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A Novel Filled Function Method and Quasi-Filled Function Method for Global Optimization*

Z.Y. WU † zhiyouwu@263.net
School of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China

H.W.J. LEE
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China

L.S. ZHANG
Department of Mathematics, Shanghai University, Shanghai 200444, China

X.M. YANG
School of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China

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Abstract. This paper gives a new definition of a filled function, which eliminates certain drawbacks of the traditional definitions. Moreover, this paper proposes a quasi-filled function to improve the efficiency of numerical computation and overcomes some drawbacks of filled functions. Then, a new filled function method and a quasi-filled function method are presented for solving a class of global optimization problems. The global optimization approaches proposed in this paper will find a global minimum of original problem by implementing a local search scheme to the proposed filled function or quasi-filled function. Illustrative examples are provided to demonstrate the efficiency and reliability of the proposed scheme.

Keywords: filled function, quasi-filled function, global optimization

1. Introduction

Consider the following unconstrained programming problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in R^n. \end{aligned} \tag{1.1}$$

Many results devoted to global optimization are available in the literature. See, for example, [3–10] and [13]. In order to ensure the ability to escape from local minimum, many global optimization algorithms would include in their consideration a subproblem of transcending local optimality, namely: given a local minimizer x^*, find a better feasible solution, or showing that x^* is a global minimizer upon termination. Among

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†To whom correspondence should be addressed.

all the different types of global optimization algorithms available in the literature, one popular approach is called the modified function approach. In this approach, the resolution to the subproblem under concerned is to replace the original cost function with a modified function. This replacement procedure, in principle, should ensure any local search applied to the modified function starting from x^*, would lead to a lower minimum of the original cost function, if there exists one. Thus, global minimizer can be obtained just by implementing local search methods to the modified function and the original function. However, it is rather difficult to construct such a modified function.

Filled function method is a typical modified function method. The concept of filled function was firstly introduced in [3]. The filled function proposed in [3] is as the following:

$$p_{x,x^*,r,\rho} = \frac{1}{r + f(x)} \exp\left(-\frac{\|x - x^*\|}{\rho^2}\right), \tag{1.2}$$

where x^* is the current local minimizer of original problem. This filled function has some drawbacks, then [4–8], [10] and [13] proposed some other filled functions, which improved the filled function (1.2). And then [14] revised the concept of filled function as follows.

A function $p(x, x^*)$ is said to be a filled function of $f(x)$ at the local minimizer x^* if it satisfies the following:

(P_1): x^* is a maximizer of $p(x, x^*)$ and the whole basin B^* of $f(x)$ at x^* becomes a part of a hill of $p(x, x^*)$;
(P_2): $p(x, x^*)$ has no minimizer or saddle point in any basin of $f(x)$ higher that B^*;
(P_3): if $f(x)$ has a basin B_2^* at x_2^* that is lower than B^*, then there is a point x' that minimize $p(x, x^*)$ on the line through x^* and x'', for any x'' in some neighborhoods of x_2^*.

The Property (P_3) revised the Property (3) in [3]. It was much stronger since a minimizer was required for lines connecting the current minimizer with every point in some neighborhoods of a next better minimizer, not just for one such point as required in Property (3) in [3]. But the Property (P_3) is still not very satisfying, since a minimizer was still required for lines and the filled function proposed in [14] is not differentiable.

Then, [9] proposed another continuously differentiable filled function, which assured the existence of a local minimizer in a lower basin but not just on lines. But the filled function presented in [9] maybe obtain its local minimizer on the boundary of $\triangle$, where $\triangle$ was decided in [9]. The points on the boundary of $\triangle$ are not a better point, i.e., its value is larger than $f(x^*)$.

Wu et al. [12] gave a new modified function method to obtain a better local minimizer by implementing local search methods to a one- constrained subproblem constructed by a modified function, which overcome the drawback in [9] in some degree, but the drawback of the method in [12] is that the subproblem for finding the better point than x^* is a constrained problem.

In order to overcome the above drawbacks. In this paper, we will propose two new kinds of modified functions: a new filled function which is different from those in the above literature, and a quasi-filled function, which is different from the filled functions and is an improvement of the filled function. Two global optimization algorithms constructed upon the proposed filled function and quasi-filled function are presented. Finally, numerical examples are given to demonstrate the efficiency and effectiveness of the proposed techniques.

2. A novel definition about the filled function

We assume that the following conditions are satisfied.

Assumption 1. $f(x)$ is continuously differentiable on R^n.

Assumption 2. Let Y be the set of all local minimizers of problem (2.1). The set F defined by

$$F = \{f(x) \mid x \in Y\} \tag{2.1}$$

is a finite set.

Note that Assumption 2 only requires that the number of local minimal values of problem (1.1) is finite. The number of local minima may be infinite.

Assumption 3. $f(x)$ satisfies that $f(x) \to +\infty$ as $\|x\| \to +\infty$.

By Assumption 3, there exist a box set Ω, a point $x_1^0 \in R^n$ and a constant $f_0 > 1$ such that $f(x) \geq f(x_1^0) + f_0$ for any $x \in R^n \setminus \text{int}\Omega$, where $\text{int}\Omega$ denotes the interior of Ω.

For convenience, let Ω be defined by

$$\Omega = \{x = (x_1, \dots, x_n) \mid c_i \leq x_i \leq d_i, \quad i = 1, \dots, n\}, \tag{2.2}$$

where $c_i, d_i, i = 1, \dots, n$, are constants.

Then, the original problem (1.1) is equivalent to the following problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in \Omega. \end{aligned} \tag{2.3}$$

i.e., $\bar{x}^*$ is a global minimizer of problem (1.1) if and only if $\bar{x}^*$ is a global minimizer of problem (2.3).

To proceed further, we need the following definitions.

Definition 2.1. $B^* \subset X$ is said to be a G-basin of problem (1.1) corresponding to a local minimizer x^* if it is a connected domain with the following properties:

(i) $f(x) \geq f(x^*)$ for any $x \in B^*$;
(ii) $\bar{x} \in B^*$ is a local minimizer of problem (1.1) if and only if $f(\bar{x}) = f(x^*)$.

Note that the concept of the G-basin in this paper is not the same as the definition of basin defined by traditional ways.

Definition 2.2. Let x^1 and x^2 be two local minimizers. If $f(x^2) > f(x^1)$, then the local minimizer x^2 is said to be higher than the local minimizer x^1 and the local minimizer x^1 is said to be lower than the local minimizer x^2. If the local minimizers x^1 and x^2 are such that $f(x^1) = f(x^2)$, then the local minimizers x^1 and x^2 are said to be with the same height.

Throughout the rest of the paper, let x^* be a local minimizer of problem (1.1), B^* be the G-basin of x^*, and let L be the set which consists of all the local minimizers of problem (1.1) lower than x^*.

Definition 2.3. A differentiable function $p(x)$ is a filled function of problem (1.1) corresponding to a local minimizer x^* if it satisfies the following properties:

(i) x^* is a strictly local maximizer of $p(x)$;
(ii) For any $x \neq x^*$ satisfying $f(x) \geq f(x^*)$, x is not a stationary point of $p(x)$, i.e., $\nabla p(x) \neq 0$;
(iii) if x^* is not a global minimizer of problem (1.1), i.e., $L \neq \emptyset$, then for any $\bar{x} \in L$, $\bar{x}$ is a local minimizer of $p(x)$ and furthermore satisfies

$$p(\bar{x}) < p(x^*)$$
$$p(\bar{x}) < p(x), \text{ for any } x \in \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω.
(iv) For any $x_1, x_2 \in \Omega$ satisfying $f(x_1) \geq f(x^*)$ and $f(x_2) \geq f(x^*)$, $\|x_2 - x^*\| > (\geq)$ $\|x_1 - x^*\|$ if and only if $p(x_2) < (\leq) p(x_1)$.

