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SOME DIMENSIONAL RESULTS FOR A CLASS OF SPECIAL HOMOGENEOUS MORAN SETS

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Abstract. We construct a class of special homogeneous Moran sets, called $\{m_k\}$-quasi homogeneous Cantor sets, and discuss their Hausdorff dimensions. By adjusting the value of $\{m_k\}_{k\geqslant 1}$, we constructively prove the intermediate value theorem for the homogeneous Moran set. Moreover, we obtain a sufficient condition for the Hausdorff dimension of homogeneous Moran sets to assume the minimum value, which expands earlier works.

Keywords: homogeneous Moran set; $\{m_k\}$-Moran set; $\{m_k\}$-quasi homogeneous Cantor set; Hausdorff dimension

MSC 2010: 28A80

1. Introduction

Moran sets play an important role in fractal geometry, and are closely connected with many subjects. There are many important developments and applications in different ways, e.g., in the power systems (see [2] and its reference), in measurement of number theory (see [7], [9]), in Lipschitz equivalence (see [6]), in multifractals (see [4], [10]), in quasi homeomorphisms (see [5], [8]), etc. In these applications, the homogeneous Moran set plays a very important part. This arouses our great research interest in it. In [3], Feng, Wen and Wu using Theorem 4.10 in [1] showed the maximal value and the minimal value of the Hausdorff dimension in the family of homogeneous Moran sets. But except for the two extreme situations (the dimension of a partial homogeneous Cantor set may assume the minimum value while the dimension of a homogeneous Cantor set may assume the maximum value), the struc-

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ture of homogeneous Moran sets whose dimension assumes an intermediate value is not clear.

In this paper, we construct a class of special homogeneous Moran sets which are called $\{m_k\}$-quasi homogeneous Cantor set. And we obtain a sufficient condition that the Hausdorff dimension of homogeneous Moran sets may get the minimum value, which extends and deepens the results of the Hausdorff dimension of homogeneous Moran sets in [3]. By adjusting the value of $\{m_k\}_{k\geqslant 1}$, we constructively prove the intermediate value theorem for the homogeneous Moran sets.

The paper is organized as follows. In Section 2, we recall some preliminaries and state our main results. In Section 3, we construct the $\{m_k\}$-Moran set and the $\{m_k\}$-quasi homogeneous Cantor set, and discuss their Hausdorff dimension. The proofs of our main results are presented in Section 4.

2. Preliminaries and main results

First, we review the concept of the homogeneous Moran set. Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers and $\{c_k\}_{k\geqslant 1}$ a sequence of positive numbers satisfying $n_k \geqslant 2$ and $n_k c_k \leqslant 1$, $k \geqslant 1$. For any $k \geqslant 1$, let $D_k = \{\sigma = \sigma_1\sigma_2\ldots\sigma_k;\ 1 \leqslant \sigma_j \leqslant n_j,\ 1 \leqslant j \leqslant k\}$, $D = \bigcup\limits_{k\geqslant 0} D_k$, where $D_0 = \{\emptyset\}$. If $\sigma = \sigma_1\sigma_2\ldots\sigma_k \in D_k$, $\tau = \tau_1\tau_2\ldots\tau_m \in D_m$, let $\sigma\tau = \sigma_1\sigma_2\ldots\sigma_k\tau_1\tau_2\ldots\tau_m$.

Definition 2.1. Let $I = [0,1]$. The collection of closed subintervals $\mathcal{I} = \{I_\sigma;\ \sigma \in D\}$ of I has *homogeneous Moran structure* if it satisfies

(i) $I_\emptyset = I$;

(ii) for all $k \geqslant 1$, $\sigma \in D_{k-1}$, $I_{\sigma 1}$, $I_{\sigma 2}$, $\ldots$, $I_{\sigma n_k}$ are subintervals of I_σ, and for $i \neq j$ is $\mathring{I}_{\sigma i} \cap \mathring{I}_{\sigma j} = \emptyset$, where $\mathring{A}$ denotes the interior of A;

(iii) for any $k \geqslant 1$ and any $\sigma \in D_{k-1}$, $1 \leqslant j \leqslant n_k$, we have

$$c_k = \frac{|I_{\sigma j}|}{|I_\sigma|}, \tag{2.1}$$

where $|A|$ denotes the diameter of A.

