Comput Math Organ Theory (2017) 23:546–571
DOI 10.1007/s10588-017-9242-8

MANUSCRIPT

Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process

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Published online: 9 February 2017

Abstract Pythagorean fuzzy set, an extension of the intuitionistic fuzzy set which relax the condition of sum of their membership function to square sum of its membership functions is less than one. Under these environment and by incorporating the idea of the confidence levels of each Pythagorean fuzzy number, the present study investigated a new averaging and geometric operators namely confidence Pythagorean fuzzy weighted and ordered weighted operators along with their some desired properties. Based on its, a multi criteria decision-making method has been proposed and illustrated with an example for showing the validity and effectiveness of it. A computed results are compared with the aid of existing results.

Keywords Pythagorean fuzzy set · MCDM · Confidence levels · Aggregation operators · Decision making

1 Introduction

MCDM is one of the fast growing research active problem in these days for reaching a final decision within a reasonable time. But it is not always permissible to give the preferences in a precise manner due to various constraints and hence their corresponding results are not ideal in some circumstances. To handle it, an IFS theory (Atanassov 1986) is one of the successful and widely used by the researchers for dealing with the vagueness and impreciseness in the data. Under these environment, the various researchers pay more attention on IFSs for aggregating the different alternatives using different aggregation operators. In order to aggregate all the performance of the criteria for alternatives, weighted and ordered weighted aggregation operators (Yager 1988; Yager and Kacprzyk 1997) play an important

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role during the information fusion process. For instance, Xu and Yager (2006) presented a geometric aggregation operator while Xu (2007) presented a weighted averaging operator for aggregating the different intuitionistic fuzzy numbers. Later on, Wang and Liu (2012) extended these operators by using Einstein norm operations under IFS environment. Garg (2016d) presented a generalized improved score function to rank these numbers and applied it to the decision-making problems. Ye (2009) presented a new accuracy function for interval-valued IFS. Garg (2016h) proposed some series of interactive aggregations operators for intuitionistic fuzzy numbers (IFNs). Garg (2016a) presented a generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein norm operations for aggregating the different intuitionistic fuzzy information. Xu et al (2014) had presented the intuitionistic fuzzy Einstein Choquet integral based operators for decision making problems. Garg (2016b) presented a generalized intuitionistic fuzzy aggregation operator under the intuitionistic multiplicative preference relation instead of intuitionistic fuzzy preference relations. Apart from these, various authors have investigated the problem of the decision-making under the different environments (Nancy and Garg 2016a, b; Dalman 2016; Dalman et al. 2016; Yu 2014; Yu and Shi 2015; Kumar and Garg 2016; Ye 2007; Garg et al. 2015) and so on. A comprehensive analysis on MCDM using different approaches under IFS environment has been summarized in Yu (2015) and Xu and Zhao (2016).

From these above studies, it has been concluded that they are valid under the restrictions that sum of the grades of memberships is non greater than one. However, in day-today life, it is not always possible to give their preferences under this restriction. For instance, if a personal gives a preferences about the alternative satisfies the criteria is 0.8 while dissatisfies is 0.6. Therefore, it does not satisfies the IFS condition i.e., $0.8 + 0.6 \nleq 1$. Hence, under such circumstances, it is not possible for the decision maker to evaluate the performance and hence IFS theory have some drawbacks. In order to overcome these, Yager and Abbasov (2013) introduced Pythagorean fuzzy set (PFS) theory which is an extension of IFS theory by relaxing the conditions of $\mu + v \leq 1$ to $\mu^2 + v^2 \leq 1$, where μ and v represents the degrees of the membership/satisfication and non-membership/dis-satisfication of an element. Also, it has been observed that all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. After their pioneer work, researchers are actively working in the field of PFS to enhance it. Yager and Abbasov (2013) showed that the Pythagorean degrees are the subclasses of the complex numbers. Later on, Zhang and Xu (2014) presented a technique for finding the best alternative based on its ideal solution under the Pythagorean fuzzy environment. Yager (2014) developed various aggregation operators, namely, Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric average (PFWGA) operator, Pythagorean fuzzy weighted power average (PFWPA) operator and Pythagorean fuzzy weighted power geometric average (PFWPGA) operator to aggregate the different Pythagorean fuzzy numbers. Peng and Yang (2015) defined the some new arithmetical operations and their corresponding properties for PFNs. Garg (2016g) defined the concepts of correlation and correlation coefficients of

PFSs. Also, Garg (2016f) presented a novel accuracy function under the interval-valued pythagorean fuzzy set (IVPFS) for solving the decision-making problems. Garg (2016c, e), further, presented a generalized averaging aggregation operators under the Pythagorean fuzzy set environment by utilizing the Einstein norm operations.

Despite the popularizes of the above work, all the above studies have investigated without considering the confidence level of the attributes. In other words, all the researchers have investigated the studies by taking the assumption that decision makers are taken to be surely familiar with the evaluated objects. But in real-life situation, this type of conditions are partially fulfill. To overcome this shortcoming, the decision makers may evaluate the alternative in terms of PFNs and their corresponding confidence levels for their familiarity with the evaluation. Therefore, the present study incorporated the idea of the confidence levels into the aggregation process during the evaluation of the alternative in terms of PFNs. Based on these evaluations, some series of the averaging and geometric aggregations operators are proposed namely CPFWA, CPFOWA, CPFWG and CPFOWG along with their desired properties. Further, a MCDM method based on these operators have been proposed for solving the problems.

The rest of the manuscript has been summarized as follows. Section 2 describe the basic concept related to the Pythagorean fuzzy set and their subsequently operations. In Sect. 3, new series of aggregation operators namely CPFWA, CPFOWA, CPFWG and CPFOWG along with their properties. Section 4 presented an algorithm for solving MCDM problems under uncertainties based on the proposed operators. Section 5 gives a case study on finding the best alternative to illustrate the applicability and implementation process of the proposed approach. Finally, the paper ends up with some concluding remarks in Sect. 6.

2 Basic concepts

Definition 1 A Pythagorean fuzzy set (PFS) A is defined as a set of ordered pairs of membership and non-membership over a universal set X and is given as (Yager 2013)

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

where $\mu_A, \nu_A : X \longrightarrow [0,1]$ represent the degrees of membership and non-membership of the element $x \in X$ such that $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$. Corresponding to its membership functions, the degree of indeterminacy is given as $\pi_A(x) = \sqrt{1 - (\mu_A(x))^2 - (\nu_A(x))^2}$. For convenience, Zhang and Xu (2014) called $\langle \mu_A(x), \nu_A(x)\rangle$ a PFN denoted by $A = \langle \mu_A, \nu_A\rangle$ and the score function of A is defined as follows:

$$sc(A) = (\mu_A)^2 - (\nu_A)^2 \tag{1}$$

where $sc(A) \in [-1, 1]$, while, an accuracy function of A be defined as follows

$$ac(A) = (\mu_A)^2 + (\nu_A)^2 \tag{2}$$

where $ac(A) \in [0, 1]$.

Based on these functions, a prioritized comparison method for any two PFNs A and B is defined as follows.

Definition 2 Let A and B be any two PFNs.

1. If $sc(A) < sc(B)$, then $A \prec B$;
2. If $sc(A) > sc(B)$, then $A \succ B$;
3. If $sc(A) = sc(B)$,
   - (i) If $ac(A) < ac(B)$, then $A \prec B$.
   - (ii) If $ac(A) > ac(B)$, then $A \succ B$.
   - (iii) If $ac(A) = ac(B)$, then $A \sim B$.

Definition 3 Basic operations: For three PFNs $\alpha = \langle \mu, \nu \rangle$, $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$ and a real positive number λ, Yager (2013); Yager and Abbasov (2013) defined the basic operations under the algebraic norm operations and are defined as follows.

- $\alpha_1 \oplus \alpha_2 = \left\langle \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2 \right\rangle$.
- $\alpha_1 \otimes \alpha_2 = \left\langle \mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2} \right\rangle$.
- $\lambda \cdot \alpha = \left\langle \sqrt{1 - (1 - \mu^2)^{\lambda}}, \nu^{\lambda} \right\rangle$.
- $\alpha^{\lambda} = \left\langle \mu^{\lambda}, \sqrt{1 - (1 - \nu^2)^{\lambda}} \right\rangle$.

Based on these operations, an averaging and geometric aggregation operators namely as PFWA and PFWG respectively have been proposed for a collection of PFNs $\alpha_j (1 \le j \le n)$ as follows (Yager 2014).

$$PFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_j^2)^{\omega_j}}, \prod_{j=1}^{n} \nu_j^{\omega_j} \right\rangle \tag{3}$$

$$\text{and } PFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle \prod_{j=1}^{n} \mu_j^{\omega_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_j^2)^{\omega_j}} \right\rangle \tag{4}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the associated normalized weight vector.