By Definition 2.3, we know that if $p(x)$ is a filled function satisfying Definition 2.3, then we have the following results:

1. x^* is a global minimizer of problem (1.1) if and only if x^* is the only stationary point of $p(x)$ and x^* is a strictly local maximizer of $p(x)$.
2. x^* is not a global minimizer of problem (1.1) if and only if there exists at least a point $\bar{x} \in \text{int}\Omega$ such that $\bar{x}$ is a local minimizer of $p(x)$ and satisfies $f(\bar{x}) < f(x^*)$, $p(\bar{x}) < p(x^*)$ and $p(\bar{x}) < p(x)$ for any $x \in \partial\Omega$, where $\text{int}\Omega$ denotes the interior of Ω.

Remark 2.1. Note that Definition 2.3 about the filled function is different from those definitions mentioned in [3–10, 13] and [14]. This novel definition is stronger than those mentioned in the above literature.

3. A novel filled function and its properties

In this section, we will introduce a filled function which satisfies the conditions of Definition 2.3.

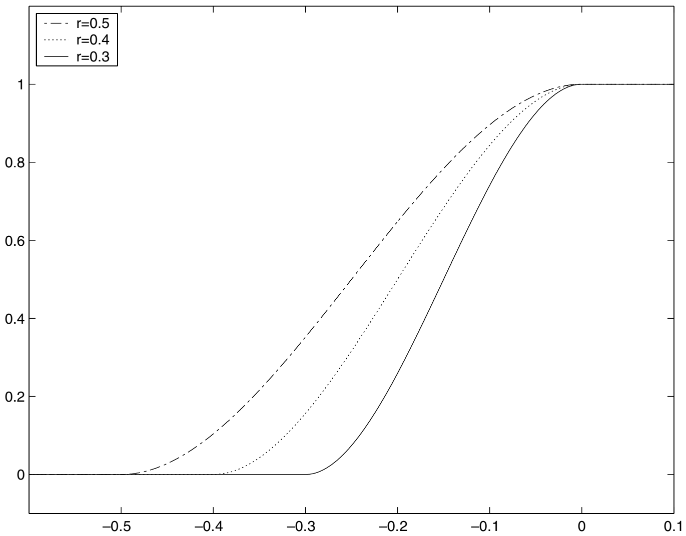

Figure 1 The behavior of $g_r(t)$ with $r = 0.5$, 0.4 and 0.3, respectively.

For any $r > 0$, let

$$g_r(t) = \begin{cases} 1, & > 0 \\ -\frac{2}{r^3}t^3 - \frac{3}{r^2}t^2 + 1, & -r < t \le 0 \\ 0, & t \le -r \end{cases} \tag{3.4}$$

and

$$f_r(t) = \begin{cases} t + r & t \le -r \\ \frac{r-2}{r^3}t^3 + \frac{r-3}{r^2}t^2 + 1, & -r < t \le 0 \cdot \\ 1 & t > 0 \end{cases} \tag{3.5}$$

The figure 1 and figure 2 are the behaviors of $g_r(t)$ and $f_r(t)$ with $r = 0.5, \sim 0.4$ and 0.3, respectively. We can easily verify that $g_r(t)$ and $f_r(t)$ are continuously differentiable on R. And

$$g_r'(t) = \begin{cases} 0, & t > 0 \\ -\frac{6}{r^3}t^2 - \frac{6}{r^2}t, & -r < t \le 0 \\ 0, & t \le -r \end{cases} \tag{3.6}$$

and

$$f_r'(t) = \begin{cases} 1 & t \le -r \\ \frac{3r-6}{r^3}t^2 + \frac{2r-6}{r^2}t, & -r < t \le 0 \cdot \\ 0 & t > 0 \end{cases} \tag{3.7}$$

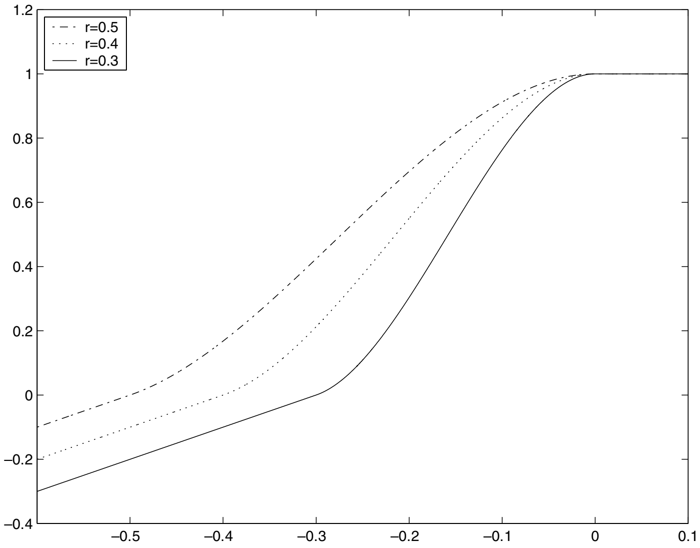

Figure 2 The behavior of $f_r(t)$ with $r = 0.5$, 0.4 and 0.3, respectively.

Let

$$H_{q,r,x^*}(x) = q\left(\exp\left(-\frac{\|x - x^*\|^2}{q}\right)g_r(f(x) - f(x^*)) + f_r(f(x) - f(x^*))\right), \quad (3.8)$$

where $r > 0, q > 0$ are parameters, x^* is the current local minimum.

Remark 3.1. Even though the function $H_{q,r,x^*}(x)$ includes an exponential term $\exp(-\frac{\|x-x^*\|^2}{q})$, but when q is very large, $\frac{\|x-x^*\|^2}{q}$ is very small, thus the exponential term $\exp(-\frac{\|x-x^*\|^2}{q})$ will not result any computational problem.

Theorem 3.1. *Suppose x^* is a local minimizer of $f(x)$, then x^* is a strictly local maximizer of $H_{q,r,x^*}(x)$ for any $r > 0, q > 0$.*

Proof: Let B^* be a G-basin of $f(x)$ at minimizer x^*, that is, for any $x \in B^*$, we have $f(x) \geq f(x^*)$. Therefore, $g_r(f(x) - f(x^*)) = g_r(f(x^*) - f(x^*)) = 1$ and $f_r(f(x) - f(x^*)) = f_r(f(x^*) - f(x^*)) = 1$ for any $r > 0$. And for any $x \neq x^*, q > 0$, $\exp(-\frac{\|x-x^*\|^2}{q}) < \exp(-\frac{\|x^*-x^*\|^2}{q})$. Thus,

$$H_{q,r,x^*}(x) < H_{q,r,x^*}(x^*)$$

for any $x \in B^*$ and $x \neq x^*$. Hence, x^* is a strictly local maximizer of $H_{q,r,x^*}(x)$ for any $r > 0, q > 0$. □

For any $x \in \Omega$, define

$$d(x) = x - x^*. \tag{3.9}$$

Theorem 3.2. *Suppose Assumption 1 is satisfied. If $x \in \Omega$ is such that $f(x) \geq f(x^*)$ and $x \neq x^*$, then x is not a stationary point of $H_{q,r,x^*}(x)$; and*

$$\nabla^T H_{q,r,x^*}(x) d(x) < 0$$

for any $r > 0, q > 0$.

Proof: By (3.8), we have

$$\nabla H_{q,r,x^*}(x) = \exp\left(-\frac{\|x - x^*\|^2}{q}\right)[-2(x - x^*)g_r(f(x) - f(x^*)) + qg_r'(f(x) - f(x^*))\nabla f(x)] + qf_r'(f(x) - f(x^*))\nabla f(x). \tag{3.10}$$

For any x satisfying $f(x) \geq f(x^*)$, for any $r > 0, q > 0$, we have

$$\nabla H_{q,r,x^*}(x) = -2\exp\left(-\frac{\|x - x^*\|^2}{q}\right)(x - x^*).$$

Thus, for any $x \in \Omega$ satisfying $f(x) \geq f(x^*)$ and $x \neq x^*$, we have $\nabla H_{q,r,x^*}(x) \neq 0$ for any $r > 0, q > 0$, i.e., x is not a stationary point of $H_{q,r,x^*}(x)$. By (3.9), we have

$$\nabla^T H_{q,r,x^*}(x) d(x) = -2\exp\left(-\frac{\|x - x^*\|^2}{q}\right)\|x - x^*\|^2 < 0.$$

□

Let

$$\beta_0 = \min_{y_1, y_2 \in F, y_1 \neq y_2} |y_1 - y_2|, \tag{3.11}$$

where F is decided by (2.1). By Assumption 2, we have that $\beta_0 > 0$.

