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On Generalizations of the Frank-Wolfe Theorem to Convex and Quasi-Convex Programmes

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Abstract. In this paper we are concerned with the problem of boundedness and the existence of optimal solutions to the constrained optimization problem.

We present necessary and sufficient conditions for boundedness of either a faithfully convex or a quasi-convex polynomial function over the feasible set defined by a system of faithfully convex inequality constraints and/or quasi-convex polynomial inequalities, where the faithfully convex functions satisfy some mild assumption. The conditions are provided in the form of an algorithm, terminating after a finite number of iterations, the implementation of which requires the identification of implicit equality constraints in a homogeneous linear system. We prove that the optimal solution set of the considered problem is nonempty, this way extending the attainability result well known as the so-called Frank-Wolfe theorem. Finally we show that our extension of the Frank-Wolfe theorem immediately implies continuity of the solution set defined by the considered system of (quasi)convex inequalities.

Keywords: convex and quasi-convex constrained programs, existence of optimal solutions, cone of recession

1. Introduction

We consider the problem

$$\text{minimize } f_0(x) \tag{1}$$

$$\text{subject to:} \quad x \in \mathcal{R} = \{x \in \mathbb{R}^n \mid f_i(x) \le 0, i \in J = \{1, 2, \ldots, m\}\} \tag{2}$$

where $f_i(x)$, $i \in J^1 = \{1, 2, \ldots, m_1\}$, are faithfully convex functions and $f_i(x)$, $i \in J^2 = \{m_1 + 1, \ldots, m\}$, are quasi-convex polynomials with unbounded level sets, while the objective function f_0 belongs to either of the two classes of functions. As shown by Rockafellar in [19], every faithfully convex function f_i can be represented in the form $f_i(x) = F_i(c_i + B_i x) + \langle a_i, x\rangle + d_i$, where F_i are strictly convex functions, and $B_i \in \mathbb{R}^{p_i \times n}$, $c_i \in \mathbb{R}^{p_i}$, $a_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$, $i \in J^1 \cup \{0\}$. We assume that in case the functions f_i are faithfully convex they satisfy the following condition:

$$s \in 0^+ f_i \Rightarrow B_i s = 0, \quad i \in J^1 \cup \{0\}, \tag{3}$$

where $0^+ f_i$ denotes the cone of recession of f_i. The problem (1) and (2) either has a finite infimum (although there may be no such point at which $f_0(x)$ achieves its minimum), or the function is unbounded from below [1, 2, 5, 6, 13, 14]. It is well known [1], that the minimum of the problem in the case when all f_i, $i \in J \cup \{0\}$ are convex may be not achieved only if the objective function and the feasible region have a common direction of recession.

Most of the algorithms for convex constrained optimization problem are based upon assumption that the set of optimal solutions is nonempty and compact, although there are many convex programs which do not satisfy the assumptions. In 1956 Frank and Wolfe proved that when the objective function is quadratic and the feasible region is linear, the set of optimal solutions is nonempty provided the problem is bounded below. This result is an extension of a well known fundamental theorem of linear programming [7, 9]. Many other authors generalized the Frank-Wolfe theorem to broader classes of functions [11, 18, 21]. In particular Perold in [18] extended the Frank-Wolfe theorem to a class of non-quadratic objective functions and linear constraints.

More recently Luo and Zhang [11] extended the Frank-Wolfe theorem to various classes of general convex/non-convex quadratic constraint systems. In particular, the attainability of an infimum of the quadratically constrained convex quadratic programmes, which was shown in [11] to be a consequence of the continuity of the feasible set, also follows from the result for l_p programming established by Terlaky in [21]. Very recently Belousov and Klatte in [4] generalized the result on attainability to convex polynomial objective function and convex polynomial sets, which was earlier demonstrated in a 1977 Belousov book. However, we have to remark that as pointed out by a Referee, the above mentioned attainability results can be deduced from the attainability theorem given by Bank and Mandel in [3] for the minimization problem where the objective and constraint functions are quasi-convex polynomials.

In this paper we present necessary and sufficient conditions for boundedness of a quasi-convex constrained problem, where the objective and constraint functions are either faithfully convex functions affine along any direction of recession or quasi-convex polynomials. The conditions are provided in the form of an algorithm, terminating after at most $\min\{m-1, n-1\}$ iterations, which in each iteration requires the identification of implicit equality constraints in a homogeneous linear system. We also show that the optimal solution set of the considered problem is nonempty, this way generalizing the attainability results proved by Luo and Zhang in [11] for a quasi-convex quadratic objective function and convex quadratic constraints and by Belousov and Klatte in [4] for convex polynomial functions. We note that Theorem 4.1 in this paper extends the result established by Bank and Mandel in [3] for quasi-convex polynomial objective function and constraints to problems including also faithfully convex functions satisfying condition (3).

The results in [3, 4, 11] are extensions of the well known Frank-Wolfe theorem for quadratic objective function and linear constraints.

We also show that our result on attainability of the infimum of the (quasi-)convex program allows us to extend perturbation results given by Luo and Zhang [11] for parametric convex quadratic programs and by Belousov and Klatte [4] for parametric convex polynomials to the broader class of parametric convex and quasi-convex programs.

2. Auxiliary results on unboundedness of the (quasi-)convex programmes

Let us represent $f_i(x) = F_i(c_i + B_i x) + \langle a_i, x\rangle + d_i,\ i \in J^1 \cup \{0\}$, where F_i are strictly convex functions, and B_i are full row rank matrices.

It is well known that if a faithfully convex function is constant along some line segment, then it is constant along any line parallel to the line segment. The set of vectors with the latter property forms what is called a constancy space of $f(x)$, which is denoted by $D_f^{=}$, [10, 20]. A vector s is called a direction of recession of $f(x)$ if for every x the function $f(x + ts)$ is a nonincreasing function of t [10, 20]. The set of all vectors of recession of the function f forms a convex cone, called the cone of recession, denoted by $0^+ f$. The constancy space $D_f^{=}$ of $f(x)$ may be defined in terms of the set $0^+ f$ [20], as

$$D_f^{=} = \{y \in \mathbb{R}^n | y \in 0^+ f \wedge -y \in 0^+ f\}.$$

We define the region

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid f_i(x) \le 0, i \in J^1\}, \tag{4}$$

where the functions $f_i(x),\ i \in J^1 = \{1, 2, \ldots, m_1\}$, are faithfully convex functions satisfying condition (3).