We call $E = \bigcap\limits_{k\geqslant 1} \bigcup\limits_{\sigma\in D_k} I_\sigma$ a homogeneous Moran set. We use $\mathcal{M}(\{n_k\}_{k\geqslant 1}, \{c_k\}_{k\geqslant 1})$ to denote the collection of homogeneous Moran sets generated by the above homogeneous Moran structure, simply written as $\mathcal{M}$, and we call $E_k = \{I_\sigma; \sigma \in D_k\}$ the k-order fundamental intervals of E; I is called the original interval of E.

If the left endpoint of $I_{\sigma 1}$ is the same as the left endpoint of I_σ, and the right endpoint of $I_{\sigma n_k}$ is the same as the right endpoint of I_σ, and the gaps between $I_{\sigma i}$,

$1 \leqslant i \leqslant n_k$, are equal, then E is called a homogeneous Cantor set, simply written as C.

If the left endpoint of $I_{\sigma 1}$ is the same as the left endpoint of I_σ, and the left endpoint of $I_{\sigma(i+1)}$ is the same as the right endpoint of $I_{\sigma i}$, $1 \leqslant i \leqslant n_k - 1$, then E is called a partial homogeneous Cantor set, simply written as C^*.

Homogeneous Cantor sets C and partial homogeneous Cantor sets C^* are two kinds of special but very important homogeneous Moran sets.

The following result is due to Feng [3]:

Theorem 2.2. *Suppose $E \in \mathcal{M}$, then we have*

$$t_* \leqslant \dim_H E \leqslant s_*,$$

where

$$\dim_H C^* = t_*, \quad \dim_H C = s_*,$$

and

$$t_* = \liminf_{k\to\infty} \frac{\ln n_1 n_2 \dots n_{k-1}}{-\ln c_1 c_2 \dots c_k n_k}, \quad s_* = \liminf_{k\to\infty} \frac{\ln n_1 n_2 \dots n_k}{-\ln c_1 c_2 \dots c_k}.$$

However, except for the extreme situations, i.e., when the dimension of the partial homogeneous Cantor set may assume the minimum value while the dimension of the homogeneous Cantor set may assume the maximum value, the structure of a homogeneous Moran set whose dimension assumes the intermediate value is not clear.

In this paper, we construct a class of special homogeneous Moran sets which are called $\{m_k\}$-quasi homogeneous Cantor sets and prove the following intermediate value theorem.

Theorem 2.3. *Suppose $t_* \leqslant s \leqslant s_*$. Then there exists an $\{m_k\}$-quasi homogeneous Cantor set $\mathcal{F}$ such that*

$$\dim_H \mathcal{F} = s. \tag{2.2}$$

In addition, we note that, for any homogeneous Moran set E, if m_{I_σ} is the number of the connected components contained in the k-order fundamental intervals I_σ, $\sigma \in D_k$, then we obtain the following useful result:

Theorem 2.4. *Suppose $E \in \mathcal{M}$. If $\sup_{\sigma\in D}\{m_{I_\sigma}\} < \infty$, then we have*

$$\dim_H E = t_*. \tag{2.3}$$

From this theorem we know that partial homogeneous Cantor set is only a small part of homogeneous Moran set whose dimension may assume the minimum value. However, Corollary 3.3 shows that $\sup_{\sigma\in D}\{m_{I_\sigma}\}<\infty$ is not a necessary condition.

3. Construction of $\{m_k\}$-Moran set and $\{m_k\}$-quasi homogeneous Cantor set

Definition 3.1. Suppose $E\in\mathcal{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1})$, E_k is the k-order fundamental interval of E and $\{m_k\}_{k\geqslant1}$ is a sequence of positive integers satisfying $1\leqslant m_k\leqslant n_k$, $k\geqslant1$. If the k-order fundamental intervals $I_{\sigma1},I_{\sigma2},\dots,I_{\sigma n_k}$ contained in I_σ, for all $I_\sigma\in E_{k-1}$, arbitrarily connect forming m_k connected components, written as $J_{\sigma1},J_{\sigma2},\dots,J_{\sigma m_k}$, then such a homogeneous Moran set E is called $\{m_k\}$-Moran set. We use $\mathcal{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1},\{m_k\})$ to denote the collection of $\{m_k\}$-Moran sets.