Theorem 3.3. *Suppose Assumption 1–3 are satisfied. If x^* is not a global minimizer of problem* (1.1) *($L \neq \emptyset$), then for any $\bar{x} \in L$, for any $q > 0$, $\bar{x}$ is a local minimizer of $H_{q,r,x^*}(x)$ and satisfies*

$$H_{q,r,x^*}(\bar{x}) < H_{q,r,x^*}(x^*)$$
$$H_{q,r,x^*}(\bar{x}) < H_{q,r,x^*}(x) \quad \text{for any } x \in \partial\Omega$$

when the parameter r is chosen such that

$$r \leq \frac{\beta_0}{2}. \tag{3.12}$$

Obviously, $\bar{x}$ is a stationary point of $H_{q,r,x^*}(x)$.

Furthermore, for any x satisfying $f(x) < f(x^*) - 2r$, x is a local minimizer of $H_{q,r,x^*}(x)$ and satisfies

$$H_{q,r,x^*}(x) < H_{q,r,x^*}(x^*)$$
$$H_{q,r,x^*}(x) < H_{q,r,x^*}(y) \quad \textit{for any } y \in \partial\Omega,$$

which must be a stationary point of $H_{q,r,x^*}(x)$.

Proof: Since $\beta_0 = \min_{y_1,y_2 \in F, y_1 \neq y_2} |y_1 - y_2|$ and since $L \neq \emptyset$, then for any $\bar{x} \in L$, we have that $\beta_0 \leq f(x^*) - f(\bar{x})$. Thus, we have $f(\bar{x}) - f(x^*) \leq -\beta_0 \leq -2r < -r$ when $r \leq \frac{\beta_0}{2}$. By the continuity of $f(x)$, there exists a neighborhood $N(\bar{x}, \delta_0)$ of $\bar{x}$, such that $f(x) - f(x^*) < -r$ for any $x \in N(\bar{x}, \delta_0)$, where $N(\bar{x}, \delta_0) = \{x \in R^n \mid \|x - \bar{x}\| < \delta_0\}$. Thus, we have that $g_r(f(x) - f(x^*)) = 0$ for any $x \in N(\bar{x}, \delta_0)$. Since $\bar{x}$ is a local minimizer of $f(x)$, then there exists a positive number $\delta_1 > 0$, such that $f(\bar{x}) \leq f(x)$ for any $x \in N(\bar{x}, \delta_1)$. Let $\delta = \min\{\delta_0, \delta_1\}$. For any $x \in N(\bar{x}, \delta)$, we have that

$$H_{q,r,x^*}(x) = q(0 + f(x) - f(x^*) + r) \geq q(f(\bar{x}) - f(x^*) + r) = H_{q,r,x^*}(\bar{x}).$$

Thus, $\bar{x}$ is a local minimizer of $H_{q,r,x^*}(x)$ and satisfies

$$H_{q,r,x^*}(\bar{x}) = q(f(\bar{x}) - f(x^*) + r) < 0 < H_{q,r,x^*}(x^*) = q(1 + 1) = 2q$$
$$\begin{aligned} H_{q,r,x^*}(\bar{x}) &= q(f(\bar{x}) - f(x^*) + r) < 0 < q < q\left(\exp\left(-\frac{\|x - x^*\|^2}{q}\right) + 1\right) \\ &= H_{q,r,x^*}(x) \quad \text{for any } x \in \partial\Omega. \end{aligned}$$

Similarly, we can prove that for any point x satisfying $f(x) \leq f(x^*) - 2r$, x is a local minimizer of $H_{q,r,x^*}(x)$ and satisfies

$$H_{q,r,x^*}(x) < H_{q,r,x^*}(x^*)$$
$$H_{q,r,x^*}(x) < H_{q,r,x^*}(y) \text{ for any } y \in \partial\Omega,$$

which must be a stationary point of $H_{q,r,x^*}(x)$. □

Theorem 3.4. *For any $x_1, x_2 \in \Omega$ satisfying $f(x_1) \geq f(x^*)$, $f(x_2) \geq f(x^*)$, $\|x_2 - x^*\| > (\geq)\|x_1 - x^*\|$ if and only if $H_{q,r,x^*}(x_2) < (\leq) H_{q,r,x^*}(x_1)$ for any $r > 0, q > 0$.*

Proof: For any $x_1, x_2 \in \Omega$ satisfying $f(x_1) \geq f(x^*)$, $f(x_2) \geq f(x^*)$, we have that

$$H_{q,r,x^*}(x_2) = q\left(\exp\left(-\frac{\|x_2 - x^*\|^2}{q}\right) + 1\right),$$
$$H_{q,r,x^*}(x_1) = q\left(\exp\left(-\frac{\|x_1 - x^*\|^2}{q}\right) + 1\right).$$

Thus, for any $r > 0, q > 0$, $\|x_2 - x^*\| > (\geq)\|x_1 - x^*\|$ if and only if $H_{q,r,x^*}(x_2) < (\leq)$ $H_{q,r,x^*}(x_1)$. □

Remark 3.2. By Theorem 3.1–3.4, we know that $H_{q,r,x^*}(x)$ is a filled function according to our novel Definition 2.3 when r satisfies condition (3.12). Note that even if the function $H_{q,r,x^*}(x)$ includes an exponential term $\exp(-\frac{\|x-x^*\|^2}{q})$, but $\exp(-\frac{\|x-x^*\|^2}{q})$ will not bring any computational problem. In fact, since the parameter q does not need to satisfy any condition, thus we can take q is very large. The very large q is just used to make $\exp(-\frac{\|x-x^*\|^2}{q})$ contribute a little to the function $H_{q,r,x^*}(x)$ and improve the decreasing rapidity of $H_{q,r,x^*}(x)$ at x when x satisfies $f(x) \geq f(x^*)$. Thus we can take q is very large, then $\exp(-\frac{\|x-x^*\|^2}{q})$ will not bring any computational problem. The term $f_r(f(x) - f(x^*))$ takes advantage of the superiority of those points which values are less than $f(x^*)$ since the less of $f(x)$ is, the less of $f_r(f(x) - f(x^*))$ is when $f(x) < f(x^*)$.

Moreover, we can verify that $g(x) = \exp(-\|x - x^*\|^2)g_r(f(x) - f(x^*))$ and $h(x) = q\exp(-\frac{\|x-x^*\|^2}{q})g_r(f(x) - f(x^*))$ are also filled functions satisfying Definition 2.3. But from the above discussions, we can easily know that the computational efficiency of function $H_{q,r,x^*}(x)$ is better than those of functions $g(x)$ and $h(x)$. Thus, using the filled function $H_{q,r,x^*}(x)$, we can obtain a filled function method, referred to as *Algorithm FFM*.

Algorithm FFM:

Step 0. Choose sufficiently small positive numbers $\mu > 0$ and $\varepsilon > 0 (\varepsilon < \mu)$ as the tolerance parameters for terminating the minimization process of problem (1.1). Choose a sufficiently large number $M > 0$ as the tolerance number q, an integer $k_0 \geq 2n$, where n is the number of variables, a set of unit directions $e_i, i = 1, \ldots, k_0$ such that $e_i, i = 1, \ldots, k_0$ almost uniformly distribute over the unit sphere $B = \{x \in R^n \mid \|x\| = 1\}$, an initial point $x_1^0 \in \Omega$, and initial values $q_0 \geq 1$ and $r_0 \leq 1$ for the parameters q and r, respectively. Set $k := 1$.

Step 1. Let x_k^* be the local minimizer of problem (1.1) starting from the initial point x_k^0. Take a positive number $\delta_0 > 0$, let $\delta = \delta_0, i = 1$.

Step 2. Let $\bar{x}_k^* = x_k^* + \delta e_i$. If $f(\bar{x}_k^*) < f(x_k^*)$, then set $x_{k+1}^0 := \bar{x}_k^*$, $k := k + 1$ and go to *Step 1*; otherwise, go to *Step 3*.

