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REMARKS ON THE BEHAVIOUR OF HIGHER-ORDER DERIVATIONS ON THE GLUING OF DIFFERENTIAL SPACES

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Abstract. This paper is about some geometric properties of the gluing of order k in the category of Sikorski differential spaces, where k is assumed to be an arbitrary natural number. Differential spaces are one of possible generalizations of the concept of an infinitely differentiable manifold. It is known that in many (very important) mathematical models, the manifold structure breaks down. Therefore it is important to introduce a more general concept. In this paper, in particular, the behaviour of $k^{\rm th}$ order tangent spaces, their dimensions, and other geometric properties, are described in the context of the process of gluing differential spaces. At the end some examples are given. The paper is self-consistent, i.e., a short review of the differential spaces theory is presented at the beginning.

Keywords: gluing of differential space; higher-order differential geometry; Sikorski differential space

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1. Introduction

The concept of a *differential space* emerged in 1960s and the whole idea can be summarized in the following way. First of all, the notion a smooth manifold is presented not in terms of maps and atlases, but rather as an algebra of functions on a given set [10], [26], [29]. Then the topology and the differential structure is recovered from this algebra. So, if some properties of this algebra are weakened, one would obtain a generalized space.

It should be noticed that there are many ways of generalizing the notion of a smooth manifold and of course, Sikorski's one, is not the only one. Beside the concept initially proposed in [36], [35] (which will be studied in this paper), there

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are also some others. Those, which are especially close to the Sikorski differential spaces are, for example, Spallek's spaces [41] and Mostow's spaces [24]. Yet, much in this direction was done also by Chen [4], Kriegl and Michor [15], Mallios and his collaborators (Rosinger, Zafiris, Vassiliou, and others) [19], [18], [20], [42], Nestruev [26], Souriau [40] and others. In [1] the reader should find a brief overview of various generalizations.

This paper is aimed at differential spaces in the sense of Sikorski [36], [35]. Yet, this concept was studied extensively by the Polish group mainly in the context of its applications to the problem of spacetime singularity (see, for example, [11], [13] and references therein). The usefulness of differential spaces is clearly seen when one tries to build the differential geometry over "singular" spaces, where the classical formalism breaks down. Indeed, the efforts of the Polish group gathered around Heller and Sasin in 1990s were focused on the endeavour to find a suitable category, which would allow to describe a spacetime together with its singularities as a single object. On the other hand, Śniatycki and his collaborators used differential spaces in the geometric quantization, in the problem of reduction of symmetries [38], [37] and in describing the mechanics on fractals [8].

In the similar period Mallios and his collaborators developed the "abstract differential geometry" starting from the theory of Banach spaces and topological algebras. Indeed, Mallios' differential triads over topological spaces are just another attempt to build a generalized differential calculus [19], [18].

As this paper is concerned with the gluing procedure, let us mention that the gluing of smooth manifolds is well known. For example, it is briefly explained in [2]. The general process of gluing in the case of Sikorski differential spaces is described in [31], [30]. A similar technique, but in the category of Spallek's differential spaces is presented in [33]. The last paper is also heavily focused on cosmological models.

Finally, let us mention that the basic concepts of the higher-order differential geometry in a classical sense can be found, for example, in [43]. Much can also be found in [3], [5], [21], [23]. For historical reasons, of course, the original paper [27] should be consulted, and also [28] presents an interesting discussion. More information about the basic concepts of the differential spaces theory with some examples can be found, for example, in the expository paper [6]. The last paper describes also a certain gluing technique, called therein the *generator gluing technique*. However, this method is strictly a global one. Yet, there exists no consistent description of an arbitrary-order gluing technique of differential spaces.