Lemma 2.1 ([15]). *The region $\mathcal{F}$ defined in* (4)*, where $f_i,\ i \in J$ are faithfully convex and satisfy condition* (3)*, is unbounded if and only if there exists a vector $s \neq 0$ satisfying the following conditions*:

$$\begin{aligned} &B_i s = 0, \quad \forall i \in J^1,\\ &\langle a_i, s\rangle \le 0, \quad \forall i \in J^1. \end{aligned} \tag{5}$$

Lemma 2.2. *If the faithfully convex functions $f_i(x),\ i \in J^1 \cup \{0\}$ satisfy condition* (3)*, then the function $f_0(x)$ is unbounded from below along a half-line in $\mathcal{F}$ if and only if there exists a vector $s \neq 0$ satisfying the following conditions:*

$$\begin{aligned} &\langle a_0, s\rangle < 0\\ &B_i s = 0, \quad \forall i \in J^1 \cup \{0\},\\ &\langle a_i, s\rangle \le 0, \quad \forall i \in J^1. \end{aligned}$$

Proof: The backward part of the proof follows immediately, since $f_0(x)$ is unbounded below along any half-line with a direction vector s satisfying the above set of conditions. To prove the forward part, let us assume that $f_0(x)$ is unbounded along a half-line $x(t) = x_0 + ts,\ t \ge 0$. Since $x(t) \in \mathcal{F}$, then by Lemma 2.1, conditions (5) are satisfied for $i \in J^1$. Since for some α, $x(t) \in L_\alpha(f_0) = \{x \in \mathbb{R}^n | f_0(x) \le \alpha\}$, $\forall t \ge 0$, then $s \in 0^+ f_0$, which by assumption (3) implies that $B_0 s = 0$. Consequently $f_0(x(t)) = F_0(c_0 + B_0(x_0 + ts)) + \langle a_0, x_0 + ts\rangle + d_i = f_0(x_0) + \langle a_0, s\rangle t \to -\infty$, when $t \to \infty$, which yields the inequality $\langle a_0, s\rangle < 0$, completing the proof of the lemma. □

We remark that convex polynomials of n-variables (see Corollary 4.1) and convex functions of the form

$$f(x) = \frac{1}{p}(\langle x, Ax\rangle)^{\frac{p}{2}} + \langle b, x\rangle + d,$$

where $p \geq 1$, and A is positive semidefinite belong to the class of faithfully convex functions satisfying condition (3).

We define a quasi-convex function by using its equivalent characterization given in [8].

Definition 2.1. Consider a function $f(x)$ on a convex set $S \subset \mathbb{R}^n$, and the level sets $L_\lambda(f) = \{x \mid x \in S, f(x) \leq \lambda\}$. Then $f(x)$ is called quasi-convex on S if $L_\lambda(f)$ is convex for each $\lambda \in \mathbb{R}$.

A polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ with degree μ is called a form of order μ if $p(tx) = t^\mu p(x)$, $\forall x \in \mathbb{R}^n$, $\forall t \in \mathbb{R}$. Each polynomial function $p(x)$ of degree $\mu \geq 1$ can be represented as a sum of forms $q_j(x)$, $j = 0, 1, \ldots, \mu$:

$$p(x) = q_\mu(x) + \cdots + q_1(x) + q_0,$$

where degree of $q_j(x)$ is j.

Lemma 2.3 ([3]). *Let $p(x)$, $x \in \mathbb{R}^n$ be a quasi-convex polynomial function. Then if $p(x) = q_\mu(x) + q_{\mu-1}(x) + \cdots + q_0$, where the functions $q_i(x)$, $i = 1, \ldots, \mu$, are forms of the order i, then the form $q_\mu(x)$ is quasi-convex.*

Lemma 2.4. *If a quasi-convex polynomial function $f_0(x)$ is unbounded below* (*or constant*) *along some half-line, then it is unbounded below* (*respectively constant*) *along any half-line with the same direction vector.*

Proof: The proof follows immediately from Lemma 1(ii) in [3], stating that for two fixed vectors x_i, $i = 1, 2$, and a vector $s \in \mathbb{R}^n$, the functions $f_0(x_i + ts)$ of one variable t, are quasi-convex polynomials of the same degree with identical coefficients corresponding to the highest degree term, provided that $f_0(x)$ is a quasi-convex polynomial function. As noted in [3], the result is not true for arbitrary quasi-convex functions. □

Similar result has been proved in [16] for arbitrary convex, (not necessarily polynomial) functions.

Lemma 2.5 ([3]). *Let $q_\mu(x)$, $x \in \mathbb{R}^n$, be a quasi-convex form of order $\mu \geq 1$. Further, let us define*

$$L = \{x \in \mathbb{R}^n \mid q_\mu(x) = 0\}$$

and

$$H = \{x \in \mathbb{R}^n \mid q_\mu(x) \leq 0\}.$$

Then the following statements hold:

(i) *L is a linear subspace of $\mathbb{R}^n$.*

(ii) *$L = H$ if order μ is an even number.*

(iii) *If the order μ is an odd number, then L is a hyperplane and H is a half-space, both in $\mathbb{R}^n$.*

Lemma 2.6 ([3]). *Let $p(x) = q_\mu(x) + \cdots + q_1(x) + q_0$, $x \in \mathbb{R}^n$, be a quasi-convex polynomial which has the degree $deg(p) = \mu \geq 1$. Let us define the sets*

$$K(p) = \{u \in \mathbb{R}^n \mid \sup\{p(tu) \mid t \geq 0\} < \infty\},$$

$$R(p) = \{u \in \mathbb{R}^n \mid \sup\{p(tu) \mid t \in \mathbb{R}\} < \infty\},$$

and an index

$$i_0 = \begin{cases} \max\{i \mid \exists u \in \mathbb{R}^n : q_\mu(u) = q_{\mu-1}(u) = \cdots = q_{i+1}(u) = 0, q_i(u) < 0\}, \\ \quad \text{if it exists} \\ 1, \quad \text{otherwise.} \end{cases}$$

Then we have

(i) $K(p) = \{u \in \mathbb{R}^n \mid q_\mu(u) = \cdots = q_{i_0+1}(u) = 0, q_{i_0}(u) \leq 0\}$, $R(p) = \{u \in \mathbb{R}^n \mid q_\mu(u) = \cdots = q_{i_0}(u) = 0\}$.

