Czechoslovak Mathematical Journal, 66 (141) (2016), 913–924

CONNECTIVITY, TOUGHNESS, SPANNING TREES OF BOUNDED DEGREE, AND THE SPECTRUM OF REGULAR GRAPHS

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(Received January 1, 2016)

Dedicated to the memory of Professor Miroslav Fiedler

Abstract. The eigenvalues of graphs are related to many of its combinatorial properties. In his fundamental work, Fiedler showed the close connections between the Laplacian eigenvalues and eigenvectors of a graph and its vertex-connectivity and edge-connectivity.

We present some new results describing the connections between the spectrum of a regular graph and other combinatorial parameters such as its generalized connectivity, toughness, and the existence of spanning trees with bounded degree.

Keywords: spectral graph theory; eigenvalue; connectivity; toughness; spanning k-tree

MSC 2010: 05C50, 05C40, 05C42, 05E99, 05C05, 15A18

1. Introduction

The spectrum of a graph is related to many important combinatorial parameters. In his fundamental and ground-breaking works, Fiedler [17], [16] determined close connections between the Laplacian eigenvalues and eigenvectors of a graph and combinatorial parameters such as its vertex-connectivity or edge-connectivity. Fiedler's work has stimulated tremendous progress and growth in spectral graph theory since then.

In this paper, we study the connections between the spectrum of a regular graph and other combinatorial parameters such as generalized connectivity, toughness and the existence of spanning trees with bounded degree.

The research of the first author was partially supported by the National Security Agency grant H98230-13-0267 and the National Science Foundation grant DMS-1600768.

Throughout this paper, we consider only finite, undirected and simple graphs. Given a graph $G=(V,E)$ of order n, we denote by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$ the eigenvalues of its adjacency matrix. When the graph G is clear from the context, we use λ_i to denote $\lambda_i(G)$. We also use the notation $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$. If G is d-regular, then $\lambda_1 = d$ and the multiplicity of d equals the number of components of G. We use $\kappa(G)$, $\kappa'(G)$ and $c(G)$ to denote the vertex-connectivity, the edge-connectivity and the number of components of a graph G, respectively. For any undefined graph theoretic notions, see Bondy and Murty [3] or Brouwer and Haemers [6].

One of the well-known results of Fiedler in [17] implies that the vertex-connectivity of a d-regular graph is at least $d-\lambda_2$. This result was improved in certain ranges by Krivelevich and Sudakov in [24] who showed that the vertex-connectivity of a d-regular graph is at least $d - 36\lambda^2/d$. Given an integer $l \geqslant 2$, Chartrand, Kapoor, Lesniak and Lick in [8] defined the *l-connectivity* $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices. Thus, $\kappa_l(G)=0$ if and only if $c(G) \geqslant l$ or $|V(G)| \leqslant l-1$. Note that $\kappa_2(G) = \kappa(G)$. For $k \geqslant 1$, a graph G is called *(k,l)-connected* if $\kappa_l \geqslant k$. See [8], [14], [23], [32] for more about l-connectivity and (k,l)-connected graphs. In particular, a structural characterization of $(2,l)$-connected graphs is presented in [23], as a generalization of the standard characterization of 2-connected graphs (see [3], Chapter 5).

Our results relating the generalized connectivity to the spectrum of a regular graph are below.

Theorem 1.1. *Let l, k be integers such as $l \geqslant k \geqslant 2$. For any connected d-regular graph G with $|V(G)| \geqslant k+l-1$, $d \geqslant 3$ and edge connectivity κ', if $\kappa' = d$, or if $\kappa' < d$ and*

$$\lambda_{\lceil (l-k+1)d/(d-\kappa') \rceil}(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $\kappa_l(G) \geqslant k$.

Corollary 1.2. *Let $l \geqslant 2$. For any connected d-regular graph G with $|V(G)| \geqslant l+1$ and $d \geqslant 3$, if*

$$\lambda_l(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $\kappa_l(G) \geqslant 2$.

Corollary 1.3. *For any connected d-regular graph G with $d \geqslant 3$, if*

$$\lambda_2(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $\kappa(G) \geqslant 2$.

Corollary 1.3 is a slight improvement of previous results of Krivelevich and Sudakov [24], Theorem 4.1, and Fiedler [17], Theorem 4.1.