To keep clarity of the exposition in this paper, we will try to show crucial steps in such a way that the reader non-familiar with differential spaces should be able to keep the track. However, for results already known, only references will be given. This paper is organized in the following way. In Section 2 we define the differential spaces and all necessary notions and facts from the higher-order geometry over such spaces. Next, in Section 3 we describe the main part of this paper, i.e., the gluing technique of arbitrary order. Finally, in Section 4 we give some examples of the technique described in Section 3. In particular, theorems proved in Section 3 and Section 4 are new results. They provide explanations including the behaviour of a topology, the dimension of a tangent space, and vector fields. We discuss also the generators of the glued differential space.

2. Higher-order geometry on differential spaces

Now, let us briefly review the fundamental concepts and definitions of differential spaces theory. Let M be a nonempty set. Let C be a family of some real functions on M, i.e., $C := \{f_1, \dots, f_k, \dots;\ f_k\colon M \to \mathbb{R} \text{ for all } k\}$. The weakest topology, for which all functions from C are continuous, is called the topology induced by C on M, and it will be denoted by τ_C.

Now, let f be a function defined on a subset $A \subset M$, and suppose that we have already some family C of functions on M (as above, defining the topology on M). Then we have also the topology on A induced from τ_C, i.e., $\tau_A := \{U \cap A;\ U \in \tau_C\}$. If for all $p \in A$ exist $B \ni p$ and $g \in C$ that $f|_B = g|_B$ where $B \in \tau_A$, then f is called a *local C-function*. In other words, f is a local C-function if at every point $p \in A$ there exists a function $g \in C$ and an open neighbourhood B (with respect to τ_C) of p, such that $f|_B = g|_B$.

Definition 2.1. The set of all local C-functions on a given set $A \subset M$ is denoted by C_A.

The *superposition closure* of a family of functions C, denoted by $\operatorname{sc} C$, is defined in the following way: $\operatorname{sc} C := \{\omega \circ (f_1, \dots, f_n);\ n \in \mathbb{N},\ \omega \in C^\infty(\mathbb{R}^n),\ f_1, \dots, f_n \in C\}$.

It is not hard to check (see, for example, [34]) that the following relations hold: $C|_A \subset C_A$, $(C_A)_A = C_A$ and $C \subset \operatorname{sc}(C_M) \subset (\operatorname{sc} C)_M$. As an example for the first relation, consider $C = C^\infty(\mathbb{R})$ and $A = (0,1) \subset \mathbb{R} = M$. The function x^{-1} restricted to A is not a restriction of any function from C, but it belongs to C_A.

Finally, suppose that some family of functions C is given on a nonempty set M. Let C be such that: $C = C_M$ and $C = \operatorname{sc} C$. Then C is called a *differential structure on M*.

Definition 2.2. Let M be a nonempty set and let C be a differential structure on M. A pair (M, C) is called the *Sikorski differential space*.

Now, suppose that we have a family of functions C_0 defined on M. Then, of course, $C = (\operatorname{sc} C_0)_M$ is a differential structure on M. Such a differential structure is called *generated by* C_0 and functions from C_0 are called *generators*. Notice that generators are not unique. For example, we can always take the whole C as generators, but in many concrete cases, it is reasonable to consider the smallest possible collection of generators. If there exists a finite C_0 which generates C, then (M, C) is called *finitely generated*.

Of course, C is a sub-algebra of $\mathbb{R}^M$. Moreover, for two differential spaces (M, C) and (N, D), a mapping $F\colon M \to N$ is called *smooth*, if $f \circ F \in C$ for all $f \in D$, and it is called a *diffeomorphism*, if it is bijective and both F and F^{-1} are smooth (in the above sense).

Notice that a differential space $(M, C^\infty(M))$ is a smooth (in a classical sense) manifold. However, we can consider some other differential structures on M. For example, a pair (M, D), where D consists of all continuous functions on M, is a differential space. Also we can insert as a generator not only a non-smooth (in a classical sense) function, but even a non-continuous (with respect to the standard topology) one. The first modification affects just the differential structure. Therefore, in the differential spaces category the notion of a smoothness is understood in a more general sense (as explained above). The second case changes also the topology of the space. Therefore, differential spaces provide some general techniques, for which both manifolds and "singular" spaces are subcases.

