Comput Optim Appl (2010) 45: 89–109
DOI 10.1007/s10589-008-9173-x

Some projection-like methods for the generalized Nash equilibria

Jianzhong Zhang · Biao Qu · Naihua Xiu

Received: 11 January 2006 / Revised: 30 January 2008 / Published online: 8 March 2008

Abstract A generalized Nash game is an m-person noncooperative game in which each player's strategy depends on the rivals' strategies. Based on a quasi-variational inequality formulation for the generalized Nash game, we present two projection-like methods for solving the generalized Nash equilibria in this paper. It is shown that under certain assumptions, these methods are globally convergent. Preliminary computational experience is also reported.

Keywords Generalized Nash equilibrium · Quasi-variational inequality · Projection-like method · Convergence

1 Introduction

Game theory is a mathematical theory of socio-economic phenomena exhibiting interaction among decision-makers, called players, whose actions affect each other. The fundamental assumptions that underlie the theory are that players pursue well-defined exogenous objectives and take into account their knowledge, or expectations, of other

This research was supported by Hong Kong University Grant Council under the CERG Project CityU and CUHK 103105, and the National Natural Science Foundation of China (No. 70471002, 10571106, 10701047).

J. Zhang (✉)
Department of Mathematics, City University of Hong Kong, Hong Kong, Hong Kong
e-mail: mazhang@alumni.cityu.edu.hk

B. Qu
Institute of Operations Research, Qufu Normal University, Qufu, China

B. Qu · N. Xiu
Department of Applied Mathematics, Beijing Jiaotong University, Beijing, China

players' behavior. So far the theory has been applied in the fields of economics, political science, evolutionary biology, computer science, statistics, accounting, social psychology, law, and branches of philosophy such as epistemology and ethics.

A game is a model of strategic interaction among a number of players, which includes the constraints on the actions that players can take and the players' interest, but does not specify the actions that players do take. A solution is a systematic description of the outcomes that may emerge in a game. The most commonly encountered solution concept in game theory is that of Nash equilibrium (NE) named after John Nash. This notion expresses a kind of optimal collective strategy in a game, where no player has anything to gain by changing only his or her own strategy. If each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium. The concept of Nash equilibrium is not exactly original to Nash (e.g., Antoine Augustin Cournot [4] showed how to find what we now call the Nash equilibrium of the Cournot duopoly game). However, Nash showed for the first time in his dissertation [14, 15] that Nash equilibria must exist for all finite games with any number of players. Until Nash, this had only been proved for 2-player zero-sum games by John von Neumann and Oskar Morgenstern [24]. This result has been generalized by Arrow and Debreu [1] and McKenzie [13] to an abstract economy in which each player's strategy space may depend on the strategy of the other players (a situation which may also occur in coalition games). The Nash equilibrium for this case was called generalized Nash equilibrium (GNE). Generally speaking, the generalized Nash equilibrium problem (GNEP) is an extension of the standard Nash equilibrium problem, in which each player's strategy set is dependent on the rival players' strategies.

It is by now a well-known fact that the Nash equilibrium problem in which each player solves a convex program can be formulated and solved as a finite-dimensional variational inequality (VI), to which a host of computational methods are applicable [5]. The connection between the generalized Nash games and quasi-variational inequalities (QVIs) was recognized by Bensoussan [2] as early as 1974 who studied these problems with quadratic functionals in a Hilbert space. Harker [7] revised these problems in Euclidean spaces. Robinson [19, 20] discussed an application of a generalized Nash problem in a two-sided game model of combat. Kocvara and Outrata [10] discussed a class of QVIs with applications to engineering. Wei and Smeers [26] introduced a QVI formulation of a spatial oligopolistic electricity model with Cournot generators and regulated transmission prices. Cash [3] studied a special class of generalized Nash equilibrium problem with applications to Cournot oligopoly problem. Pang [17] recently analyzed the computational resolution of the generalized Nash game by a penalization method for the noncooperative multi-leader-follower games. To our knowledge, there is only a handful of papers that address QVI in finite dimensions in terms of existence of solutions and solution methods. As being pointed in [18], "the study of the QVI to date is in its infancy at best". So, computing a generalized Nash equilibrium remains a challenging task up-to-date. As such, it is interesting to develop efficient computational methods for solving a GNEP by the QVI formulation.

The main purpose of this paper is to investigate some alternative approaches to solving a GNEP. It is well known that the class of projection methods is one of the fundamental tools for solving convex optimization problems and monotone variational inequality problems. The projection type of methods possesses some advantages. Firstly, it is easy to implement (especially, for the optimization problem or the variational inequality problem with simple bound constraints), uses little storage, and can readily exploit any sparsity or separable structure in the corresponding mapping or the constrained set of the problem. Secondly, it is able to drop and add many constraints from and to the active set at each iteration. Due to its structural and theoretical advantages, various projection-type methods [8, 9, 12, 21, 25, 27, 28], such as the basic projection algorithm, the extragradient algorithm and its variants, and the hyperplane projection algorithm, have been designed to solve different convex optimization problems or monotone variational inequality problems. Interested readers may consult the monograph by Facchinei and Pang [5]. In this paper, based on the QVI reformulation of a GNEP, we present two projection-like algorithms for solving a GNEP. The global convergence of these algorithms is proved under certain conditions. Some numerical results are given to demonstrate the viability of the two algorithms.

The rest of this paper is organized as follows. Section 2 states the QVI formulation for the generalized Nash game. Section 3 describes some definitions, background material on projection and basic assumptions employed in the sequel. Section 4 gives a projection-like algorithm and shows its convergence. Section 5 presents another algorithm that may calculate fewer number of projections in each iteration, and proves its convergence. The numerical results are given in Sect. 6.

2 QVI formulation

The generalized Nash game can be defined as follows [7]. Let N be the set of m players, where m is finite, $X^i \subseteq \Re^{n_i}$ be the strategy set of player i $(i = 1, \dots, m)$, that we assume to be convex, and $X = \prod_{j\in N} X^j \subseteq \Re^n$, where $n = \sum_{i=1}^m n_i$, and $X^{N\setminus i} = \prod_{j\in N, j\neq i} X^j$, that is, X represents the full Cartesian product of the strategy sets, and $X^{N\setminus i}$ means this full set except the ith player's feasible region.