(ii) *If the polynomial function $p(x)$ is convex, then an index i_0 defined above has a value* 1.

(iii) *$K(p)$ is the recession cone $0^+ p$ of all non-empty level sets $L_\alpha(p) = \{x \in \mathbb{R}^n \mid p(x) \leq \alpha\}$ of $p(x)$.*

We note that it follows immediately from Lemma 2.6 that $R(p)$ is a constancy space $D_p^=$ of p.

We observe that the properties stated in Lemma 2.6 are not true for a general class of quasi-convex functions, which may have different cones of recession for different level sets.

Lemma 2.7. *Let $f_0(x) = q^0_{\mu_0}(x) + \cdots + q^0_1(x) + q^0_0$, $x \in \mathbb{R}^n$, be a quasi-convex polynomial function of degree $\mu_0 \geq 1$, and the index i_0 be defined as in Lemma 2.6, that is*

$$i_0 = \begin{cases} \max\{i \mid \exists u \in \mathbb{R}^n : q^0_{\mu_0}(u) = q^0_{\mu_0-1}(u) = \cdots = q^0_{i+1}(u) = 0, q^0_i(u) < 0\}, \\ \quad \text{if it exists} \\ 1, \quad \text{otherwise.} \end{cases}$$

Then $f_0(x)$ is unbounded from below along a half-line in $\mathcal{F}$ if and only if there exists a vector s satisfying the following conditions:

$$q^0_\tau(s) = 0, \quad \tau = \mu_0, \mu_0 - 1, \ldots, i_0 + 1, \quad q^0_{i_0}(s) < 0, \tag{6}$$

$$B_i s = 0, \quad \forall i \in J^1, \tag{7}$$

$$\langle a_i, s\rangle \leq 0, \quad \forall i \in J^1. \tag{8}$$

Proof: To prove the backward part of the lemma let us assume that s satisfies conditions (6)–(8). We observe that since s satisfies conditions (6), $f_0(x)$ is unbounded below on $\mathbb{R}^n$ along the half-line $x(t) = ts,\ t \geq 0$. Let $\bar{x} \in \mathcal{F}$. Then it follows from Lemma 2.4 that $f_0(x)$ is unbounded below along the half-line $\bar{x}(t) = \bar{x} + ts,\ t \geq 0$. Since s satisfies conditions (7) and (8) it follows that $\bar{x}(t) \subset \mathcal{F}$.

To prove the forward part, let us assume that $f_0(x)$ is unbounded below along a feasible half-line $x_0(t) = x_0 + ts,\ t \geq 0$. Since $x_0(t) \subset \mathcal{F}$, then by Lemma 2.1, conditions (7) and (8) are satisfied. Since the function $f_0(x_0(t)),\ t \geq 0$, is a quasi-convex polynomial of one variable and $f_0(x)$ is unbounded below along the half-line $x_0(t),\ t \geq 0$, then $f_0(x)$ is bounded from above along this half-line. Therefore for some α, $x_0(t) \in L_\alpha(f_0) = \{x \in \mathbb{R}^n | f_0(x) \leq \alpha\}$, $\forall t \geq 0$, which by Lemmas 2.4 and 2.6(iii) implies that $s \in 0^+ f_0$. Thus it follows from Lemma 2.6(i) that $\exists i_0$, (satisfying conditions of this lemma), such that equations $q^0_{\mu_0}(s) = \cdots = q^0_{i_0+1}(s) = 0,\ q^0_{i_0}(s) \leq 0$ hold. Let us suppose that

$$q^0_{\mu_0}(s) = \cdots = q^0_{i_0+1}(s) = 0 \quad \text{and} \quad q^0_{i_0}(s) = 0. \tag{9}$$

Since $f_0(x)$ is unbounded below along the half-line $x_0(t)$ then by Lemma 2.4 it is also unbounded below along the half-line $x(t) = ts$. But Eqs. (9) imply that $s \in D^{=}_{f_0}$, that is that $f_0(x)$ is constant along the half-line $x(t) = ts,\ t \geq 0$, and consequently it is is also constant along the half-line $x_0(t) = x_0 + ts,\ t \geq 0$. This contradicts an earlier assumption that $f_0(x_0(t)),\ t \geq 0$, is unbounded below, which proves that $q_{i_0}(s) < 0$, completing the proof of the lemma. □

We need to make the following remarks. We observe that by Lemma 2.5, $\{s \in \mathbb{R}^n \mid q^0_{\mu_0}(s) = 0\}$ is a linear subspace of $\mathbb{R}^n$, the set $\{s \in \mathbb{R}^n | q^0_{\mu_0}(s) = q^0_{\mu_0-1}(s) = 0\}$ is a linear subspace of $\{s \in \mathbb{R}^n \mid q_{\mu_0}(s) = 0\}$ and so on. It follows from Lemmas 2.5 and 2.6 that the set defined by the system

$$q^0_\tau(s) = 0,\ \tau = \mu_0, \mu_0 - 1, \ldots, i_0 + 1; \quad q^0_{i_0}(s) \leq 0 \tag{10}$$

is a linear half-space of $\mathbb{R}^\kappa$, for some $\kappa \leq n$, which implies that it can be replaced by a system of linear equations and one linear inequality. Therefore the problem to determine whether there exists s satisfying $q^0_{i_0}(s) < 0$ and (10), can be replaced with a LP problem with a homogeneous system of linear equations. More specifically, the latter problem is equivalent to the LP problem to determine whether or not a linear inequality $\langle a_0, s\rangle \leq 0$, is an implicit equality in the system

$$\langle a_0, s\rangle \leq 0, \quad B_0 s = 0,$$

where $B_0 s = 0$ is a system of linear equations representing a linear subspace defined by $q^0_\tau(s) = 0,\ \tau = \mu_0, \ldots, i_0 + 1$ and the inequality $\langle a_0, s\rangle \leq 0$ represents a half-space defined by the system (10).

Each polynomial $f_i(x)$, $i = m_1 + 1, \ldots, m$, of degree $\mu_i \geq 1$, will be represented as a sum of the forms $q^i_j(x)$, $j = 0, 1, 2, \ldots, \mu_i$:

$$f_i(x) = q^i_{\mu_i}(x) + \cdots + q^i_1(x) + q^i_0,$$

where q^i_j, $j = 0, 1, \ldots, \mu_i$, is a form of degree j satisfying $q^i_j(tx) = t^j q^i_j(x)$, $\forall x \in \mathbb{R}^n$, $\forall t \in \mathbb{R}$.