Step 3. Let

$$H_{q,r,x_k^*}(x) = q\left(\exp\left(-\frac{\|x - x_k^*\|^2}{q}\right)g_r(f(x) - f(x_k^*)) + f_r(f(x) - f(x_k^*))\right),$$

where $g_r(t)$ and $f_r(t)$ are decided by (3.4) and (3.5), respectively. Solve the problem:

$$\begin{array}{ll} \min & H_{q,r,x_k^*}(x) \\ & x \in \Omega \end{array} \tag{3.13}$$

by a local search method starting from the initial point $\bar{x}_k^*$.

If the minimizer of problem (3.13) attained at the boundary of Ω, go to *Step 4*, else if one of the following conditions holds at some point $y_k^* \in \Omega$:

(1) $(y_k^* - x_k^*)^T \nabla H_{q,r,x_k^*}(y_k^*) \geq 0$;
(2) $f(y_k^*) < f(x_k^*)$;
(3) $\|\nabla H_{q,r,x_k^*}(y_k^*)\| \leq \varepsilon$;

then, set $x_{k+1}^0 := y_k^*$, $k := k+1$ and go to *Step 1*; otherwise, continue the minimization of $H_{q,r,x^*}(x)$ on Ω.

Step 4. If $q < M$, then increase q(such as, let $q := 10q$), go to *Step 2*; otherwise, let $q := q_0$, go to *Step 5*.

Step 5. If $\delta > \mu$, then decrease δ(such as, let $\delta := \frac{\delta}{2}$) and let $q = q_0$, go to *Step 2*; otherwise, let $q := q_0$, $\delta := \delta_0$, go to *Step 6*.

Step 6. If $i < k_0$, then let $i := i+1$, go to *Step 2*; otherwise, go to *Step 7*.

Step 7. If $r > \mu$, then decrease r(such as, let $r := \frac{r}{10}$), and set $i := 1$, $q := q_0$, $\delta := \delta_0$, then go to *Step 2*; otherwise, stop and x_k^* is a global minimizer of problem (1.1).

In this section, we have given a new definition of a filled function and a new filled function satisfying the given definition and a new filled function method, referred to as Algorithm FFM. By the given Algorithm FFM, we can obtain a global minimizer or an approximate global minimizer of original problem. But the local minimizer of problem (3.13) obtained in *Step 3* is very easily on the boundary of Ω, in which case, we have to change the parameters q, r or change the starting point $\bar{x}_k^*$. Thus, the k_0 maybe have to be chosen very large. In order to improve numerical computational efficiency, in the next section, we will introduce another modified function, which possesses almost all properties set in Definition 2.3 on one hand, and demonstrate computational superiority on the other, which local minimizer on Ω must be in the interior of Ω. This function is referred to as the quasi-filled function.

4. Quasi-filled function and its properties

By Assumption 3, we know that there exist a point x_1^0 and a positive number $f_0 > 1$, such that $f(x) \geq f(x_1^0) + f_0$ for any $x \in \partial\Omega$. If x^* is a local minimizer of $f(x)$ on Ω starting from x_1^0, then $f(x^*) \leq f(x_1^0)$. Thus, $f(x) \geq f(x^*) + f_0$ for any $x \in \partial\Omega$.

For any $r > 0$ and given $c > 0$, let

$$g_{r,c}(t) = \begin{cases} c, & t \geq 0 \\ -\dfrac{2c}{r^3}t^3 - \dfrac{3c}{r^2}t^2 + c, & -r < t \leq 0 \\ 0, & t \leq -r \end{cases} \tag{4.14}$$

$$h_{r,c}(t) = \begin{cases} t+r & t \leq -r \\ \dfrac{r-2}{r^3}t^3 + \dfrac{r-3}{r^2}t^2 + 1, & -r < t \leq 0 \\ 1 & 0 < t \leq 1 \\ -\dfrac{4c-2}{r^3}t^3 + \dfrac{(6c-3)(r+2)}{r^3}t^2 & 1 \leq t \leq 1+r \\ \quad -\dfrac{(6c-3)(2+2r)}{r^3}t + \dfrac{4c-2+(6c-3)r}{r^3} + 1 & \\ 2c & t > 1+r \end{cases} \tag{4.15}$$

The figure 1 is the behavior of $g_{r,c}(t)$ with $c = 1$ and $r = 0.5, 0.4, 0.3$, respectively. The figure 3 is the behavior of $h_{r,c}(t)$ with $c = 1$ and $r = 0.5, 0.4, 0.3$, respectively.

Obviously, $g_{r,c}(t)$ and $h_{r,c}(t)$ are continuously differentiable on R. And

$$g'_{r,c}(t) = \begin{cases} 0, & t \geq 0 \\ -\frac{6c}{r^3}t^2 - \frac{6c}{r^2}t, & -r < t \leq 0 \\ 0, & t \leq -r \end{cases}$$

$$h'_{r,c}(t) = \begin{cases} 1 & t \leq -r \\ \frac{3r-6}{r^3}t^2 + \frac{2r-6}{r^2}t, & -r < t \leq 0 \\ 0 & 0 < t \leq 1 \\ -\frac{12c-6}{r^3}t^2 + \frac{(12c-6)(r+2)}{r^3}t & 1 \leq t \leq 1+r \\ \quad -\frac{(6c-3)(2+2r)}{r^3} & \\ 0 & t > 1+r \end{cases}.$$

Let

$$F_{q,r,c,x^*}(x) = q\left(\exp\left(-\frac{\|x-x^*\|^2}{q}\right) g_{r,c}(f(x)-f(x^*)) + h_{r,c}(f(x)-f(x^*))\right). \tag{4.16}$$

Consider the following programming problem:

$$\begin{aligned} \min \quad & F_{q,r,c,x^*}(x) \\ & x \in \Omega. \end{aligned} \tag{4.17}$$

Then, we have the following results:

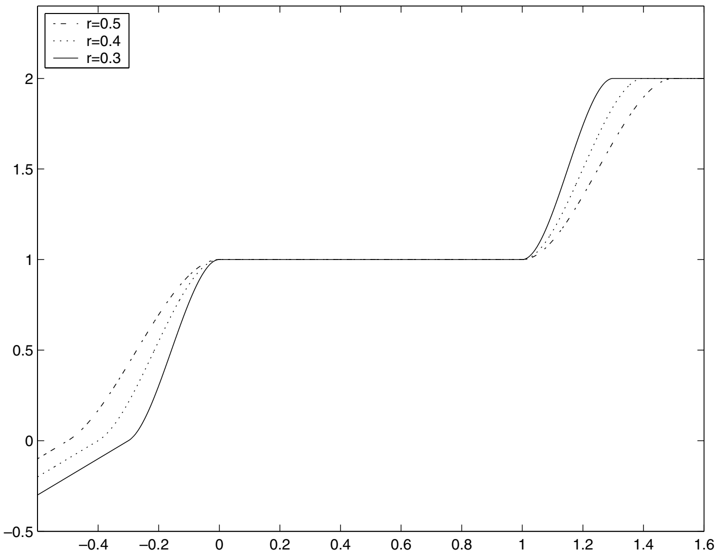

Figure 3 The behavior of $h_{r,c}(t)$ with with $c = 1$ and $r = 0.5, 0.4, 0.3$, respectively.