Suppose $E\in\mathcal{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1},\{m_k\})$, let J_σ be the connected components contained in the $(k-1)$-order fundamental intervals. Let $A_k=\{J_\sigma;\ \sigma\in\Sigma_k\}$, where $\Sigma_k=\{\sigma=i_1i_2\dots i_{k-1}j_k;\ 1\leqslant i_s\leqslant n_s,\ 1\leqslant s\leqslant k-1,\ 1\leqslant j_k\leqslant m_k\}$. Obviously, $\sharp\mathcal{A}_k=n_1n_2\dots n_{k-1}m_k$. Let $l_*=\liminf_{k\to\infty}\ln(n_1n_2\dots n_{k-1}m_k)/-\ln(c_1c_2\dots c_k\times n_k/m_k)$, $\delta_k=c_1c_2\dots c_k$, $N_k=n_1n_2\dots n_k$.

Theorem 3.2. *Suppose $E\in\mathcal{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1},\{m_k\})$, then we have*

$$\dim_H E\leqslant l_*. \tag{3.1}$$

Proof. For any $l_*<l<1$, there exist subsequence $\{k_i\}_{i\geqslant1}$ and a positive integer N such that for any $i>N$, we have $l\geqslant\ln(n_1n_2\dots n_{k_i-1}m_{k_i})/-\ln(c_1c_2\dots c_{k_i}\times n_{k_i}/m_{k_i})$, i.e.,

$$n_1n_2\dots n_{k_i-1}m_{k_i}\Big(c_1c_2\dots c_{k_i}\frac{n_{k_i}}{m_{k_i}}\Big)^l\leqslant1. \tag{3.2}$$

Obviously, $\mathcal{A}_{k_i}$ is a $\delta_{k_i}n_{k_i}$-cover of E, by the definition of Hausdorff measure,

$$H^l_{\delta_{k_i}n_{k_i}}(E)\leqslant\sum_{J_\sigma\in\mathcal{A}_{k_i}}|J_\sigma|^l=\sum_{I\in E_{k_i-1}}\sum_{j=1}^{m_{k_i}}|J_j|^l. \tag{3.3}$$

Since $\sum_{j=1}^{m_{k_i}}|J_j|\equiv n_{k_i}\delta_{k_i}$, $0<l<1$, by Jensen's inequality, we have

$$H^l_{\delta_{k_i}n_{k_i}}(E)\leqslant\sum_{I\in E_{k_i-1}}m_{k_i}\Big(\frac{n_{k_i}\delta_{k_i}}{m_{k_i}}\Big)^l=n_1n_2\dots n_{k_i-1}m_{k_i}\Big(\delta_{k_i}\frac{n_{k_i}}{m_{k_i}}\Big)^l\leqslant1. \tag{3.4}$$

As $i \to \infty$, $\delta_{k_i} n_{k_i} \to 0$, we get $H^l(E) \leqslant 1 < \infty$. By the definition of Hausdorff dimension, we have $\dim_H E \leqslant l$. By the arbitrariness of l, we have $\dim_H E \leqslant l_*$. $\square$

Corollary 3.3. *If* $\lim\limits_{k\to\infty} \ln(m_k)/-\ln(\delta_k n_k) = 0$, *then for any* $E \in M(\{n_k\}_{k\geqslant 1}, \{c_k\}_{k\geqslant 1}, \{m_k\})$, *we have*

$$\dim_H E = t_*, \tag{3.5}$$

where t_* *is defined as in Theorem* 2.2.