Let $K^i : X^{N\setminus i} \to X^i$ be a point-to-set mapping which represents the ability of all players $j \neq i$ to affect the feasible strategies of player i. Then

$$K^i(z) \subseteq X^i, \quad \forall z \in X^{N\setminus i}.$$

Let K denote the mapping formed from the K^i, $\forall i \in N$; i.e., for all $x \in X$,

$$K(x) = \prod_{i\in N} K^i(x^{N\setminus i}), \tag{1}$$

where $x^{N\setminus i}$ represents the vector x with the ith subvector x^i removed. Finally, let the cost function for player i be represented by the function $u^i : \mathrm{gr}K^i \to \Re$, where $\mathrm{gr}K^i$ denotes the graph of the mapping K^i and has dimension n. The generalized Nash game is thus defined by the data $\{X^i, K^i, u^i\}_{i\in N}$, and an equilibrium of this game, called a generalized Nash equilibrium (GNE), is defined as a point $x^* = (x^{*1}, x^{*2}, \dots, x^{*m}) \in X$ such that

$$x^{*i} \in K^i(x^{*N\setminus i}), \quad \forall i \in N,$$
$$u^i(x^*) \le u^i(y^i, x^{*N\setminus i}), \quad \forall y^i \in K^i(x^{*N\setminus i}),\ i \in N. \tag{2}$$

In a word, x^* is a GNE if it is feasible with respect to each mapping K^i and if it is a minimizer of each player's cost function over the feasible set (2).

Assume that $u^i(\cdot, x^{N\setminus i})$ is convex and continuously differentiable in $\Re^{n_i}$ and that $K^i(x^{N\setminus i}) \subseteq X^i$ is a closed and convex subset of $\Re^{n_i}$ for each $i \in N$. Then from the analysis above, the generalized Nash game is to find a tuple $x^* \in \Re^n$, that is a GNE, such that for each $i = 1, \dots, m$, x^{*i} is an optimal solution of the convex optimization problem in the variable x^i with $x^{N\setminus i}$ fixed at $x^{*N\setminus i}$:

$$\min_{s.t.\ x^i \in K^i(x^{*N\setminus i})} u^i(x^i, x^{*N\setminus i}).$$

Defining

$$F(x) = (\nabla_{x^i} u^i(x))_{i=1}^m \in \Re^n$$

and by (1), we see that x^* is a GNE if and only if $x^* \in K(x^*)$ and

$$\langle F(x^*), y - x^* \rangle \ge 0, \quad \forall y \in K(x^*). \tag{3}$$

The problem (3) is an instance of the quasi-variational inequality problem (QVI).

3 Preliminaries

In this section, we state some definitions, background material on projection and basic assumptions which will be used later.

For a given nonempty closed convex set Ω in $\Re^n$, the orthogonal projection from $\Re^n$ onto Ω is defined by

$$P_\Omega(x) = \operatorname{argmin}\{\|x - y\| \,|\, y \in \Omega\}, \quad x \in \Re^n.$$

It has the following well-known properties.

Lemma 3.1 ([29]) *Let Ω be a nonempty closed convex subset in $\Re^n$. Then a vector w is the projection of the vector x onto Ω if and only if*

$$\langle w - x, z - w \rangle \ge 0, \quad \text{for all } z \in \Omega.$$

Furthermore, P_Ω is nonexpansive, that is, for all $x, y \in \Re^n$,

$$\|P_\Omega(x) - P_\Omega(y)\| \le \|x - y\|.$$

Let M be a mapping from $\Re^n$ into $\Re^n$. For any $x \in \Re^n$ and $\alpha > 0$, define

$$x(\alpha) = P_\Omega(x - \alpha M(x)), \qquad e(x, \alpha) = x - x(\alpha).$$

Lemma 3.2 ([6, 22]) *For any given vector x and its mapping $M(x)$ in $\Re^n$,*

(a) $\|x - x(\alpha)\|$ *is nondecreasing with respect to* $\alpha > 0$;
(b) $\frac{\|x - x(\alpha)\|}{\alpha}$ *is nonincreasing with respect to* $\alpha > 0$.

From Lemma 3.2, we immediately conclude a useful lemma.

Lemma 3.3 *Let M be a continuous mapping from $\Re^n$ into $\Re^n$. For any $x \in \Re^n$ and $\alpha > 0$, we have*

$$\min\{1, \alpha\}\|e(x, 1)\| \le \|e(x, \alpha)\| \le \max\{1, \alpha\}\|e(x, 1)\|.$$

It is easy to get a necessary and sufficient condition for a point x^* to be a solution of the QVI problem (3) from Lemma 3.1.

Lemma 3.4 *A point x^* is a solution of the QVI problem* (3) *if and only if*

$$r_{K(x^*)}(x^*) \stackrel{\text{def}}{=} \|x^* - P_{K(x^*)}(x^* - F(x^*))\| = 0.$$

Now we give some concepts for the continuity of the point-to-set mapping K defined by (1).

Definition 3.1 Let $\bar{x} \in X$. The mapping $K(\cdot)$ is said to be

(a) upper semicontinuous (or closed) at $\bar{x}$ if

$$\left.\begin{array}{r} \{x_k\} \in X \text{ and } x_k \to \bar{x} \ (k \to \infty) \\ y_k \in K(x_k) \\ y_k \to \bar{y} \ (k \to \infty) \end{array}\right\} \quad \Rightarrow \quad \bar{y} \in K(\bar{x});$$

(b) lower semicontinuous at $\bar{x}$ if $x_k \in X$ and $x_k \to \bar{x}$ implies that for any $\bar{y} \in K(\bar{x})$, there exists a sequence $\{y_k\}$ with $y_k \in K(x_k)$, such that $y_k \to \bar{y}$ $(k \to \infty)$;
(c) continuous at $\bar{x}$ if it is both upper semicontinuous and lower semicontinuous at $\bar{x}$;
(d) continuous on X if and only if it is continuous at every point of X.

Definition 3.2 A point-to-point mapping $F(\cdot)$ is said to be

(a) pseudo monotone on X if for all vectors x and y in X,

$$\langle F(y), x - y\rangle \ge 0 \quad \Rightarrow \quad \langle F(x), x - y\rangle \ge 0;$$

(b) monotone on X if

$$\langle F(x) - F(y), x - y\rangle \ge 0, \quad \forall x, y \in X.$$

Definition 3.3 Let $\bar{x} \in X$. A point-to-point mapping $F(\cdot)$ is said to be

(a) monotone at $\bar{x}$, if for any $x \in X$,

$$\langle F(x) - F(\bar{x}), x - \bar{x} \rangle \geq 0;$$

(b) strictly monotone at $\bar{x}$ if

$$\langle F(x) - F(\bar{x}), x - \bar{x} \rangle > 0, \quad \forall x \in X \text{ and } x \neq \bar{x}.$$

We suppose the following assumption for problem (3) is satisfied.

Assumption (H)

(i) $S^* \stackrel{\text{def}}{=} \{x \in S | \langle F(x), y - x \rangle \geq 0, \forall y \in \bar{S}\} \neq \emptyset$, where $S = \bigcap_{x \in X} K(x)$ and $\bar{S} = \bigcup_{x \in X} K(x)$;

(ii) $F(\cdot)$ is monotone (or pseudo monotone) on X;

(iii) $K(\cdot)$ is continuous on X.