Let us now define the convex region

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in J^2\},$$

where the functions $f_i(x)$, $i \in J^2 = \{m_1 + 1, \ldots, m\}$, are quasi-convex polynomials.

We will prove the following lemma.

Lemma 2.8. *The nonempty feasible region $\mathcal{P}$ is unbounded if and only if there exists a vector s, $(s \neq 0)$, satisfying the following conditions*:

$$q^l_\tau(s) = 0, \quad \tau = \mu_l, \mu_l - 1, \ldots, i_l + 1; \quad q^l_{i_l}(s) \leq 0, \quad l = m_1 + 1, \ldots, m, \tag{11}$$

where $\mu_{m_1+1}, \ldots, \mu_m$ denote the degrees of the polynomials $f_{m_1+1}(x), \ldots, f_m(x)$ respectively and i_l, $l = m_1 + 1, \ldots, m$ is defined as

$$i_l = \begin{cases} \max\{i \mid \exists u \in \mathbb{R}^n : q^l_{\mu_l}(u) = \cdots = q^l_{i+1}(u) = 0, q^l_i(u) < 0\}, & \text{if it exists} \\ 1, \quad \text{otherwise}, \end{cases}$$

for $l = m_1 + 1, \ldots, m$.

Proof: It follows from the Definition 2.1 that the region $\mathcal{P}$ is convex. Thus the region $\mathcal{P}$ is unbounded iff it contains a half-line. We will show first the backward part of the lemma. Let $x_0 \in \mathcal{P}$ and $s \neq 0$ satisfies conditions (11). Then it follows from Lemma 2.6 that $s \in \bigcap_{i=m_1+1}^m 0^+ f_i$. As indicated in [3] all functions $f_i(x_0 + ts)$, $t \geq 0$, are nonincreasing functions of t, that is $x_0 + ts \in \mathcal{P}$, $\forall t \geq 0$, which completes the first part of the proof. To prove the forward part of the proof suppose that the set $\mathcal{P}$ contains a half-line, let say $x(t) = \bar{x} + ts$, $t \geq 0$. Thus $\bar{x} + ts \in \bigcap_{i=m_1+1}^m L_0(f_i) = \{x \mid f_i(\bar{x} + ts) \leq 0, i \in J^2\}$, which by Lemma 2.6, yields that $s \in \bigcap_{i=m_1+1}^m 0^+ f_i$, and that s satisfies conditions (11). ☐

Let us define the feasible region $\mathcal{P}(\lambda) = \{x \in \mathbb{R}^n \mid f_i(x) \leq \lambda_i, i \in J^2\}$, where $\lambda_i \in \mathbb{R}, \forall i$. It follows immediately from the Lemma 2.8 that the following corollary holds.

Corollary 2.1. *The nonempty feasible region $\mathcal{P}$ is unbounded iff the nonempty region $\mathcal{P}(\lambda)$ is unbounded,* $\forall \lambda \in \mathbb{R}$.

Lemma 2.9. *The quasi-convex polynomial function $f_0(x)$ is unbounded from below along a half-line in $\mathcal{P}$ if and only if there exists a vector s satisfying the following* *conditions*:

$$q_\tau^0(s) = 0, \quad \tau = \mu_0, \mu_0 - 1, \ldots, i_0 + 1; \quad q_{i_0}^0(s) < 0 \tag{12}$$

$$q_\tau^l(s) = 0, \quad \tau = \mu_l, \mu_l - 1, \ldots, i_l + 1; \quad q_{i_l}^l(s) \leq 0, \ l = m_1 + 1, \ldots, m, \tag{13}$$

where μ_l denote the degree of the quasi-convex polynomials $f_l(x)$, $l = 0, m_1 + 1, \ldots, m$ and i_l, $l = 0, m_1 + 1, \ldots, m$ are defined as in the Lemma 2.8.

Proof: To prove the backward part of the lemma we observe that $f_0(x)$ is unbounded below on $\mathbb{R}^n$ along the half-line $x(t) = ts$, $t \geq 0$, if s satisfies conditions (12). Since s also satisfies conditions (13), then $s \in \bigcap_{i=m_1+1}^m 0^+ f_i$. Let $\bar{x} \in \mathcal{P}$. It follows that $\bar{x} + ts \in \mathcal{P}$, $\forall t \geq 0$ and the Lemma 2.4 implies that $f_0(x)$ is unbounded below on $\mathcal{P}$ along the half-line $\bar{x}(t) = \bar{x} + ts$, $t \geq 0$.

To prove the forward part, let us assume that $f_0(x)$ is unbounded below along a feasible half-line $x_0(t) = x_0 + ts$, $t \geq 0$. Since $x_0(t) \subset \mathcal{P}$, then by Lemma 2.6, conditions (13) are satisfied. Since for some α, $x_0(t) \in L_\alpha(f_0) = \{x \in \mathbb{R}^n \mid f_0(x) \leq \alpha\}$, $\forall t \geq 0$, then by Lemma 2.6(iii) it follows that $s \in 0^+ f_0$, which by part (i) of the same lemma implies that $\exists i_0$, (where i_0 satisfies conditions of the Lemma 2.6), such that equations $q_{\mu_0}^0(s) = \cdots = q_{i_0+1}^0(s) = 0$, $q_{i_0}^0(s) \leq 0$, hold. Let us suppose that $q_{\mu_0}^0(s) = \cdots = q_{i_0+1}^0(s) = 0$ and $q_{i_0}^0(s) = 0$. Then by Lemma 2.6, $s \in D_{f_0}^{=}$, which contradicts the assumption that $f_0(x)$ is unbounded below along the half-line $x_0(t)$, $t \geq 0$. This proves that $q_{i_0}^0(s) < 0$, i.e. s satisfies conditions (12), which completes the proof of the lemma. □

Clearly, a very similar result to the one stated in the Lemma 2.9 can be obtained for a faithfully convex objective function satisfying condition (3), defined over the region $\mathcal{P}$.