Theorem 1 *Consider three PFNs* $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$, $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ *and* $\alpha_2 = \langle \mu_2, \nu_2 \rangle$ *and a real* $\lambda > 0$ *then* $\alpha_3 = \alpha^{\lambda}$, $\alpha_4 = \lambda\alpha$, $\alpha_5 = \alpha_1 \oplus \alpha_2$ and $\alpha_6 = \alpha_1 \otimes \alpha_2$ *are all PFNs.*

Proof Since $\alpha = \langle \mu, v \rangle$ be a PFN which means that $\mu_\alpha^2 + v_\alpha^2 \leq 1$. Therefore, $1 - v_\alpha^2 \geq \mu_\alpha^2 \geq 0$ and hence $(1 - v_\alpha^2)^\lambda \geq (\mu_\alpha^2)^\lambda \geq 0$. Thus, we have

$$\left((\mu_\alpha)^\lambda\right)^2 + \left(\sqrt{1-(1-(v_\alpha)^2)^\lambda}\right)^2 \leq 1.$$

Furthermore,

$$\left((\mu_\alpha)^\lambda\right)^2 + \left(\sqrt{1-(1-(v_\alpha)^2)^\lambda}\right)^2 = 0$$

iff $\mu_\alpha = v_\alpha = 0$
and

$$\left((\mu_\alpha)^\lambda\right)^2 + \left(\sqrt{1-(1-(v_\alpha)^2)^\lambda}\right)^2 = 1$$

iff $(\mu_\alpha)^2 + (v_\alpha)^2 = 1$.

Thus, $\alpha_3 = \alpha^\lambda$ is PFN. Similarly, we can prove that α_4, α_5 and α_6 are PFNs. □

Theorem 2 *Let* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then*

(i) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$
(ii) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$
(iii) $\lambda \cdot (\alpha_1 \oplus \alpha_2) = \lambda \cdot \alpha_1 \oplus \lambda \cdot \alpha_2$
(iv) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$
(v) $\lambda_1 \cdot \alpha \oplus \lambda_2 \cdot \alpha = (\lambda_1 + \lambda_2) \cdot \alpha$
(vi) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1 + \lambda_2}$

Theorem 3 *Let* $\alpha_1 = \langle \mu_1, v_1 \rangle$ *and* $\alpha_2 = \langle \mu_2, v_2 \rangle$ *be two PFNs then*

(i) $\alpha_1^c \wedge \alpha_2^c = (\alpha_1 \vee \alpha_2)^c$
(ii) $\alpha_1^c \vee \alpha_2^c = (\alpha_1 \wedge \alpha_2)^c$
(iii) $\alpha_1^c \oplus \alpha_2^c = (\alpha_1 \otimes \alpha_2)^c$
(iv) $\alpha_1^c \otimes \alpha_2^c = (\alpha_1 \oplus \alpha_2)^c$
(v) $(\alpha_1 \vee \alpha_2) \oplus (\alpha_1 \wedge \alpha_2) = \alpha_1 \oplus \alpha_2$
(vi) $(\alpha_1 \vee \alpha_2) \otimes (\alpha_1 \wedge \alpha_2) = \alpha_1 \otimes \alpha_2$

Proof The proof is trivial. □

3 Pythagorean fuzzy information aggregation operations with confidence levels

In the existing literature, all the researchers have investigated the studies by taking the assumption that decision makers are taken to be surely familiar with the evaluated objects. But in real-life situation, this type of conditions are partially fulfill. To overcome this shortcoming, in this section, we are presenting a series of an averaging and geometric aggregation operators with different confidence levels for their familiarity with the evaluation.

3.1 Averaging operator

Definition 4 Let Ω be a collection of PFNs $\alpha_1, \alpha_2, \ldots, \alpha_n$ and η_j be the confidence levels of PFN α_j such that $0 \le \eta_j \le 1$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of these PFNs such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$ and let $CPFWA : \Omega^n \longrightarrow \Omega$. If

$$CPFWA(\langle\alpha_1, \eta_1\rangle, \langle\alpha_2, \eta_2\rangle, \ldots, \langle\alpha_n, \eta_n\rangle) = \omega_1(\eta_1\alpha_1) \oplus \omega_2(\eta_2\alpha_2) \oplus \ldots \oplus \omega_n(\eta_n\alpha_n)$$

then CPFWA is called confidence Pythagorean fuzzy weighted averaging operator.

Theorem 4 *Let $\alpha_j = \langle\mu_j, v_j\rangle, j = 1, 2, \ldots, n$ be 'n' PFNs and η_j be its confidence levels then the aggregated value by CPFWA operator is also PFN and*

$$\begin{aligned} CPFWA(\langle\alpha_1, \eta_1\rangle, \langle\alpha_2, \eta_2\rangle, \ldots, \langle\alpha_n, \eta_n\rangle) &= \bigoplus_{j=1}^n \omega_j(\eta_j\alpha_j) \\ &= \left\langle \sqrt{1 - \prod_{j=1}^n (1 - \mu_j^2)^{\eta_j\omega_j}}, \prod_{j=1}^n (v_j)^{\eta_j\omega_j} \right\rangle \end{aligned} \tag{5}$$

where ω_j is the weight vector associate with α_j such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

Proof This result has been proved by using induction on n.

For $n = 2$, we have

$$CPFWA(\langle\alpha_1, \eta_1\rangle, \langle\alpha_2, \eta_2\rangle) = \omega_1(\eta_1\alpha_1) \oplus \omega_2(\eta_2\alpha_2)$$

According to Theorem 1, we can see that both $\eta_1\alpha_1$ and $\eta_2\alpha_2$ are PFNs, and the value of $\omega_1(\eta_1\alpha_1) \oplus \omega_2(\eta_2\alpha_2)$ is PFN. So, we have

$$\begin{aligned} \eta_1\alpha_1 &= \left\langle \sqrt{1 - (1 - \mu_1^2)^{\eta_1}}, v_1^{\eta_1} \right\rangle = \left\langle a_1, b_1 \right\rangle \\ \Rightarrow \omega_1(\eta_1\alpha_1) &= \left\langle \sqrt{1 - (1 - a_1^2)^{\omega_1}}, b_1^{\omega_1} \right\rangle = \left\langle \sqrt{1 - (1 - \mu_1^2)^{\eta_1\omega_1}}, v_1^{\eta_1\omega_1} \right\rangle \end{aligned}$$

and

$$\begin{aligned} \eta_2\alpha_2 &= \left\langle \sqrt{1 - (1 - \mu_2^2)^{\eta_2}}, v_2^{\eta_2} \right\rangle = \langle a_2, b_2 \rangle \\ \Rightarrow \omega_2(\eta_2\alpha_2) &= \left\langle \sqrt{1 - (1 - a_2^2)^{\omega_2}}, b_2^{\omega_2} \right\rangle = \left\langle \sqrt{1 - (1 - \mu_2^2)^{\eta_2\omega_2}}, v_2^{\eta_2\omega_2} \right\rangle \end{aligned}$$

Thus,

$$CPFWA(\left\langle \alpha_1, \eta_1 \right\rangle, \langle \alpha_2, \eta_2 \right\rangle) = \omega_1(\eta_1 \alpha_1) \oplus \omega_2(\eta_2 \alpha_2)$$
$$= \left\langle \sqrt{1-(1-\mu_1^2)^{\eta_1\omega_1} + 1 - (1-\mu_2^2)^{\eta_2\omega_2} - (1-(1-\mu_1^2)^{\eta_1\omega_1})(1-(1-\mu_2^2)^{\eta_2\omega_2})}, \nu_1^{\eta_1\omega_1}\nu_2^{\eta_2\omega_2} \right\rangle$$
$$= \left\langle \sqrt{1-(1-\mu_1^2)^{\eta_1\omega_1}(1-\mu_2^2)^{\eta_2\omega_2}}, \nu_1^{\eta_1\omega_1}\nu_2^{\eta_2\omega_2} \right\rangle$$

which is true.