The toughness $t(G)$ of a connected graph G is defined as $t(G) = \min\{|S| \times (c(G-S))^{-1}\}$, where the minimum is taken over all proper subsets $S \subset V(G)$ such that $c(G-S) > 1$. A graph G is *t-tough* if $t(G) \geqslant t$. This parameter was introduced by Chvátal [9] in 1973 and is closely related to many graph properties, including Hamiltonicity, pancyclicity and spanning trees, see [2]. By the definitions of toughness and generalized connectivity, for a noncomplete connected graph G we have $t(G) = \min\limits_{2 \leqslant l \leqslant \alpha} \{\kappa_l(G)/l\}$ where α is the independence number of G (see also [14]).

The relationship between the toughness of a regular graph and the eigenvalues has been considered by many researchers, among which Alon [1] is the first.

Theorem 1.4 (Alon [1]). *For any connected d-regular graph G,*

$$t(G) > \frac{1}{3}\Big(\frac{d^2}{d\lambda + \lambda^2} - 1\Big).$$

Around the same time, Brouwer [5] independently discovered a slightly better bound of $t(G)$.

Theorem 1.5 (Brouwer [5]). *For any connected d-regular graph G,*

$$t(G) > \frac{d}{\lambda} - 2.$$

Brouwer in [4] conjectured that the lower bound from the previous theorem can be improved to $d/\lambda - 1$ for any connected d-regular graph G. For the special case of toughness 1, Liu and Chen in [27] improved Brouwer's previous result.

Theorem 1.6 (Liu and Chen [27]). *For any connected d-regular graph G, if*

$$\lambda_2(G) < \begin{cases} d - 1 + \dfrac{3}{d+1} & \text{if } d \text{ is even,} \\ d - 1 + \dfrac{2}{d+1} & \text{if } d \text{ is odd,} \end{cases}$$

then $t(G) \geqslant 1$.

Recently, Cioabă and Wong in [12] further improved the above result.

Theorem 1.7 (Cioabă and Wong [12]). *For any connected d-regular graph G, if*

$$\lambda_2(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $t(G) \geqslant 1$.

Moreover, Cioabă and Wong in [12] showed that the previous result is the best possible by constructing d-regular graphs whose second largest eigenvalues equal the right-hand-side of the inequality from the previous theorem, but with toughness less than 1. An immediate corollary of the previous result is the following.

Corollary 1.8 (Cioabă and Wong [12]). *For any bipartite connected d-regular graph G, if*

$$\lambda_2(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $t(G) = 1$.

These authors also found the second largest eigenvalue condition for $t(G) \geqslant \tau$, where $\tau \leqslant \kappa'/d$ is a positive number.

Theorem 1.9 (Cioabă and Wong [12]). *Let G be a connected d-regular graph with edge connectivity κ' and $d \geqslant 3$. Suppose that τ is a positive number such that $\tau \leqslant \kappa'/d$. If $\lambda_2(G) < d - \tau d/(d+1)$, then $t(G) \geqslant \tau$.*

In this paper, we continue investigating the relationship between toughness of a regular graph and its eigenvalues. The following theorems are the main results. As $\lceil d/(d-\kappa')\rceil \geqslant 2$, Theorem 1.10 is an improvement of Theorem 1.7. For bipartite regular graphs, Theorem 1.11 improves Corollary 1.8. We shall also mention that in Theorem 1.9 the eigenvalue condition is not needed, see Theorem 1.12. As an application of Theorem 1.12, Corollary 1.13 confirms a conjecture of Brouwer [4] when $\kappa' < d$.

Theorem 1.10. *Let G be a connected d-regular graph with $d \geqslant 3$ and edge connectivity κ'. If $\kappa' = d$, or if $\kappa' < d$ and*

$$\lambda_{\lceil d/(d-\kappa')\rceil}(G) < \begin{cases} \dfrac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \dfrac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $t(G) \geqslant 1$.