Using arguments similar to those following Lemma 2.7, we can deduce that the system (13) can be replaced with the linear system

$$\langle a_j, s \rangle \leq 0, \quad j \in J^2, \tag{14}$$

$$B_j s = 0, \quad j \in J^2, \tag{15}$$

where $B_j s = 0$ is a system of linear equations representing a linear subspace defined by $\{s \mid q_\tau^j(s) = 0, \ \tau = \mu_j, \ldots, i_j + 1\}$, $j = 0, m_1 + 1, \ldots, m$, and the inequality $\langle a_j, s \rangle \leq 0$, represents a half-space of $\{s \mid q_\tau^j(s) = 0, \ \tau = \mu_j, \ldots, i_j + 1\}$, defined by the inequality $q_{i_l}^l(s) \leq 0$. In particular in Lemma 2.9, the problem to determine whether or not there exists vector s satisfying the system (12) and (13), can be replaced with a LP problem to determine whether or not a linear inequality $\langle a_j, s \rangle \leq 0$ is an implicit equality in the system (14) and (15).

Incorporating this unifying representation and taking into account that $\mathcal{R} = \mathcal{F} \cap \mathcal{P}$, where both $\mathcal{F}$ and $\mathcal{P}$ are convex regions, allows us to combine results proved in Lemmas 2.2, 2.7 and 2.9 in the following theorem.

Theorem 2.1. *Let us assume that the functions $f_i(x)$, $i \in J \cup \{0\}$ are either faithfully convex satisfying condition* (3)*, or quasi-convex polynomials. Then the function $f_0(x)$ is unbounded from below along a half-line in $\mathcal{R}$ defined in* (2) *if and only if there exists a* *vector s satisfying the following conditions:*

$$\langle a_0, s\rangle < 0$$

$$B_i s = 0, \quad \forall i \in J \cup \{0\},$$

$$\langle a_i, s\rangle \leq 0, \quad \forall i \in J.$$

3. Algorithm to determine boundedness of the (quasi-)convex constrained programmes

In [6, 12] we have presented an algorithm to determine whether or not a convex quadratic objective function is bounded from below over the region defined by convex quadratic constraints. In every iteration, the algorithm requires the identification of implicit equality constraints in a homogeneous linear system. The algorithm terminates after at most $\min\{m-1, n-1\}$ iterations, indicating whether or not the objective function is bounded from below. The proof of the algorithm shows not only that in a case the problem is bounded below, the objective function attains its minimum over the feasible region, but also it demonstrates that the algorithm can be used to reduce the number of variables and constraints in the quadratically constrained quadratic programming problem (QCQP), that is whenever QCQP is bounded below then its solution can be constructed from the solution to the reduced problem with a bounded feasible region. In this section we will show that the approach presented in [6,12] can be extended to certain classes of convex and quasi-convex functions, which in particular contain all convex polynomial functions.

We consider the optimization problem

$$\text{minimize} \quad f_0(x) \tag{16}$$

$$\text{subject to} : x \in \mathcal{R} = \{x \in \mathbb{R}^n | f_i(x) \leq 0, i \in J = \{1, 2, \ldots, m\}\} \tag{17}$$

where $f_i(x),\ i = 1, \ldots, m_1$, are faithfully convex functions satisfying conditions (3), and $f_i(x),\ i = m_1 + 1, \ldots, m$ are quasi-convex polynomial functions with unbounded level sets. The objective function $f_0(x)$ belongs to either of the two classes of functions. We show that if the objective function is bounded below over the feasible region then the optimal solution set is nonempty. To this end we will present necessary and sufficient conditions for boundedness of the objective function over the convex region $\mathcal{R}$, defined in (17).

We note that in order to be able to use a uniform notation when dealing with two different types of functions, the system (13) has been replaced with the corresponding homogeneous linear system (14) and (15). Analogously, as indicated in the remark following Lemma 2.7, when the objective function $f_0(x)$ is a quasi-convex polynomial, the problem of determining whether there exists a vector s satisfying $q^0_{i_0}(s) < 0$ and (10), has been replaced with a problem to determine whether or not a linear inequality $\langle a_0, s\rangle \leq 0$ is an implicit equality in the system $\langle a_0, s\rangle \leq 0,\ B_0 s = 0$.

We will prove that the following algorithm can be used to determine whether the problem (16) and (17) is bounded below.

Algorithm A

Step 1. Set $k = 0$, $J_0 = J \cup \{0\}$, and go to Step 2.
Step 2. Determine if there exists $s \in \mathbb{R}^n$ such that $\langle a_0, s\rangle < 0$ for s satisfying

$$\langle a_i, s\rangle \le 0, \quad i \in J_k, \tag{18}$$

$$B_i s = 0, \quad i \in J_k, \tag{19}$$

If such an s exists, then stop with the message that $f_0(x)$ is unbounded from below over $\mathcal{R}$, otherwise go to Step 3.
Step 3. Find the index set $J_{k+1} \subseteq J_k$ of all implicit equality constraints in the system (18) and (19). If $J_{k+1} = J_k$, then stop with the message that $f_0(x)$ is bounded below over $\mathcal{R}$. Otherwise, replace $k := k + 1$ and go to Step 2.

To show that the algorithm terminates in finitely many steps we will use the following notation. For each index set J_k we define the region $\mathcal{R}(J_k) = \{x \in \mathbb{R}^n \mid f_i(x) \le 0,\ i \in J_k \setminus \{0\}\}$, and $\mathcal{R}_\gamma(J_k) = \{x \in \mathcal{R}(J_k) \mid f_0(x) \le \gamma\}$, where $\gamma >\inf \{f_0(x) \mid x \in \mathcal{R}\}$. The matrix whose columns are the vectors a_i, $i \in J_k$, and the columns of the matrices B_i^T, $i \in J_k$ is denoted by $A(J_k)$. The column space of $A(J_k)$ is denoted by $C(A(J_k))$, and the null space of $A^T(J_k)$, (which is an orthogonal complement of $C(A(J_k))$, is denoted by $\mathcal{N}(A^T(J_k))$. Furthermore, for a vector $v \in \mathcal{R}_\gamma(J_k)$ we define the linear manifold

$$\mathcal{R}_\gamma(J_k, v) = \{v + s \mid s \in C(A(J_k))\}.$$

Lemma 3.1. *If Algorithm A terminates in Step* 3, *then* $\mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$ *is bounded for all* $v \in \mathcal{R}_\gamma(J_k)$.