It is not easy to test the part (i) of Assumption (H) in practice. But it gives a sufficient condition to guarantee that the solution set of QVI (3) is nonempty. When for all $x \in X$, $K(x) \equiv K$, i.e., $K(x)$ is a constant set, the QVI reduces to the classical VI. In this case, the part (i) of Assumption (H) is to say that the solution set of the VI problem is nonempty. So, in some sense, the part (i) is a generalization of nonemptyness of the solution set from VI to QVI problems.

4 The first projection-like algorithm

In this section, we will give a projection-like algorithm and prove its convergence. First we formally state the algorithm.

Algorithm 1

Step 1 Given constants $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$, and $\rho \in (0, 2)$. Take $x_{-1} \in X$. Choose arbitrarily an $x_0 \in K(x_{-1})$. Set $k = 0$.

Step 2 If $r_{K(x_k)}(x_k) = 0$ then stop. Otherwise, let

$$\bar{x}_k = P_{K(x_k)}(x_k - \alpha_k F(x_k)),$$

where $\alpha_k = \gamma l^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\alpha_k \langle F(x_k) - F(\bar{x}_k), x_k - \bar{x}_k \rangle \leq \mu \|x_k - \bar{x}_k\|^2. \tag{4}$$

Step 3 Set

$$x_{k+1} = P_{K(x_k)}(x_k - \beta_k(x_k - \bar{x}_k + \alpha_k F(\bar{x}_k))),$$

where β_k is given by

$$\beta_k = \rho(1 - \mu) \frac{\|x_k - \bar{x}_k\|^2}{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2}.$$

Step 4 Set $k := k + 1$ and go to Step 2.

In order to better understand the above algorithm, we make the following remarks.

(1) Algorithm 1 and the subsequent Algorithm 2 are projection-like algorithms that make two projections per iteration. We may regard $\bar{x}_k$ as a predictor and x_{k+1} as a corrector. This class of technique is often used to solve monotone VIs, see for example [8, 9, 11, 12, 21, 28]. Although in each iteration the amount of computation is nearly doubled, the benefit is significant because these algorithms can be convergent using inexact line search and weakening the request that F is strongly monotone to monotone (or pseudo monotone) only. Compared with the projection-like algorithm proposed in [28], Algorithm 1 uses the projection $P_{K(x_k)}$ formed by the current iterative point x_k, rather than a fixed projection P_K. The new iterative point x_{k+1} produced is in $K(x_k)$, while x_k is not necessarily in $K(x_k)$, which prevents us from using some existing results about the projection and makes the convergence proof of the algorithm difficult.

(2) If $r_{K(x_k)}(x_k) = 0$, then from Lemma 3.4, we get that x_k is a solution to the QVI problem (3), that is, x_k is a GNE.

(3) By the following Lemmas 4.1 and 4.2, we know that Algorithm 1 is well defined.

Lemma 4.1 *Let $x \in X$ be arbitrary. Define*

$$x_{K(x)}(\alpha) = P_{K(x)}(x - \alpha F(x)).$$

Then for any $\mu \in (0, 1)$, when α is a sufficiently small positive number, we have

$$\alpha\langle F(x) - F(x_{K(x)}(\alpha)), x - x_{K(x)}(\alpha)\rangle \le \mu\|x - x_{K(x)}(\alpha)\|^2. \tag{5}$$

Proof First we know that $x_{K(x)}(\alpha) = P_{K(x)}(x - \alpha F(x)) \to P_{K(x)}(x)$ as $\alpha \to 0$.

If there exists an $\tilde{\alpha} > 0$ such that $x = x_{K(x)}(\tilde{\alpha})$, then from the first result of Lemma 3.1, we obtain that $x = x_{K(x)}(\alpha)$ for all $\alpha > 0$. In this case, (5) holds for all $\alpha > 0$ trivially.

If $x \ne x_{K(x)}(\alpha)$ for all $\alpha > 0$, we shall prove that (5) holds for all sufficiently small $\alpha > 0$ by contradiction. If the conclusion does not hold, then there exists a positive sequence $\{\alpha_i\}$ $(i = 1, 2, \ldots)$ which tends to zero such that for any α_i,

$$\alpha_i\langle F(x) - F(x_{K(x)}(\alpha_i)), x - x_{K(x)}(\alpha_i)\rangle > \mu\|x - x_{K(x)}(\alpha_i)\|^2.$$

Using the above inequality and Cauchy-Schwartz inequality, we have

$$\alpha_i\|F(x) - F(x_{K(x)}(\alpha_i))\| > \mu\|x - x_{K(x)}(\alpha_i)\|. \tag{6}$$

Two cases are to be considered.

Case 1: $x \notin K(x)$. Then $\alpha_i\|F(x) - F(x_{K(x)}(\alpha_i))\|$ would tend to zero while $\mu\|x - x_{K(x)}(\alpha_i)\|$ would tend to a positive number as $\alpha_i \to 0$, which contradicts (6).

Case 2: $x \in K(x)$. In this case, it is obvious that $x = P_{K(x)}(x)$. Since F is continuous and $x_{K(x)}(\alpha_i) \to P_{K(x)}(x) = x$ as $\alpha_i \to 0$, $\|F(x) - F(x_{K(x)}(\alpha_i))\|$ would tend to zero while $\frac{\|x-x_{K(x)}(\alpha_i)\|}{\alpha_i}$ will be not smaller than the positive number $||x - x_{K(x)}(1)||$ as $\alpha_i \to 0$ by the conclusion (b) of Lemma 3.2, which also contradicts (6).

The proof is completed. □

Lemma 4.2 *If* $r_{K(x_k)}(x_k) \neq 0$, *then*

$$\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\| \neq 0.$$

Proof Let x^* be an element of the set S^*. Then by the definition of S^* and the fact that $x_k \in K(x_{k-1})$ and $\bar{x}_k \in K(x_k)$, we have

$$\langle F(x^*), x_k - x^* \rangle \geq 0, \tag{7}$$

and

$$\langle F(x^*), \bar{x}_k - x^* \rangle \geq 0. \tag{8}$$

By (7), (8) and the monotonicity (or the pseudo monotonicity) of F, we have

$$\langle F(x_k), x_k - x^* \rangle \geq 0, \tag{9}$$

and

$$\langle F(\bar{x}_k), \bar{x}_k - x^* \rangle \geq 0. \tag{10}$$

Since $\bar{x}_k = P_{K(x_k)}(x_k - \alpha_k F(x_k))$, by Lemma 3.1 and the fact that $x^* \in K(x_k)$, we get

$$\langle x_k - \alpha_k F(x_k) - \bar{x}_k, \bar{x}_k - x^* \rangle \geq 0. \tag{11}$$