Theorem 4.1. *If x^* is a local minimizer of original problem* (1.1), *then for any $r > 0, q > 0, c > 0$, x^* is a strictly local maximizer of function $F_{q,r,c,x^*}(x)$ on Ω.*

Proof: Since x^* is a local minimizer of continuous function $f(x)$, then there exists a $\delta > 0$, such that $0 \leq f(x) - f(x^*) \leq 1$ for any $x \in N(x^*, \delta)$. Therefore, by (4.16), we have $\exp(-\frac{\|x-x^*\|^2}{q})g_{r,c}(f(x) - f(x^*)) < \exp(-\frac{\|x^*-x^*\|^2}{q})g_{r,c}(f(x^*) - f(x^*))$ and $h_{r,c}(f(x) - f(x^*)) = h_{r,c}(f(x^*) - f(x^*))$ for any $x \in N(x^*, \delta) \setminus \{x^*\}$. Thus, $F_{q,r,c,x^*}(x) < F_{q,r,c,x^*}(x^*)$ for any $x \in N(x^*, \delta) \setminus \{x^*\}$. Thus x^* is a strictly local maximizer of function $F_{q,r,c,x^*}(x)$. □

Theorem 4.2. *Suppose Assumption 1 is satisfied. For any $r > 0$, if $x \in \Omega$ and $x \neq x^*$ satisfies $0 \leq f(x) - f(x^*) \leq 1$ or $f(x) - f(x^*) \geq 1 + r$, then x is not a stationary point of $F_{q,r,c,x^*}(x)$. Otherwise, if x is a stationary point of f(x), then x is not a stationary point of $F_{q,r,c,x^*}(x)$. And $\nabla^T F(q, r, c, x^*)(x)d(x) < 0$ for any x satisfying the above conditions.*

Proof: By (4.16), we have

$$\nabla F_{q,r,c,x^*}(x) = \exp\left(-\frac{\|x - x^*\|^2}{q}\right)[-2(x - x^*)g_{r,c}(f(x) - f(x^*)) + qg'_{r,c}(f(x) - f(x^*))\nabla f(x)] + qh'_{r,c}(f(x) - f(x^*))\nabla f(x).$$

Thus for any x satisfying $f(x) \geq f(x^*)$, we can rewrite the above to get

$$\nabla F_{q,r,c,x^*}(x) = -2c\exp\left(-\frac{\|x - x^*\|^2}{q}\right)(x - x^*) + qh'_{r,c}(f(x) - f(x^*))\nabla f(x). \tag{4.18}$$

By (4.18), if $x \neq x^*$ satisfies $0 \leq f(x) - f(x^*) \leq 1$ or $f(x) - f(x^*) \geq 1 + r$, we can conclude

$$\nabla F_{q,r,c,x^*}(x) = -2c\exp\left(-\frac{\|x - x^*\|^2}{q}\right)(x - x^*) \neq 0.$$

Otherwise, if x satisfies $1 < f(x) - f(x^*) < 1 + r$ and $\nabla f(x) = 0$, we also conclude

$$\nabla F_{q,r,c,x^*}(x) = -2c\exp\left(-\frac{\|x - x^*\|^2}{q}\right)(x - x^*) \neq 0.$$

Hence, x is not a stationary point of function $F_{q,r,c,x^*}(x)$. And we have that $\nabla^T F_{q,r,c,x^*}(x)d(x) < 0$ for any x satisfies the above conditions. □

Theorem 4.3 *Suppose Assumption* 1−3 *are satisfied. If x^* is not a global minimizer of problem* (1.1), *i.e., $L \neq \emptyset$, then for any $\bar{x} \in L$, when $r \leq \frac{\beta_0}{2}$, $\bar{x}$ is a local minimizer of $F_{q,r,c,x^*}(x)$ and satisfies*

$$F_{q,r,c,x^*}(\bar{x}) < F_{q,r,c,x^*}(x^*),\ F_{q,r,c,x^*}(\bar{x}) < F_{q,r,c,x^*}(x) \text{ for any } x \in \partial\Omega,$$

where β_0 is decided by (3.11). *Obviously, $\bar{x}$ is a stationary point of $F_{q,r,c,x^*}(x)$.*

Furthermore, for any x satisfying $f(x) < f(x^*) - 2r$, x is a local minimizer of $F_{q,r,c,x^*}(x)$ and satisfies

$$F_{q,r,c,x^*}(x) < F_{q,r,c,x^*}(x^*)$$
$$F_{q,r,c,x^*}(x) < F_{q,r,c,x^*}(y) \textit{ for any } y \in \partial\Omega,$$

which must be a stationary point of $F_{q,r,c,x^*}(x)$.

Proof: Since $\beta_0 = \min_{y_1,y_2 \in F, y_1 \neq y_2} |y_1 - y_2|$ and since $L \neq \emptyset$, then for any $\bar{x} \in L$, we have that $f(\bar{x}) - f(x^*) \leq -\beta_0 \leq -2r < -r$ when $r \leq \frac{\beta_0}{2}$. By the continuity of $f(x)$, there exists a neighborhood $N(\bar{x}, \delta_0)$ of $\bar{x}$, such that $f(x) - f(x^*) < -r$ for any $x \in N(\bar{x}, \delta_0)$, where $N(\bar{x}, \delta_0) = \{x \in R^n \mid \|x - \bar{x}\| < \delta_0\}$. Thus, we have that $g_{r,c}(f(x) - f(x^*)) = 0$ for any $x \in N(\bar{x}, \delta_0)$. Since $\bar{x}$ is a local minimizer of $f(x)$, there exists a positive number $\delta_1 > 0$, such that $f(\bar{x}) \leq f(x)$ for any $x \in N(\bar{x}, \delta_1)$. Let $\delta = \min\{\delta_0, \delta_1\}$, then we have that

$$F_{q,r,c,x^*}(x) = q(0 + f(x) - f(x^*) + r) \geq q(f(\bar{x}) - f(x^*) + r) = F_{q,r,c,x^*}(\bar{x})$$

for any $x \in N(\bar{x}, \delta)$. Thus, $\bar{x}$ is a local minimizer of $F_{q,r,c,x^*}(x)$ and satisfies

$$\begin{aligned} F_{q,r,c,x^*}(\bar{x}) &= q(f(\bar{x}) - f(x^*) + r) < 0 < F_{q,r,c,x^*}(x^*) = q(c+1) \\ F_{q,r,c,x^*}(\bar{x}) &= q(f(\bar{x}) - f(x^*) + r) < 0 < q < q\left(c \exp\left(-\frac{\|x - x^*\|^2}{q}\right) + 1\right) \\ &< F_{q,r,c,x^*}(x) \text{ for any } x \in \partial\Omega, \\ &\text{since } f(x) - f(x^*) > 1 \text{ for any } x \in \partial\Omega \text{ by Assumption 3.} \end{aligned}$$

Similarly, we can prove that for any $r > 0$, for any point x satisfying $f(x) \leq f(x^*) - 2r$, x is a local minimizer of $F_{q,r,c,x^*}(x)$ and satisfies

$$F_{q,r,c,x^*}(x) < F_{q,r,c,x^*}(x^*)$$
$$F_{q,r,c,x^*}(x) < F_{q,r,c,x^*}(y) \text{ for any } y \in \partial\Omega,$$

which must be a stationary point of $F_{q,r,c,x^*}(x)$. □

Note that the second part of Theorem 4.3 illustrates that function $F_{q,r,c,x^*}(x)$ has many local minimizers if x^* is not a global minimizer of problem (1.1).

Theorem 4.4. *For any x_0 satisfying $f(x_0) - f(x^*) \leq 1$, the local minimizer $\bar{x}$ of problem* (4.17) *starting from x_0 is in the interior of Ω when r and c satisfy the following conditions, respectively.*

$$r \leq f_0 - 1 \quad \textit{and} \quad c \geq 1, \tag{4.19}$$

where $f_0 > 1$ is decided by Assumption 3.

Proof: For any $x \in \partial\Omega$, since $f(x) \geq f(x^*) + f_0 \geq f(x^*) + 1 + r$ when $r \leq f_0 - 1$, then we have

$$\begin{aligned} F_{q,r,c,x^*}(x) &= q\left(\exp\left(-\frac{\|x - x^*\|^2}{q}\right) g_{r,c}(f(x) - f(x^*)) + h_{r,c}(f(x) - f(x^*))\right) \\ &> q h_{r,c}(1 + r) \\ &= 2qc. \end{aligned}$$

For any x_0 satisfying $f(x_0) \leq f(x^*) + 1$,

$$\begin{aligned} F_{q,r,c,x^*}(x_0) &= q\left(\exp\left(-\frac{\|x_0 - x^*\|^2}{q}\right) g_{r,c}(f(x_0) - f(x^*)) + h_{r,c}(f(x_0) - f(x^*))\right) \\ &\leq q(c + h_{r,c}(1)) \\ &\leq q(c + 1). \end{aligned}$$

Thus when $c \geq 1$, we have $q(c + 1) \leq 2qc$. Therefore, any local minimizer of problem (4.17) starting from x_0 satisfying $f(x_0) \leq f(x^*) + 1$ can not be obtained on the boundary of Ω when r and c satisfy $r \leq f_0 - 1$ and $c \geq 1$, respectively. □

By the given quasi-filled function $F_{q,r,c,x^*}(x)$, we can propose a method to obtain a global minimizer of original problem (1.1) via implementing local search schemes. This method is referred to as Algorithm QFFM.