Proof: If $\mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$ is unbounded then it follows from convexity of $\mathcal{R}_\gamma(J_k)$, (which for constraints with indices in $J_k \cap J^2$ is a consequence of the Definition 2.1) that there exists $s \in C(A(J_k))$ such that $v + ts \in \mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$, $\forall t \ge 0$. Since the set $\mathcal{R}_\gamma(J_k)$ is convex and $v + ts \in \mathcal{R}_\gamma(J_k)$, $\forall t \ge 0$, then it follows from Lemma 2.1 that $B_i s = 0$, $\langle a_i, s\rangle \le 0$, $i \in (J^1 \cap J_k) \setminus \{0\}$, while Lemma 2.8 yields that $B_i s = 0$, $\langle a_i, s\rangle \le 0, i \in (J^2 \cap J_k) \setminus \{0\}$. Furthermore, since $f_0(v + ts) \le \gamma$, $\forall t \ge 0$, then in case $f_0(x)$ is quasi-convex polynomial we have by Lemma 2.6 that $s \in 0^+ f_0$. Thus for the quasi-convex polynomial objective function, the vector s satisfies the system (10), which by the remarks at the end of the previous section, can be replaced by a linear homogeneous system $B_0 s = 0$, $\langle a_0, s\rangle \le 0$. In case when $f_0(x)$ is faithfully convex it follows immediately from the inequality $f_0(v + ts) \le \gamma$, $\forall t \ge 0$, that $s \in 0^+ f_0$, which by the condition (3) implies that s satisfies the system $B_0 s = 0$, $\langle a_0, s\rangle \le 0$ as well. But termination in Step 3 implies that $s \in \mathcal{N}(A^T(J_k))$, which is a contradiction since $s \ne 0$ and $s \in C(A(J_k))$. Consequently the set $\mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$ is bounded. □

Lemma 3.2. *If Algorithm A terminates in Step* 3, *then* $f_0(x)$ *is bounded from below over* $\mathcal{R}(J_k)$.

Proof: Clearly, it follows from Lemma 3.1 that the convex region $\mathcal{R}_\gamma(J_k)\cap\mathcal{R}_\gamma(J_k,v)$ is bounded for all $v\in\mathcal{R}_\gamma(J_k)$, and consequently $f_0(x)$ is bounded below over this region for all $v\in\mathcal{R}_\gamma(J_k)$. Since $\min\{f_0(x)\,|\,x\in\mathcal{R}_\gamma(J_k)\}=\min\{f_0(x)\,|\,x\in\mathcal{R}(J_k)\}$, we need only to show that for $v_1,\ v_2\in\mathcal{R}_\gamma(J_k),\ v_1\neq v_2$, it holds that $\min\{f_0(x)\,|\,x\in\mathcal{R}_\gamma(J_k)\cap\mathcal{R}_\gamma(J_k,v_1)\}=\min\{f_0(x)\,|\,x\in\mathcal{R}_\gamma(J_k)\cap\mathcal{R}_\gamma(J_k,v_2)\}$. To this end let W_k be a matrix whose columns form an orthogonal basis for $C(A(J_k))$, and let us write $v_i=v_i^C+v_i^N$, where $v_i^C\in C(A(J_k))$, and $v_i^N\in N(A^T(J_k))$, $i=1,2$. Thus for $i=1,2$, any $x^i\in\mathcal{R}_\gamma(J_k,v_i)$ can be expressed as $x^i=v_i^N+W_k\xi_i$, for some vector ξ_i. Using the change of variables $x=v_i^N+W_k\xi$, where $v_i^N\in\mathcal{N}(A^T(J_k))$, yields $B_0v_i^N=0$, $\langle a_0,v_i^N\rangle=0$, which in case the function $f_0(x)$ is a quasi-convex polynomial yields $q_\tau^0(v_i^N)=0$, $\tau=\mu_0,\dots,i_0$, which on the other hand by Lemma 2.6 implies that $v_i^N\in D_{f_0}^=$, and consequently $f_0(x)=f_0(v_i^N+W_k\xi)=f_0(W_k\xi)$ for both types of the objective function. Similarly it follows from the relation $v_i^N\in\mathcal{N}(A^T(J_k))$ that $v_i^N\in D_{f_j}^=$, $j\in J_k\setminus\{0\}$, which leads to the equality

$$\min\{f_0(x)\,|\,x\in\mathcal{R}_\gamma(J_k)\cap\mathcal{R}_\gamma(J_k,v_i)\}=\min\big\{f_0\big(v_i^N+W_k\xi\big)\big|\,f_j\big(v_i^N+W_k\xi\big)\le 0,$$
$$j\in J_k\setminus\{0\}\}=\min\{f_0(W_k\xi)\,|\,f_j(W_k\xi)\le 0,\quad j\in J_k\setminus\{0\}\},\quad i=1,2,\tag{20}$$

which completes the proof of the lemma. □

Lemma 3.3. *If $f_0(x)$ is bounded from below over $\mathcal{R}(J_k)$, then $f_0(x)$ is bounded from below over $\mathcal{R}(J_{k+1})$.*

Proof: It follows from Algorithm A that $\forall i\in J_k\setminus J_{k+1}$, the constraint $\langle a_i,s\rangle\le 0$, is not an implicit equality in the system (18) and (19), while $\forall i\in J_{k+1}$ the constraint $\langle a_i,s\rangle\le 0$ is an implicit equality in (18) and (19). Thus, $\forall i\in J_k\setminus J_{k+1}$ there exists an s_i satisfying (18) and (19) with $\langle a_i,s_i\rangle<0$ and with $\langle a_j,s_i\rangle=0$, $\forall j\in J_{k+1}$. Furthermore, since $f_0(x)$ is bounded from below over $\mathcal{R}(J_k)$ it follows from Theorem 2.1 that $\langle a_0,s_i\rangle=0$. Let

$$\hat{s}=\sum_{i\in J_k\setminus J_{k+1}}s_i.$$

It follows that $\hat{s}$ satisfies (18) and (19) with $\langle a_i,\hat{s}\rangle<0$, $i\in J_k\setminus J_{k+1}$, and $\langle a_i,\hat{s}\rangle=0$, $i\in J_{k+1}$. Since $0\in J_{k+1}$, then $B_0\hat{s}=0$, $\langle a_0,\hat{s}\rangle=0$. Thus in case when $f_0(x)$ is quasi-convex polynomial we have $q_\tau^l(\hat{s})=0$, $\tau=\mu_l,\mu_{l-1}\dots,i_l+1,i_l$, $l\in(J^2\cap J_{k+1})\cup\{0\}$, which by Lemma 2.6 gives that $\hat{s}\in D_{f_i}^=$, $i\in(J^2\cap J_{k+1})\cup\{0\}$. We note that the latter conclusion follows immediately for faithfully convex objective function and constraints, that is for constraints with indices $i\in(J^1\cap J_{k+1})\cup\{0\}$.