Assume Eq. (5) holds for $n = k$, i.e.,

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_k, \eta_k \rangle) = \left\langle \sqrt{1 - \prod_{j=1}^{k}(1-\mu_j^2)^{\eta_j\omega_j}}, \prod_{j=1}^{k} \nu_j^{\eta_j\omega_j} \right\rangle$$

Now, by using the operational laws as PFNs for $n = k + 1$ we have,

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_{k+1}, \eta_{k+1} \rangle) =$$
$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_k, \eta_k \rangle) \oplus \omega_{k+1}(\eta_{k+1}\alpha_{k+1})$$
$$= \left\langle \sqrt{1 - \prod_{j=1}^{k}(1-\mu_j^2)^{\eta_j\omega_j}}, \prod_{j=1}^{k} \nu_j^{\eta_j\omega_j} \right\rangle \oplus \left\langle \sqrt{1-(1-\mu_{k+1}^2)^{\eta_{k+1}\omega_{k+1}}}, \nu_{k+1}^{\eta_{k+1}\omega_{k+1}} \right\rangle$$
$$= \left\langle \sqrt{1 - \prod_{j=1}^{k+1}(1-\mu_j^2)^{\eta_j\omega_j}}, \prod_{j=1}^{k+1} \nu_j^{\eta_j\omega_j} \right\rangle$$

i.e., when $n = k + 1$, Eq. (5) also holds.

Hence, Eq. (5) holds for any n.

Next, in order to show *CPFWA* is PFN.

As $\alpha_j = \langle \mu_j, \nu_j \rangle$ for all j is PFN, thus $0 \le \mu_j, \nu_j \le 1$ and $\mu_j^2 + \nu_j^2 \le 1$. Therefore, $0 \le 1 - \mu_j^2 \le 1$ which implies that $0 \le \prod_{j=1}^{n}(1-\mu_j^2)^{\eta_j\omega_j} \le 1$ and hence $0 \le \sqrt{1 - \prod_{j=1}^{n}(1-\mu_j^2)^{\eta_j\omega_j}} \le 1$ and $0 \le \prod_{j=1}^{n} \nu_j^{\eta_j\omega_j} \le 1$.

Again,

$$\left(\sqrt{1 - \prod_{j=1}^{n}(1-\mu_j^2)^{\eta_j\omega_j}}\right)^2 + \left(\prod_{j=1}^{n} \nu_j^{\eta_j\omega_j}\right)^2 = 1 - \prod_{j=1}^{n}(1-\mu_j^2)^{\eta_j\omega_j} + \prod_{j=1}^{n} \nu_j^{2\eta_j\omega_j}$$
$$\le 1 - \prod_{j=1}^{n} \nu_j^{2\eta_j\omega_j} + \prod_{j=1}^{n} \nu_j^{2\eta_j\omega_j} = 1$$

Hence, *CPFWA* operator is PFN and therefore proof is completed. □

Remark 1 If all $\eta_j = 1$ then the CPFWA reduces to PFWA operator (Yager 2014)

$$PFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_j^2)^{\omega_j}}, \prod_{j=1}^{n} v_j^{\omega_j} \right\rangle$$

Example 1 Let $\alpha_1 = \langle (0.5, 0.7), 0.7 \rangle$, $\alpha_2 = \langle (0.8, 0.4), 0.8 \rangle$ and $\alpha_3 = \langle (0.3, 0.6), 0.85 \rangle$ be three PFNs with confidence levels and $\omega = (0.25, 0.40, 0.35)^T$ be their corresponding weight vectors then

$$\prod_{j=1}^{n} \left(1 - \mu_j^2\right)^{\eta_j \omega_j} = \left(1 - 0.5^2\right)^{0.7 \times 0.25} \times \left(1 - 0.8^2\right)^{0.8 \times 0.40} \times \left(1 - 0.3^2\right)^{0.85 \times 0.35} = 0.6668$$

$$\prod_{j=1}^{n} (v_j)^{\eta_j \omega_j} = (0.7)^{0.7 \times 0.25} \times (0.4)^{0.8 \times 0.40} \times (0.6)^{0.85 \times 0.35} = 0.6019$$

Thus, by Eq. (5), we get

$$\begin{aligned} CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \langle \alpha_3, \eta_3 \rangle) &= \left\langle \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_j^2)^{\eta_j \omega_j}}, \prod_{j=1}^{n} v_j^{\eta_j \omega_j} \right\rangle \\ &= \left\langle 0.5773, 0.6019 \right\rangle \end{aligned}$$

For a collections of PFNs, the proposed aggregation operator CPFWA satisfies the following properties.

Property 1 (Idempotency) *If $\alpha_j = \alpha_0 = \langle (\mu_0, v_0), \eta_0 \rangle$ for all j, i.e., $\mu_j = \mu_0, v_j = v_0$ and $\eta_j = \eta_0$ then*

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \eta_0 \alpha_0$$

Proof Since $\alpha_j = \alpha_0 = \langle (\mu_0, v_0), \eta_0 \rangle$ for all j and $\sum_{j=1}^{n} \omega_i = 1$, so by Theorem 4,

$$\begin{aligned} CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) &= \left\langle \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_0^2)^{\eta_0 \omega_j}}, \prod_{j=1}^{n} v_0^{\eta_0 \omega_j} \right\rangle \\ &= \left\langle \sqrt{1 - (1 - \mu_0^2)^{\sum_{j=1}^{n} \eta_0 \omega_j}}, v_0^{\sum_{j=1}^{n} \eta_0 \omega_j} \right\rangle \\ &= \left\langle \sqrt{1 - (1 - \mu_0^2)^{\eta_0}}, v_0^{\eta_0} \right\rangle \\ &= \eta_0 \alpha_0 \end{aligned}$$

Hence proof is complete. □

Property 2 (Boundedness) *Let* $\alpha^- = \left\langle \min_j\{\eta_j\mu_j\}, \max_j\{\eta_j\nu_j\}\right\rangle$ *and* $\alpha^+ = \left\langle \max_j\{\eta_j\mu_j\}, \min_j\{\eta_j\nu_j\}\right\rangle$ *then*

$$\alpha^- \leq CPFWA(\langle\alpha_1,\eta_1\rangle, \langle\alpha_2,\eta_2\rangle, \ldots, \langle\alpha_n,\eta_n\rangle) \leq \alpha^+ \tag{6}$$

Proof As $\min_j\{\mu_j\} \leq \mu_j \leq \max_j\{\mu_j\}$ for $j = 1, 2, \ldots, n$ this implies, $1 - (\max_j\{\mu_j\})^2 \leq 1 - \mu_j^2 \leq 1 - (\min_j\{\mu_j\})^2$ then for all j, we have

$$\prod_{j=1}^{n}\left(1 - (\max_j\{\mu_j\})^2\right)^{\eta_j\omega_j} \leq \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j} \leq \prod_{j=1}^{n}\left(1 - (\min_j\{\mu_j\})^2\right)^{\eta_j\omega_j}$$

$$\Rightarrow\left(1 - (\max_j\{\mu_j\})^2\right)^{\sum_{j=1}^{n}\eta_j\omega_j} \leq \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j} \leq \left(1 - (\min_j\{\mu_j\})^2\right)^{\sum_{j=1}^{n}\eta_j\omega_j}$$

$$\Rightarrow(1 - (\max_j\{\mu_j\})^2)^{\eta_j} \leq \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j} \leq (1 - (\min_j\{\mu_j\})^2)^{\eta_j}$$

$$\Rightarrow 1 - (1 - (\min_j\{\mu_j\})^2)^{\eta_j} \leq 1 - \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j} \leq 1 - (1 - (\max_j\{\mu_j\})^2)^{\eta_j}$$

$$\Rightarrow\sqrt{1 - (1 - (\min_j\{\mu_j\})^2)^{\eta_j}} \leq \sqrt{1 - \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j}} \leq \sqrt{1 - (1 - (\max_j\{\mu_j\})^2)^{\eta_j}}$$

$$i.e., \min_j\{\eta_j\nu_j\} \leq \sqrt{1 - \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j\omega_j}} \leq \max_j\{\eta_j\mu_j\}.$$

Furthermore, $\min_j\{\nu_j\} \leq \nu_j \leq \max_j\{\nu_j\}$ for all $j = 1, 2, \ldots, n$ this implies that $(\min_j\{\nu_j\})^{\eta_j\omega_j} \leq (\nu_j)^{\eta_j\omega_j} \leq (\max_j\{\nu_j\})^{\eta_j\omega_j}$ and hence $(\min_j\{\nu_j\})^{\eta_j} \leq \prod_{j=1}^{n}(\nu_j)^{\eta_j\omega_j}$ $\leq (\max_j\{\nu_j\})^{\eta_j}$, i.e., $\min_j\{\eta_j\nu_j\} \leq \prod_{j=1}^{n}(\nu_j)^{\eta_j\omega_j} \leq \max_j\{\eta_j\nu_j\}$.

