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Covering a compact polygonal set by identical circles

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Abstract The paper considers the problem of covering compact polygonal set by identical circles of minimal radius. A mathematical model of the problem based on Voronoi polygons is constructed and its characteristics are investigated. On the ground of the characteristics a modification of the Zoutendijk feasible directions method is developed to search local minima. A special approach is suggested to choose starting points. Many computational examples are given.

Keywords Covering · Circles · Optimization · Mathematical modeling · Voronoi polygons

1 Introduction

There is a non-empty bounded canonical-closed connected polygonal set $P \subset R^2$ (a set P is canonical if $P = \mathrm{cl}(\mathrm{int}(P))$, where $\mathrm{int}(P)$ is the interior of P and $\mathrm{cl}(P)$ is the closure of P [1]), R^2 is the Euclidean arithmetic 2-dimensional space. P is specified by sides lying on straight lines whose equations are

$$\chi_j(x, y) = a_j x + b_j y + c_j = 0, \quad j = 1, 2, \dots, q. \tag{1}$$

If P is convex then it is specified by the inequality system

$$\chi_j(x, y) = a_j x + b_j y + c_j \geq 0, \quad j = 1, 2, \dots, q. \tag{2}$$

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Furthermore, let Λ be a family of identical circles $C_i \subset R^2$ of radius r and center coordinates $u_i = (x_i, y_i), i = 1, 2, \ldots, n$. Later on a circle C_i translated by a vector u_i is designated as $C_i(u_i)$. Thus a vector $u = (u_1, u_2, \ldots, u_n) \in R^{2n}$ defines a location of circles $C_i, i = 1, 2, \ldots, n$, in R^2 and forms a family $\Lambda(u)$.

Definition 1 The family $\Lambda(u)$ is called a covering of P if the following relation is satisfied

$$P \cap \left(\bigcup_{i=1}^{n} C_i(u_i) \right) = P. \tag{3}$$

Definition 2 A subfamily $\Lambda_k(v)$ of $\Lambda(u)$ is called a subcovering of P if there exists a vector $v = (u_1, u_2, \ldots, u_k) \in R^{2k}$ s.t. $k \le n$ that ensures a covering of P, i.e. the following relation holds true

$$P \cap \left(\bigcup_{j=1}^{k} C_i(u_{i_j}) \right) = P. \tag{4}$$

Then there arises the following covering problem.

Problem 1 Find a subfamily $\Lambda_k(v)$ of $\Lambda(u)$ consisting of a minimal number of circles.

It is easily seen that the problem solution may be reduced to a sequence of the following k subproblems.

Let $A(S)$ be the area of a set S, $v = (u_1, u_2, \ldots, u_k)$, $k = 1, 2, \ldots, m \le n$. Then $A_k(v) = A(P \setminus \bigcup_{i=1}^{k} C_i(u_i))$ is the area of the not covered part of P.

Problem 2 For each k find $\min_v A_k(v)$.

Then it is evident that a solution of Problem 1 is a vector $v = (u_1, u_2, \ldots, u_k)$ such that $A_k(v) = 0$ and $A_{k-1}(v) \neq 0$ under $v = (u_1, u_2, \ldots, u_{k-1})$. So, when substituting Problem 1 with Problem 2, function $A_k(v)$ can be replaced by a simpler function that enables to find out whether P is covered. To this end we construct a function depending on variables $u_1, u_2, \ldots, u_n$ and its values are an area estimation of the not covered part of P.

The problem of determining good coverings of a region with n equal circles is poorly investigated in the literature as compared with packing problems. Most results in the literature on coverings of geometrical shapes concern the problem of covering a sphere by circles (circular caps) [2, 3, 6, 14].

The earliest computer results for circle coverings in the plane were obtained by Zahn, Jr. [4], who searched for coverings of a circle by equal circles using discretization.

In 1995 Tarnai and Gaspar published the first computational results on coverings of a square with at most 10 circles [5]. A locally optimal covering was found by simulating a system consisting of shrinking, tensioned bars and pin joints.

Melissen and Schuur [7], using the simulated annealing algorithm, improved the coverings with 6 and 8 circles and presented a new covering with 11 circles. Circle centers were placed in a uniform grid points. The grid was gradually refined. The objective function value (maximal radius) was computed with the help of the Voronoi polygons for the centers of the circles. For the obtained topology of the covering a polynomial equation root gives the solution.

Lengyel and Veres [8] extended the results of [5] and gave coverings of a square with up to 23 circles.

Other related results include coverings of a rectangle with at most 7 circles [9] and [10]; coverings of an equilateral triangle (with optimality proofs in the smallest cases) [11] and [12]. A survey on circle coverings can be found in [13], Chap. 5.

Nurmela and Östergård [14] present an algorithm for finding good circle coverings of the square. They proposed a method for numerically determining the structure of a covering. Using these algorithms, they improved the previous coverings of a square with 12 to 21 circles [8] and presented new coverings with 24 to 30 circles. Nurmela and Östergård also obtained a covering of a triangle with at most 36 circles [12].

This paper continues the investigations started in [15] and [16].

In the second section of this paper, we construct a mathematical model of the problem. Section 3 considers the basic properties of the constructed mathematical model. On the ground of the properties a solving strategy is proposed in Sect. 4. Section 5 is devoted to a local extremum search method being a modification of the Zoutendijk feasible directions method. Furthermore, in this section an approach to a search of starting points is offered. In Sect. 6 numerical examples are given. In Sect. 7, the performance is analyzed and some conclusions are made.

2 Mathematical model

Let $C_i(u_i)$ be specified by inequalities

$$\psi_i(X, u_i) = (x - x_i)^2 + (y - y_i)^2 - r^2 \leq 0, \quad i = 1, 2, \ldots, n. \tag{5}$$

We note that function $\psi_i(X, 0)$ specifies a paraboloid of revolution.

We take a rectangle P and circles C_1 and C_2. It is easily seen that if $u^0 = (u_1^0, u_2^0) = (-a, 0, a, 0)$ is such that the circles do not cover P (Fig. 1(a)) then $\psi_i(X, u_i^0) \geq 0$, $i = 1, 2$, on no covered part of P (hatched part of P). If $P \cap (C_1(u_1^0) \cup C_2(u_2^0)) = P$ (Fig. 1(b)) then $\psi_i(X, u_i^0) \leq 0$, $i = 1, 2$, on P.

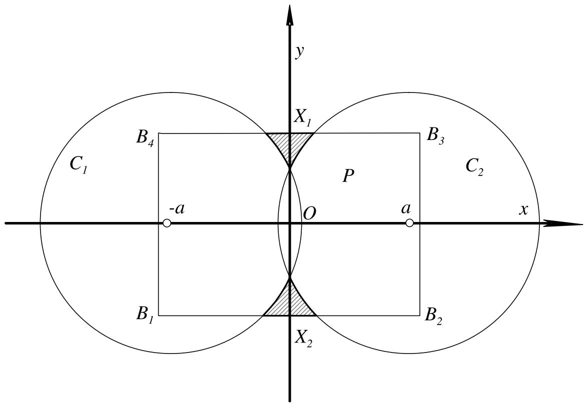
This leads to a construction of the piecewise smooth function [18]

$$F(X) = \min\{\psi_1(X, u_1^0), \psi_2(X, u_2^0)\}. \tag{6}$$

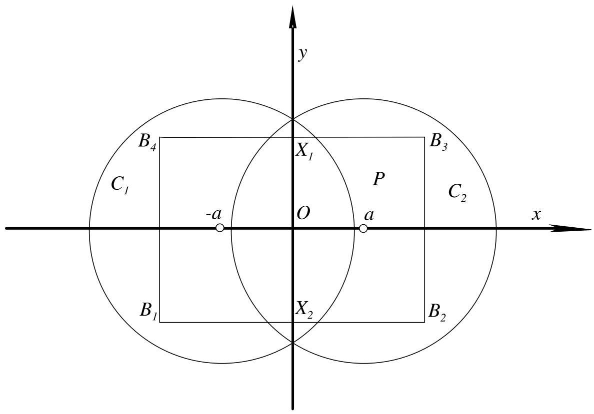
If $\psi_1(X, u_1^0) = \psi_2(X, u_2^0)$ then, in fact, $\psi_1(X, u_1^0) - \psi_2(X, u_2^0) = 0$ is reduced to the equation $x = 0$, i.e. in the plane $y0z$ the equation $\psi_1(X, u_1^0) = \psi_2(X, u_2^0)$ is satisfied. This means that

$$F(X) = \begin{cases} \psi_1(X, u_1^0) & \text{if } x < 0, \\ \psi_2(X, u_2^0) & \text{if } x > 0, \\ \psi_1(X, u_1^0) = \psi_2(X, u_2^0) & \text{if } x = 0. \end{cases} \tag{7}$$