Let

$$x^*=\operatorname{argmin}\{f_0(x)\,|\,x\in\mathcal{R}(J_k)\}.$$

Then $\forall t>0$, we have

$$f_0(x^*+t\hat{s})=f_0(x^*),$$
$$f_i(x^*+t\hat{s})=f_i(x^*)\le b_i,\quad i\in J_{k+1}\setminus\{0\}.$$

We note that

$$f_i(x^* + t\hat{s}) = F_i(c_i + B_i(x^* + t\hat{s})) + \langle a_i, x^* + t\hat{s}\rangle + d_i < F_i(c_i + B_i x^*) \\ + \langle a_i, x^*\rangle + d_i = f_i(x^*) \le 0, \quad i \in (J_k \setminus J_{k+1}) \cap J^1.$$

Furthermore, since the system $B_i\hat{s} = 0,\ \langle a_i, \hat{s}\rangle < 0,\ i \in J^2 \cap (J_k \setminus J_{k+1})$, implies that

$$q^l_\tau(\hat{s}) = 0, \quad \tau = \mu_l, \mu_{l-1} \ldots, i_l + 1, \quad \text{and} \quad q^l_{i_l}(\hat{s}) < 0,\ l \in J^2 \cap (J_k \setminus J_{k+1}),$$

then by Lemma 2.6, $\hat{s} \in 0^+ f_i,\ i \in J^2 \cap (J_k \setminus J_{k+1})$. Therefore, $f_i(x^* + t\hat{s}) \le f_i(x^*) \le 0,\ \forall i \in J^2 \cap (J_k \setminus J_{k+1})$.

Since

$$f_0(x^* + t\hat{s}) = \min\{f_0(x) \mid x \in \mathcal{R}(J_k)\}, \quad t > 0, \tag{21}$$

and since the constraints with $i \in J_k \setminus J_{k+1}$ are not active at $x^* + t\hat{s}$, it follows that

$$f_0(x^* + t\hat{s}) = \min\{f_0(x) \mid x \in \mathcal{R}(J_{k+1})\}, \quad t \ge 0, \tag{22}$$

Therefore $f_0(x)$ is bounded from below over $\mathcal{R}(J_{k+1})$. □

In the next corollary we show that the algorithm can be used to reduce the number of variables and constraints in the problem (16) and (17).

Corollary 3.1. *If Algorithm B terminates in Step* 3 *with the index set* J_k, *then*

$$\min\{f_0(W_k\xi) \mid f_i(W_k\xi) \le 0, i \in J_k \setminus \{0\}\} = \min\{f_0(x) \mid x \in \mathcal{R}\}, \tag{23}$$

where $|J_k| \le m - k$ *and* $\xi \in \mathbb{R}^N$, *with* $N \le n - k$. *Furthermore, the solution to the problem* (16) and (17) *can be constructed from a solution to the reduced problem in* (23).

Proof: From (20) we have that $\min\{f_0(W_k\xi) \mid f_i(W_k\xi) \le 0, i \in J_k \setminus \{0\}\} = \min\{f_0(x) \mid x \in \mathcal{R}_\gamma(J_k)\} = \min\{f_0(x) \mid x \in \mathcal{R}(J_k)\}$. Let $\hat{\xi} = \operatorname{argmin}\{f_0(W_k\xi) \mid f_i(W_k\xi) \le 0, i \in J_k \setminus \{0\}\}$ and set $\hat{x}_k = W_k\hat{\xi}$, so that

$$\hat{x}_k \in \operatorname{argmin}\{f_0(x) \mid x \in \mathcal{R}(J_k)\}.$$

It follows from the proof of the Lemma 3.3 and Eqs. (21) and (22) that $\exists t_k \in \mathbb{R}$, and $\hat{s}_k \in \mathbb{R}^n$ such that $\hat{x}_k + t_k\hat{s}_k \in \mathcal{R}(J_{k-1})$, and

$$f_0(\hat{x}_k + t_k\hat{s}_k) = \min\{f_0(x) \mid x \in \mathcal{R}(J_{k-1})\}.$$

Repeating this step $(k-1)-$ times we end with $\hat{x} = (\hat{x}_1 + t_1\hat{s}_1) \in \mathcal{R}(J_0)$ and

$$f_0(\hat{x}) = \min\{f_0(x) \mid x \in \mathcal{R}(J_0)\}.$$

Since in each nonterminating iteration of Step 3 we have $J_{k+1} \subsetneq J_k$, then the number of elements in J_k decreases at least by one, which yields $|J_k| \leq m - k$. To show that $N \leq n - k$ we observe that $\forall j \in J_k \setminus J_{k+1}$, the vector $a_j \notin C(A(J_{k+1}))$ since for some s, $\langle a_j, s \rangle < 0$ and $B_i s = 0$, $\langle a_i, s \rangle = 0$, $\forall i \in J_{k+1}$. Thus the number of columns of the matrix W_k decreases at least by one in each iteration of the Algorithm. This completes the proof of the corollary. □

Using the results stated in the Theorem 2.1 and Lemmas 3.2, 3.3 and Corollary 3.1 we are now able to provide sufficient and necessary conditions for boundedness of the problem (16) and (17), stated in the following theorem.

Theorem 3.1. *Algorithm A terminates in Step* 2 *iff* f_0 *is unbounded from below over* $\mathcal{R}$, *and it terminates in Step* 3 *iff* f_0 *is bounded from below over* $\mathcal{R}$.

Proof: If Algorithm A terminates in Step 2 then by Theorem 2.1 $f_0(x)$ is unbounded below over $\mathcal{R}(J_k)$. Now applying the contrapositive of Lemma 3.3 k-times, yields that $f_0(x)$ is unbounded below over $\mathcal{R}$. If Algorithm A terminates in Step 3 then Lemma 3.2 implies that $f_0(x)$ is bounded below over $\mathcal{R}(J_k)$. Since $\mathcal{R} \subset \mathcal{R}(J_k)$, therefore $f_0(x)$ is also bounded below over $\mathcal{R}$. Furthermore, since by Corollary 3.1 the algorithm terminates after at most $\min\{m-1, n-1\}$ iterations, and it terminates in either Step 2 or Step 3, the hypotheses of the theorem follows. □

4. Extension of the Frank-Wolfe type theorem to (quasi-)convex objective function and constraints

In this section we establish an attainability result for the problem (16) and (17), proved in Theorem 4.1, as well as we show the continuity of the solution set defined by the system of inequalities in (17).