Let $\alpha = CPFWA(\langle\alpha_1,\eta_1\rangle, \langle\alpha_2,\eta_2\rangle, \ldots, \langle\alpha_n,\eta_n\rangle) = \langle\mu_\alpha, \nu_\alpha\rangle$. Then, we have $\min_j\{\eta_j\mu_j\} \leq \mu_\alpha \leq \max_j\{\eta_j\mu_j\}$ and $\min_j\{\eta_j\nu_j\} \leq \nu_\alpha \leq \max_j\{\eta_j\nu_j\}$. So by definition of score function, we have

$$sc(\alpha) = \mu_\alpha^2 - \nu_\alpha^2 \leq (\max_j\{\eta_j\mu_j\})^2 - (\min_j\{\eta_j\nu_j\})^2 = sc(\alpha^+)$$
$$sc(\alpha) = \mu_\alpha^2 - \nu_\alpha^2 \geq (\min_j\{\eta_j\mu_j\})^2 - (\max_j\{\eta_j\nu_j\})^2 = sc(\alpha^-)$$

In that direction, three cases are considered.

Case 1: If $sc(\alpha) < sc(\alpha^+)$ and $sc(\alpha) > sc(\alpha^-)$ then it follows from Definition 2 that

$$\alpha^- < CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) < \alpha^+$$

Case 2: If $sc(\alpha) = sc(\alpha^+)$ i.e., $\mu_\alpha^2 - \nu_\alpha^2 = (\max_j\{\eta_j \mu_j\})^2 - (\min_j\{\eta_j \nu_j\})^2$, then by above inequalities, we have $\mu_\alpha = \max_j\{\eta_j \mu_j\}$ and $\nu_\alpha = \min_j\{\eta_j \nu_j\}$. Thus

$$ac(\alpha) = \mu_\alpha^2 + \nu_\alpha^2 = (\max_j\{\eta_j \mu_j\})^2 + (\min_j\{\eta_j \nu_j\})^2$$

then it follows from Definition 2 that

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \alpha^+$$

Case 3: If $sc(\alpha) = sc(\alpha^-)$ i.e., $\mu_\alpha^2 - \nu_\alpha^2 = (\min_j\{\eta_j \mu_j\})^2 - (\max_j\{\eta_j \nu_j\})^2$, then we get $\mu_\alpha = \min_j\{\eta_j \mu_j\}$ and $\nu_\alpha = \max_j\{\eta_j \nu_j\}$. Thus

$$ac(\alpha) = \mu_\alpha^2 + \nu_\alpha^2 = (\min_j\{\eta_j \mu_j\})^2 + (\max_j\{\eta_j \nu_j\})^2$$

then it follows from Definition 2 that

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \alpha^-$$

Hence, Eq. (8) holds. □

Property 3 (Monotonicity) *If α_j and β_j be two different collections of PFNs such that $\alpha_j \leq \beta_j$ for all j then*

$$CPFWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) \leq CPFWA(\langle \beta_1, \eta_1 \rangle, \langle \beta_2, \eta_2 \rangle, \ldots, \langle \beta_n, \eta_n \rangle)$$

Proof Proof of this property is similar to that of above, so we omit here. □

3.2 Ordered weighted averaging operator

Definition 5 Let Ω be a family of PFNs α_j, η_j be its confidence levels such that $0 \leq \eta_j \leq 1$. A confidence Pythagorean fuzzy ordered weighted averaging (CPFOWA) operator is a mapping $\Omega^n \longrightarrow \Omega$:

$$CPFOWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \omega_1(\eta_{\delta(1)} \alpha_{\delta(1)}) \oplus \omega_2(\eta_{\delta(1)} \alpha_{\delta(1)}) \oplus \ldots \oplus \omega_n(\eta_{\delta(1)} \alpha_{\delta(1)})$$

where $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\alpha_{\delta(j-1)} \geq \alpha_{\delta(j)}$ for any j.

Theorem 5 *Let $\alpha_j = \langle \mu_j, v_j \rangle$ be n' PFNs and η_j be its confidence levels then the aggregated value by CPFOWA operator is also PFN and given by*

$$CPFOWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \left\langle \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\delta(j)}^2)^{\eta_{\delta(j)}\omega_j}}, \prod_{j=1}^{n} (v_{\delta(j)})^{\eta_{\delta(j)}\omega_j} \right\rangle \quad (7)$$

Proof Proof of this Theorem is similar to that of Theorem 4, so we omit here. □

Example 2 Let $\alpha_1 = \langle (0.3, 0.9), 0.75 \rangle$, $\alpha_2 = \langle (0.5, 0.8), 0.80 \rangle$, $\alpha_3 = \langle (0.8, 0.4), 0.7 \rangle$ and $\alpha_4 = \langle (0.7, 0.5), 0.90 \rangle$ be four PFNs and $\omega = (0.2, 0.3, 0.1, 0.4)^T$ be their corresponding weight vectors. Then score values of each PFN is $sc(\alpha_1) = 0.3^2 - 0.9^2 = -0.72$, $sc(\alpha_2) = 0.5^2 - 0.8^2 = -0.39$, $sc(\alpha_3) = 0.8^2 - 0.4^2 = 0.48$ and $sc(\alpha_4) = 0.7^2 - 0.5^2 = 0.24$. Thus $\alpha_3 > \alpha_4 > \alpha_2 > \alpha_1$ and therefore $\alpha_{\delta(1)} = \alpha_3$, $\alpha_{\delta(2)} = \alpha_4$, $\alpha_{\delta(3)} = \alpha_2$ and $\alpha_{\delta(4)} = \alpha_1$. Now, we have

$$\begin{aligned}
\prod_{j=1}^{4} \left(1 - \mu_{\delta(j)}^2\right)^{\eta_{\delta(j)}\omega_j} &= \left(1 - 0.8^2\right)^{0.7\times0.2} \times \left(1 - 0.7^2\right)^{0.9\times0.3} \times \left(1 - 0.5^2\right)^{0.8\times0.1} \\
&\quad \times \left(1 - 0.3^2\right)^{0.75\times0.4} \\
&= 0.6865 \\
\prod_{j=1}^{4} (v_{\delta(j)})^{\eta_{\delta(j)}\omega_j} &= (0.4)^{0.7\times0.2} \times (0.5)^{0.9\times0.3} \times (0.8)^{0.8\times0.1} \times (0.9)^{0.75\times0.4} \\
&= 0.6943
\end{aligned}$$

So, by Eq. (7), we get

$$\begin{aligned}
CPFOWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \langle \alpha_3, \eta_3 \rangle, \langle \alpha_4, \eta_4 \rangle) &= \left\langle \sqrt{1 - \prod_{j=1}^{4} (1 - \mu_{\delta(j)}^2)^{\eta_{\delta(j)}\omega_j}}, \prod_{j=1}^{4} (v_{\delta(j)})^{\eta_{\delta(j)}\omega_j} \right\rangle \\
&= \langle 0.5599, 0.6943 \rangle
\end{aligned}$$

CPFOWA operator follows the same properties as that of CPFWA operator which have been stated, without proof, as follows for the collections of PFNs α_j, $(j = 1, 2, \ldots, n)$.

Property 4

(i) (Idempotency) *If $\alpha_j = \alpha_0 = \langle (\mu_0, v_0), \eta_0 \rangle$ for all j, i.e., $\mu_j = \mu_0, v_j = v_0$ and $\eta_j = \eta_0$ then*

$$CPFOWA(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_n, \eta_n \rangle) = \eta_0 \alpha_0$$

(ii) (Boundedness) *Let* $\alpha^- = \left\langle \min_j\{\eta_j\mu_j\}, \max_j\{\eta_j v_j\}\right\rangle$ *and* $\alpha^+ = \left\langle \max_j\{\eta_j\mu_j\}, \min_j\{\eta_j v_j\}\right\rangle$ *then*

$$\alpha^- \leq CPFOWA(\langle\alpha_1,\eta_1\rangle, \langle\alpha_2,\eta_2\rangle, \ldots, \langle\alpha_n,\eta_n\rangle) \leq \alpha^+ \tag{8}$$

(iii) (Monotonicity) *For collections of two different PFNs* $\alpha_j = \langle\mu_{\alpha_j}, v_{\alpha_j}\rangle$ *and* $\beta_j = \langle\mu_{\beta_j}, v_{\beta_j}\rangle$ *which satisfies the relation* $\alpha_j \leq \beta_j$ *for all* $j = 1, 2, \ldots, n$ *i.e., if* $\mu_{\alpha_j} \leq \mu_{\beta_j}$ *and* $v_{\alpha_j} \geq v_{\beta_j}$ *for all j, then*

$$CPFOWA(\langle\alpha_1,\eta_1\rangle, \langle\alpha_2,\eta_2\rangle, \ldots, \langle\alpha_n,\eta_n\rangle) \leq CPFOWA(\langle\beta_1,\eta_1\rangle, \langle\beta_2,\eta_2\rangle, \ldots, \langle\beta_n,\eta_n\rangle)$$

3.3 Geometric operator

In this section, a confidence Pythagorean fuzzy information aggregation from the geometric mean has been presented over the families of the PFNs denoted by Ω and their corresponding aggregation operator.