Theorem 2.1 below is well-known (see, for example, [6]).

Theorem 2.1. *If (M, C) is a Hausdorff differential space, with $C = (\mathrm{sc}\; C_0)_M$ and $C_0 = \{f_1, \dots, f_k\}$, then $F = (f_1, \dots, f_k)$ is a diffeomorphism from (M, C) to $(F(M), C^\infty(F(M))$, where $F(M) \subset \mathbb{R}^n$ and $C^\infty(F(M) := (C^\infty(\mathbb{R}^k)|_{F(M)})_{F(M)} = (\mathrm{sc}\,\{\pi_1|_{F(M)}, \dots, \pi_k|_{F(M)}\})_{F(M)}$, where $\pi_i\colon \mathbb{R}^k \ni (x_1, \dots, x_i, \dots, x_k) \mapsto x_i \in \mathbb{R}$ are the projections for $i = 1, \dots, k$.*

Further, we will also need to consider functions vanishing at a given point and functions which, together with their derivatives up to order k, vanish at that point. This is an algebraic counterpart of the notion of tangency of order k. Therefore, let (M, C) be a differential space, and $p \in M$. Then $\mathfrak{m}_p := \{f \in C;\ f(p) = 0\}$ is an ideal. Similarly, let $\mathfrak{m}_p^k$ be the k^{th} power of the ideal $\mathfrak{m}_p$. In other words, $\mathfrak{m}_p^k$ is generated by $f_1 f_2 \dots f_k$, where $f_1, f_2, \dots, f_k \in \mathfrak{m}_p$.

Definition 2.3. Let $k \in \mathbb{N}$. A linear mapping $v\colon C \to \mathbb{R}$ is called the k^{th} *order tangent vector to the differential space* (M, C) *at the point* $p \in M$, if the following conditions hold:

(1) $v(r) = 0$ for all $r \in \mathbb{R} \subset C$.
(2) $v(\mathfrak{m}_p^{k+1}) = 0$. The set of all k^{th} order tangent vectors to (M, C) at $p \in M$ is denoted by $T_p^k M$.

Let us observe that $T_p^k M$ possesses a natural structure of a linear space. Moreover, it is obvious that, for $k = 1$, the above definition is equivalent with the classical one, i.e., 1^{st} order tangent vectors are exactly $\mathbb{R}$-linear operators, satisfying the Leibniz rule. Of course, for an arbitrary $k \in \mathbb{N}$ it is true that $\mathfrak{m}_p^{k+1} \subset \mathfrak{m}_p^k$ and, therefore, $T_p^k M \subset T_p^{k+1} M$.

Further, we will need to consider a certain linear functional. Namely, let $v \in T_p^k M$ be an arbitrary k^{th} order tangent vector, and define $l_v([f]) := v(f)$ for an arbitrary $f \in \mathfrak{m}_p$ and $[f] \in \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$. It is clear that l_v is a linear functional.

Lemma 2.1. *The mapping* $I\colon T_p^k M \ni v \mapsto l_v \in (\mathfrak{m}_p/\mathfrak{m}_p^{k+1})^*$, *defined by* $I(v) := l_v$, *is an isomorphism of linear spaces.*

Proof. First, notice that I is linear. If $l_v = 0$, then $v = 0$. Therefore I is a monomorphism. Now, let $l \in (\mathfrak{m}_p/\mathfrak{m}_p^{k+1})^*$ and consider the mapping $v_l\colon C \to \mathbb{R}$ defined by $v_l(f) := l([f - f(p)])$ for an arbitrary $f \in C$. It is easy to notice that $v_l \in T_p^k M$ and $I(v_l) = l$. Therefore I is also an epimorphism. $\square$

Corollary 2.1. *Let* (M, C) *be a differential space,* $k \in \mathbb{N}$ *and* $f \in C$. *If* $v(f) = 0$ *for any* $v \in T_p^k M$, *then* $f - f(p) \in \mathfrak{m}_p^{k+1}$.