Proof. If $\lim\limits_{k\to\infty} \ln m_k/-\ln(\delta_k n_k) = 0$, then

$$\begin{aligned} l_* &= \liminf_{k\to\infty} \frac{\ln n_1 n_2 \ldots n_{k-1} m_k}{-\ln(\delta_k n_k/m_k)} \\ &= \liminf_{k\to\infty} \frac{\ln n_1 n_2 \ldots n_{k-1}/-\ln(\delta_k n_k) + \ln m_k/-\ln(\delta_k n_k)}{1 + \ln m_k/-\ln(\delta_k n_k)} = t_*. \end{aligned}$$

And since $t_* \leqslant \dim_H E \leqslant l_*$ is the known result, we have $\dim_H E = t_*$. $\square$

In order to verify that the maximum value l_* can be obtained, in the following we introduce a special class of $\{m_k\}$-Moran sets, called $\{m_k\}$-quasi homogeneous Cantor sets. Let $a_k = [n_k/m_k] + 1$, where $[a]$ denotes the largest integer less than or equal to a.

Definition 3.4. Suppose $\mathcal{F} \in \mathcal{M}(\{n_k\}_{k\geqslant 1}, \{c_k\}_{k>1}, \{m_k\})$. Let I be the $(k-1)$-order fundamental interval of $\mathcal{F}$, and let $J_1, J_2, \ldots, J_{m_k}$ be the connected components contained in I. If the gaps between J_i, $1 \leqslant i \leqslant m_k$, are equal and the left endpoint of J_1 is the same as the left endpoint of I, the right endpoint of J_{m_k} is the same as the right endpoint of I, and

$$\max\{|J_i| - |J_j|;\ 1 \leqslant i,\ j \leqslant m_k\} = \delta_k,$$

then $\mathcal{F}$ is called an $\{m_k\}$-quasi homogeneous Cantor set.

Remark 3.5. In the special case, if $m_k \equiv n_k$, then $\mathcal{F}$ is a homogeneous Cantor set; if $m_k \equiv 1$, then $\mathcal{F}$ is a partial homogeneous Cantor set.

Remark 3.6. By the definition of $\{m_k\}$-quasi homogeneous Cantor set, we can see that if for any $1 \leqslant i \leqslant m_k$, J_i is composed of a_k or $a_k - 1$, k-order fundamental intervals arbitrarily connected, then $|J_i| = \delta_k a_k$ or $\delta_k(a_k - 1)$.

4. Proof of main results

First, we introduce a very important theorem, that is, the mass distribution principle:

Lemma 4.1 ([1]). *Suppose μ is the mass distribution on $\mathcal{F}$, and for some s, there exist $C > 0$ and $\delta > 0$ such that for any U satisfying $|U| \leqslant \delta$ we have*

$$\mu(U) \leqslant C|U|^s.$$

Then $\dim_H F \geqslant s$.

Theorem 4.2. *Suppose $\mathcal{F}$ is an $\{m_k\}$-quasi homogeneous Cantor set. Then*

$$\dim_H \mathcal{F} = l_*.$$

Proof. By Theorem 3.2, $\dim_H \mathcal{F} \leqslant l_*$. It suffices to prove that $\dim_H \mathcal{F} \geqslant l_*$. For any $0 < l < l_*$, there exists N such that when $k > N$, then

$$n_1 n_2 \dots n_{k-1} m_k \Big(\delta_k \frac{n_k}{m_k}\Big)^l \geqslant 1. \tag{4.1}$$

There exists a natural measure μ on the homogeneous Moran set $\mathcal{F}$ and for any k-order fundamental intervals I,

$$\mu(I) = \frac{1}{n_1 n_2 \dots n_k}. \tag{4.2}$$

In order to apply Lemma 4.1, in the following we will show that there exists a constant $C > 0$ such that for any U satisfying $|U| = \delta < \delta_N$ we have

$$\mu(U) \leqslant C|U|^l. \tag{4.3}$$

It is easily checked that there exists a unique $k > N$ such that $a_k\delta_k \leqslant \delta < a_{k-1}\delta_{k-1}$. Suppose that e is the number of elements in the intersection of U and $\mathcal{A}_k$, let $J_{\sigma i}$, $J_{\sigma(i+1)}$ be the two adjacent connected components contained in the $(k-1)$-order fundamental intervals I_σ. By the definition of an $\{m_k\}$-quasi homogeneous Cantor set, the gaps between them are independent of σ and i, written as η_k. We divide the proof into two cases as follows:

Case 1. $\delta_k a_k \leqslant \delta < \eta_k$. In this case, obviously, $e \leqslant 3$. Since

$$\frac{n_k}{m_k} \leqslant a_k \leqslant \frac{n_k}{m_k} + 1 \leqslant 2\frac{n_k}{m_k},$$

this implies

$$\mu(U) \leqslant 3a_k \frac{1}{n_1 n_2 \dots n_k} \leqslant \frac{6}{n_1 n_2 \dots n_{k-1} m_k} \leqslant 6\Big(\delta_k \frac{n_k}{m_k}\Big)^l \leqslant 6(\delta_k a_k)^l \leqslant 6\delta^l. \tag{4.4}$$

Case 2. $\max\{\delta_k a_k, \eta_k\} \leqslant \delta < \delta_{k-1} a_{k-1}$.