Theorem 1.11. *For any bipartite connected d-regular graph G with $\kappa' < d$, if $\lambda_{\lceil d/(d-\kappa')\rceil}(G) < d - (d-1)/2d$, then $t(G) = 1$.*

Theorem 1.12. *Let G be a connected d-regular graph with edge connectivity κ'. Then $t(G) \geqslant \kappa'/d$.*

Corollary 1.13. *For any connected d-regular graph G with $d \geqslant 3$ and edge connectivity $\kappa' < d$, $t(G) > d/\lambda_2 - 1 \geqslant d/\lambda - 1$.*

Recently, there has been a lot of activity concerning connections between the eigenvalues of a graph and the maximum number of edge-disjoint spanning trees that can be packed in the graph [13], [21], [19], [22], [26], [29], [28], [35]. Another interesting problem would be to see how the eigenvalues of a graph influence the types of spanning trees contained in it. For an integer $k \geqslant 2$, a *k-tree* is a tree with the maximum degree at most k. This topic is related to connected factors. A *$[1,k]$-factor* is a spanning subgraph in which each vertex has the degree at least one and at most k. By definition, a graph G has a spanning k-tree if and only if G has a connected $[1,k]$-factor. For more about degree bounded trees, we refer the readers to survey [33]. For spectral conditions of k-factors in regular graphs, see [11], [20], [31], [30]. In his PhD Dissertation, Wong [35] proved the following sufficient spectral condition for the existence of spanning k-trees in regular graphs for $k \geqslant 3$.

Theorem 1.14 (Wong [35])**.** *Let $k \geqslant 3$ and let G be a connected d-regular graph. If $\lambda_4 < d - d/((k-2)(d+1))$, then G has a spanning k-tree.*

In this paper, we improve this result.

Theorem 1.15. *Let $k \geqslant 3$ and let G be a connected d-regular graph with edge connectivity κ'. Let $l = d - (k-2)\kappa'$. Each of the following statements holds.*

(i) *If $l \leqslant 0$, then G has a spanning k-tree.*

(ii) *If $l > 0$ and $\lambda_{\lceil 3d/l\rceil} < d - d/((k-2)(d+1))$, then G has a spanning k-tree.*

Note that eigenvalue conditions for the existence of spanning 2-trees (Hamiltonian paths) and Hamiltonian cycles have been obtained by Krivelevich and Sudakov in [25] and Butler and Chung in [7].

2. Preliminaries

In this section, we present some eigenvalue interlacing results to be used in our arguments. For a real and symmetric matrix M of order n and a natural number $1 \leqslant i \leqslant n$, we denote by $\lambda_i(M)$ the i-th largest eigenvalue of M. The following interlacing theorem can be found in many textbooks, for example [6], page 35, or [18], page 193, and is usually referred to as Cauchy eigenvalue interlacing.

Theorem 2.1. *Let A be a real symmetric $n \times n$ matrix and B a principal $m \times m$ submatrix of A. Then $\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{n-m+i}(A)$ for $1 \leqslant i \leqslant m$.*

Corollary 2.2. *Let $S_1, S_2, \ldots, S_p$ be disjoint subsets of $V(G)$ with $e(S_i, S_j) = 0$ for $i \neq j$. For $1 \leqslant i \leqslant p$ let $G[S_i]$ denote the subgraph of G induced by S_i. Then*

$$\lambda_p(G) \geqslant \lambda_p\left(G\left[\bigcup_{i=1}^{p} S_i\right]\right) \geqslant \min_{1 \leqslant i \leqslant p} \{\lambda_1(G[S_i])\}.$$

Let $d \geqslant 3$ be an integer, and let $\mathcal{X}(d)$ denote the family of all connected irregular graphs with maximum degree d, order $n \geqslant d+1$ and size m with $2m \geqslant dn - d + 1$ that have at least two vertices of degree d if d is odd, and at least three vertices of degree d if d is even. If $t \geqslant 2$ is an even integer, let M_t denote the disjoint union of $t/2$ edges. If G and H are two vertex disjoint graphs, the join $G \vee H$ of G and H is the graph obtained by taking the union of G and H and adding all the edges between the vertex set of G and the vertex set of H. The complement of G is denoted by $\overline{G}$. For $d \geqslant 3$, define X_d as $\overline{M}_{d-1} \vee K_2$ if d is odd and $\overline{M}_{d-2} \vee K_3$ if d is even.