Fig. 1 Covering by two circles



(a) $\exists X \in P \ \psi_i(X, u_i^0) > 0$



(b) $\forall X \in P \ \psi_i(X, u_i^0) \leq 0$

It follows from (7) that in the plane $y0z$ function $F(X) = y^2 - r^2 + a^2$. Thus, $F(X)$ reaches local minima at points $(-a, 0)$ and $(a, 0)$ ($(0, 0)$ is a saddle point). Since $\psi_1(X, u_1)$ and $\psi_2(X, u_2)$ are convex, then $F(X)$ on P can reach local maxima only at points X^1 and X^2, being the intersection of y-axis and the sides of P (Fig. 1(a), (b)) and at vertices B_j, $j = 1, 2, 3, 4$, of P. Hence, if $\max\{F(X^1), F(X^2), F(B_j), j = 1, 2, 3, 4)\} = \alpha \leq 0$ then P is inside $C_1(u_1^0) \cup C_2(u_2^0)$ and if $\alpha > 0$ then $C_1(u_1^0) \cup C_2(u_2^0)$ is not a covering of P.

This allows drawing the following important conclusion: to verify whether $C_1(u_1^0) \cup C_2(u_2^0)$ is a covering of P it is sufficient to calculate $\max F(X)$ s.t. $X \in P$.

On the ground of the above arguments we build the function

$$\Gamma_k^r(Z) = \Gamma_k^r(X, u) = \min\{\psi_i(X, u_i),\ i = 1, 2, \ldots, k \leq n\}. \tag{8}$$

It is evident if $u = u^*$ is such that $\Gamma_k^r(X, u^*) \leq 0$ for any $X \in P$ then $\Lambda(u^*)$ is a covering of P. If $u = u^*$ is such that $\Gamma_k^r(X, u^*) > 0$ for at least one $X \in P$ then $\Lambda_k(u^*)$ is not a covering of P. In other words, if $u = u^*$ is such that $\max \Gamma_k^r(X, u^*) =$

$\alpha \leq 0$ s.t. $X \in P$ then $\Lambda(u^*)$ is a covering of P; if $u = u^*$ is such that $\alpha > 0$ then $\Lambda(u^*)$ is not a covering of P. In addition, α is a measure of area $A_k(u^*)$ of no covered part of P. Furthermore, if $\alpha \to 0$ then $A_k \to 0$. This means that Problem 2 may be reduced to a search of vector u^* which ensures minimal value of max $\Gamma_k^r(X, u^*)$ s.t. $X \in P$.

Whence Problem 2 may be reduced to the equivalent problem

$$\Gamma_k^r(X^*, u^*) = \min_{u \in G} \max_{X \in P} \Gamma_k^r(X, u) \tag{9}$$

where $G = G_1 \times G_2 \times \cdots \times G_k \subset R^{2k}$; $G_i = P \oplus C_i^\varepsilon$, $i = 1, 2, \ldots, k$; C_i^ε is a circle of radius $\varepsilon < r$; $\oplus$ is the Minkowski sum operation [17]. Such a choice of G_i ensures participation of C_i in covering P (see conclusions of items 6 and 7, Sect. 3).

Since the radii of circles C_i, $i = 1, 2, \ldots, k$, are identical, the Problem 1 can be reduced to the sequence of the following ones

$$\Gamma_k(X^*, u^*) = \min_{u \in G} \max_{X \in P} \Gamma_k(X, u), \quad k = 1, 2, \ldots, \tag{10}$$

where

$$\begin{aligned} &\Gamma_k(X, u) = \min\{\varphi_i(X, u_i),\ i = 1, 2, \ldots, k \leq n\}, \\ &\varphi_i(X, u_i) = (x - x_i)^2 + (y - y_i)^2, \end{aligned} \tag{11}$$

i.e. if $u = u^*$ is such that $\max_{X \in P} \Gamma_k(X, u^{*,k}) = \alpha \leq r^2$ and $\max_{X \in P} \Gamma_{k-1}(X, u^{*,k-1}) = \alpha > r^2$ then $\Lambda(u^*)$ is a covering of P.

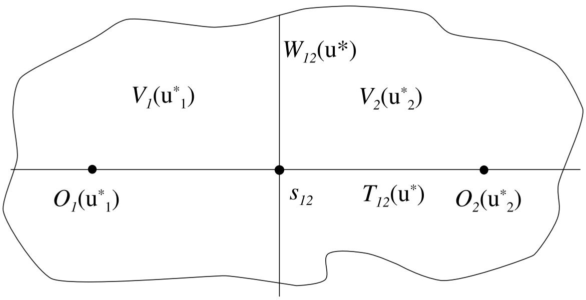
3 Characteristics of the mathematical model

1. Since $\varphi_i(X, u_i)$, $i = 1, 2, \ldots, n$, are everywhere defined and smooth, it follows that $\Gamma_k(Z)$ is everywhere defined and piecewise smooth [18]. Thus on any compact set $\Gamma_k(Z) < \infty$ and $\nabla \Gamma_k(Z)$ (gradient of $\Gamma_k(Z)$) exists almost everywhere.
2. Since $\varphi_i(X, u_i)$, $i = 1, 2, \ldots, n$, are bounded from below and unbounded from above, $\Gamma_k(Z)$ is bounded from below and unbounded from above and by previous item $\Gamma_k(Z) < \infty$ on any compact set. It is evident that min $\Gamma_k(Z) = 0$.
3. $\Gamma_k(Z)$ has a ravine type.
4. Let $k = 2$ and $\Gamma_2(Z) = \min\{\varphi_1(X, u_1), \varphi_2(X, u_2)\}$. If $u^* = (u_1^*, u_2^*)$ is such that $u_1^* = u_2^*$ then $F_2(X) = \varphi_1(X, u_1^*) = \varphi_2(X, u_2^*)$. So we consider the case $u_1^* \neq u_2^*$. Let intersection points of axes of paraboloids $z = \varphi_i(X, u_i^*)$ and plane $x0y$ be denoted by $O_i(u_i^*)$, $i = 1, 2$.