(i) If $m_{k-1} = n_{k-1}$, then $a_{k-1} = 1$. Hence U intersects two elements in $\mathcal{A}_{k-1}$ at most. If $m_{k-1} < n_{k-1}$, then $a_{k-1} \geqslant 2$. Hence $2\delta_{k-1}(a_{k-1} - 1) \geqslant a_{k-1}\delta_{k-1} > \delta$. Thus U intersects three elements in $\mathcal{A}_{k-1}$ at most and hence $e \leqslant 3a_{k-1}m_k$.

(ii) Since U intersects e elements in $\mathcal{A}_k$, it follows that

$$\frac{\delta_k a_k}{2}(e-2) \leqslant \delta, \qquad \frac{e-2}{2}\eta_k \leqslant (e-1)\eta_k \leqslant \delta,$$

namely, $((e-2)/2)\max\{\delta_k a_k, \eta_k\} \leqslant \delta$, so

$$e \leqslant \frac{2\delta}{\max\{\delta_k a_k, \eta_k\}} + 2 \leqslant \frac{4\delta}{\max\{\delta_k a_k, \eta_k\}}.$$

Integrating (i) and (ii), we have

$$\begin{aligned}\mu(U) &\leqslant \min\Big\{3a_{k-1}m_k, \frac{4\delta}{\max\{\delta_k a_k, \eta_k\}}\Big\} \frac{a_k}{n_1 n_2 \dots n_k} \\ &\leqslant 4(m_k a_{k-1})^{1-l}\Big(\frac{\delta}{\max\{\delta_k a_k, \eta_k\}}\Big)^l \frac{a_k}{n_1 n_2 \dots n_k} \\ &= 4\delta^l \frac{a_k m_k a_{k-1}^{1-l}}{n_1 n_2 \dots n_k (m_k \max\{\delta_k a_k, \eta_k\})^l}.\end{aligned}$$

The second inequality above is established because

$$\min\{x, y\} = (\min\{x, y\})^{1-l}(\min\{x, y\})^l \leqslant x^{1-l} y^l.$$

Since

$$\begin{aligned}\delta_{k-1} = n_k\delta_k + (m_k - 1)\eta_k &\leqslant m_k\Big(\frac{n_k}{m_k}\delta_k + \eta_k\Big) \\ &\leqslant m_k(a_k\delta_k + \eta_k) \leqslant 2m_k \max\{\delta_k a_k, \eta_k\},\end{aligned}$$

we have

$$\begin{aligned}\mu(U) &\leqslant 4 \cdot 2^l \delta^l \frac{a_k m_k a_{k-1}^{1-l}}{n_1 n_2 \dots n_k (\delta_{k-1})^l} = 4 \cdot 2^l \delta^l \frac{a_{k-1} a_k m_k}{n_1 n_2 \dots n_k (\delta_{k-1} a_{k-1})^l} \\ &\leqslant 16 \cdot 2^l \delta^l \frac{1}{n_1 n_2 \dots n_{k-2} m_{k-1} (\delta_{k-1} a_{k-1})^l} \leqslant 16 \cdot 2^l \delta^l.\end{aligned} \tag{4.5}$$

By Lemma 4.1, we have $\dim_H \mathcal{F} \geqslant l$. By the arbitrariness of l, we have $\dim_H \mathcal{F} \geqslant l_*$. Hence we completed the proof of Theorem 4.2. $\square$