Lemma 2.3 (Cioabă and Wong [12]). *Let $d \geqslant 3$ be an integer and $H \in \mathcal{X}(d)$. Then*

$$\lambda_1(H) \geqslant \theta(d) = \begin{cases} \frac{1}{2}(d - 2 + \sqrt{d^2+12}) & \text{if } d \text{ is even,} \\ \frac{1}{2}(d - 2 + \sqrt{d^2+8}) & \text{if } d \text{ is odd.} \end{cases}$$

Equality occurs if and only if $G = X_d$.

Theorem 2.4 (Cioabă [10]). *Let k and d be two integers such that $d \geqslant k \geqslant 2$. If G is a d-regular graph with $\lambda_2(G) < d - 2(k-1)/(d+1)$, then $\kappa'(G) \geqslant k$.*

Corollary 2.5. *Let G be a d-regular graph with $d \geqslant 2$ and edge connectivity $\kappa' < d$. Then $\lambda_2(G) \geqslant d - 2\kappa'/(d+1)$.*

Proof. Let $k = \kappa' + 1$ in the contrapositive of Theorem 2.4. □

In this section, we prove Theorem 1.1. Corollaries 1.2 and 1.3 obviously follow from Theorem 1.1.

P r o o f of Theorem 1.1. We prove it by contradiction, i.e., we assume that $\kappa_l(G) < k$. By definition, there exists a subset $S \subset V(G)$ with $|S| \leqslant k-1$ such that $c(G-S) \geqslant l$. Let $s = |S|$, $c = c(G-S)$ and let $H_1, H_2, \ldots, H_c$ be the components of $G-S$. For $1 \leqslant i \leqslant c$ let $n_i = |V(H_i)|$ and let t_i be the number of edges between H_i and S. Then $t_i \geqslant \kappa'$ for $1 \leqslant i \leqslant c$. Since G is d-regular, $\sum\limits_{i=1}^{c} t_i \leqslant ds \leqslant d(k-1)$.

As $d(k-1) \geqslant \sum\limits_{i=1}^{c} t_i \geqslant c\kappa' \geqslant l\kappa'$, we have $ld - d(k-1) \leqslant ld - l\kappa'$. If $\kappa' = d$, then the previous inequality is impossible, a contradiction. Thus, we may assume that $\kappa' < d$, and hence $l \geqslant (l-k+1)d/(d-\kappa')$. We claim that there are at least $\lceil (l-k+1)d/(d-\kappa')\rceil$ indices i such that $t_i < d$. Otherwise, there would be at most $\lceil (l-k+1)d/(d-\kappa')\rceil - 1$ indices i such that $t_i < d$. In other words, there would be at least $c - \lceil (l-k+1)d/(d-\kappa')\rceil + 1$ indices i with $t_i \geqslant d$. Thus,

$$\begin{aligned}\sum_{i=1}^{c} t_i &\geqslant \Big(c - \Big\lceil \frac{(l-k+1)d}{d-\kappa'}\Big\rceil + 1\Big)d + \Big(\Big\lceil \frac{(l-k+1)d}{d-\kappa'}\Big\rceil - 1\Big)\kappa' \\ &= cd - \Big(\Big\lceil \frac{(l-k+1)d}{d-\kappa'}\Big\rceil - 1\Big)(d-\kappa') \\ &> cd - \frac{(l-k+1)d}{d-\kappa'}(d-\kappa') \\ &= cd - (l-k+1)d = (c-l)d + (k-1)d \geqslant ds,\end{aligned}$$

contrary to $\sum\limits_{i=1}^{c} t_i \leqslant ds$. Hence, there are at least $\lceil (l-k+1)d/(d-\kappa')\rceil$ indices i such that $t_i < d$. Without loss of generality, we may assume these indices are $1, 2, \ldots, \lceil (l-k+1)d/(d-\kappa')\rceil$.

For $1 \leqslant i \leqslant \lceil (l-k+1)d/(d-\kappa')\rceil$, $n_i \geqslant d+1$. Otherwise, if $n_i \leqslant d$, then $dn_i = t_i + 2|E(H_i)| \leqslant t_i + n_i(n_i-1) \leqslant t_i + d(n_i-1)$, which implies $t_i \geqslant d$, contrary to $t_i < d$.