   We designate a set of points X on which $\varphi_1(X, u_1^*) < \varphi_2(X, u_2^*)$ as $V_1(u_1^*)$ and a set of points X on which $\varphi_1(X, u_1^*) > \varphi_2(X, u_2^*)$ as $V_2(u_2^*)$. Then $\Gamma_2(Z) = \varphi_1(X, u_1^*)$ on $V_1(u_1^*)$ and $\Gamma_2(Z) = \varphi_2(X, u_2^*)$ on $V_2(u_2^*)$. Let us now consider a set of points X that satisfy the equation

$$\varphi_1(X, u_1^*) - \varphi_2(X, u_2^*) = 0.$$

Fig. 2 A separating line for two points



After simple transformations we obtain the equation of a straight line $W_{12}(u^*)$

$$xa + yb + c = 0 \tag{12}$$

where $a = 2(x_2^* - x_1^*)$, $b = 2(y_2^* - y_1^*)$ and $c = -(x_2^{*2} - x_1^{*2}) - (y_2^{*2} - y_1^{*2})$. It is easily verified that $W_{12}(u^*)$ is orthogonal to a straight line $T_{12}(u^*)$ passing through points $O_i(u_i^*)$, $i = 1, 2$, (Fig. 2) whose equation is

$$x(y_2^* - y_1^*) - y(x_2^* - x_1^*) + y_1^*(x_2^* - x_1^*) - x_1^*(y_2^* - y_1^*) = 0.$$

Having solved (12) with respect to y and replacing y in $\varphi_1(X, u_1^*)$ or $\varphi_2(X, u_2^*)$ we obtain the equation of a parabola

$$z = x^2\left(1 + \frac{a^2}{b^2}\right) + 2x\left(\frac{ac}{b^2} - x_1^* + \frac{ay_1^*}{b}\right) + \left(\frac{c}{b} + y_1^*\right)^2 + x_1^{*2}$$

lying in a plane specified by (12). It follows from the equation that the minimal value of the parabola is achieved at the point s_{12}, being the intersection of $W_{12}(u^*)$ and $T_{12}(u^*)$ (Fig. 2), i.e. s_{12} is the middle of segment $[O_1(u_1^*), O_2(u_2^*)]$. Thus point s_{12} is a local minimum of $F_2(X)$ on the set defined by (12). This means that $F_2(X)$ increases along straight line $W_{12}(u^*)$ from point s_{12} and $F_2(X)$ decreases along the straight line $T_{12}(u^*)$ from point s_{12}. Hence point s_{12} is a saddle point and points $O_i(u_i^*)$, $i = 1, 2$, are local minima, i.e. $F_2(X)$ has no local maxima and is unbounded.

Furthermore, straight line $W_{12}(u^*)$ divides the plane R^2 into two half-planes $V_1(u^*)$ and $V_2(u^*)$ (Fig. 2) so that for any point $X \in V_1(u_1^*) \setminus W_{12}(u^*)$ the inequality $\rho(X, u_1^*) < \rho(X, u_2^*)$ where $\rho(X, u_1^*) = \|X - u_1^*\|$, is true and for any point $X \in V_2(u_1^*) \setminus W_{12}(u^*)$ the inequality $\rho(X, u_2^*) < \rho(X, u_1^*)$ always holds true. This means that half-planes $V_1(u_1^*)$ and $V_2(u_2^*)$ are Voronoi polygons [20] and $F_2(X)$ is specified as follows

$$F_2(X) = \begin{cases} \varphi_1(X, u_1^*) & \text{if } X \in V_1(u_1^*) \setminus W_{12}(u^*), \\ \varphi_2(X, u_2^*) & \text{if } X \in V_2(u_2^*) \setminus W_{12}(u^*), \\ \varphi_1(X, u_1^*) = \varphi_2(X, u_2^*) & \text{if } X \in W_{12}(u^*). \end{cases}$$

Thus $F_2(X)$ can reach local maxima only at vertices of P and intersection points $W_{12}(u^*)$ and $\mathrm{fr}(P)$ (frontier of P [1]).

If straight line $W_{12}(u^*)$ does not intersect P then either circle C_1 or circle C_2 does not take part in a covering of P.

5. Let $k = 3$ and $\Gamma_3(Z) = \min\{\varphi_1(X, u_1), \varphi_2(X, u_2), \varphi_3(X, u_3)\}$. We assume $F_3(X) = \min\{\varphi_1(X, u_1^*), \varphi_2(X, u_2^*), \varphi_3(X, u_3^*)\}$ where $u_1 \neq u_2$, $u_1 \neq u_3$ and $u_2 \neq u_3$.

   The mutual position of points $O_i(u_i^*)$, $i = 1, 2, 3$, in plane R^2 is exhausted by the following three cases:

   a. $O_i(u_i^*)$, $i = 1, 2, 3$, lie on the same straight line $T(u^*)$ (Fig. 3(a)).
   b. $O_i(u_i^*)$, $i = 1, 2, 3$, are vertices of an acute-angled triangle (Fig. 3(b)).
   c. $O_i(u_i^*)$, $i = 1, 2, 3$, are vertices of either a rectangle or an obtuse-angled triangle (Fig. 3(c)).

Case a. For the sake of definiteness, points $O_i(u_i^*)$, $i = 1, 2, 3$, are placed on a straight line $T(u^*)$ as shown in Fig. 3(a). It is evident that equations

$$\varphi_1(X, u_1^*) - \varphi_2(X, u_2^*) = 0,$$
$$\varphi_2(X, u_2^*) - \varphi_3(X, u_3^*) = 0$$

specify straight lines $W_{12}(u_1^*, u_2^*)$ and $W_{23}(u_2^*, u_3^*)$ respectively, and after a simple transformation the equations take the form

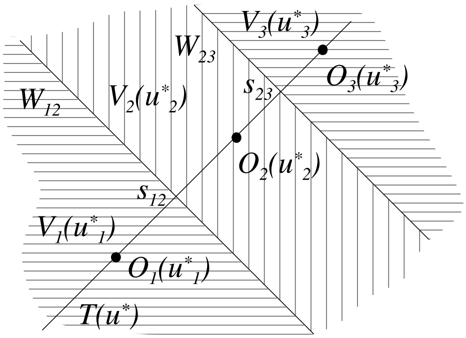
$$2x(x_2^* - x_1^*) + 2y(y_2^* - y_1^*) - (x_2^{*2} - x_1^{*2}) - (y_2^{*2} - y_1^{*2}) = 0,$$
$$2x(x_3^* - x_2^*) + 2y(y_3^* - y_2^*) - (x_3^{*2} - x_2^{*2}) - (y_3^{*2} - y_2^{*2}) = 0.$$

It follows from the previous item that W_{12} is orthogonal to T_{13} and their intersection point s_{12} is the middle of segment $[O_1(u_1^*), O_2(u_2^*)]$; W_{23} is orthogonal to T_{13} and their intersection point s_{23} is the middle of segment $[O_2(u_2^*), O_3(u_3^*)]$ (Fig. 3(a)). Thus straight lines W_{12} and W_{23} divide the plane R^2 into three parts $V_1(u_1^*)$, $V_2(u_2^*)$ and $V_3(u_3^*)$.
Whence

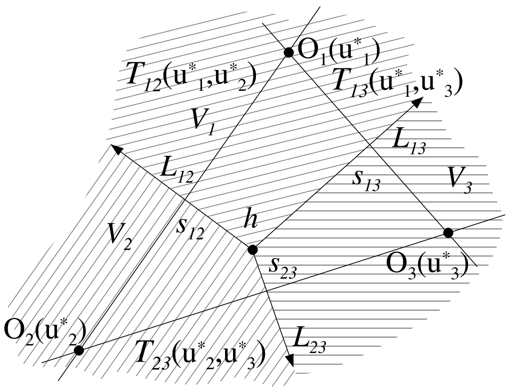
$$F_3(X) = \begin{cases} \varphi_1(X, u_1^*) & \text{if } X \in V_1 \setminus W_{12}, \\ \varphi_2(X, u_2^*) & \text{if } X \in V_2 \setminus (W_{12} \cup W_{23}), \\ \varphi_3(X, u_3^*) & \text{if } X \in V_3 \setminus W_{23}, \\ \varphi_1(X, u_1^*) = \varphi_2(X, u_2^*) & \text{if } X \in W_{12}, \\ \varphi_2(X, u_2^*) = \varphi_3(X, u_3^*) & \text{if } X \in W_{23}. \end{cases} \tag{13}$$

The function is unbounded from above, does not possess local maxima, has three local minima $X^{*i} = u_i^*$, $i = 1, 2, 3$, and two saddle points s_{12} and s_{23}. Thus $F_3(X)$ can reach local maxima on P only at vertices of P and intersection points of W_{12} and $\mathrm{fr}(P)$, W_{23} and $\mathrm{fr}(P)$. It is easily seen that sets V_1, V_2 and V_3 are Voronoi polygons [20].