Definition 6 Let $\alpha_j = \langle\mu_j, v_j\rangle$ be 'n' PFNs and η_j be its confidence levels. A confidence Pythagorean fuzzy weighted geometric (CPFWG) operator is a mapping $\Omega^n \longrightarrow \Omega$ such that

$$CPFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigotimes_{j=1}^{n}(\alpha_j^{\eta_j})^{\omega_j} = (\alpha_1^{\eta_1})^{\omega_1} \otimes (\alpha_2^{\eta_2})^{\omega_2} \otimes \ldots \otimes (\alpha_n^{\eta_n})^{\omega_n} \tag{9}$$

where $(\omega_1, \omega_2, \ldots, \omega_n)^T$ be the normalized weight vector of α_j.

Theorem 6 *Let* $\alpha_j = \left\langle\mu_j, v_j\right\rangle$ *be a collection of 'n' PFNs and* η_j *be its confidence levels such that* $0 \leq \eta_j \leq 1$ *for* $j = 1, 2, \ldots, n$ *then aggregated value by CPFWG operator is also PFN and*

$$CPFWG(\langle\alpha_1,\eta_1\rangle, \langle\alpha_2,\eta_2\rangle, \ldots, \langle\alpha_n,\eta_n\rangle) = \bigotimes_{j=1}^{n}(\alpha_j^{\eta_j})^{\omega_j} = \left\langle \prod_{j=1}^{n}(\mu_j)^{\eta_j\omega_j}, \sqrt{1 - \prod_{j=1}^{n}(1 - v_j^2)^{\eta_j\omega_j}}\right\rangle \tag{10}$$

where ω_j *is the weight vector associate with* α_j *such that* $\omega_j \in [0,1]$ *and* $\sum_{j=1}^{n}\omega_j = 1$.

Proof For $n = 2$, we have

$$CPFWG(\left\langle\alpha_1,\eta_1\right\rangle, \left\langle\alpha_2,\eta_2\right\rangle) = (\alpha_1^{\eta_1})^{\omega_1} \otimes (\alpha_2^{\eta_2})^{\omega_2}$$

According to Theorem 1, we can see that both $\alpha_1^{\eta_1}$ and $\alpha_2^{\eta_2}$ are PFNs, and the value of $(\alpha_1^{\eta_1})^{\omega_1} \otimes (\alpha_2^{\eta_2})^{\omega_2}$ is PFN. So,

$$\alpha_1^{\eta_1} = \left\langle \mu_1^{\eta_1}, \sqrt{1-(1-v_1^2)^{\eta_1}} \right\rangle = \langle a_1, b_1 \rangle$$
$$\Rightarrow (\alpha_1^{\eta_1})^{\omega_1} = \left\langle a_1^{\omega_1}, \sqrt{1-(1-b_1^2)^{\omega_1}} \right\rangle = \left\langle \mu_1^{\eta_1\omega_1}, \sqrt{1-(1-v_1^2)^{\eta_1\omega_1}} \right\rangle$$

and

$$\alpha_2^{\eta_2} = \left\langle \mu_2^{\eta_2}, \sqrt{1-(1-v_2^2)^{\eta_2}} \right\rangle = \langle a_2, b_2 \rangle$$
$$\Rightarrow (\alpha_2^{\eta_2})^{\omega_2} = \left\langle a_2^{\omega_2}, \sqrt{1-(1-b_2^2)^{\omega_2}} \right\rangle = \left\langle \mu_2^{\eta_2\omega_2}, \sqrt{1-(1-v_2^2)^{\eta_2\omega_2}} \right\rangle$$

Then

$$CPFWG(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle) = (\alpha_1^{\eta_1})^{\omega_1} \otimes (\alpha_2^{\eta_2})^{\omega_2}$$
$$= \left\langle \mu_1^{\eta_1\omega_1}\mu_2^{\eta_2\omega_2}, \sqrt{1-(1-v_1^2)^{\eta_1\omega_1} + 1-(1-v_2^2)^{\eta_2\omega_2} - (1-(1-v_1^2)^{\eta_1\omega_1})(1-(1-v_2^2)^{\eta_2\omega_2})} \right\rangle$$
$$= \left\langle \mu_1^{\eta_1\omega_1}\mu_2^{\eta_2\omega_2}, \sqrt{1-(1-v_1^2)^{\eta_1\omega_1}(1-v_2^2)^{\eta_2\omega_2}} \right\rangle$$

which is true for $n = 2$.

Assume Eq. (10) holds for $n = k$, i.e.,

$$CPFWG(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_k, \eta_k \rangle) = \left\langle \prod_{j=1}^{k} \mu_j^{\eta_j\omega_j}, \sqrt{1-\prod_{j=1}^{k}(1-v_j^2)^{\eta_j\omega_j}} \right\rangle$$

Now for $n = k+1$,

$$CPFWG(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_{k+1}, \eta_{k+1} \rangle) =$$
$$CPFWG(\langle \alpha_1, \eta_1 \rangle, \langle \alpha_2, \eta_2 \rangle, \ldots, \langle \alpha_k, \eta_k \rangle) \otimes (\alpha_{k+1}^{\eta_{k+1}})^{\omega_{k+1}}$$
$$= \left\langle \prod_{j=1}^{k} \mu_j^{\eta_j\omega_j}, \sqrt{1-\prod_{j=1}^{k}(1-v_j^2)^{\eta_j\omega_j}} \right\rangle \otimes \left\langle \mu_{k+1}^{\eta_{k+1}\omega_{k+1}}, \sqrt{1-(1-v_{k+1}^2)^{\eta_{k+1}\omega_{k+1}}} \right\rangle$$
$$= \left\langle \prod_{j=1}^{k+1} \mu_j^{\eta_j\omega_j}, \sqrt{1-\prod_{j=1}^{k+1}(1-v_j^2)^{\eta_j\omega_j}} \right\rangle$$

i.e. Eq. (10) holds for $n = k+1$.

Hence, Eq. (10) holds for any n.

Next, in order to show *CPFWG* is an PFN.

As $\alpha_j = \langle \mu_j, v_j \rangle$ for all j is an PFN, thus $0 \le \mu_j, v_j \le 1$ and $\mu_j^2 + v_j^2 \le 1$. Thus $0 \le \prod_{j=1}^{n}(1-v_j^2)^{\eta_j\omega_j} \le 1$ and hence $0 \le \sqrt{1-\prod_{j=1}^{n}(1-v_j^2)^{\eta_j\omega_j}} \le 1$ and

$0 \le \prod_{j=1}^{n} \mu_j^{\eta_j \omega_j} \le 1.$

Again,

$$\left(\sqrt{1-\prod_{j=1}^{n}(1-v_j^2)^{\eta_j\omega_j}}\right)^2+\left(\prod_{j=1}^{n}\mu_j^{\eta_j\omega_j}\right)^2=1-\prod_{j=1}^{n}(1-v_j^2)^{\eta_j\omega_j}+\prod_{j=1}^{n}\mu_j^{2\eta_j\omega_j}$$
$$\le 1-\prod_{j=1}^{n}\mu_j^{2\eta_j\omega_j}+\prod_{j=1}^{n}\mu_j^{2\eta_j\omega_j}=1$$

Hence, *CPFWG* is PFN. □

Remark 2 If all $\eta_j = 1$ then the CPFWG reduces to PFWG operator (Yager 2014)

$$PFWG(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle \prod_{j=1}^{n}\mu_j^{\omega_j},\sqrt{1-\prod_{j=1}^{n}(1-v_j^2)^{\omega_j}}\right\rangle$$

Example 3 Let $\alpha_1 = \langle(0.4, 0.7), 0.7\rangle$, $\alpha_2 = \langle(0.7, 0.5), 0.8)$ and $\alpha_3 = \langle(0.8, 0.4), 0.7\rangle$ be three PFNs and $\omega = (0.4, 0.3, 0.3)^T$ be its associated weight vectors then by utilizing CPFWG operator, we have

$$\prod_{j=1}^{3}(\mu_j^{\eta_j})^{\omega_j}=(0.4^{0.7})^{0.4}\times(0.7^{0.8})^{0.3}\times(0.8^{0.7})^{0.3}=0.6777$$
$$\prod_{j=1}^{3}(1-v_j^2)^{\eta_j\omega_j}=(1-0.7^2)^{0.7\times0.4}\times(1-0.5^2)^{0.8\times0.3}\times(1-0.4^2)^{0.7\times0.3}=0.7451$$

Thus, $CPFWG(\alpha_1,\alpha_2,\alpha_3)=\langle 0.6777,\sqrt{1-0.7451}\rangle=\langle 0.6777, 0.5048\rangle$.