Algorithm QFFM:

Step 0. Choose a sufficiently small positive number $\mu > 0$ as the tolerance parameter for terminating the minimization process of problem (1.1). Choose a sufficiently large positive number $M > 0$ as the tolerance numbers q and c (in the following examples, we take $\mu = 10^{-10}$ and $M = 10^{10}$). Choose an integer $k_0 \geq 2n$ and unit directions $e_i, i = 1, \ldots, k_0$, where n is the number of variables (in the following examples, we take $k_0 = 2n$ and e_i as the positive and negative coordinate directions). Pick an initial point $x_1^0 \in \Omega$ and initial parameters: $r_0 > 0, c_0 > 0, q_0 > 0$ (in the following examples, we take $r_0 = 1, c_0 = 1$ and $q_0 = 10^5$). Take a positive number $\delta_0 > 0$ (in the following examples, we take $\delta_0 = \frac{1}{2}$). Let $k := 1$.

Step 1. Let x_k^* be the local minimizer of problem (1.1) starting from the initial point x_k^0. If $f(x_k^*) \geq f(x_{k-1}^*)$(where $x_0^* = x_1^0$), then go to *Step 4*; otherwise, Let $q := q_0, c := c_0, r := r_0, \delta := \delta_0$ and $i := 1$, go to *Step 2*;

Step 2. Let $\bar{x}_k^* = x_k^* + \delta e_i$. If $f(\bar{x}_k^*) < f(x_k^*)$, then set $x_{k+1}^0 := \bar{x}_k^*, k := k + 1$ and go to *Step 1*; if $f(x_k^*) \leq f(\bar{x}_k^*) \leq f(x_k^*) + 1$, then go to *Step 3*; otherwise, let $\delta := \frac{\delta}{2}$, go to *Step 2*.

Step 3. Let

$$F_{q,r,c,x_k^*}(x) = q\left[\exp\left(-\frac{\|x - x_k^*\|^2}{q}\right) g_{r,c}(f(x) - f(x_k^*)) + h_{r,c}(f(x) - f(x_k^*))\right],$$

where $g_{r,c}(t)$ and $h_{r,c}(t)$ are decided by (4.14) and (4.15), respectively. Solve the problem:

$$\begin{array}{ll} \min & F_{q,r,c,x_k^*}(x) \\ & x \in \Omega \end{array} \tag{4.20}$$

by a local search method starting from the initial point $\bar{x}_k^*$. Let $\bar{x}_{q,r,c,x_k^*}$ be the local minimizer obtained. Then, set $x_{k+1}^0 := \bar{x}_{q,r,c,x_k^*}$, $k := k+1$ and go to *Step 1.*

Step 4. If $q < M$, then increase q (in the following examples, we let $q := 10q$), then goto *Step 3*; otherwise, go to *Step 5*.

Step 5. If $c < M$, then increase c (in the following examples, we let $c := 10c$), and let $q = q_0$, go to *Step 3*; otherwise, go to *Step 6*.

Step 6. If $i < k_0$, then let $i := i+1$, $q := q_0$, $c := c_0$, $\delta := \delta_0$, go to *Step 2*; otherwise, go to *Step 7*.

Step 7. If $r > \mu$, then decrease r (in the following examples, we let $r := \frac{r}{10}$, let $i := 1, q := q_0, c := c_0, \delta := \delta_0$ go to *Step 2*; otherwise, stop and x_k^* is a global minimizer of problem (1.1).

5. Numerical results

In this section, we will use the Algorithm QFFM to solve a set of well-recognized test functions, where the optimization subroutine within the optimization Toolbox in Matlab 6.1 is used as the local search scheme to obtain local minimizers of problem (1.1) and problem (4.20).

In the following examples, we take $M = 10^{10}$, $\mu = 10^{-10}$, $k_0 = 2n$ and the direction $e_i, i = 1, \ldots, k_0$ are the positive and negative coordinate directions in *Step 0*, and take the initial q, c, r, δ as $q_0 = 10^5, c_0 = 1, \delta_0 = \frac{1}{2}$ and $r_0 = 1$ in *Step 1*, respectively and take $q := 10q, c := 10c, r := r/10$ in *Step 4*, *Step 5* and *Step 7*, respectively. The numerical results of each example obtained are presented in the respective tables, where the following notations are adopted.

k—the number of local minima obtained by Algorithm QFFM that satisfy $f(x_k^*) < f(x_{k-1}^*)$.

x_k^0—the initial point for the kth local minimization process of problem (1.1), $x_{k+1}^0 := \bar{x}_{q,r,c,x_k^*}$.

x_k^*—the local minimizer of $f(x)$ starting from x_k^0.

$e_i, i = 1, \ldots, n$—$e_i \in R^n$ and the ith element is 1, the others are 0.

$e_i, i = n+1, \ldots, 2n$—$e_i \in R^n$ and the $(i-n)$th element is -1, the others is 0.

$\bar{x}_{q,r,c,x_k^*}$—The local minimizer of problem (4.20) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from $\bar{x}_k^*$ (all the $\bar{x}_{q,r,c,x_k^*}$ given in these tables satisfy $f(x_{k+1}^*) < f(x_k^*)$, where x_{k+1}^* is a local minimizer of function $f(x)$ starting from $\bar{x}_{q,r,c,x_k^*}$).

Example 5.1 (Goldstein-Price (G-P) ($n = 2$) [2])

$$\begin{aligned} \min_{x,y} \quad f_G(x,y) = {} & [1 + (x+y+1)^2(19 - 14x + 3x^2 - 14y + 6xy + 3y^2)] \\ & \times [30 + (2x - 3y)^2(18 - 32x + 12x^2 + 48y - 36xy + 27y^2)] \end{aligned} \tag{5.1}$$

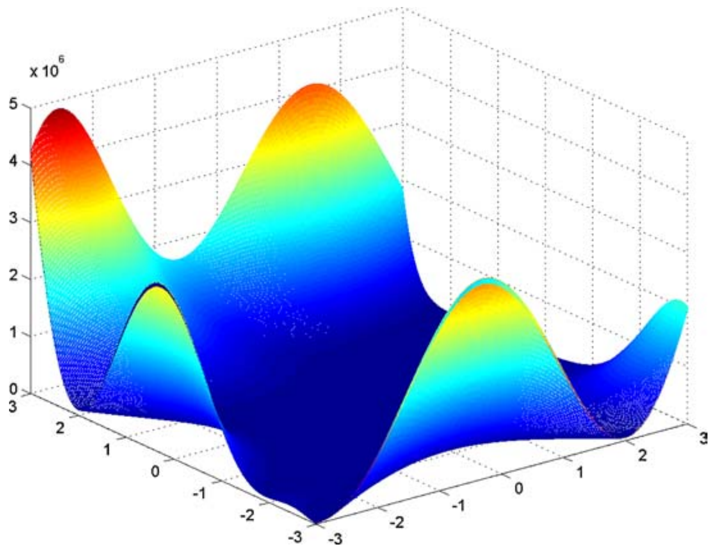

Figure 4 The behavior of Goldstein-Price.

$$\text{s.t.} \quad -3 \le x \le 3, -3 \le y \le 3.$$

From figure 4, we see that there are many local minima of this problem G-P. We take the initial point $x_1^0 = (1, 1)$. The first local minimizer of problem (5.1) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from $x_1^0 = (1, 1)$ is $x_1^* = (1.8000, 0.2000)$ with the local minimal value of $f(x_1^*) = 84.0000$.

Table 1 gives the numerical results obtained by Algorithm QFFM for problem G-P.