Thus, by (4), (9), (10) and (11), we obtain

$$\begin{aligned}
&\langle x_k - \bar{x}_k + \alpha_k F(\bar{x}_k), x_k - x^* \rangle \\
&\quad = \langle x_k - \bar{x}_k, x_k - x^* \rangle + \alpha_k \langle F(\bar{x}_k), x_k - x^* \rangle \\
&\quad = \langle x_k - \bar{x}_k - \alpha_k F(x_k) + \alpha_k F(x_k), x_k - x^* \rangle + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k + \bar{x}_k - x^* \rangle \\
&\quad = \langle x_k - \bar{x}_k - \alpha_k F(x_k), x_k - x^* \rangle + \alpha_k \langle F(x_k), x_k - x^* \rangle + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\qquad + \alpha_k \langle F(\bar{x}_k), \bar{x}_k - x^* \rangle \\
&\quad \geq \langle x_k - \bar{x}_k - \alpha_k F(x_k), x_k - x^* \rangle + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\quad = \langle x_k - \bar{x}_k - \alpha_k F(x_k), x_k - \bar{x}_k + \bar{x}_k - x^* \rangle + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\quad = \langle x_k - \bar{x}_k - \alpha_k F(x_k), x_k - \bar{x}_k \rangle + \langle x_k - \bar{x}_k - \alpha_k F(x_k), \bar{x}_k - x^* \rangle \\
&\qquad + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\quad \geq \langle x_k - \bar{x}_k - \alpha_k F(x_k), x_k - \bar{x}_k \rangle + \alpha_k \langle F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\quad = \langle x_k - \bar{x}_k - \alpha_k (F(x_k) - F(\bar{x}_k)), x_k - \bar{x}_k \rangle \\
&\quad = \|x_k - \bar{x}_k\|^2 - \alpha_k \langle F(x_k) - F(\bar{x}_k), x_k - \bar{x}_k \rangle \\
&\quad \geq (1 - \mu) \|x_k - \bar{x}_k\|^2.
\end{aligned} \tag{12}$$

If $r_{K(x_k)}(x_k) \neq 0$, then again by the first result of Lemma 3.1, we know that $x_k \neq \bar{x}_k$. Thus, from (12), $x_k - \bar{x}_k + \alpha_k F(\bar{x}_k) \neq 0$.

This completes the proof. □

Now, we establish the convergence of Algorithm 1.

Theorem 4.1 *Suppose Assumption* (H) *holds. Let* $\{x_k\}$ *be a sequence generated by Algorithm* 1. *Then* $\{x_k\}$ *is bounded, and any accumulation point of* $\{x_k\}$ *is a solution to the QVI problem* (3).

Proof Let x^* be an element of the set S^*. Then by Lemma 3.1, (12) and the definition of β_k in Algorithm 1, we have

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\quad = \|P_{K(x_k)}(x_k - \beta_k(x_k - \bar{x}_k + \alpha_k F(\bar{x}_k))) - x^*\|^2 \\
&\quad \le \|x_k - x^* - \beta_k(x_k - \bar{x}_k + \alpha_k F(\bar{x}_k))\|^2 \\
&\quad = \|x_k - x^*\|^2 - 2\beta_k\langle x_k - x^*, x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\rangle + \beta_k^2\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2 \\
&\quad \le \|x_k - x^*\|^2 - 2(1-\mu)\beta_k\|x_k - \bar{x}_k\|^2 + \beta_k^2\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2 \\
&\quad = \|x_k - x^*\|^2 - 2\rho(1-\mu)^2\frac{\|x_k - \bar{x}_k\|^4}{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2} \\
&\qquad + \rho^2(1-\mu)^2\frac{\|x_k - \bar{x}_k\|^4}{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2} \\
&\quad = \|x_k - x^*\|^2 - \rho(2-\rho)(1-\mu)^2\frac{\|x_k - \bar{x}_k\|^4}{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|^2},
\end{aligned} \tag{13}$$

which implies that the sequence $\{\|x_k - x^*\|\}$ is monotonically decreasing and hence convergent. So, $\{x_k\}$ is bounded. Consequently we get from (13) that

$$\lim_{k\to\infty} \frac{\|x_k - \bar{x}_k\|^2}{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k))\|} = 0. \tag{14}$$

Moreover, it is easy to show that $\{\bar{x}_k\}$ is bounded. In fact,

$$\begin{aligned}
\|\bar{x}_k\| &= \|P_{K(x_k)}(x_k - \alpha_k F(x_k))\| \\
&= \|P_{K(x_k)}(x_k - \alpha_k F(x_k)) + x^* - P_{K(x_k)}(x^*)\| \\
&\le \|x^*\| + \|x_k - x^* - \alpha_k F(x_k)\| \\
&\le \|x^*\| + \|x_k - x^*\| + \alpha_k\|F(x_k)\|,
\end{aligned}$$

which, together with the boundedness of $\{x_k\}$, deduces the desired result. So the sequence $\{\|x_k - \bar{x}_k + \alpha_k F(\bar{x}_k)\|\}$ is also bounded. Thus, from (14) we have

$$\lim_{k\to\infty} \|x_k - \bar{x}_k\| = 0. \tag{15}$$

Assume that $\bar{x}$ is an accumulation point of $\{x_k\}$. Then there exists a subsequence $\{x_k\}_{k\in\aleph}$, where $\aleph \subseteq \{0, 1, \ldots\}$, such that

$$\lim_{k\in\aleph, k\to\infty} x_k = \bar{x}.$$

We are ready to prove that $\bar{x}$ is a solution to the QVI problem (3).

First we show that $\bar{x} \in K(\bar{x})$. From (15) we have

$$\lim_{k\in\aleph, k\to\infty} \bar{x}_k = \bar{x},$$

which, together with the upper semicontinuity of $K(\cdot)$ and the fact that $\bar{x}_k \in K(x_k)$, deduces the desired result.

Next we need to show $\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \forall y \in K(\bar{x})$. To do so, we first prove that there exists at least a subsequence $\{\|e_k(x_k, 1)\|\}_{k\in\bar{\aleph}}$ (where $\bar{\aleph} \subseteq \aleph$) of $\{\|e_k(x_k, 1)\|\}$ such that

$$\lim_{k\in\bar{\aleph}, k\to\infty} \|e_k(x_k, 1)\| = 0, \tag{16}$$

where $e_k(x_k, \alpha) = x_k - P_{K(x_k)}(x_k - \alpha F(x_k))$.

Two cases are to be considered.