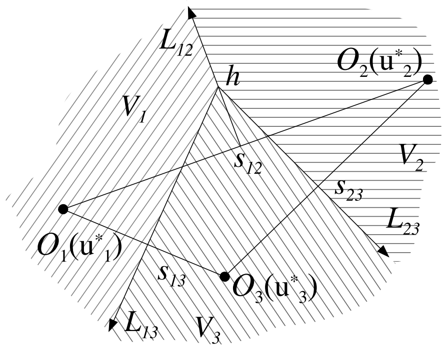
Case b. For the sake of definiteness, points $O_i(u_i^*)$, $i = 1, 2, 3$, are placed on lines T_{12}, T_{13} and T_{23}, as shown in Fig. 3(b) and form a triangle H. In this case, the equations



(a) Three centers lie on the same straight line



(b) Three centers are the vertices of an acute-angled triangle



(c) Three centers are the vertices of an obtuse-angled triangle

Fig. 3 Three centers

$$\varphi_1(X, u_1^*) - \varphi_2(X, u_2^*) = 0,$$
$$\varphi_2(X, u_1^*) - \varphi_3(X, u_3^*) = 0,$$
$$\varphi_3(X, u_3^*) - \varphi_1(X, u_1^*) = 0.$$

are reduced to the following ones

$$\begin{aligned} &2x(x_2^* - x_1^*) + 2y(y_2^* - y_1^*) - (x_2^{*2} - x_1^{*2}) - (y_2^{*2} - y_1^{*2}) = 0, \\ &2x(x_3^* - x_2^*) + 2y(y_3^* - y_2^*) - (x_3^{*2} - x_2^{*2}) - (y_3^{*2} - y_2^{*2}) = 0, \\ &2x(x_3^* - x_1^*) + 2y(y_3^* - y_1^*) - (x_3^{*2} - x_1^{*2}) - (y_3^{*2} - y_1^{*2}) = 0 \end{aligned} \tag{14}$$

which specify straight lines $W_{12}(u_1^*, u_2^*)$, $W_{23}(u_2^*, u_3^*)$ and $W_{31}(u_1^*, u_3^*)$ on plane R^2 respectively (Fig. 3(b)). Let $T_{12}(u_1^*, u_2^*)$, $T_{23}(u_2^*, u_3^*)$ and $T_{31}(u_1^*, u_3^*)$ be straight lines passing through pairs of points $\{O_1(u_1^*), O_2(u_2^*)\}, \{O_2(u_2^*), O_3(u_3^*)\}$ and $\{O_1(u_1^*), O_3(u_3^*)\}$ respectively. It is easily verified that W_{12}, W_{23} and W_{31} and T_{12}, T_{23} and T_{31} are orthogonal respectively. In addition, their intersection points s_{12}, s_{23} and s_{31} are the middles of triangle sides $[O_1(u_1^*), O_2(u_2^*)]$, $[O_2(u_2^*), O_3(u_3^*)]$ and $[O_1(u_1^*), O_3(u_3^*)]$ respectively. This means that straight lines W_{12}, W_{23} and W_{31} intersect at the same point h. Let L_{12}, L_{23} and L_{13} be rays emanating from point h to sides H along straight lines W_{12}, W_{23} and W_{31} respectively. The rays divide the plane R^2 into three sets $V_1(u_1^*), V_2(u_2^*)$ and $V_3(u_3^*)$ (Fig. 3(b)). In so doing, $\mathrm{fr}(V_1) = L_{12} \cup L_{13}$, $\mathrm{fr}(V_2) = L_{12} \cup L_{23}$ and $\mathrm{fr}(V_3) = L_{13} \cup L_{23}$. Thus

$$F_3(X) = \begin{cases} \varphi_1(X, u_1^*) & \text{if } X \in V_1 \setminus L_{12} \cup L_{13}, \\ \varphi_2(X, u_2^*) & \text{if } X \in V_2 \setminus L_{12} \cup L_{23}, \\ \varphi_3(X, u_3^*) & \text{if } X \in V_3 \setminus L_{13} \cup L_{23}, \\ \varphi_1(X, u_1^*) = \varphi_2(X, u_2^*) & \text{if } X \in L_{12}, \\ \varphi_2(X, u_2^*) = \varphi_3(X, u_3^*) & \text{if } X \in L_{23}, \\ \varphi_1(X, u_1^*) = \varphi_3(X, u_3^*) & \text{if } X \in L_{13}. \end{cases}$$

The function is unbounded from above, reaches a local maximum at the point h, has three local minima $X^{*i} = u_i^*$, $i = 1, 2, 3$, and three saddle points s_{12}, s_{23} and s_{13}, and can reach local maxima on P only at the point h, vertices of P and intersection points of rays L_{12}, L_{23}, L_{13} and $\mathrm{fr}(P)$.
Indeed, since point $(h, F_3(h)) \in R^3$ is an intersection point of three parabolas whose minima are attained at points s_{12}, s_{23} and s_{13} being points of $F_3(X)$, it follows that the point h is a local maximum of $F_3(X)$ (Fig. 3(b)).
It is evident that sets V_1, V_2 and V_3 are Voronoi polygons [20] and h is a common vertex of Voronoi polygons V_1, V_2 and V_3.
If one of rays L_{12}, L_{23} or L_{13} does not intersect P then one of circles C_i, $i = 1, 2, 3$, does not take part in covering P when P is convex.

Case c. Based on the above, we can confirm that sets $V_1(u_1^*)$, $V_2(u_2^*)$ and $V_3(u_3^*)$ are Voronoi polygons [20] for points $O_i(u_i^*)$, $i = 1, 2, 3$, (Fig. 3(c)). Since $O_i(u_i^*)$, $i = 1, 2, 3$, are vertices of an obtuse-angled triangle, it follows that points s_{13} and s_{23} are saddle points of $F_3(X)$, point s_{12} is not a saddle and the point h is not a local maximum of $F_3(X)$. Hence, $F_3(X)$ can reach local maxima on P only at vertices of P and intersection points of rays L_{12}, L_{23}, L_{13} and $\mathrm{fr}(P)$.

Thus we can draw the following conclusion: $F_3(X)$ is unbounded and reaches local maximum at vertex h of $V_1(u_1^*)$, $V_2(u_2^*)$ and $V_3(u_3^*)$ if points $O_i(u_i^*)$, $i =$

$1, 2, 3$, are vertices of acute-angled triangle H and, hence, rays L_{12}, L_{23} and L_{31} specified by (14) form Voronoi polygons $V_1(u_1^*)$, $V_2(u_2^*)$ and $V_3(u_3^*)$.

6. We assume $F_k(X) = \Gamma_k(X, u^*) = \min\{\varphi_i(X, u_i^*), i = 1, 2, \ldots, k\}$ where $u_i \neq u_j$, $i \neq j$, $0 < i < j = 2, 3, \ldots, k$. Thus the number of Voronoi polygons is k.

   On the ground of the properties of $F_2(X)$ and $F_3(X)$, we can draw the following conclusions:

   a. $F_k(X)$ is unbounded and can reach local maxima at vertices of $V_i(u_i^*)$, $i = 1, 2, \ldots, k$, being interior points of convex hull Π_k of $O_i(u_i^*)$, $i = 1, 2, \ldots, k$.

      Indeed, let $B_{1k}(u^*)$ be a set of vertices of $V_i(u_i^*)$, $i = 1, 2, \ldots, k$. We suppose $h_j \in B_{1k}(u^*)$ is the common vertex of V_{i-1}, V_i and V_{i+1} and $h_j \notin \Pi_k$. V_{i-1}, V_i and V_{i+1} intersect $\mathrm{fr}(\Pi_k)$. Since $F_k(X)$ increases along ray $L_{i,(i+1)}$, h_j is not a local maximum of $F_k(X)$.

   b. $F_k(X)$ on the set P has a finite set $B(u^*)$ of local maxima X^{i*}, $i = 1, 2, \ldots, m(u^*)$.