From Table 1, we see that a global minimizer of problem (5.1) obtained by Algorithm QFFM is:

$$x^* = (-0.0000 - 1.0000)$$

Table 1. Results for G-P by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, c, r	$\bar{x}_{q,r,c,x_k^*}$
1	(1,1)	(1.8000, 0.2000)	84.0000		
				$1/2^7, e_1, 10^5, 1, 1$	(1.8155, 0.2004)
2	(1.8155, 0.2004)	(−0.0000 −1.0000)	3.0000		
				for any $c \le 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \le 10^{10}$ and $r \ge 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \ge 3.0000$		

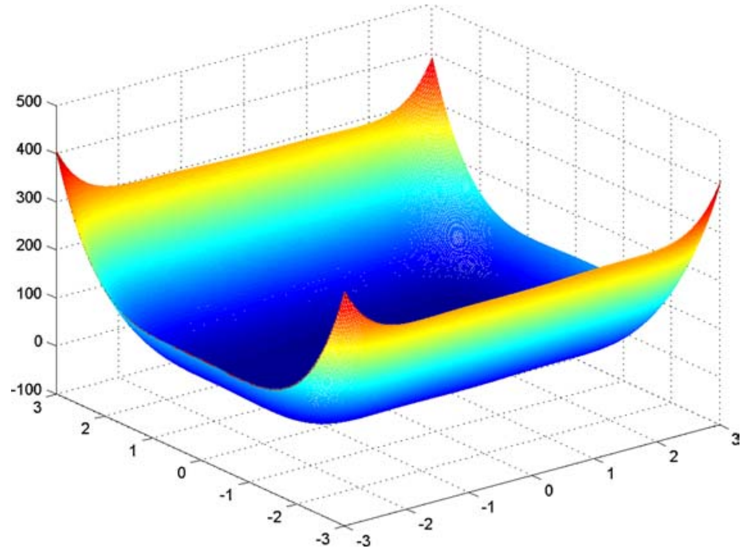

Figure 5 The behavior of Six-Hump Camel-back.

with global optimal value $f(x^*) = 3.0000$.

Example 5.2 (Six-Hump Camel-back ($n = 2$) [1])

$$\min \ f_S(x) = 4x_1^2 - 2.1x_1^4 + x_1^6/3 - x_1x_2 - 4x_2^2 + 4x_2^4 \qquad (5.2)$$
$$-3 \leq x_i \leq 3, i = 1, 2.$$

From figure 5, we see that there are many local minima of this problem. We take the initial point $x_1^0 = (0, 0)$. The first local minimizer of problem (5.2) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from $x_1^0 = (0, 0)$ is $x_1^* = (0, 0)$ with the local minimal value $f(x_1^*) = 0$.

Table 2 gives the numerical results obtained by Algorithm QFFM for Six-Hump Camel-back.

Table 2. Results for Six-Hump Camel-back by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, c, r	$\bar{x}_{q,r,c,x_k^*}$
1	(0 0)	(0 0)	0		
				$1/2, e_1, 10^5, 1, 1$	$(0.5412, -0.0000)$
2	(0.0313, 0)	(0.0898, 0.7127)	-1.0316		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq -1.0316$		

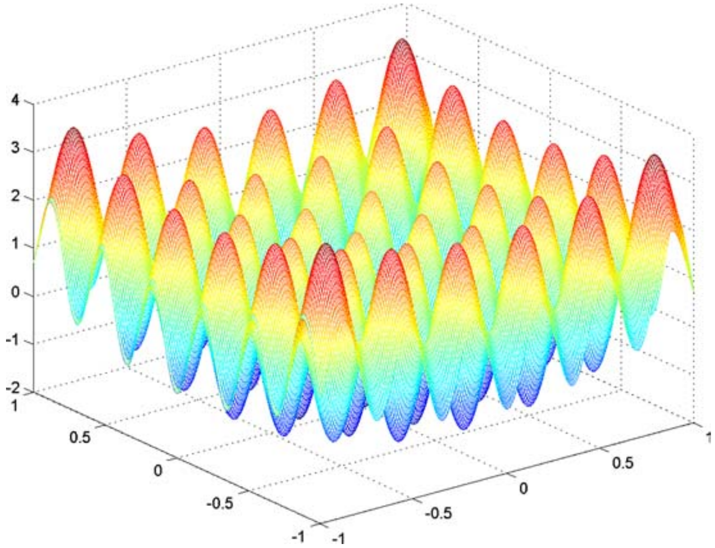

Figure 6 The behavior of Rastrigin.

From Table 2, we see that a global minimizer of problem (5.2) obtained by Algorithm QFFM is:

$$x^* = (0.0898, 0.7127)$$

with global optimal value $f(x^*) = -1.0316$.

Example 5.3 Rastrigin ($n = 2$) [11]

$$\begin{aligned} \min \quad & f_R(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2) \\ \text{s.t.} \quad & -2 \leq x_i \leq 2, i = 1, 2. \end{aligned} \tag{5.3}$$

From figure 6, we see that there are many local minima of this problem. We take the initial point $x_1^0 = (1, 1)$. The first local minimizr of problem (5.3) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from $x_1^0 = (1, 1)$ is $x_1^* = (1.0408, 1.0408)$ with the local minimal value $f(x_1^*) = 0.1798$.

Table 3 gives the numerical results obtained by Algorithm QFFM for problem (5.3). From Table 3, we see that a global minimizer of problem (5.3) obtained by Algorithm QFFM is:

$$x^* = 1.0 \times 10^{-6}(0.1434, -0.3560)$$

with the global minimal value $f(x^*) = -2.0000$.

Table 3. Results for Rastrigin by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, c, r	$\bar{x}_{q,r,c,x_k^*}$
1	(1 1)	(1.0408, 1.0408)	0.1798		
				$1/2^4, e_1, 10^5, 1, 1$	(1.1241, 1.0408)
2	(1.1241, 1.0408)	(0.3469, 1.0408)	−0.7890		
				$1/2^4, e_1, 10^5, 1, 1$	(0.7132, 1.0407)
3	(0.7132, 1.0407)	(−0.0000, 1.0408)	−0.9101		
				$1/2^4, e_1, 10^5, 10^4, 1$	(0.7515 , 1.0405)
4	(0.7515, 1.0405)	(0.0000, 0.6938)	−1.5156		
				$1/2^4, e_1, 10^5, 10^4, 1$	(0.7512, 0.6936)
5	(0.7512, 0.6936)	$1.0 \times 10^{-6}(0.1434, -0.3560)$	−2.0000		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq -2.0000$		

Example 5.4 (two-dimensions Shubert III function ($n = 2$) (Shubert, 1972) [3])

$$\begin{aligned} \min\ f_S(x) = &\left(\sum_{i=1}^{5} i\cos[(i+1)x_1 + i]\right)\left(\sum_{i=1}^{5} i\cos[(i+1)x_2 + i]\right) \\ &+ [(x_1 + 1.42513)^2 + (x_2 + 0.80032)^2] \\ &\text{s.t.} -10 \leq x_i \leq 10, i = 1, 2. \end{aligned} \tag{5.4}$$

From figure 7, we see that there are many local minima of this problem (there are about 760 minimums). We take the initial point $x_1^0 = (1, 1)$. The first local minimizer of problem (5.4) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from $x_1^0 = (1, 1)$ is $x_1^* = (-0.8017, 2.7818)$ with the local minimal value $f(x_1^*) = -25.0600$.

Table 4 gives the numerical results obtained by Algorithm QFFM for the two-dimensions Shubert III function. From Table 4, we see that a global minimizer of problem (5.4) obtained by Algorithm QFFM is:

$$x^* = (-1.4251, -0.8003)$$

with the global minimal value $f(x^*) = -186.7309$.