Since $dn_i = t_i + 2|E(H_i)|$ for $1 \leqslant i \leqslant \lceil (l-k+1)d/(d-\kappa')\rceil$, if d is even, hence t_i is also even, and thus $t_i \leqslant d-2$. If d is odd, then $t_i \leqslant d-1$. As $n_i \geqslant d+1$, each H_i contains at least three vertices of degree d if d is even, and at least two vertices of degree d if d is odd. Thus, $H_i \in \mathcal{X}_d$ for $1 \leqslant i \leqslant \lceil (l-k+1)d/(d-\kappa')\rceil$. By Corollary 2.2 and Lemma 2.3, $\lambda_{\lceil (l-k+1)d/(d-\kappa')\rceil}(G) \geqslant \min\limits_{1 \leqslant i \leqslant \lceil (l-k+1)d/(d-\kappa')\rceil} \{\lambda_1(H_i)\} \geqslant \theta(d)$, contrary to the assumption. This completes the proof. $\square$

In this section, we prove Theorems 1.10, 1.11, 1.12 and Corollary 1.13.

Proof of Theorem 1.10. We prove it by contradiction, i.e., we assume that $t(G) < 1$. By definition, there exists a subset $S \subset V(G)$ such that $|S|/(c(G-S)) < 1$. Let $s = |S|$, $c = c(G-S)$ and let $H_1, H_2, \ldots, H_c$ be the components of $G-S$. For $1 \leqslant i \leqslant c$ let $n_i = |V(H_i)|$ and let t_i be the number of edges between H_i and S. Then $s < c$ and $t_i \geqslant \kappa'$ for $1 \leqslant i \leqslant c$. Since G is d-regular, $\sum\limits_{i=1}^{c} t_i \leqslant ds$.

As $c\kappa' \leqslant \sum\limits_{i=1}^{c} t_i \leqslant ds \leqslant d(c-1)$, we have $c(d-\kappa') \geqslant d$. If $\kappa' = d$, then we get a contradiction. Thus, we may assume that $\kappa' < d$, and so $c \geqslant d/(d-\kappa')$. We claim that there are at least $\lceil d/(d-\kappa')\rceil$ indices i such that $t_i < d$. Otherwise, there would be at most $\lceil d/(d-\kappa')\rceil - 1$ indices i such that $t_i < d$. In other words, there would be at least $c - \lceil d/(d-\kappa')\rceil + 1$ indices i with $t_i \geqslant d$. Thus,

$$\begin{aligned}\sum_{i=1}^{c} t_i &\geqslant \Big(c - \Big\lceil \frac{d}{d-\kappa'} \Big\rceil + 1\Big)d + \Big(\Big\lceil \frac{d}{d-\kappa'} \Big\rceil - 1\Big)\kappa' \\ &= cd - \Big(\Big\lceil \frac{d}{d-\kappa'} \Big\rceil - 1\Big)(d-\kappa') \\ &> cd - \frac{d}{d-\kappa'}(d-\kappa') = cd - d \geqslant ds,\end{aligned}$$

contrary to $\sum\limits_{i=1}^{c} t_i \leqslant ds$. Thus, there are at least $\lceil d/(d-\kappa')\rceil$ indices i such that $t_i < d$. Without loss of generality, we may assume these indices are $1, 2, \ldots, \lceil d/(d-\kappa')\rceil$.

For $1 \leqslant i \leqslant \lceil d/(d-\kappa')\rceil$ we have $n_i \geqslant d+1$. Otherwise, if $n_i \leqslant d$, then $dn_i = t_i + 2|E(H_i)| \leqslant t_i + n_i(n_i - 1) \leqslant t_i + d(n_i - 1)$, which implies $t_i \geqslant d$, contrary to $t_i < d$.