      Let $B_{1k}^*(u^*) = B_{1k}(u^*) \backslash K(u^*)$ where $K(u^*)$ is the set of Voronoi vertices outside $\mathrm{int}(\Pi_k)$, and set $B^*(u^*) = B_{1k}(u^*) \cup B_{2k}(u^*) \cup B_{3k}$, $B(u^*) = B_{1k}^*(u^*) \cup B_{2k}(u^*) \cup B_{3k}$ where $B_{2k}(u^*)$ is a set of intersection points of sides of $V_i(u_i^*)$, $i = 1, 2, \ldots, k$, and $\mathrm{fr}(P)$, B_{3k} is a set of vertices of P. By previous item and boundedness of P we have $B(u^*) \subset B^*(u^*)$.

   c.
$$F_k(X^*) = \max_{X \in P} F_k(X) = \max_{X \in B(u^*)} F_k(X). \tag{15}$$

      It is easily seen that $F_k(X^*)$ is the radius of the circles covering P.

      Hence, if $F_k(X^*) \leq r^2$ then $\Lambda_k(u^*)$ is a covering of P, and if $F_k(X^*) > r^2$ then $\Lambda_k(u^*)$ is not a covering of P.

      In what follows we suppose that $B_{qk}(u^*)$ consists of $m_q(u^*)$ points, $q = 1, 2, 3$.

7. The construction of sets $B_{1k}(u)$, $B_{2k}(u)$ and B_{3k} implies the following results:

   if $X \in B_{1k}(u)$ then $\Gamma_k(X, u)$ is specified by at least three functions of family $\{\varphi_i(X, u_i^*), i = 1, 2, \ldots, k\}$, i.e. $\Gamma_k(X, u) = \min\{\varphi_{i_j}(X, u_{i_j}^*), j = 1, 2, \ldots, \tau \geq 3\}$ and at least $\varphi_{i_1}(X, u_{i_1}^*) = \varphi_{i_2}(X, u_{i_2}^*) = \varphi_{i_3}(X, u_{i_3}^*)$;

   if $X \in B_{2k}(u)$ then $\Gamma_k(X, u)$ is specified by at least two functions of family $\{\varphi_i(X, u_i^*), i = 1, 2, \ldots, k\}$, i.e. $\Gamma_k(X, u) = \min\{\varphi_{i_j}(X, u_{i_j}^*), j = 1, 2, \ldots, \tau \geq 2\}$ and at least $\varphi_{i_1}(X, u_{i_1}^*) = \varphi_{i_2}(X, u_{i_2}^*)$;

   if $X \in B_{3k}$ then $\Gamma_k(X, u)$ is specified by at least one function of family $\{\varphi_i(X, u_i^*), i = 1, 2, \ldots, k\}$, i.e. $\Gamma_k(X, u) = \min\{\varphi_{i_j}(X, u_{i_j}^*), j = 1, 2, \ldots, \tau \geq 1\}$.

4 Solving strategy

1. Firstly, we choose a value of k by using any estimate of the number of circles to cover P. Obviously, the better is the estimate the faster Problem 1 is solved.
2. In a random way or by using some special rule we select $u^0 = (u_1^0, u_2^0, \ldots, u_k^0) \in G$ so that $u_i^0 \neq u_j^0$, $0 < i < j = 2, 3, \ldots, k$.
3. Construct Voronoi polygons V_i for points $O_i(u_i^0)$, $i = 2, 3, \ldots, k$.

4. Construct the convex hull H of points $O_i(u_i^0)$, $i = 2, 3, \ldots, k$.
5. Form the set $B^0(u^0)$.
6. Construct the function $F_k(X) = \min\{\varphi_i(X, u_i^0), i = 1, 2, \ldots, k\}$ (11) and compute $F_k(X^0) = \max_{X \in B^0} F_k(X)$.
7. If $F_k(X^0) < r^2$ then we take $k - 1$ and return to item 2; if $F_k(X^0) > r^2$ then we pass to the next item.
8. Making use of a gradient method (see the next section) we define a vector $z = (z_1, z_2, \ldots, z_m) \in R^{2k+2}$ such that vector $Z^1 = Z^0 + \xi z = (X^0, u^0) + \xi z$ ensures $\Gamma(Z^1) = \max_{X \in B^1} \Gamma(X, u^1)$ and $\Gamma(Z^1) < \Gamma(Z^0)$. After that, if $\Gamma(Z^1) > r^2$ we take points $O_i(u_i^1)$, $i = 1, 2, \ldots, k$, and return to item 3 with $u_i^0 = u_i^1$, $i = 1, 2, \ldots, k$. If $\Gamma(Z^1) \leq r^2$ the problem has been solved and its solution is equal to k. If z does not exist and $\Gamma(Z^1) > r^2$ then we either pass to item 1 or take $k + 1$ and pass to item 1.

5 Search of local minima

Using the properties of $\Gamma_k(Z)$, the following way of searching for local extrema of problem (10) is offered.

The iterative process starts with iteration number $s = 0$.

Voronoi polygons are constructed for points $O_i(u_i^s)$, $i = 1, 2, \ldots, k$, and set $B^s = B(u^s)$ is formed.

Let $Z^s = (X^s, u^s)$ be a point such that $F_k(X^s) = \max_{X \in B^s} F_k(X) = \alpha^s$. We form set $Q \subset M^s \times J^s$ of pairs (t, j), $t \in M^s \subset \{1, 2, \ldots, m^s\}$, $m^s \leq m(u^s) = m_1(u^s) + m_2(u^s) + m_3(u^s)$, $j \in J^s = \{i_1, i_2, \ldots, i_{\gamma^s}\} \subset \{1, 2, \ldots, k\}$ such that $X^{st} \in B$ and the following equations are satisfied

$$\varphi_j(X^{st}, u_j^s) \geq \alpha^s - \delta^s \tag{16}$$

where $\delta^s \geq 0$ and is chosen according to the rules discussed later. At $s = 0$ $\delta^s = \delta_\diamond = \frac{\delta_\diamond}{2^l}$, $l = 0$.

The set of points X^{st}, $t = 1, 2, \ldots, m^s$, is denoted by $A^s = A(u^s)$.

On the ground of functions (16) the following function is constructed

$$\Omega_k(X, u_\star^s) = \min\{\varphi_j(X, u_j),\ j \in J^s\} \tag{17}$$

where $u_\star^s = (u_{i_1}, u_{i_2}, \ldots, u_{i_{\gamma^s}})$.

Let $Q = Q_1 \cup Q_2 \cup Q_3$ where $Q_q = \{(t, j) \in Q | X^{st} \in B_{qk}(u^s)\}$, $q = 1, 2, 3$.

Let M_i^s be projections of Q_i, $i = 1, 2, 3$ onto M^s, i.e. $X^{st} \in B_{qk}^s(u^s)$ when $t \in M_q^s$, $q = 1, 2, 3$.

Suppose that variables X^{st}, $t \in M_1^s \cup M_2^s$ are independent ones. Renumber X^{st} for simplicity starting from points of M_1, then points of M_2. They form a vector $\mu = (\mu_1, \mu_2, \ldots, \mu_{n^s}) \in R^{2\,\mathrm{card}(M_1 \cup M_2)}$. Since points of B_{3k}^s are vertices of P, they cannot change their placement.

On the ground of $\varphi_j(X, u_j)$, $j \in J^s$ the following functions can be constructed

$$\omega_{ti}(\mu_t, u_i) = \varphi_i(\mu_t, u_i), \quad (t, i) \in Q_1 \cup Q_2 \tag{18}$$

and

$$w_{ti}(u_i) = \varphi_i(X^{st}, u_i), \quad (t, i) \in Q_3. \tag{19}$$

This enables to build the function

$$\Phi_k(Y) = \Phi_k(\mu, u_\star^s) = \max\{\omega_{tj}(\mu_t, u_j),\ (t, j) \in Q_1 \cup Q_2,\ w_{tj}(u_j), \quad (t, j) \in Q_3\}, \tag{20}$$

where $Y \in R^d$, $d = 2(\mathrm{card}(M_1) + \mathrm{card}(M_2) + \gamma^s)$.