Case 1: $\inf_{k\in\aleph}\{\alpha_k\} = \alpha_{\min} > 0$. Then from Lemma 3.3, we have

$$\|e_k(x_k, 1)\| \leq \frac{\|x_k - \bar{x}_k\|}{\min\{1, \alpha_k\}},$$

which, together with (15), implies that

$$\lim_{k\in\aleph, k\to\infty} \|e_k(x_k, 1)\| \leq \lim_{k\in\aleph, k\to\infty} \frac{\|x_k - \bar{x}_k\|}{\min\{1, \alpha_{\min}\}} = 0.$$

Case 2: $\inf_{k\in\aleph}\{\alpha_k\} = \alpha_{\min} = 0$. Since $\alpha_{\min} = 0$, there must exist a subsequence $\{\alpha_k\}_{k\in\bar{\aleph}}$, where $\bar{\aleph} \subseteq \aleph$, such that $\lim_{k\in\bar{\aleph}, k\to\infty} \alpha_k = 0$. Thus, for all sufficiently small α_k, $\frac{\alpha_k}{l}$ must violate the search rule (4), that is

$$\frac{\alpha_k}{l}\left\langle x_k - x_k\left(\frac{\alpha_k}{l}\right), F(x_k) - F\left(x_k\left(\frac{\alpha_k}{l}\right)\right)\right\rangle > \mu\left\|x_k - x_k\left(\frac{\alpha_k}{l}\right)\right\|^2,$$

where $x_k(\frac{\alpha_k}{l}) = P_{K(x_k)}(x_k - \frac{\alpha_k}{l}F(x_k))$. Again from Cauchy-Schwartz inequality and Lemma 3.3 we get

$$\mu\|e_k(x_k, 1)\| \leq \mu\frac{\|x_k - x_k(\frac{\alpha_k}{l})\|}{\frac{\alpha_k}{l}} < \left\|F(x_k) - F\left(x_k\left(\frac{\alpha_k}{l}\right)\right)\right\|,$$

that is,

$$\|e_k(x_k, 1)\| < \frac{1}{\mu}\left\|F(x_k) - F\left(x_k\left(\frac{\alpha_k}{l}\right)\right)\right\|.$$

Moreover, we can get that

$$\begin{aligned}\left\|x_k - x_k\left(\frac{\alpha_k}{l}\right)\right\| &= \left\|x_k - \bar{x}_k + \bar{x}_k - P_{K(x_k)}\left[x_k - \frac{\alpha_k}{l}F(x_k)\right]\right\| \\ &\le \|x_k - \bar{x}_k\| + \left\|\frac{\alpha_k}{l}F(x_k) - \alpha_k F(x_k)\right\| \\ &= \|x_k - \bar{x}_k\| + \left(\frac{1}{l} - 1\right)\alpha_k \|F(x_k)\| \\ &\to 0 \quad (k \in \bar{\aleph}, k \to \infty)\end{aligned}$$

which reduces that

$$\lim_{k\in\bar{\aleph}, k\to\infty} \left\|F(x_k) - F\left(x_k\left(\frac{\alpha_k}{l}\right)\right)\right\| = 0.$$

Thus, we have

$$\lim_{k\in\bar{\aleph}, k\to\infty} \|e_k(x_k, 1)\| = 0.$$

From the analysis above, we get the desired conclusion.

Now, we continue to prove the main result.

Since $K(\cdot)$ is lower semicontinuous, for any $y \in K(\bar{x})$, there exists a sequence $\{y_k\}$ with $y_k \in K(x_k)$ such that

$$\lim_{k\in\bar{\aleph}, k\to\infty} y_k = y.$$

From the fact that $x_k - e_k(x_k, 1) = P_{K(x_k)}(x_k - F(x_k))$, we have

$$\langle F(x_k) - e_k(x_k, 1), y_k - x_k + e_k(x_k, 1)\rangle \ge 0,$$

that is,

$$\langle F(x_k), y_k - x_k\rangle + \langle F(x_k), e_k(x_k, 1)\rangle - \langle e_k(x_k, 1), y_k - x_k\rangle - \|e_k(x_k, 1)\|^2 \ge 0.$$

Letting $k \to \infty (k \in \bar{\aleph})$, due to (16) and the boundedness of $\{x_k\}$ and $\{y_k\}$, we deduce that

$$\langle F(\bar{x}), y - \bar{x}\rangle \ge 0.$$

From the arbitrariness of y, we conclude that $\bar{x}$ is a solution to problem (3).

This completes the proof. □

Theorem 4.2 *Suppose Assumption* (H) *holds. Let* $\{x_k\}$ *be a sequence generated by Algorithm* 1. *If* F *is strictly monotone at an accumulation point of* $\{x_k\}$, *say* $\bar{x}$, *then*

$$\lim_{k\to\infty} x_k = \bar{x}.$$

Proof Since $\bar{x}$ is an accumulation point of $\{x_k\}$, there exists a subsequence $\{x_k\}_{k\in\aleph}$, where $\aleph \subseteq \{0, 1, \dots\}$, such that

$$\lim_{k\in\aleph, k\to\infty} x_k = \bar{x}. \tag{17}$$

From Theorem 4.1, we know that $\bar{x}$ is a solution to problem (3).

Let x^* be an element of the set S^*. Then from the definition of S^*, we have

$$\langle F(x^*), \bar{x}_k - x^* \rangle \geq 0.$$

Letting $k \to \infty$ $(k \in \aleph)$, taking (17) and (15) into account, we get

$$\langle F(x^*), \bar{x} - x^* \rangle \geq 0, \tag{18}$$

which, together with the monotonicity (or the pseudo monotonicity) of F, implies that

$$\langle F(\bar{x}), \bar{x} - x^* \rangle \geq 0. \tag{19}$$

On the other hand, from the fact that $x^* \in K(x_k)$, using the upper semicontinuity of $K(\cdot)$ and (17), we obtain

$$x^* \in K(\bar{x}).$$

Because $\bar{x}$ is a solution to the QVI problem (3), we have

$$\langle F(\bar{x}), x^* - \bar{x} \rangle \geq 0. \tag{20}$$

Thus, from (19) and (20), we obtain

$$\langle F(\bar{x}), x^* - \bar{x} \rangle = 0. \tag{21}$$

From the monotonicity (or the pseudo monotonicity) of F and (21), we have

$$\langle F(x^*), x^* - \bar{x} \rangle \geq 0. \tag{22}$$

(22) and (18) give that

$$\langle F(x^*), x^* - \bar{x} \rangle = 0.$$

Thus,

$$\langle F(\bar{x}), x^* - \bar{x} \rangle = \langle F(x^*), x^* - \bar{x} \rangle = 0.$$

It follows that

$$\langle F(x^*) - F(\bar{x}), x^* - \bar{x} \rangle = 0.$$

From the assumption that F is strictly monotone at $\bar{x}$, we get that $\bar{x} = x^* \in S^*$.