Lemma 4.3. *Suppose $t_* \leqslant s \leqslant s_*$, then there exists a sequence of positive integers $\{m_k\}_{k\geqslant 1}$ such that*

$$l_* = s. \tag{4.6}$$

P r o o f. It suffices to show that when $t_* < s < s_*$, the conclusion is established. Let $\delta_k = c_1c_2\dots c_k$, $N_k = n_1n_2\dots n_k$. By the definition of an $\{m_k\}$-Moran set, for any $k \geqslant 1$ we have $1 \leqslant m_k \leqslant n_k$. Since $s < s_* = \liminf\limits_{k\to\infty} \ln N_k/{-\ln\delta_k}$, there exists $N > 0$ such that for any $k \geqslant N$ we have $s \leqslant \ln N_k/{-\ln\delta_k}$, namely, $N_k\delta_k^s \geqslant 1$. Due to $s < 1$, we get

$$n_k(N_k\delta_k^s)^{1/(s-1)} \leqslant n_k. \tag{4.7}$$

▷ When $k < N$, we take $m_k \equiv 1$;
▷ when $k \geqslant N$ and $n_k(N_k\delta_k^s)^{1/(s-1)} \leqslant 1$, we take $m_k = [n_k(N_k\delta_k^s)^{1/(s-1)}] + 1$;
▷ when $k \geqslant N$ and $n_k(N_k\delta_k^s)^{1/(s-1)} \geqslant 1$, we take $m_k = [n_k(N_k\delta_k^s)^{1/(s-1)}]$.

Hence by the above proof, the sequence of positive integers $\{m_k\}_{k\geqslant 1}$ which we have constructed satisfies $1 \leqslant m_k \leqslant n_k$.

And when $k \geqslant N$, we have $m_k - n_k(N_k\delta_k^s)^{1/(s-1)} \leqslant 1$. Hence

$$\begin{aligned} l_* &= \liminf_{k\to\infty} \frac{\ln N_{k-1}m_k}{-\ln\delta_k \dfrac{n_k}{m_k}} = \liminf_{k\to\infty} \frac{\ln N_{k-1}n_k(N_k\delta_k^s)^{1/(s-1)}}{-\ln\delta_k n_k \cdot {n_k}^{-1}(N_k\delta_k^s)^{1/(s-1)}} \\ &= \liminf_{k\to\infty} \frac{\ln N_k {N_k}^{1/(s-1)}{\delta_k}^{s/(s-1)}}{-\ln\delta_k{\delta_k}^{s/(1-s)}{N_k}^{1/(1-s)}} = \liminf_{k\to\infty} \frac{s\ln N_k\cdot\delta_k}{(s-1)^{-1}\ln N_k\cdot\delta_k} = s. \end{aligned}$$

By Theorem 4.2 and Lemma 4.3, we can get Theorem 2.3 easily. □

Finally, we prove Theorem 2.4 as follows.

P r o o f of Theorem 2.4. It suffices to show that $\dim_H E \leqslant t_*$. For any $t_* < t < 1$, there exists a subsequence $\{k_i\}_{i\geqslant 1}$ and a positive integer N such that for any $i > N$ we have $n_1n_2\dots n_{k_i-1}(\delta_{k_i}n_{k_i})^t \leqslant 1$.

Obviously A_{k_i} is a $\delta_{k_i}n_{k_i}$-cover of E; by the definition of Hausdorff measure,

$$\begin{aligned} H^t_{\delta_{k_i}n_{k_i}}(E) &\leqslant \sum_{\sigma\in D_{k_i-1}}\sum_{j=1}^{m_\sigma}|J_{\sigma j}|^t \leqslant m_\sigma n_1n_2\dots n_{k_i-1}(\delta_{k_i}n_{k_i})^t \\ &\leqslant mn_1n_2\dots n_{k_i-1}(\delta_{k_i}n_{k_i})^t \leqslant m, \end{aligned}$$

where $m = \sup\limits_{\sigma\in D} m_\sigma$. As $i\to\infty$, then $\delta_{k_i}n_{k_i} \leqslant \delta_{k_i-1} \to 0$. Thus we get that $\mathcal{H}^t(E) \leqslant m < \infty$. By the definition of Hausdorff dimension, we have $\dim_H E \leqslant t$, so $\dim_H E \leqslant t_*$.

Hence we have completed the proof of Theorem 2.4. □

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