Property 5 Let α_j and β_j be collections of two different PFNs and $0 \le \eta_j \le 1$ be its confidence levels of them, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vectors of PFNs such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$; then

(i) (Idempotency) *If $\alpha_j = \alpha_0 = \langle(\mu_0, v_0), \eta_0\rangle$ for all j then*

$$CPFWG(\langle\alpha_1,\eta_1\rangle,\langle\alpha_2,\eta_2\rangle,\ldots,\langle\alpha_n,\eta_n\rangle)=\alpha_0^{\eta_0}$$

(ii) (Boundedness) *Let $\alpha^- = \left\langle \min_j\{\mu_j^{\eta_j}\}, \max_j\{v_j^{\eta_j}\}\right\rangle$ and $\alpha^+ = \left\langle \max_j\{\mu_j^{\eta_j}\}, \min_j\{v_j^{\eta_j}\}\right\rangle$ then*

$$\alpha^- \le CPFWG(\langle\alpha_1,\eta_1\rangle,\langle\alpha_2,\eta_2\rangle,\ldots,\langle\alpha_n,\eta_n\rangle)\le\alpha^+$$

(iii) (Monotonicity) *If* $\alpha_j \leq \beta_j$ *for all j then*

$$CPFWG(\langle\alpha_1,\eta_1\rangle,\langle\alpha_2,\eta_2\rangle,\ldots,\langle\alpha_n,\eta_n\rangle) \leq CPFWG(\langle\beta_1,\eta_1\rangle,\langle\beta_2,\eta_2\rangle,\ldots,\langle\beta_n,\eta_n\rangle)$$

3.4 Ordered weighted geometric operator

Definition 7 Suppose Ω be a family of PFNs $\alpha_j = \langle\mu_j, \nu_j\rangle$ and η_j be the confidence levels such that $0 \leq \eta_j \leq 1$ for $j = 1, 2, \ldots, n$ and $CPFOWG : \Omega^n \longrightarrow \Omega$, if

$$CPFOWG(\langle\alpha_1,\eta_1\rangle,\langle\alpha_2,\eta_2\rangle,\ldots,\langle\alpha_n,\eta_n\rangle) = (\alpha_{\delta(1)}^{\eta_{\delta(1)}})^{\omega_1} \otimes (\alpha_{\delta(2)}^{\eta_{\delta(2)}})^{\omega_2} \otimes \ldots \otimes (\alpha_{\delta(n)}^{\eta_{\delta(n)}})^{\omega_n}$$

where δ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\alpha_{\delta(j-1)} \geq \alpha_{\delta(j)}$ for any j. Then CPFOWG is called confidence pythagorean fuzzy ordered weighted geometric operator.

Theorem 7 *The aggregate value by CPFOWG operator for PFNs* $\alpha_j = \langle\mu_j, \nu_j\rangle$ *is again PFN and given by*

$$\begin{aligned} CPFOWG(\left\langle\alpha_1,\eta_1\right\rangle,\langle\alpha_2,\eta_2\rangle,\ldots,\langle\alpha_n,\eta_n\rangle) &= \bigotimes_{j=1}^{n}(\alpha_{\delta(j)}^{\eta_{\delta(j)}})^{\omega_j} \\ &= \left\langle \prod_{j=1}^{n}(\mu_{\delta(j)})^{\eta_{\delta(j)}\omega_j}, \sqrt{1-\prod_{j=1}^{n}(1-\nu_{\delta(j)}^2)^{\eta_{\delta(j)}\omega_j}} \right\rangle \end{aligned} \tag{11}$$

where η_j *is confidence levels of and* ω_j *is the normalized weight vector of* α_j.

Proof Proof of this theorem is similar to that of Theorem 6. □

Example 4 Consider the data given in Example 3 then by score function sc, we have $sc(\alpha_1) = 0.4^2 - 0.7^2 = -0.33$, $sc(\alpha_2) = 0.7^2 - 0.5^2 = 0.24$ and $sc(\alpha_3) = 0.8^2 - 0.4^2 = 0.48$. Thus $sc(\alpha_3) \geq sc(\alpha_2) \geq sc(\alpha_1)$ we have $\alpha_3 \geq \alpha_2 \geq \alpha_1$. So $\alpha_{\delta(1)} = \alpha_3$, $\alpha_{\delta(2)} = \alpha_2$ and $\alpha_{\delta(3)} = \alpha_1$. Therefore,

$$\prod_{j=1}^{3}(\mu_j^{\eta_{\delta(j)}})^{\omega_{\delta(j)}} = (0.8^{0.7})^{0.4} \times (0.7^{0.8})^{0.3} \times (0.4^{0.7})^{0.3} = 0.7114$$

$$\prod_{j=1}^{3}(1-\nu_{\delta(j)}^2)^{\eta_{\delta(j)}\omega_j} = (1-0.4^2)^{0.7\times0.4} \times (1-0.5^2)^{0.8\times0.3} \times (1-0.7^2)^{0.7\times0.3} = 0.7716$$

Thus, $CPFOWG(\alpha_1, \alpha_2, \alpha_3) = \langle 0.7114, 0.4779\rangle$.

As similar to CPFWG operator, CPFOWG operators follows the same properties as given in Property 5.

4 Group decision making approach under confidence levels

Consider a decision-making problem with a collection of m different alternatives $A = \{A_1, A_2, \ldots, A_m\}$ and n criteria $C = \{C_1, C_2, \ldots, C_n\}$ whose weight vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Assume that there are k set of decision makers denoted by $D = \{D_1, D_2, \ldots, D_k\}$, whose weight vector is $\xi = (\xi_1, \xi_2, \ldots, \xi_k)^T$ satisfying $\xi_q > 0, q = 1, 2, \ldots, k$ and $\sum_{q=1}^{k} \xi_q = 1$ which are evaluating each alternative A_i w.r.t. the criteria C_j in terms of PFNs. Then following are the steps utilize for finding the best alternative under the set of feasible ones.

Step 1: Collect the information related to each alternative A_i under the different criteria C_j from each decision maker and are summarized in the form of PFNs $D^q = \langle \mu_{ij}^q, v_{ij}^q \rangle_{m\times n}$ for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, k$ as

$$D^q_{m\times n} = \begin{pmatrix} \langle(\mu_{11}^q, v_{11}^q), \eta_{11}^q\rangle & \langle(\mu_{12}^q, v_{12}^q), \eta_{12}^q\rangle & \cdots & \langle(\mu_{1n}, v_{1n}), \eta_{1n}^q\rangle \\ \langle(\mu_{21}^q, v_{21}^q), \eta_{21}^q\rangle & \langle(\mu_{22}^q, v_{22}^q), \eta_{22}^q\rangle & \cdots & \langle(\mu_{2n}, v_{2n}), \eta_{2n}^q\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle(\mu_{m1}^q, v_{m1}^q), \eta_{m1}^q\rangle & \langle(\mu_{m2}^q, v_{m2}^q), \eta_{m2}^q\rangle & \cdots & \langle(\mu_{mn}, v_{mn}), \eta_{mn}^q\rangle \end{pmatrix}$$

where $\eta_{ij}^q, (0 \leq \eta_{ij}^q \leq 1)$ be the confidence levels provided by the decision makers that they are familiar with the topic.

Step 2: Different types of criteria are normalized by using the following transformation.

$$r_{ij} = \begin{cases} \alpha_{ij}^c; & j \in B \\ \alpha_{ij}; & j \in C \end{cases} \tag{12}$$

where α_{ij}^c is the complement of α_{ij} and B, C represent the benefit and cost type criteria respectively.

Step 3: Aggregate all the individuals pythagorean fuzzy decision matrix D^q into the collective pythagorean decision matrix by utilizing the CPFWA operator

$$\alpha_{ij} = CPFWA(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^k) = \left\langle \sqrt{1 - \prod_{q=1}^{k} \{1 - (\mu_{ij}^q)^2\}^{\eta_{ij}^q \xi_q}}, \prod_{q=1}^{k} (v_{ij}^q)^{\eta_{ij}^q \xi_q} \right\rangle$$

or the CPFWG operator

$$\alpha_{ij} = CPFWG(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^k) = \left\langle \prod_{q=1}^{k} (\mu_{ij}^q)^{\eta_{ij}^q \xi_q}, \sqrt{1 - \prod_{q=1}^{k} \{1 - (v_{ij}^q)^2\}^{\eta_{ij}^q \xi_q}} \right\rangle$$

Step 4: Aggregate the pythagorean fuzzy numbers α_{ij} (as obtained from Step 3) by using PFWA operator:

$$\alpha_i = PFWA(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) = \left\langle \sqrt{1 - \prod_{j=1}^{n}\{1 - (\mu_{ij})^2\}^{\omega_j}}, \prod_{j=1}^{n}(\nu_{ij})^{\omega_j} \right\rangle$$

or the PFWG operator

$$\alpha_i = PFWG(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) = \left\langle \prod_{j=1}^{n}(\mu_{ij})^{\omega_j}, \sqrt{1 - \prod_{j=1}^{n}\{1 - (\nu_{ij})^2\}^{\omega_j}} \right\rangle$$

Step 5: Rank all the alternative based on the score function.