Since $dn_i = t_i + 2|E(H_i)|$ for $1 \leqslant i \leqslant \lceil d/(d-\kappa')\rceil$, so if d is even, then t_i is also even, and thus $t_i \leqslant d-2$. If d is odd, then $t_i \leqslant d-1$. As $n_i \geqslant d+1$, each H_i contains at least three vertices of degree d if d is even, and at least two vertices of degree d if d is odd. Thus, $H_i \in \mathcal{X}_d$ for $1 \leqslant i \leqslant \lceil d/(d-\kappa')\rceil$. By Corollary 2.2 and Lemma 2.3, $\lambda_{\lceil d/(d-\kappa')\rceil}(G) \geqslant \min\limits_{1 \leqslant i \leqslant \lceil d/(d-\kappa')\rceil} \{\lambda_1(H_i)\} \geqslant \theta(d)$, contrary to the assumption. This completes the proof. □

Lemma 4.1. *For any bipartite regular graph G, $t(G) \leqslant 1$.*

Proof. Let S be the set of vertices of one part of the bipartition. Then this equation $c(G-S) = |S|$ holds. Thus, $t(G) \leqslant |S|/(c(G-S)) = 1$. □

P r o o f of Theorem 1.11. We prove it by contradiction, i.e., we assume that $t(G) \neq 1$. By Lemma 4.1, $t(G) < 1$. By definition, there exists a subset $S \subset V(G)$ such that $|S|/(c(G-S)) < 1$. An argument similar to that in the proof of Theorem 1.10 shows that there are at least $\lceil d/(d-\kappa')\rceil$ components H_i of $G-S$ such that $t_i < d$, where t_i is the number of edges between H_i and S for $1, 2, \ldots, \lceil d/(d-\kappa')\rceil$. Let $n_i = |V(H_i)|$ and let $m_i = |E(H_i)|$ for $1, 2, \ldots, \lceil d/(d-\kappa')\rceil$. Then $2m_i = dn_i - t_i \geqslant dn_i - d + 1$. As each H_i is also bipartite, $m_i \leqslant n_i^2/4$. Thus, $n_i^2/2 \geqslant 2m_i \geqslant dn_i - d + 1$, which implies that $n_i^2 - 2dn_i + 2d - 2 \geqslant 0$. Hence, $n_i \geqslant 2d$. By Corollary 2.2,

$$\begin{aligned}\lambda_{\lceil d/(d-\kappa')\rceil}(G) &\geqslant \min_{1\leqslant i\leqslant \lceil d/(d-\kappa')\rceil}\{\lambda_1(H_i)\} \geqslant \min_{1\leqslant i\leqslant \lceil d/(d-\kappa')\rceil}\left\{\frac{2m_i}{n_i}\right\}\\ &\geqslant \frac{dn_i - d + 1}{n_i} \geqslant d - \frac{d-1}{2d},\end{aligned}$$

contrary to the assumption. This completes the proof. □

P r o o f of Theorem 1.12. Suppose that S is a vertex-cut of G. Let $s = |S|$, $c = c(G-S)$ and let $H_1, H_2, \ldots, H_c$ be the components of $G-S$. For $1 \leqslant i \leqslant c$ let $n_i = |V(H_i)|$ and let t_i be the number of edges between H_i and S. Then $t_i \geqslant \kappa'$ for $1 \leqslant i \leqslant c$. As G is d-regular, $\sum\limits_{i=1}^{c} t_i \leqslant ds$. Thus, $c\kappa' \leqslant \sum\limits_{i=1}^{c} t_i \leqslant ds$, which implies that $s/c \geqslant \kappa'/d$. Hence, $t(G) \geqslant \kappa'/d$. □

P r o o f of Corollary 1.13. By Corollary 2.5, $\lambda_2 \geqslant d - 2\kappa'/(d+1)$, which implies that $2\kappa'/(\lambda_2(d+1)) \geqslant d/\lambda_2 - 1$. If $d \geqslant 4$, then $\lambda_2 \geqslant d - 2\kappa'/(d+1) > 2$. If $d = 3$, then $\kappa' \leqslant 2$, and thus $\lambda_2 \geqslant d - 2\kappa'/(d+1) \geqslant 2$. By Theorem 1.12,

$$t(G) \geqslant \frac{\kappa'}{d} > \frac{\kappa'/d}{(\lambda_2/2)(1+1/d)} = \frac{2\kappa'}{\lambda_2(d+1)} \geqslant \frac{d}{\lambda_2} - 1,$$

which completes the proof. □

5. Spectrum and spanning k-trees in regular graphs

In this section, we prove Theorem 1.15. We will use the following sufficient condition of the existence of a spanning k-tree obtained by Win [34], which was also proved by Ellingham and Zha [15] with a new proof later.