Proof. If $v(f) = 0$ for an arbitrary $v \in T_p^k M$, then for an arbitrary $l \in (\mathfrak{m}_p/\mathfrak{m}_p^{k+1})^*$, it is true that $l([f - f(p)]) = v_l(f) = 0$. As a result, $[f - f(p)] = 0$, so $f - f(p) \in \mathfrak{m}_p^{k+1}$. $\square$

Similarly, as in the classical case, we can introduce the differentials of arbitrary order of a function $f \in C$. Namely, let $v \in T_p^k M$, where $p \in M$ and $k \in \mathbb{N}$. Then the mapping $d_p^k f\colon T_p^k M \to \mathbb{R}$, defined by $d_p^k f(v) := v(f)$, is called the k^{th} *order differential of* f *at the point* $p \in M$.

Notice that for any $k \in \mathbb{N}$, a natural differential structure, denoted by $T^k C$, can be generated on $T^k M := \bigsqcup_{p \in M} T_p^k M$ (see, for example, [25]). In particular, $T^k C$ is generated by $\{d^k f;\ f \in C\} \cup \{f \circ \pi^k;\ f \in C\}$, where $\pi^k\colon T^k M \to M$ is the natural projection and $d^k f\colon T^k M \to \mathbb{R}$ is defined by $(d^k f)(V) := V(f)$, $V \in T^k M$.

Moreover, for 1^{st} order tangent vectors we have a well known (see, for example, [32]) formula for computations. In particular, if $v\colon C \to \mathbb{R}$ is a tangent vector at p (i.e., a linear mapping, satisfying the Leibniz rule), then for an arbitrary $n \in \mathbb{N}$ and $\omega \in C^\infty(\mathbb{R}^n)$ and a collection of functions $f_1, \dots, f_n$ from the generators of the structure C, the formula

$$v(\omega \circ (f_1, \dots, f_n)) = \sum_{i=1}^{n} \frac{\partial \omega}{\partial x_i}(f_1(p), \dots, f_n(p)) v(f_i) \tag{2.1}$$

holds.

Notice that equation (2.1) allows to recover the behaviour of tangent vectors on the superposition closure from the behaviour on the generators. Nevertheless, equation (2.1) cannot be used, in full generality, to prolong derivations of an algebra to its superposition closure (see [22] and [9] on this concern), and we do not discuss it further here. However, we can generalize the above formula for k^{th} order vectors.

Lemma 2.2. *Let (M, C) be a differential space generated by C_0, and $p \in M$. If $g \in \mathfrak{m}_p$ is such that $d^k_p g = 0$, then there exists an open neighbourhood $U \ni p$, and $f_1, \ldots, f_n \in C_0$, $\omega_{i_1 \ldots i_{k+1}} \in C^\infty(\mathbb{R}^n)$, where $n \in \mathbb{N}$ and $i_1, \ldots, i_{k+1} = 1, \ldots, n$, such that*

$$(2.2)\quad g|_U = \Big(\sum_{i_1, \ldots, i_{k+1}=1}^{n} (f_{i_1} - f_{i_1}(p)) \ldots (f_{i_{k+1}} - f_{i_{k+1}}(p)) \omega_{i_1 \ldots i_{k+1}} \circ (f_1, \ldots, f_n) \Big)\Big|_U .$$