5 Illustrative example

Considering a decision-making problem with customers' choice to buy a four wheeler vehicle from five different types say $A = \{A_1, A_2, A_3, A_4, A_5\}$. In order to make this process, six factors $C = \{C_1, C_2, \ldots, C_6\}$ are considered which stands for "the consumption petrol", "the safety factor", "the degree of comfort", "the design", "the mileage", and "the price". The weight vector corresponding to these six criteria $C_j (j = 1, 2, \ldots, 6)$ is $\omega = (0.15, 0.25, 0.14, 0.16, 0.20, 0.10)^T$. Then following are the analysis conducted for finding the best alternative among the feasible ones by using CPFWA and CPFWG operators.

5.1 By CPFWA operator

Step 1: Three decision makers, $D^q (q = 1, 2, 3)$, whose weight vector is $\xi = (0.35, 0.35, 0.30)^T$, have rating these alternatives $A_i (i = 1, 2, 3, 4, 5)$ w.r.t. the criteria C_j in terms of PFNs $\alpha_{ij} = \langle (\mu_{ij}^q, \nu_{ij}^q), \eta_{ij}^q \rangle$ for $i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 6$, and their corresponding summary are listed in Tables 1, 2 and 3 respectively.

Step 2: Since all the attributes is of same type, so there is no need to normalization.

Step 3.: Utilize CPFWA operator as given in Eq. (5) to aggregate these three individual preference decision making matrix into the collective Pythagorean decision matrix $D = (\alpha_{ij})_{5 \times 6}$. The results corresponding to it has been summarized in Table 4.

Step 4: Aggregate all these different rating by using PFWA operator to get the overall value of the alternative A_i, $(i = 1, 2, 3, 4, 5)$ as

$$\alpha_1 = \langle 0.5891, 0.3469 \rangle \quad ; \quad \alpha_2 = \langle 0.5527, 0.3205 \rangle \quad ; \quad \alpha_3 = \langle 0.6692, 0.2518 \rangle$$
$$\alpha_4 = \langle 0.5669, 0.3804 \rangle \quad ; \quad \alpha_5 = \langle 0.6267, 0.3636 \rangle$$

Step 5: Score values corresponding to them are $sc(\alpha_1) = 0.2267$, $sc(\alpha_2) = 0.2028$, $sc(\alpha_3) = 0.3844$, $sc(\alpha_4) = 0.1766$ and $sc(\alpha_5) = 0.2605$. Since $sc(\alpha_3) > sc(\alpha_5) > sc(\alpha_1) > sc(\alpha_2) > sc(\alpha_4)$ thus we have $A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$. Hence A_3 is the best alternative.

5.2 By CPFWG operator

Based on CPFWG operator, the main steps are as follows.

Step 3: Utilize CPFWG operator as given in Eq. (10) to aggregate all preferences of the decision maker $D^q (q = 1, 2, 3)$ into the single one decision matrix $D = (\alpha_{ij})_{5\times 6}$ and their result is summarized in Table 5.

Step 4: Based on the Table 5 and by utilizing PFWG operator, the overall preference value of i th alternative is computed as

$$\alpha_1 = \langle 0.6183, 0.3289 \rangle; \quad \alpha_2 = \langle 0.5958, 0.3247 \rangle; \quad \alpha_3 = \langle 0.6944, 0.2507 \rangle$$
$$\alpha_4 = \langle 0.5705, 0.3501 \rangle; \quad \alpha_5 = \langle 0.6369, 0.3525 \rangle$$

Step 5: Finally, score value of α_i are $sc(\alpha_1) = 0.2741$, $sc(\alpha_2) = 0.2496$, $sc(\alpha_3) = 0.4194$, $sc(\alpha_4) = 0.2029$ and $sc(\alpha_5) = 0.2814$ thus we have $A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$. Hence A_3 is the best alternative.

On the other hand, if we conduct the analysis based on the different studies as proposed by the various authors (Xu and Yager 2006; Xu 2007; Wang and Liu 2012; Ye 2009; Yager and Abbasov 2013; Garg 2016e, f) by considering that all the decision makers are taken to be surely familiar ($\eta_j = 1$, for all j) with the evaluated objects then their subsequently results are summarized in Table 6. From this comparison table, it has been concluded that the best alternative obtained by the proposed approach coincides with these existing studies. Therefore, the considered approach can be taken as an alternative way to solve these types of problem in a more profitable way. Furthermore, it has also been observed that the nature of the relative score values follows the same trend (increasing or decreasing) and hence proposed approach is conservative in nature.

According to the above comparison analysis, the proposed method for addressing the decision-making problems has the following merits with respect to the existing ones.

(i) As discussed above, the classical, fuzzy and intuitionistic fuzzy sets all are the special cases of the Pythagorean fuzzy set. Since, so far, authors have used the IFS which is characterized by the degree of the membership and non-membership of a particular element such that their sum is less than or

Table 1 Pythagorean fuzzy decision matrix given by expert D^1

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	$\langle(0.4, 0.5), 0.70\rangle$	$\langle(0.5, 0.3), 0.75\rangle$	$\langle(0.3, 0.7), 0.70\rangle$	$\langle(0.8, 0.1), 0.80\rangle$	$\langle(0.6, 0.2), 0.70\rangle$	$\langle(0.5, 0.3), 0.70\rangle$
A_2	$\langle(0.6, 0.3), 0.75\rangle$	$\langle(0.5, 0.2), 0.80\rangle$	$\langle(0.6, 0.1), 0.75\rangle$	$\langle(0.7, 0.1), 0.80\rangle$	$\langle(0.3, 0.6), 0.70\rangle$	$\langle(0.4, 0.3), 0.70\rangle$
A_3	$\langle(0.4, 0.4), 0.75\rangle$	$\langle(0.8, 0.1), 0.75\rangle$	$\langle(0.5, 0.1), 0.80\rangle$	$\langle(0.6, 0.3), 0.70\rangle$	$\langle(0.4, 0.5), 0.80\rangle$	$\langle(0.3, 0.2), 0.70\rangle$
A_4	$\langle(0.2, 0.4), 0.70\rangle$	$\langle(0.4, 0.3), 0.80\rangle$	$\langle(0.9, 0.1), 0.75\rangle$	$\langle(0.7, 0.2), 0.70\rangle$	$\langle(0.2, 0.5), 0.75\rangle$	$\langle(0.7, 0.1), 0.75\rangle$
A_5	$\langle(0.6, 0.2), 0.80\rangle$	$\langle(0.3, 0.6), 0.85\rangle$	$\langle(0.4, 0.5), 0.80\rangle$	$\langle(0.3, 0.5), 0.75\rangle$	$\langle(0.6, 0.4), 0.70\rangle$	$\langle(0.9, 0.1), 0.75\rangle$

Table 2 Pythagorean fuzzy decision matrix given by expert D^2

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	$\langle(0.5, 0.4), 0.80\rangle$	$\langle(0.3, 0.4), 0.85\rangle$	$\langle(0.7, 0.2), 0.70\rangle$	$\langle(0.9, 0.1), 0.75\rangle$	$\langle(0.5, 0.5), 0.80\rangle$	$\langle(0.3, 0.6), 0.80\rangle$
A_2	$\langle(0.2, 0.4), 0.70\rangle$	$\langle(0.5, 0.2), 0.90\rangle$	$\langle(0.8, 0.1), 0.85\rangle$	$\langle(0.7, 0.1), 0.85\rangle$	$\langle(0.5, 0.5), 0.90\rangle$	$\langle(0.7, 0.2), 0.80\rangle$
A_3	$\langle(0.8, 0.2), 0.75\rangle$	$\langle(0.6, 0.3), 0.85\rangle$	$\langle(0.7, 0.1), 0.85\rangle$	$\langle(0.7, 0.1), 0.90\rangle$	$\langle(0.9, 0.1), 0.85\rangle$	$\langle(0.7, 0.2), 0.85\rangle$
A_4	$\langle(0.8, 0.1), 0.85\rangle$	$\langle(0.4, 0.5), 0.70\rangle$	$\langle(0.8, 0.2), 0.70\rangle$	$\langle(0.4, 0.6), 0.80\rangle$	$\langle(0.5, 0.5), 0.80\rangle$	$\langle(0.5, 0.4), 0.75\rangle$
A_5	$\langle(0.7, 0.2), 0.90\rangle$	$\langle(0.6, 0.4), 0.85\rangle$	$\langle(0.9, 0.1), 0.90\rangle$	$\langle(0.6, 0.4), 0.85\rangle$	$\langle(0.8, 0.2), 0.90\rangle$	$\langle(0.4, 0.5), 0.70\rangle$