Theorem 4.1. *Let* f_i, $i \in J \cup \{0\}$ *be either faithfully convex functions satisfying condition* (3) *or quasi-convex polynomial functions. Then if the objective function* f_0 *is bounded from below on the nonempty feasible set*

$$\mathcal{R} = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0,\ i \in J\},$$

then infimum of the problem (16) and (17) *is attained.*

Proof: The Theorem 3.1 states that if f_0 is bounded below on $\mathcal{R}$ then Algorithm A terminates in Step 3 with the set of indices J_k, such that $J_k = J_{k+1}$. By Corollary 3.1 we have

$$\min\{f_0(W_k\xi) \mid f_i(W_k\xi) \leq 0, i \in J_k \setminus \{0\}\} = \min\{f_0(x) \mid x \in \mathcal{R}\}.$$

It follows from the Eq. (20) that

$$\min\{f_0(x) \mid x \in \mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)\} = \min\{f_0(W_k\xi) \mid f_j(W_k\xi) \leq 0,\ j \in J_k \setminus \{0\}\},$$

$\forall v \in \mathcal{R}_\gamma(J_k)$. However by Lemma 3.1 the set $\mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$ is bounded, which means that the minimum of $f_0(x)$ over $\mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)$ is attained. Furthermore, it has been demonstrated in the Corollary 3.1, that the solution to the problem (16) and (17) can be constructed from the solution to the problem $\min\{f_0(W_k\xi) \mid f_i(W_k\xi) \leq 0, i \in J_k \setminus \{0\}\}$, and consequently by the Eq. (20) it can also be constructed from the solution to the problem

$$\min\{f_0(x) \mid x \in \mathcal{R}_\gamma(J_k) \cap \mathcal{R}_\gamma(J_k, v)\}.$$

This completes the proof of the theorem. □

We will show using Lemma 2.6, that as a special case of Theorem 4.1 we obtain the following result for convex polynomial functions proved in [4]:

Corollary 4.1. *If f_i, $i \in J \cup \{0\}$ are convex polynomials then if f_0 is bounded below on $\mathcal{R}$, then the function f_0 attains its infimum over the region $\mathcal{R}$.*

We note that Theorem 4.1 generalizes similar result proved by Luo and Zhang in [11] for quasi-convex quadratic objective function and convex quadratic constraints, although our proof was entirely different than the approach presented in [11].

Remark 1. The result proved in the Theorem 4.1 can not be generalized to convex functions not satisfying condition (3). For example the function $f_0(x, y) = y$ has an unattained infimum over $\mathcal{R} = \{(x, y) | e^{-x} - y \leq 0,\ x \geq 1\}$ equal 0, although the sets $\mathcal{R}(\epsilon_k) = \{(x, y) | e^{-x} - y \leq 0,\ x \geq 1,\ y \leq \epsilon_k\}$ are nonempty for $\epsilon_k > 0$.

We note, thanks to the remark provided by a Referee, that an attainability result similar to the one stated in Theorem 4.1 for faithfully convex and/or quasi-convex polynomial objective function and constraints, was established earlier for (solely) quasi-convex polynomial functions by Bank and Mandel in [3]. However, while our proof is based upon the algorithm to determine unboundedness of the problem (16) and (17), their proof is based upon an entirely different idea.

Corollary 4.2. *Let $f_i(x)$, $i \in J$ be either faithfully convex functions satisfying conditions* (3) *or quasi-convex polynomial functions, and let us define for $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$,*

$$\mathcal{R}(\epsilon) = \{x \in \mathbb{R}^n \mid f_i(x) \leq \epsilon_i, i \in J\}.$$

Suppose that the sets $\mathcal{R}(\epsilon^k)$ are nonempty for some sequence $\{\epsilon^k\}$ of nonnegative vectors ϵ^k approaching zero. Then the set $\mathcal{R}$ is also nonempty.

Proof: Proof is similar to the proof of Corollary 1 in [4] and is based upon the result stated in Theorem 4.1. □

Corollary 4.3. *Let $f_i(x)$, $i \in J \cup \{0\}$ be either faithfully convex functions satisfying conditions* (3) *or quasi-convex polynomial functions and let us define for the sequence $\{\epsilon_k\}$, where $\epsilon_k > 0$, and $\epsilon_k \to 0$, the sets*

$$\mathcal{R}_0(\epsilon_k) = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in J,\ f_0(x) \leq f^* + \epsilon_k\},$$

where $f^* = \inf\{f_0(x) \mid x \in \mathcal{R}\}$ *is a finite number. Then if the sets* $\mathcal{R}_0(\epsilon_k)$ *are nonempty* $\forall k$, *then the problem* (16) *and* (17) *has an optimal solution.*

Proof: Proof is similar to the proof of Corollary 2 in [4] and is based upon the result stated in Corollary 4.2. □

Remark 2. Corollaries 4.2 and 4.3 similarly to the Theorem 4.1 can not be generalized to faithfully convex functions not satisfying condition (3). For example the set $\mathcal{R}$ defined by convex constraint $e^{-x} \leq \epsilon_k$ is feasible for any $\epsilon_k > 0$, but is infeasible for $\epsilon_k = 0$. Of course the function $f_0(x) = e^{-x}$ has an unattained unconstrained infimum equal 0.

Corollary 4.4. *The image of an affine mapping of a convex set* $\mathcal{R}$ *defined in* (17), *where the functions* f_i, $i \in J$, *are either faithfully convex and satisfy conditions* (3) *or quasi-convex polynomials is a closed set.*

Proof: Proof is similar to the proof of Corollary 1 in [11] and is based upon the result proved in Theorem 4.1. □

Remark 3. To show that Corollary 4.4 also cannot be extended to arbitrary convex functions let us consider the set $\mathcal{R}$ defined by the constraints: $x_1 \geq 0$ and $e^{-x_1} - x_2 \leq 0$, and the linear mapping $T(x_1, x_2) = x_2$. The image $T(\mathcal{R})$ of the set $\mathcal{R}$ is $(0, \infty)$, which is clearly not a closed set.

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