4e–2/3
1.0	100	4/3.7e–2/4	5/4.1e–2/189	3/5.6e–2/4
1.1	60	2/3.6e–2/1	3/3.6e–2/76	1/5.3e–2/2
1.1	80	4/3.4e–2/2	4/3.5e–2/191	2/5.2e–2/4
1.1	100	4/3.1e–2/5	6/3.1e–2/404	3/4.9e–2/6

than (2). To confirm this, we also solved the SDP relaxation based on (10) (not shown in the tables) and found that it usually outperforms sparse-SOS relaxation in terms of both the number of accurately positioned sensors and RMSD.

Finally, in all experiments, we see that sparse-SOS takes significantly more time to solve than ESDP, while ESDP and SDP are comparable in terms of speed.

7 Conclusion and further remarks

In this paper, we showed that Nie's sparse-SOS relaxation is stronger than the edge-based semidefinite programming (ESDP) relaxation for noiseless sensor network localization, which is also illustrated by our numerical results. We also extended the trace test for accuracy to the sparse-SOS relaxation. Furthermore, we showed that the SOS relaxation is stronger than the SDP relaxation proposed by Biswas and Ye. We would like to remark that there are other ways of recovering sensor positions from the convex relaxations other than those considered in this paper. For example, in [6, Sect. 4.1], for the SDP relaxation, they also considered obtaining sensor positions via a best rank-2 approximation of $Z \in \mathcal{S}_{\mathrm{sdp}}$. This approach usually performs better in terms of reducing the objective function considered in their work. It would be interesting to extend our analysis to study this way of recovering sensor positions.

In view of Theorems 2 and 3, it is also worth investigating efficient algorithms to solve for a relative interior solution of (15); from our computational results, it is clear that applying a standard interior point solver is not efficient. Since (15) has a partial separable structure, one direction is to look for a distributed algorithm, like the LPCGD algorithm in [15], to solve (15). A distributed algorithm is important for applications like real time tracking. Since each edge is related to a 15 by 15 matrix in sparse-SOS relaxation, it should take more time to solve (15) than to solve (4). However, sparse-SOS relaxation is stronger than ESDP relaxation by Theorem 2: this is a tradeoff between solution accuracy and solution time.

A possible approach to save solution time and yet get higher accuracy would be to use this stronger convex relaxation to refine the solution obtained from solving ESDP relaxation. Taking advantage of the existing trace test for ESDP, we take an ESDP solution, fix those sensors with small trace as new anchors, and run sparse-SOS relaxation in the remaining reduced network. The advantage of this approach is that we would still have an accuracy certificate for the refined solution (the trace test), which is not common for existing refinement heuristics. Moreover, since sparse-SOS relaxation is solved on the reduced network, the time taken to solve the problem should be smaller compared to solving sparse-SOS relaxation on the whole network.

Acknowledgements The authors would like to thank Maryam Fazel and Rekha Thomas for careful proofreading and useful comments, and Alberto Chiecchio for suggesting Example 5. We would also like to thank the anonymous referees for their many helpful suggestions. The second author would like to thank Henry Wolkowicz for discussion of the EDM approach to SDP relaxation. The first author was partially supported by the National Science Foundation Focused Research Group grant DMS-0757371 and by Fundação para a Ciência e Tecnologia.

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