Thus we may use $\bar{x}$ in place of x^* in (13), and obtain that $\{\|x_k - \bar{x}\|\}$ is convergent. Because there exists a subsequence $\{\|x_k - \bar{x}\|\}_{k\in\aleph}$ converging to 0,

$$\lim_{k\to\infty} x_k = \bar{x}.$$

This completes the proof. □

5 Another algorithm

In Sect.4 we have given an algorithm which requires an Armijo-like line search procedure to obtain the step size and asks a projection at each trial point until the step length is determined. This sometimes is computationally expensive. To overcome this defect, adopting the technology of [25], we now give another algorithm in which only two projections are needed in each iteration.

Algorithm 2

Step 1. Given constants $l \in (0, 1)$, $\mu \in (0, 1)$, and $\rho \in (0, 2)$. Take $x_0 \in X$. Set $k = 0$.
Step 2. If $r_{K(x_k)}(x_k) = 0$ then stop. Otherwise, let

$$z_k = P_{K(x_k)}(x_k - F(x_k)),$$
$$y_k = (1 - \alpha_k)x_k + \alpha_k z_k,$$

where $\alpha_k = l^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\langle F(x_k) - F((1 - l^m)x_k + l^m z_k), x_k - z_k\rangle \le \mu\|x_k - z_k\|^2. \tag{23}$$

Step 3. Set

$$x_{k+1} = P_{K(x_k)}(x_k - \beta_k d_k),$$

where d_k and β_k are given by

$$d_k = x_k - z_k + \frac{F(y_k)}{\alpha_k}$$

and

$$\beta_k = \rho(1 - \mu)\frac{\|x_k - z_k\|^2}{\|d_k\|^2},$$

respectively.
Step 4. Set $k := k + 1$ and go to Step 2.

In this section, the following assumption is imposed:

$$\text{(A)} \quad x \in K(x), \quad \forall x \in X.$$

In order to prove the feasibility of Algorithm 2, we need the following two lemmas.

Lemma 5.1 *Let $x \in X$ be arbitrary. For any $\alpha \in (0, 1)$, define*

$$z = P_{K(x)}(x - F(x)), \qquad y(\alpha) = (1 - \alpha)x + \alpha z.$$

Then for any given $\mu \in (0, 1)$, when $\alpha > 0$ is sufficiently small, we have

$$\langle F(x) - F(y(\alpha)), x - z\rangle \le \mu\|x - z\|^2.$$

Proof It is easy to get the desired result by the continuity of the mapping $F(\cdot)$. □

This lemma shows that Step 2 is well defined.

Lemma 5.2 *Suppose Assumptions* (H) *and* (A) *hold. If* $r_{K(x_k)}(x_k) \neq 0$, *then* $d_k \neq 0$.

Proof Let x^* be an element of the set S^*. Then by the definition of S^* and the fact that $x_k \in K(x_k)$ and $y_k \in K(y_k)$, we have

$$\langle F(x^*), x_k - x^* \rangle \geq 0, \tag{24}$$

and

$$\langle F(x^*), y_k - x^* \rangle \geq 0. \tag{25}$$

By (24), (25) and the monotonicity (or the pseudo monotonicity) of F, we have

$$\langle F(x_k), x_k - x^* \rangle \geq 0, \tag{26}$$

and

$$\langle F(y_k), y_k - x^* \rangle \geq 0. \tag{27}$$

Since $z_k = P_{K(x_k)}(x_k - F(x_k))$, by Lemma 3.1 and the fact that $x^* \in K(x_k)$, we get

$$\langle x_k - F(x_k) - z_k, z_k - x^* \rangle \geq 0. \tag{28}$$

Thus, by (23), (26), (27) and (28), we obtain

$$\begin{aligned}
\langle d_k, x_k - x^* \rangle &= \left\langle x_k - z_k + \frac{F(y_k)}{\alpha_k}, x_k - x^* \right\rangle \\
&= \langle x_k - z_k, x_k - x^* \rangle + \left\langle \frac{F(y_k)}{\alpha_k}, x_k - x^* \right\rangle \\
&= \langle x_k - F(x_k) - z_k, x_k - x^* \rangle + \langle F(x_k), x_k - x^* \rangle \\
&\quad + \frac{1}{\alpha_k}\langle F(y_k), x_k - y_k \rangle + \frac{1}{\alpha_k}\langle F(y_k), y_k - x^* \rangle \\
&\geq \langle x_k - F(x_k) - z_k, x_k - z_k \rangle + \langle x_k - F(x_k) - z_k, z_k - x^* \rangle \\
&\quad + \frac{1}{\alpha_k}\langle F(y_k), x_k - y_k \rangle \\
&\geq \langle x_k - F(x_k) - z_k, x_k - z_k \rangle + \frac{1}{\alpha_k}\langle F(y_k), \alpha_k(x_k - z_k) \rangle \\
&= \langle x_k - F(x_k) - z_k, x_k - z_k \rangle + \langle F(y_k), x_k - z_k \rangle \\
&= \langle x_k - z_k - (F(x_k) - F(y_k)), x_k - z_k \rangle \\
&= \|x_k - z_k\|^2 - \langle F(x_k) - F(y_k), x_k - z_k \rangle \\
&\geq (1 - \mu)\|x_k - z_k\|^2.
\end{aligned} \tag{29}$$

If $r_{K(x_k)}(x_k) \neq 0$, that is, $x_k \neq z_k$, then from (29), $d_k \neq 0$.
This completes the proof. □

Now, we establish the convergence of Algorithm 2.

Theorem 5.1 *Suppose Assumptions* (H) *and* (A) *hold. Let* $\{x_k\}$ *be a sequence generated by Algorithm* 2. *Then* $\{x_k\}$ *is bounded, and any accumulation point of* $\{x_k\}$ *is a solution to problem* (3).

Proof Let x^* be an element of the set S^*. Then by Lemma 3.1, (29) and the definition of β_k in Algorithm 2, we have

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|P_{K(x_k)}(x_k - \beta_k d_k) - x^*\|^2 \\ &\leq \|x_k - x^* - \beta_k d_k\|^2 \\ &= \|x_k - x^*\|^2 - 2\beta_k \langle x_k - x^*, d_k\rangle + \beta_k^2 \|d_k\|^2 \\ &\leq \|x_k - x^*\|^2 - 2\beta_k (1-\mu)\|x_k - z_k\|^2 + \beta_k^2 \|d_k\|^2 \\ &= \|x_k - x^*\|^2 - 2\rho(1-\mu)^2 \frac{\|x_k - z_k\|^4}{\|d_k\|^2} + \rho^2 (1-\mu)^2 \frac{\|x_k - z_k\|^4}{\|d_k\|^2} \\ &= \|x_k - x^*\|^2 - \rho(2-\rho)(1-\mu)^2 \frac{\|x_k - z_k\|^4}{\|d_k\|^2} \\ &\leq \|x_k - x^*\|^2, \end{aligned} \tag{30}$$

which implies that the sequence $\{\|x_k - x^*\|\}$ is monotonically decreasing and hence $\{x_k\}$ is bounded. Consequently we get from (30) that

$$\lim_{k\to\infty} \frac{\|x_k - z_k\|^2}{\|d_k\|} = 0. \tag{31}$$

Moreover, it is easy to show that $\{z_k\}$ is bounded. In fact,

$$\begin{aligned} \|z_k - x^*\| &= \|P_{K(x_k)}(x_k - F(x_k)) - x^*\| \\ &\leq \|x_k - F(x_k) - x^*\| \\ &\leq \|x_k\| + \|F(x_k)\| + \|x^*\| \end{aligned}$$

which, together with the boundedness of $\{x_k\}$ and the continuity of F, deduces the boundedness of $\{z_k\}$. So $\{y_k\}$ and $\{F(y_k)\}$ are also bounded.