Let $Y^s = (X^{s1}, X^{s2}, \dots, X^{s\nu^s})$. By relations (16–19) $\Omega_k(X^s, u_\star^s) = \Phi_k(Y^s) = \alpha_s$. If Y^s is not a local minimum of $\Phi_k(Y)$ then there exists a point $Y^{s+1} = Y^s + \xi h_k^s$ where $\xi > 0$, $h_k^s = (-\nabla_p)\Phi_k(Y^s)$ is a steepest descent vector (anti-pseudo-gradient) of $\Phi_k(Y)$ at point Y^s such that $\Phi_k(Y^s) > \Phi_k(Y^{s+1}) = \alpha_{s+1} < \alpha_s$.

Since to a vector μ^{s+1} there corresponds a set $A^{s+1} = A(u^{s+1})$ consisting of points $X^{s+1,t}$, $t \in (M_1^{s+1} \cup M_2^{s+1})$ then inequalities $\varphi_j(X^{st}, u_j^s) > \varphi_j(X^{s+1,t}, u_j^{s+1})$, $(t, j) \in Q$ are true. Whence

$$\Omega_k(X^s, u^s) > \Omega_k(X^{s+1}, u^{s+1}) = \max_{X \in A^{s+1}} \Omega_k(X, u^{s+1}).$$

Hence, there exists such ξ that

$$\Gamma_k(X^s, u^s) > \Gamma_k(X^{s+1}, u^{s+1}) = \max_{X \in B^{s+1}(u^{s+1})} \Gamma_k(X, u^{s+1})$$

where $B^{s+1} = B(u^{s+1})$ constructed for points $O_i(u_i^{s+1})$, $i = 1, 2, \dots, k$. Then a search for the point (X^{s+1}, u^{s+1}) such that $\Gamma_k(X^s, u^s) > \Gamma_k(X^{s+1}, u^{s+1})$ can be reduced to solving the following problem.

Since by virtue of Sect. 3 at the point Y^s a steepest descent vector of $\Phi_k(Y)$ exists, we can always find a vector $z \in R^d$ such that $\Phi_k(Y^s) > \Phi_k(Y^s + \xi z)$ where $\xi > 0$. To this end, a modification of the Zountendijk feasible directions method [19] is used. The vector z is found by solving the following linear programming problem

$$\min \varkappa \tag{21}$$

s.t.

$$\begin{cases} (\nabla\omega_{tj}(\mu_t^s, u_j^s), z) \le \varkappa, & (t, j) \in Q_1 \cup Q_2, \\ (\nabla w_{tj}(u_j^s), z) \le \varkappa, & (t, j) \in Q_3, \\ \chi_{p(t)}(\mu_t^s) = 0, & t \in M_2^s, \\ -1 \le z_i \le 1, & i \in M_1^s \cup M_2^s \cup J^s, \end{cases} \tag{22}$$

where $\chi_{p(t)}(\mu_t^s) = 0$ is generated by an equation $\chi_p(x, y) = 0$ of a straight line passing through the side $p(t)$ of P (see inequality collection (1)) on which X^{st} lies, and ensures a placement of appropriate points X^{st}, $t \in M_2^s$ on appropriate sides of P.

If $\varkappa < 0$, the vector Y^{s+1} is formed as follows

$$Y^{s+1} = Y^s + \frac{\xi}{2^\lambda} z$$

where a starting value of λ is $\lambda = 0$ $(\lambda = 0, 1, \dots)$. Voronoi polygons are constructed for points $O_i(u_i^{s+1})$, $i = 1, 2, \dots, k$, and the set $B(u^{s+1})$ is formed. Then

$\Gamma_k(X^{s+1}, u^{s+1}) = \max_{X \in B^{s+1}(u^{s+1})} \Gamma_k(X, u^{s+1})$ is calculated. If $\Gamma_k(X^{s+1}, u^{s+1}) < \Gamma_k(X^s, u^s)$ then point (X^{s+1}, u^{s+1}) is taken as a starting point and the iterative procedure is repeated. If $\Gamma_k(X^{s+1}, u^{s+1}) \geq \Gamma_k(X^s, u^s)$ then λ is increased by 1 and a new point $Y^{s+1} = Y^s + \frac{\xi}{2^\lambda} z$ is formed and all calculations are repeated for it. The process is continued until $\frac{\xi}{2^\lambda} < \varepsilon_2 > 0$.

If $\varkappa < 0$ then we take $\delta^s = \delta_\diamond = \frac{\delta_\diamond}{2^l}$ with value of l increased by 1; inequalities (16) are selected again and the new problem for them is formed and solved.

Thus the solution process is continued until both $\delta < \varepsilon_1$ and $\xi < \varepsilon_2$. $\varepsilon_1 = 0.0001$ and $\varepsilon_2 = 0.0001$ are chosen. Note that, because of the ravine form of $\Phi_k^s(Y)$, too small values of ε_1 and ε_2 significantly degrade the convergence of the iterative process.

To obtain another local extremum, it is necessary to set a new starting point Z^0. A starting point may be formed in either a random way or based on the previous solution obtained for $k-1$ circles. Let $Z^* = (X^*, u^*) = (X^*, u_1^*, u_2^*, \ldots, u_{k-1}^*)$ be a solution of the problem (15) for $k-1$ circles and to this solution there corresponds a set $B_{k-1}^* = B(u^*) = B_{1,(k-1)}(u^*) \cup B_{2,(k-1)}(u^*) \cup B_{3,(k-1)}$. Any point $u_k^0 \in B(u^*)$ can be chosen as an initial point. A numerical trial shows that better results are obtained if u_k^0 is chosen from $B_{2,(k-1)}(u^*) \cup B_{3,(k-1)}$. Then a new starting point $Z^0 = (X^0, u^0)$ is formed as follows: $Z^0 = (X^0, u^0) = (X^0, u_1^*, u_2^*, \ldots, u_{k-1}^*, u_k^0)$ where $F_k(X^0) = \max_{X \in B^0} F_k(X) = \alpha_0$ and $B^0 = B(u^0) = B(u_1^*, u_2^*, \ldots, u_{k-1}^*, u_k^0)$. It is easily seen that the starting point always yields a local minimum Y^{*k} such that $\Phi_k(Y^{*k}) \leq \Phi_k(Y^{*(k-1)})$.

6 Numerical examples

Example 1 To cover a unit square with n identical circles of minimal radius. n varies from 31 to 72 and $n = 81, 100$. The results of covering for $n \leq 30$ coincide with ones published in [14].

The results of best coverings are shown in Table 1. Dependence of the radius on the number of circles is shown in Figs. 4 and 5.

Placement of 100 circles covering the square is shown in Fig. 6.

Example 2 There is a polygonal set P (Fig. 7) whose vertex coordinates of the outer contour are (0.0, 0.0), (0.0, −50.0), (20.0, −50.0), (20.0, −40.0), (30.0, −30.0), (40.0, −20.0), (50.0, −10.0), (60.0, 0.0), (70.0, 0.0), (75.0, 0.0), (80.0, 0.0), (90.0, 0.0), (100.0, −10.0), (110.0, −20.0), (120.0, −30.0), (130.0, −40.0), (130.0, −50.0), (150.0, −50.0), (150.0, 80.0), (130.0, 100.0), (120.0, 100.0), (80.0, 70.0), (90.0, 100.0), (0.0, 100.0) and vertex coordinates of two inner contours are (130.0, 50.0), (110.0, 20.0) (90.0, 50.0), (110.0, 80.0), (130.0, 50.0) and (45.0, 50.0), (70.0, 30.0), (20.0, 30.0), (35.0, 50.0), (20.0, 70.0), (70.0, 70.0), (45.0, 50.0) respectively.