Theorem 5.1 (Ellingham and Zha [15], Win [34]). *Let $k \geqslant 2$ and let G be a connected graph. If for any $S \subseteq V(G)$, $c(G-S) \leqslant (k-2)|S| + 2$, then G has a spanning k-tree.*

Now we are ready to prove Theorem 1.15.

Proof of Theorem 1.15. We prove it by contradiction, i.e., we assume that G does not have spanning k-trees for $k \geqslant 3$. By Theorem 5.1, there exists a subset $S \subseteq V(G)$ such that

$$c(G-S) \geqslant (k-2)|S| + 3. \tag{5.1}$$

Let $s = |S|$, $c = c(G-S)$ and let $H_1, H_2, \ldots, H_c$ be the components of $G-S$. For $1 \leqslant i \leqslant c$ let $n_i = |V(H_i)|$ and let t_i be the number of edges between H_i and S. Then $t_i \geqslant \kappa'$ for $1 \leqslant i \leqslant c$. Since G is d-regular, $c\kappa' \leqslant \sum_{i=1}^{c} t_i \leqslant ds$. By (5.1), $s \leqslant (c-3)/(k-2)$. Thus, $c\kappa' \leqslant d(c-3)/(k-2)$, which implies that

$$c(d-(k-2)\kappa') \geqslant 3d. \tag{5.2}$$

Thus, $l = d-(k-2)\kappa' > 0$, contrary to (i). This proves (i). Now, we continue to prove (ii).

By (5.2), $c \geqslant \lceil 3d/l \rceil$. We claim that there are at least $\lceil 3d/l \rceil$ indices i such that $t_i < d/(k-2)$. Otherwise, there would be at most $\lceil 3d/l \rceil - 1$ indices i such that $t_i < d/(k-2)$. In other words, there would be at least $c - \lceil 3d/l \rceil + 1$ indices i with $t_i \geqslant d/(k-2)$. Thus,

$$\begin{aligned} ds \geqslant \sum_{i=1}^{c} t_i &\geqslant \Bigl(c - \Bigl\lceil \frac{3d}{l} \Bigr\rceil + 1\Bigr)\frac{d}{k-2} + \Bigl(\Bigl\lceil \frac{3d}{l} \Bigr\rceil - 1\Bigr)\kappa' \\ &= \frac{cd}{k-2} - \Bigl(\Bigl\lceil \frac{3d}{l} \Bigr\rceil - 1\Bigr)\Bigl(\frac{d}{k-2} - \kappa'\Bigr) \\ &> \frac{cd}{k-2} - \frac{3d}{l}\Bigl(\frac{d}{k-2} - \kappa'\Bigr) \\ &= \frac{cd}{k-2} - \frac{3d}{k-2} = d\frac{c-3}{k-2} \geqslant ds, \end{aligned}$$

a contradiction. This proves that there are at least $\lceil 3d/l \rceil$ indices i such that $t_i < d/(k-2)$. Without loss of generality, we may assume these indices are $1, 2, \ldots, \lceil 3d/l \rceil$.

For $1 \leqslant i \leqslant \lceil 3d/l \rceil$, since $t_i < d/(k-2)$, it is not hard to get $n_i \geqslant d+1$ by counting the total degree of H_i. By Corollary 2.2, $\lambda_{\lceil 3d/l \rceil}(G) \geqslant \min_{1 \leqslant i \leqslant \lceil 3d/l \rceil} \{\lambda_1(H_i)\} \geqslant d - d/((k-2)(d+1))$, contrary to the assumption. This completes the proof. $\square$

6. Final remarks

In this paper, we established some new connections between the spectrum of a regular graph and its generalized connectivity, toughness or the existence of spanning k-trees. Some of our results are the best possible. For example, the constructions from [12], Section 3, show that the upper bound from Theorem 1.10 is the best possible. Also, Corollary 1.3 is the best possible when $d = 4$. To see this, construct a 4-regular graph by taking two disjoint copies of X_4 and adding a new vertex adjacent to 4 vertices (2 in each copy of X_4) of degree 3. The resulting graph is 4-regular, has vertex-connectivity 1 and its second largest eigenvalue equals the upper bound from Corollary 1.3.

It would be interesting to improve and generalize our results to general graphs and eigenvalues of Laplacian matrix, signless Laplacian or normalized Laplacian.

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