Assume that $\bar{x}$ is an accumulation point of $\{x_k\}$. Then there exists a subsequence $\{x_k\}_{k\in\aleph}$, where $\aleph \subseteq \{0, 1, \ldots\}$, such that

$$\lim_{k\in\aleph, k\to\infty} x_k = \bar{x}. \tag{32}$$

We claim that there exists at least a subsequence $\{\|x_k - z_k\|\}_{k\in\bar{\aleph}}$ (where $\bar{\aleph} \subseteq \aleph$) such that

$$\lim_{k\in\bar{\aleph},k\to\infty} \|x_k - z_k\| = 0. \tag{33}$$

We consider two cases.

Case 1: $\inf_{k\in\aleph}\{\alpha_k\} = \alpha_{\min} > 0$. In this case, from the boundedness of $\{x_k\}$, $\{z_k\}$ and $\{F(y_k)\}$, we know that the sequence $\{d_k\}_{k\in\aleph}$ is bounded. Thus from (31) we get

$$\lim_{k\in\aleph,k\to\infty} \|x_k - z_k\| = 0.$$

Case 2: $\inf_{k\in\aleph}\{\alpha_k\} = \alpha_{\min} = 0$. Since $\alpha_{\min} = 0$, there must exist a subsequence $\{\alpha_k\}_{k\in\bar{\aleph}}$, where $\bar{\aleph} \subseteq \aleph$, such that

$$\lim_{k\in\bar{\aleph},k\to\infty} \alpha_k = 0. \tag{34}$$

Thus, for all sufficiently small α_k with $k \in \bar{\aleph}$, $\frac{\alpha_k}{l}$ must violate the search rule (23), that is,

$$\left\langle F(x_k) - F\left(\left(1 - \frac{\alpha_k}{l}\right)x_k + \frac{\alpha_k}{l} z_k\right), x_k - z_k\right\rangle > \mu\|x_k - z_k\|^2.$$

Using Cauchy-Schwarz inequality, we have

$$\left\| F(x_k) - F\left(\left(1 - \frac{\alpha_k}{l}\right)x_k + \frac{\alpha_k}{l} z_k\right)\right\| > \mu\|x_k - z_k\|, \quad \forall k \in \bar{\aleph}.$$

Using (34) and the continuity of $F(\cdot)$, we obtain

$$\lim_{k\in\bar{\aleph},k\to\infty} \|x_k - z_k\| = 0.$$

So, the claim is true.

Now, we are ready to prove that $\bar{x}$ is a solution to problem (3).

First we show that $\bar{x} \in K(\bar{x})$. From (32) and (33) we have

$$\lim_{k\in\bar{\aleph},k\to\infty} z_k = \bar{x}, \tag{35}$$

which, together with the upper semicontinuity of $K(\cdot)$ and the fact that $z_k \in K(x_k)$, deduces the desired result.

Next we need to show that $\langle F(\bar{x}), w - \bar{x}\rangle \geq 0$, $\forall w \in K(\bar{x})$. Since $K(\cdot)$ is lower semicontinuous, for any $w \in K(\bar{x})$, there exists a sequence $\{w_k\}$ with $w_k \in K(x_k)$ $(k \in \bar{\aleph})$ such that

$$\lim_{k\in\bar{\aleph},k\to\infty} w_k = w.$$

From the fact that $z_k = P_{K(x_k)}(x_k - F(x_k))$, we have

$$\langle z_k - x_k + F(x_k), w_k - z_k\rangle \geq 0,$$

that is

$$\langle F(x_k), w_k - z_k\rangle + \langle z_k - x_k, w_k - z_k\rangle \geq 0.$$

Letting $k \to \infty$ ($k \in \bar{\aleph}$), taking into account (33), (35) and the boundedness of $\{z_k\}$ and $\{w_k\}$, we deduce that

$$\langle F(\bar{x}), w - \bar{x}\rangle \geq 0.$$

From the arbitrariness of w, we conclude that $\bar{x}$ is a solution to the QVI problem.

This completes the proof. □

Following the same line as what in the proof of Theorem 4.2, we have the following theorem about the convergence of the whole sequence $\{x_k\}$ generated by Algorithm 2.

Theorem 5.2 *Suppose Assumptions* (H) *and* (A) *hold. Let* $\{x_k\}$ *be a sequence generated by Algorithm* 2. *If* F *is strictly monotone at an accumulation point of* $\{x_k\}$, *say* $\bar{x}$, *then*

$$\lim_{k\to\infty} x_k = \bar{x}.$$

6 The numerical results

To give some insight into the behavior of the algorithms presented in this paper, we implemented them in MATLAB to solve two examples of generalized Nash games which have appeared in the literature. We terminate the algorithms if $r_{K(x_k)}(x_k) < \epsilon$ holds for an iterate x_k, and all experiments end successfully by satisfying this criterion.

Throughout the computational experiments, we set the parameters used in Algorithms 1 and 2 as $\epsilon = 10^{-6}$, $\gamma = 1$, $l = 0.5$, $\mu = 0.3$, and $\rho = 1.99$. In the results reported below, all CPU times are in seconds. The approximate solution is referred to the last iterative point. The symbol "/" means that the number of iterations exceeds 1000 or the CPU time exceeds 300 seconds.