The best coverings were obtained for 30 (Fig. 7) and 40 circles and the minimal radii for them are 16.6176655 and 14.07100757 respectively.

Table 1 Minimal covering circle radius

Number of circles	Radius	Number of circles	Radius	Number of circles	Radius
31	12.0371253	46	9.7200494	61	8.4159869
32	11.8379079	47	9.6297287	62	8.3538970
33	11.5807761	48	9.5277641	63	8.2543829
34	11.4398942	49	9.3706869	63	8.2543829
35	11.2733037	50	9.3088779	64	8.1521571
36	11.0167265	51	9.2363666	65	8.1120775
37	10.9179680	52	9.1156130	66	8.0646584
38	10.8008993	53	9.0664022	67	7.9991389
39	10.6317348	54	9.0082611	68	7.9501656
40	10.5466202	55	8.9229742	69	7.8973203
41	10.4481627	56	8.7430573	70	7.8426729
42	10.1849626	57	8.6835865	71	7.7684263
43	10.1367558	58	8.6216131	72	7.6601792
44	9.9973637	59	8.5409924	81	7.2131527
45	9.8872660	60	8.4346341	100	6.4812891

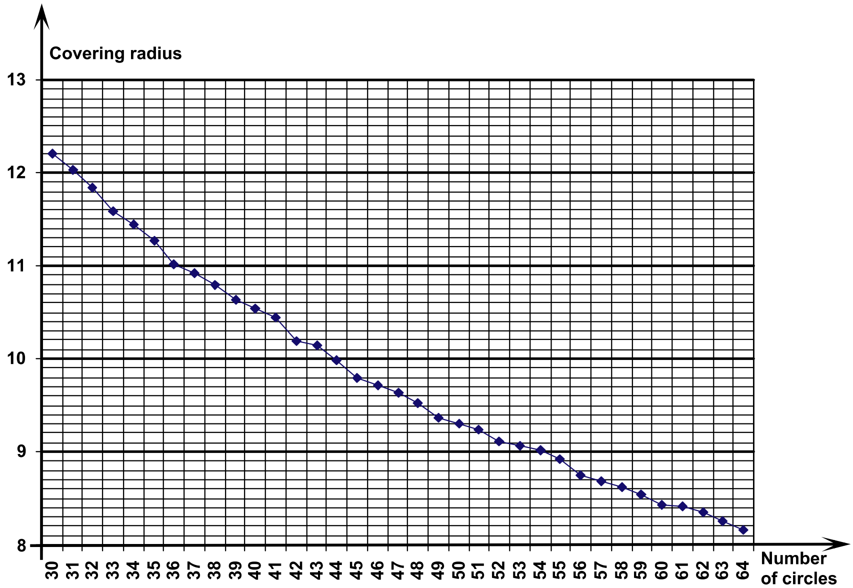


Fig. 4 Dependence of the radius on the number of circles: 30,...,64

7 Performance analysis

The approach offered to construct starting points guarantees obtaining local minima significantly better then those obtained from randomly selected starting points.

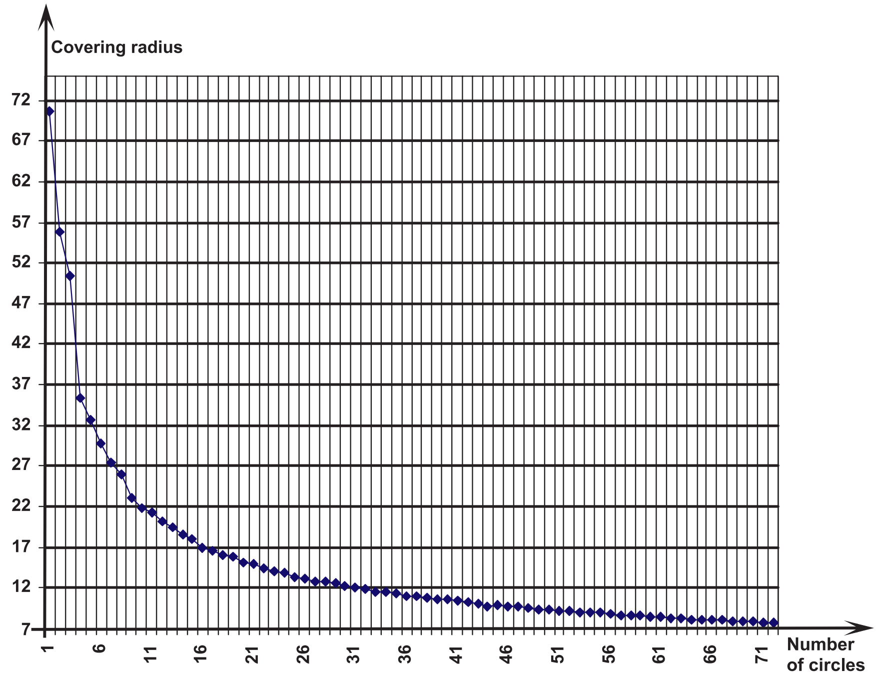


Fig. 5 Dependence of the radius on the number of circles: 1,. . . ,72

Example for $n = 31$.

Using the best result for 30 circles we can form the following starting points (only the 31-st point differs, so only its positions are indicated): (43.0188, 35.5112), ((43.2122, 47.0529), ((35.286, 53.3111), ((21.5731, 47.1612), ((21.6526, 35.6152), ((35.0104, 29.5787), ((78.3493, 64.3849), ((64.9908, 70.4223), ((56.9831, 64.4892), ((56.788, 52.9506), ((64.716, 46.6892), ((78.4284, 52.8387), ((78.0966, 29.0639), ((64.6575, 35.403), ((56.4552, 29.176), ((56.3989, 17.8864), ((63.2684, 12.3264), ((78.2859, 18.2751), ((21.3141, 11.8898), ((13.6254, 17.8751), ((0, 12.1063), ((0, 0), ((21.1934, 0), ((13.744, 29.5581), ((0, 35.75), ((36.7309, 87.6758), ((43.601, 82.1136), ((57.5236, 88.1561), ((57.3714, 100), ((36.2457, 100), ((86.2189, 46.8366), ((100, 53.1175), ((100, 64.2529), ((86.2595, 70.4427), ((35.3425, 64.6004), ((21.9036, 70.9356), ((13.6362, 64.3554), ((13.7829, 53.1631), ((86.3763, 82.1266), ((100, 87.8983), ((78.6857, 88.1109), ((100, 100), ((78.8041, 100), ((34.9395, 17.6586), ((21.7115, 81.7267), ((16.1461, 100), ((63.7545, 0), ((83.8525, 0), ((65.0614, 82.3391), ((42.4753, 11.8432), ((43.543, 70.8271), ((100, 0), ((100, 18.3025), ((0, 81.6963), ((0, 100), ((86.3636, 35.6419), ((100, 29.4199), ((0, 46.8786), ((0, 70.5762), ((42.6283, 0). For adjacent points at the frontier of the region in the above list, the middles of the segments connecting those points were also taken as initial ones for the 31-st circle center.