Example 5.5 (n-dimensions Sine-square II function (Levy and Montalvo, 1985) [4])

$$\begin{aligned} \min\ f_L(x) = &\frac{\pi}{n}\{10\sin^2(\pi y_1) + (y_n - 1)^2 \\ &+ \sum_{i=1}^{n-1}[(y_i - 1)^2(1 + 10 sin^2(\pi y_{i+1})]\}, \end{aligned} \tag{5.5}$$

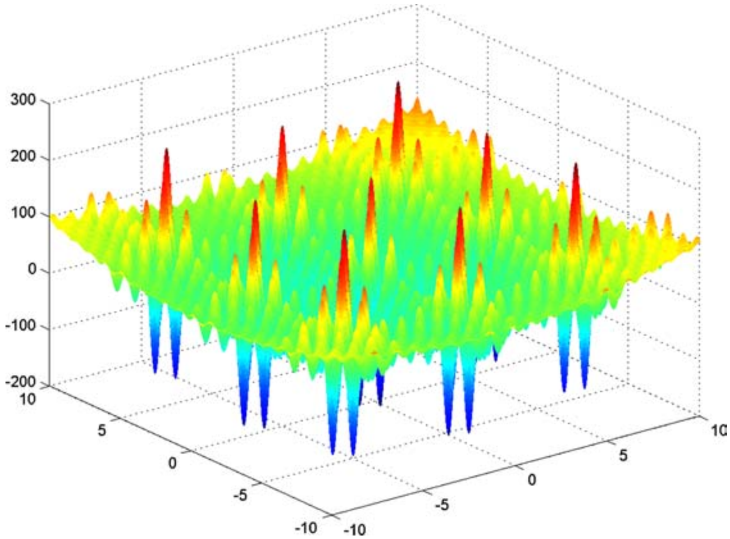

Figure 7 The behavior of two-dimension Shubert III function.

$$\text{s.t. } y_i = 1 + (x_i - 1)/4$$
$$-10 \leq x_i \leq 10, i = 1, 2, \ldots, n.$$

From figure 8, we see that there are many local minima of this problem ($n = 2$).

When $n = 2$, we take the initial point $x_1^0 = (0, 0)$, the first local minimizer of problem (5.5) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from x_1^0 is $x_1^* = (4.9599, 1.0000)$ with the local minimal value $f(x_1^*) = 1.5550$. Table 5 gives the numerical results obtained by Algorithm QFFM for two-dimensions Sine-square II function. From Table 5, we see that a global minimizer

Table 4. Results for Shubert III function by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	$\delta, e_i, , q, c, r$	$\bar{x}_{q,r,c,x_k^*}$
1	(1, 1)	(−0.8017, 2.7818)	−25.0600		
				$1/2^4, e_1, 10^5, 10, 1$	(−1.4136, 2.2210)
2	(−1.4136, 2.2210)	(−1.4251, 2.2950)	−28.0619		
				$1/2^4, e_1, 10^5, 10, 1$	(−1.4251, −0.8003)
2	(−1.4251, −0.8003)	(−1.4251, −0.8003)	−186.7309		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq -186.7309$		

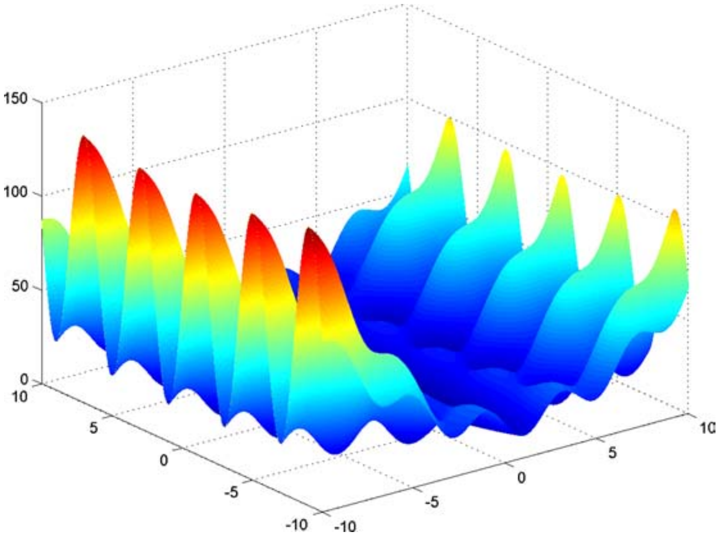

Figure 8 The behavior of two-dimensions Sine-square II function.

Table 5. Results for two-dimensions Sine-square II function by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, c, r	$\bar{x}_{q,r,c,x_k^*}$
1	(0 0)	(4.9599, 1.0000)	1.5550	$1/4$, e_1, 10^6, 1, 1	(1.0000, 1.0000)
2	(1.0000, 1.0000)	(1.0000, 1.0000)	1.6003×10^{-13}	for any $c \leq 10^{10}$ $e_i, i = 1, \dots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq 1.6003 \times 10^{-13}$		

of the two-dimensions Sine-square II function obtained by Algorithm QFFM is:

$$x^* = (1.0000, 1.0000)$$

with the global minimal value $f(x^*) = 1.6003 \times 10^{-13} \approx 0$.

When $n = 6$, we take initial point $x_1^0 = (2, \dots, 2)$(in fact, we can take any point as the initial point). The first local minimal value of problem (5.5) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from x_1^0 is $f(x_1^*) = 0.5183$. Table 6 gives the numerical results obtained by Algorithm QFFM for the 6-dimensions Sine-square II function. From Table 6, we see that a global minimizer obtained by Algorithm QFFM of the 6-dimensions Sine-square II function is:

$$x^* = (1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000),$$

with global minimal value $f(x^*) = 1.6066 \times 10^{-11} \approx 0$.

Table 6. Results for 6-dimensions Sine-square II function by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, c, r	$\bar{x}_{q,r,c,x_k^*}$
1	(2, 2, 2, 2, 2, 2)	(−2.9598, 1.0000, 1.0004, 1.0002, 0.9999, 0.9997)	0.5183		
				$1/2^2, e_1, 10^5, 10, 1$	1.0000, 1.0000, 1.0000, 1.0000, 2 1.0000, 1.0000
2	(1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)	(1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)	1.6066×10^{-11}		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq 1.6066 \times 10^{-11}$		

When $n = 20$, we take initial point x_1^0 = zeros(20, 1)(in fact, we can take any point as the initial point). The first local minimal value of problem (5.5) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from x_1^0 is $f(x_1^*) = 1.8265 \times 10^{-5}$. Table 7 gives the numerical results obtained by Algorithm QFFM for the 20-dimensions Sine-square II function. From Table 7, we see that a global minimizer of the 20-dimensions Sine-square II function obtained by Algorithm QFFM is:

$$x^* = \text{ones}(20, 1),$$

with global minimal value $f(x^*) = 2.4946 \times 10^{-12} \approx 0$.

When $n = 50$, we take initial point x_1^0 = zeros(50, 1) (in fact, we can take any point as the initial point). The first local minimal value of problem (5.5) obtained by the optimization subroutine within the optimization Toolbox in Matlab 6.1 starting from x_1^0 is $f(x_1^*) = 1.6227 \times 10^{-5}$. Table 8 gives the numerical results obtained by Algorithm QFFM for the 50-dimensions Sine-square II function.

From Table 8, we see that a global minimizer of the 50-dimensions Sine-square II function obtained by Algorithm QFFM is:

$$x^* = \text{ones}(50, 1),$$

with global minimal value $f(x^*) = 2.3380 \times 10^{-12} \approx 0$.

6. Conclusions

In this paper, we have defined a new filled function and a quasi-filled function and then developed two associated global optimization methods. Using the two proposed global optimization methods, we can obtain the required global minimizer in finite steps under

Table 7. Results for 20-dimensions Sine-square II function by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, r, c	$\bar{x}_{q,r,c,x_k^*}$
1	zeros(20,1)		1.8265×10^{-5}		
				$1/2^{32}$, e_{49}, 10^5, 10^{-10}, 1	ones(20,1)
2	ones(20,1)	ones(20,1)	2.4946×10^{-12}		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq 2.4946 \times 10^{-12}$		

Table 8. Results for 50-dimensions Sine-square II function by QFFM.

k	x_k^0	x_k^*	$f(x_k^*)$	δ, e_i, q, r, c	$\bar{x}_{q,r,c,x_k^*}$
1	zeros(50,1)		1.6227×10^{-5}		
				$1/2^{32}$, e_{49}, 10^5, 10^{-10}, 1	ones(50,1)
2	ones(50,1)	ones(50,1)	2.3380×10^{-12}		
				for any $c \leq 10^{10}$ $e_i, i = 1, \ldots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,c,x_k^*}$
	$\bar{x}_{q,r,c,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq 2.3380 \times 10^{-12}$		

the assumption that the set of local minimal objective function values is finite. From the numerical results, we see that the proposed techniques(here just the quasi-filled function method is used) are very reliable and efficient.

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