Example 1 This example was used by Harker [7, 23] and Outrata [16]. Consider a two-person game, in which each player picks a number x^i between 0 and 10 and the sum of their numbers must be less than or equal to 15. The cost functions and mappings K^i are defined by

$$u^1(x^1, x^2) = (x^1)^2 + \frac{8}{3}x^1x^2 - 34x^1,$$

$$u^2(x^1, x^2) = (x^2)^2 + \frac{5}{4}x^1x^2 - 24.25x^2,$$

$$K^1(\bar{x}^2) = \{0 \leq x^1 \leq 10, x^1 \leq 15 - \bar{x}^2\},$$

$$K^2(\bar{x}^1) = \{0 \leq x^2 \leq 10, x^2 \leq 15 - \bar{x}^1\}.$$

Table 1 The results of Example 1

Starting point	Number of iterations		CPU (s)		Approximate solution	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(0,0)^T$	201	261	0.0678	0.0759	$(5,9)^T$	$(5,9)^T$
$(10,0)^T$	/	/	83.2922	/	$(10,5)^T$	$(10,5)^T$
$(10,10)^T$	206	266	0.0755	0.0667	$(5,9)^T$	$(5,9)^T$
$(0,10)^T$	177	187	0.0595	0.0430	$(5,9)^T$	$(5,9)^T$
$(5,5)^T$	235	265	0.0771	0.0674	$(5,9)^T$	$(5,9)^T$

Now consider the QVI formulation of this problem. The function F is defined by

$$F^1(x) = 2x^1 + \frac{8}{3}x^2 - 34,$$

$$F^2(x) = 2x^2 + \frac{5}{4}x^1 - 24.25.$$

The set of GNE solutions is composed of the point $(5,9)^T$ and the line segment $[(9,6)^T, (10,5)^T]$. The test results for this example are listed in Table 1 using different starting points.

Example 2 This example is a modification of the Stackelberg-Cournot-Nash equilibria problem tested in [16]. Consider an oligopolistic market in which m firms supply a homogeneous product in a noncooperative fashion. Let $p : \mathrm{int}\,\Re_+ \to \mathrm{int}\,\Re_+$ assign to the overall quantity Q of the product on the market the unit price at which consumers will purchase this quantity of products. This function p is called the inverse demand curve. The production cost is represented by a cost function f_i, $i = 1, 2, \ldots, m$.

The function f_i $(i = 1, \ldots, m)$ and p are taken as the following forms:

$$f_i(x^i) = c_i x^i + \frac{\beta_i}{\beta_i + 1} \tau_i^{-\frac{1}{\beta_i}} (x^i)^{\frac{1+\beta_i}{\beta_i}}, \tag{36}$$

where $c_i, \beta_i, \tau_i, i = 1, 2, \ldots, m$, are given positive parameters; Furthermore, let

$$p(Q) = 5000^{\frac{1}{\eta}} Q^{-\frac{1}{\eta}}, \tag{37}$$

where the positive parameter η is termed demand elasticity.

We consider the generalized Nash equilibria of an oligopolistic market in which the productions x^i are subject to not only the mutually independent production constraints $x^i \in X_i$, $i = 1, 2, \ldots, m$, but also a joint production bound

$$\sum_{i=1}^{m} x^i \le N.$$

Table 2 Parameter specification for production costs

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
c_i	10	8	6	4	2
τ_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

Table 3 The results of Example 2

Two algorithms	CPU (s)	Number of iterations	Approximate solution				
			Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Algorithm 1	24.9760	86	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 2	27.5000	148	36.9325	41.8181	43.7066	42.6592	39.1790

Carefully speaking, a generalized Nash equilibria is a vector $(x^{*1}, x^{*2}, \dots, x^{*m})^T$ such that x^{*i}, $i = 1, 2, \dots, m$, is a solution of the optimization problem

$$\min_{s.t.x^i \in \bar{X}_i} f_i(x^i) - x^i p\left(x^i + \sum_{j=1, j\neq i}^{m} x^{*j}\right), \tag{38}$$

where

$$\bar{X}_i = \left\{x^i | x^i \in X_i, x^i + \sum_{j=1, j\neq i}^{m} x^{*j} \leq N\right\}.$$

In (38) each firm maximizes its profit subject to its production constraints and the joint production constraint under the assumption that the remaining firms stick at their equilibria productions.

Using the functions f_i and p given by (36) and (37), and following the same line of the transformation process of Sect. 2, this problem can be converted to a QVI. We take $m = 5$ in the test. The function F is defined by

$$F^i(x) = c_i + \left(\frac{x^i}{\tau_i}\right)^{\frac{1}{\beta_i}} + \left(\frac{5000}{Q}\right)^{\frac{1}{\eta}} \left(\frac{x^i}{\eta Q} - 1\right), \quad i = 1, \dots, 5,$$

where $Q = \sum_{i=1}^{5} x^i$. All firms have the same lower production bound 1 and upper production bound 150, that is,

$$X_i = \{x^i | 1 \leq x^i \leq 150\}, \quad i = 1, \dots, 5.$$

The parameters of the production cost functions are given in Table 2.

Table 3 lists the results of Algorithm 1 and Algorithm 2 when they are applied to Example 2 with initial point $x_0 = (50, 50, 50, 50, 50)^T$, $\eta = 1.1$ and $N = 700$.

Table 4 The results of Example 2 with another initial point

Two algorithms	CPU (s)	Number of iterations	Approximate solution				
			Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Algorithm 1	19.9480	99	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 2	14.8210	138	36.9325	41.8181	43.7066	42.6592	39.1790

Table 5 The results of Example 3

Starting point	Number of iterations		CPU (s)		Approximate solution	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(0, 0)^T$	244	259	0.0630	0.0470	$(5, 9)^T$	$(5, 9)^T$
$(10, 0)^T$	247	252	0.0620	0.0320	$(5, 9)^T$	$(5, 9)^T$
$(10, 10)^T$	184	199	0.0470	0.0310	$(5, 9)^T$	$(5, 9)^T$
$(0, 10)^T$	177	165	0.0310	0.0320	$(5, 9)^T$	$(5, 9)^T$
$(5, 5)^T$	235	246	0.0470	0.0630	$(5, 9)^T$	$(5, 9)^T$

When the initial point is chosen as $x_0 = (10, 10, 10, 10, 10)^T$, the result is given in Table 4.

When an additional joint constraint is added, similar numerical performance is observed. We omit the details. We notice that in Examples 1, 2 above, all joint constraints are shared by all players. How about numerical performance for the examples in which not all joint constraints are shared by all players? To answer this question, we construct a simple example which is a modification of Example 1 as follows.

Example 3 In this example, the set $K^2(\bar{x}^1) = \{0 \le x^2 \le 10, x^2 \le 15 - \bar{x}^1\}$ in Example 1 is replaced by $K^2(\bar{x}^1) = \{2 \le x^2 \le 10\}$.

Table 5 lists the results for Algorithms 1 and 2 being applied to Example 3 with different initial points. We see that in all cases CPU times are reduced comparing with Example 1.

We know that choice of parameter values affects performance of a method in practice. As this is not the main objective of the paper, we would not give detailed test in this aspect. The numerical experiments tested for the three simple problems are used to demonstrate the viability of the methods proposed in this paper. A remarkable characteristic of the algorithms is the computational simplicity, which makes the algorithms easy to be implemented.

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