Table 3 Pythagorean fuzzy decision matrix given by expert D^3

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	$\langle(0.6, 0.1), 0.75\rangle$	$\langle(0.8, 0.1), 0.75\rangle$	$\langle(0.9, 0.1), 0.80\rangle$	$\langle(0.6, 0.3), 0.85\rangle$	$\langle(0.4, 0.5), 0.80\rangle$	$\langle(0.9, 0.1), 0.75\rangle$
A_2	$\langle(0.7, 0.2), 0.80\rangle$	$\langle(0.5, 0.5), 0.75\rangle$	$\langle(0.9, 0.1), 0.80\rangle$	$\langle(0.4, 0.5), 0.85\rangle$	$\langle(0.4, 0.6), 0.75\rangle$	$\langle(0.8, 0.1), 0.80\rangle$
A_3	$\langle(0.8, 0.1), 0.85\rangle$	$\langle(0.9, 0.1), 0.85\rangle$	$\langle(0.6, 0.3), 0.90\rangle$	$\langle(0.5, 0.5), 0.75\rangle$	$\langle(0.8, 0.1), 0.80\rangle$	$\langle(0.4, 0.6), 0.70\rangle$
A_4	$\langle(0.3, 0.5), 0.75\rangle$	$\langle(0.6, 0.3), 0.85\rangle$	$\langle(0.9, 0.1), 0.90\rangle$	$\langle(0.8, 0.1), 0.80\rangle$	$\langle(0.3, 0.6), 0.75\rangle$	$\langle(0.5, 0.4), 0.70\rangle$
A_5	$\langle(0.4, 0.6), 0.8\rangle$	$\langle(0.8, 0.1), 0.80\rangle$	$\langle(0.8, 0.2), 0.75\rangle$	$\langle(0.6, 0.4), 0.75\rangle$	$\langle(0.5, 0.4), 0.75\rangle$	$\langle(0.9, 0.1), 0.80\rangle$

Table 4 Aggregative Pythagorean fuzzy decision matrix D

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	$\langle 0.4477, 0.3889\rangle$	$\langle 0.5325, 0.3306\rangle$	$\langle 0.6662, 0.3555\rangle$	$\langle 0.7526, 0.2109\rangle$	$\langle 0.4548, 0.4701\rangle$	$\langle 0.6127, 0.3844\rangle$
A_2	$\langle 0.5008, 0.3958\rangle$	$\langle 0.4584, 0.3284\rangle$	$\langle 0.7479, 0.1585\rangle$	$\langle 0.5930, 0.2217\rangle$	$\langle 0.3766, 0.6323\rangle$	$\langle 0.6156, 0.2730\rangle$
A_3	$\langle 0.6611, 0.2865\rangle$	$\langle 0.7493, 0.2123\rangle$	$\langle 0.5750, 0.1911\rangle$	$\langle 0.5660, 0.3084\rangle$	$\langle 0.7384, 0.2389\rangle$	$\langle 0.4785, 0.3752\rangle$
A_4	$\langle 0.5336, 0.3446\rangle$	$\langle 0.4308, 0.4431\rangle$	$\langle 0.8237, 0.1978\rangle$	$\langle 0.6067, 0.3362\rangle$	$\langle 0.3262, 0.6120\rangle$	$\langle 0.5182, 0.3544\rangle$
A_5	$\langle 0.5616, 0.3395\rangle$	$\langle 0.5777, 0.3764\rangle$	$\langle 0.7426, 0.2776\rangle$	$\langle 0.4768, 0.5165\rangle$	$\langle 0.6253, 0.3916\rangle$	$\langle 0.7642, 0.2653\rangle$

Table 5 Aggregative Pythagorean fuzzy decision matrix D

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	$\langle 0.5865, 0.3383 \rangle$	$\langle 0.5541, 0.2754 \rangle$	$\langle 0.6652, 0.4032 \rangle$	$\langle 0.8022, 0.1705 \rangle$	$\langle 0.5832, 0.3841 \rangle$	$\langle 0.5882, 0.3736 \rangle$
A_2	$\langle 0.5412, 0.2727 \rangle$	$\langle 0.5664, 0.2918 \rangle$	$\langle 0.7979, 0.0895 \rangle$	$\langle 0.6443, 0.2759 \rangle$	$\langle 0.4870, 0.5094 \rangle$	$\langle 0.6853, 0.1905 \rangle$
A_3	$\langle 0.7005, 0.2395 \rangle$	$\langle 0.7887, 0.1809 \rangle$	$\langle 0.6453, 0.1755 \rangle$	$\langle 0.6747, 0.2949 \rangle$	$\langle 0.7107, 0.2870 \rangle$	$\langle 0.5524, 0.3308 \rangle$
A_4	$\langle 0.4811, 0.3234 \rangle$	$\langle 0.5426, 0.3375 \rangle$	$\langle 0.8951, 0.1234 \rangle$	$\langle 0.6720, 0.3583 \rangle$	$\langle 0.4117, 0.4756 \rangle$	$\langle 0.6563, 0.2855 \rangle$
A_5	$\langle 0.6217, 0.3509 \rangle$	$\langle 0.5691, 0.4130 \rangle$	$\langle 0.7118, 0.2979 \rangle$	$\langle 0.5582, 0.3918 \rangle$	$\langle 0.7037, 0.3007 \rangle$	$\langle 0.7577, 0.2697 \rangle$

equal to one. However, in most of the day-today-life problem this condition may not be satisfied when an expert gave their preferences towards the elements. For handling this, PFS is one of the generalized theory which can handle not only incomplete information but also the indeterminate information and inconsistent information, which exist commonly in real situations. Therefore, the existing studies are more suitable than the existing ones for solving the real-life and engineering design problems.

(ii) Also, it has observed from the Table 6 that the results computed by the various existing approaches are under the environment without considering the confidence levels of the attributes during the evaluation. In other words, all these approaches have analyzed their theories with the assumption that decision maker are taken to be 100% confidence with the evaluated objects. But in real-life situation, these types of conditions are partially fulfilled.

(iii) The existing operators for IFS are a special case of the proposed operators. Furthermore, some of the existing operators for PFS are also a special case of the proposed operators. Therefore, it has been concluded that the proposed aggregation operators are more generalized and suitable to solve the real-life problems more accurately than the existing ones.

6 Conclusion

The objective of this work is to present some series of new averaging and geometric aggregation operator by considering the degree of the confidence levels of each decision makers' during evaluation. Traditionally, it has been assumed that the all

Table 6 Comparative analysis

Method	Score values					Order of alternatives
	A_1	A_2	A_3	A_4	A_5	
Xu and Yager (2006)	0.1638	0.1484	0.3452	0.0778	0.1768	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
Xu (2007)	0.4043	0.3591	0.5353	0.3349	0.3871	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
Wang and Liu (2012)	0.3602	0.3226	0.5090	0.2787	0.3487	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
Ye (2009)	0.3636	0.3218	0.3690	0.3110	0.4392	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
Yager and Abbasov (2013)	0.4067	0.3518	0.5020	0.3727	0.4331	$A_3 \succ A_5 \succ A_1 \succ A_4 \succ A_2$
Garg (2016e)	0.3385	0.2853	0.4637	0.2755	0.3489	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
Garg (2016f)	−0.2860	−0.3973	−0.1549	−0.3374	−0.2226	$A_3 \succ A_5 \succ A_1 \succ A_4 \succ A_2$
CPFWA	0.2267	0.2028	0.3844	0.1766	0.2605	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
CPFOWA	0.3131	0.1713	0.4375	0.3429	0.1883	$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$
CPFWG	0.2741	0.2496	0.4194	0.2029	0.2814	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
CPFOWG	0.3540	0.2450	0.4766	0.3588	0.2987	$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$

the decision maker give their preferences of the different alternative at the same level of confidence. But this shortcoming has been ruled out in the present manuscript by considering the confidence level factor (η) of the decision maker. Based on it, new aggregation operators namely CPFWA, CPFOWA, CPFWG and CPFOWG are proposed under PFS environment. The desirable properties corresponding to each operator has also been discussed. Furthermore, it has been observed that when $\eta = 1$ for all the preferences then the proposed aggregation operators reduces to the existing PFWA and PFWG operators. A comparative study with some existing operators has been presented which shows that the proposed operators provides an alternative ways to solve MCDM problem in a more effective manner. In future work, we may extend the proposed function to the different applications and to solve various uncertain programming problems.

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