The coordinates of the circle centers of the best covering are as follows:

(32.35431053, 39.92297145), (68.53869317, 56.33800489), (67.16543257, 22.29855019), (10.52989308, 5.831972643), (3.330406439, 23.2310947),

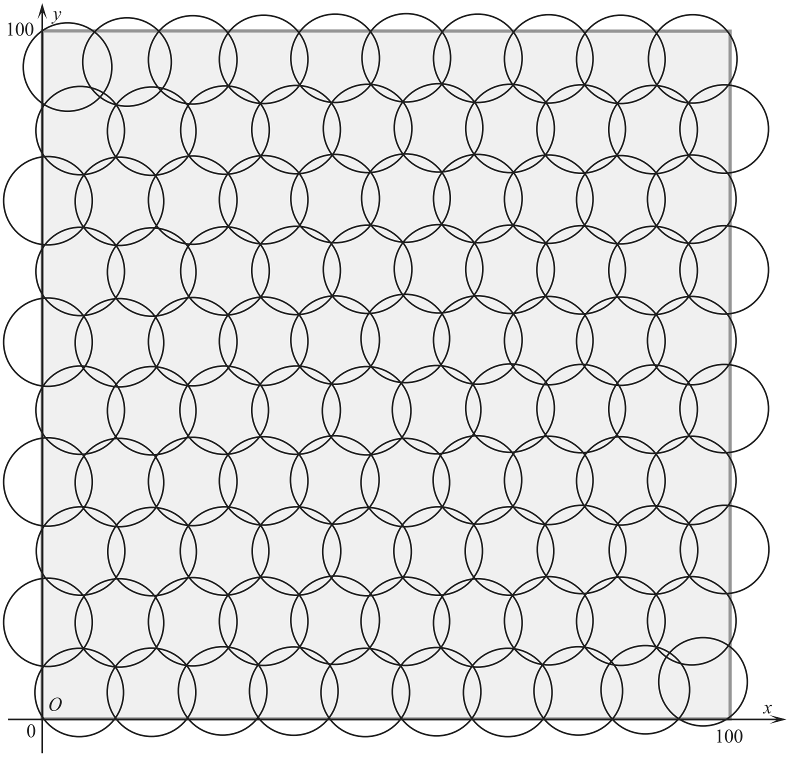


Fig. 6 Covering a square by 100 circles

(50.35025097, 93.13495433), (89.54480391, 56.74586796), (26.39821111, 56.41906944), (98.75646378, 74.6833601), (89.98126810, 93.32800150), (24.38985500, 23.23170423), (53.77894953, 39.68429103), (30.74646564, 92.894831), (73.73360172, 6.517518260), (78.87531508, 74.45439567), (45.77483663, 22.60142424), (47.48072715, 56.51409223), (39.39038091, 74.05714351), (91.92676835, 8.928215194), (10.51506602, 94.14133634), (89.09177775, 22.9457821), (58.96913142, 74.26134856), (10.6835520, 40.34371709), (18.98786310, 75.06352536), (96.05710171, 39.40806343), (52.86629601, 5.420698573), (31.58934161, 5.832582139), (70.10011945, 93.09903699), (7.496961538, 55.30652097), (75.21293172, 39.29345748), (2.47710382, 76.50326236).

The value of the least local minimum obtained is 12.03704935426.

The least local minima obtained from a random starting point is 12.2661951.

The best result for 29 circles is obtained only by a random choice of starting points.

The approach offered in the paper can be used to solve problems of covering polyhedra by identical spheres.

This approach also enables using parallel computations to obtain various local minima.

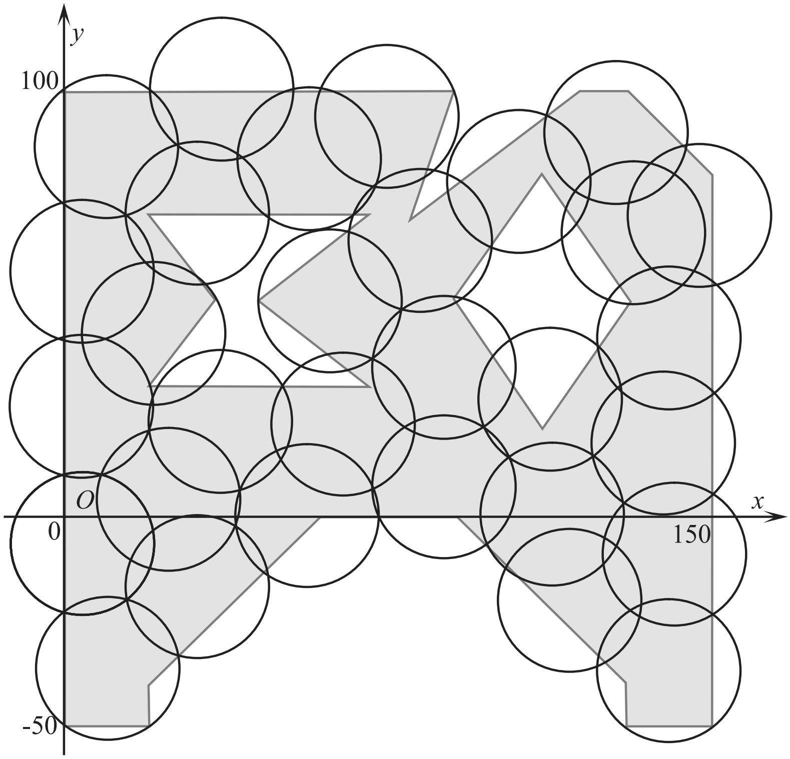


Fig. 7 Covering a non-convex region by 30 circles

References

1. Berge, Cl.: Topological Spaces. Oliver and Boyd, Edinburgh (1963)
2. Sloane, N.J.A.: with the collaboration of Hardin, R.H., Smith, W.D., et al.: Tables of spherical codes. http://www.research.att.com/~njas/packings/
3. Melissen, H.: Packing and covering with circles. PhD thesis, Universiteit Utrecht, the Netherlands (1997)
4. Zahn, C.T. Jr.: Black box maximization of circular coverage. J. Res. Nat. Bur. Stand. Sect. B **66**, 181–216 (1962)
5. Tarnai, T., Gaspar, Zs.: Covering a square by equal circles. Elem. Math. **50**, 167–170 (1995)
6. Rogers, C.A.: Packing and Covering. Cambridge University Press, Cambridge (1964)
7. Melissen, J.B.M., Schuur, P.C.: Improved coverings of a square with six and eight equal circles. Electron. J. Comb. **3**(32R), 10 (1996) (electronic)
8. Lengyel, A., Veres, I.A.: Egységnégyzet lefedése egybevágó körökkel (Covering the unit square with congruent circles). Competition essay, Technical University Budapest, Hungary (1996)
9. Heppes, A., Melissen, H.: Covering a rectangle with equal circles. Period. Math. Hung. **34**, 65–81 (1997)
10. Melissen, J.B.M., Schuur, P.C.: Covering a rectangle with six and seven circles. Discrete Appl. Math. **99**, 149–156 (2000)
11. Melissen, H.: Loosest circle coverings of an equilateral triangle. Math. Mag. **70**, 119–125 (1997)
12. Nurmela, K.J.: Conjecturally optimal coverings of an equilateral triangle with up to 36 equal circles. Exp. Math. **9**(2), 241–250 (2000)
13. Melissen, H.: Packing and covering with circles. PhD thesis, Universiteit Utrecht, the Netherlands (1997)

14. Nurmela, K.J., Östergård, P.R.J.: Covering a square with up to 30 circles. Helsinki University of Technology, Laboratory for Theoretical Computer Science. Research report (2000)
15. Stoyan, Yu.G., Patsuk, V.M.: Covering a polygonal region with equal number of circles of a given radius. Reports Of the National Academy of Sciences of Ukraine, No. 3, pp. 74–77. Kiev (2006) (in Russian)
16. Patsuk, V., Stoyan, Yu.: Covering a convex polygon with the given number of equal circles of minimal radius. In: Proc. of the Workshop on Cutting Stock Problems 2005 (WSCSP2005), pp. 59–64. Sapientia University of Miercurea-Ciuc, Romania (2006)
17. Minkowski, H.: Dichteste gitterformige Lagerung. Nachr. Ges. Wiss., pp. 311–355. Gottingen (1904)
18. Rvatchev, V.L.: Theory of R-functions and Some Applications. Naukova Dumka, Kiev (1982) (in Russian)
19. Zoutendijk, G.: Methods of Feasible Directions: a Study in Linear and Non-Linear Programming. Elsevier, Amsterdam (1970)
20. Voronoi, G.F.: Collected Works, vol. II, pp. 171–368. Academy of Sciences of USSR, Kiev (1952)