Proof. Since $g \in \mathfrak{m}_p \subseteq C = (\operatorname{sc} C_0)_M$, there exists an open neighbourhood $U \ni p$ and $f_1, \ldots, f_n \in C_0$ and $\theta \in C^\infty(\mathbb{R}^n)$, where $n \in \mathbb{N}$, such that

$$(2.3)\qquad g|_U = \theta \circ (f_1, \ldots, f_n)|_U$$

and $\theta(f_1(p), \ldots, f_n(p)) = 0$. Of course, $\theta \in \mathfrak{m}_{F(p)} \subseteq C^\infty(\mathbb{R})$, where $F = (f_1, \ldots, f_n)$. (Because of Theorem 2.1, F is a diffeomorphism from (M, C) to $(F(M), C^\infty(F(M))$, with $F(M) \subset \mathbb{R}^n$.) Moreover, locally, $0 = d^k_p(g) = d^k_p(\theta \circ (f_1, \ldots, f_n)) = d^k_p(\theta \circ F) = d^k_{F(p)}\theta = 0$. Therefore, there exist $\omega_{i_1 \ldots i_{k+1}} \in C^\infty(\mathbb{R}^n)$, $i_1, \ldots, i_{k+1} = 1, \ldots, n$, such that

$$(2.4)\quad \theta(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_{k+1}=1}^{n} (x_{i_1} - f_{i_1}(p)) \ldots (x_{i_{k+1}} - f_{i_{k+1}}(p)) \omega_{i_1 \ldots i_{k+1}}(x_1, \ldots, x_n)$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then, equation (2.2) follows from equations (2.3) and (2.4).

□

Lemma 2.3. *Let (M, C) be a differential space generated by C_0, (N, D) a differential space generated by D_0 and $p_0 \in M$. Consider an arbitrary function $f \in C \otimes D$ (with $f = \alpha \otimes \beta$, where $\alpha \in C$ and $\beta \in D$) such that $f \in \bigcap_{q \in N} \mathfrak{m}_{(p_0, q)}$ and $d^k_{(p_0,q)} f = 0$ for all $q \in N$. Then for an arbitrary $q_0 \in N$ there exist open neighbourhoods $U \in \tau_C$ of p_0 and $V \in \tau_D$ of q_0, and functions $f_1, \ldots, f_n \in C_0$, $g_1, \ldots, g_m \in D_0$ and $\omega_{i_1 \ldots i_{k+1}} \in C^\infty(\mathbb{R}^{n+m})$, where $n, m \in \mathbb{N}$ and $i_1, \ldots, i_{k+1} = 1, \ldots, n + m$, such* *that*

$$f(p,q) = \sum_{i_1,\dots,i_{k+1}=1}^{n} (f_{i_1}(p) - f_{i_1}(p_0)) \dots (f_{i_{k+1}}(p) - f_{i_{k+1}}(p_0)) \times \omega_{i_1\dots i_{k+1}}(f_1(p),\dots,f_n(p),g_1(q),\dots,g_m(q)), \tag{2.5}$$

where $(p,q) \in U \times V$.

P r o o f. From the assumptions it is clear that $f(p_0,q) = 0$ for all $q \in N$. There exist open neighbourhoods $U \ni p_0$ and $V \ni q_0$ and $\omega \in C^\infty(\mathbb{R}^{n+m})$, $f_1,\dots,f_n \in C_0$, $g_1,\dots,g_m \in D_0$ such that $f(p,q) = \omega(f_1(p),\dots,f_n(p),g_1(q),\dots,g_m(q))$ for $(p,q) \in U \times V$. Let us fix $p \in U$ and $q \in V$. Define $\theta\colon \mathbb{R} \to \mathbb{R}$ as $\theta(t) := \omega(t(f_1(p) - f_1(p_0)) + f_1(p_0),\dots,t(f_n(p) - f_n(p_0)) + f_n(p_0), g_1(q),\dots,g_m(q))$. It is clear that $\theta(0) = f(p_0,q) = 0$ and $\theta(1) = f(p,q)$. Consequently, $f(p,q) = \theta(1) - \theta(0) = \int_0^1 (\mathrm{d}\theta/\mathrm{d}t)\,\mathrm{d}t = \sum\limits_{i=1}^{n} (f_i(p) - f_i(p_0)) h_i(f_1(p),\dots,f_n(p),g_1(q),\dots,g_m(q))$, where

$$h_i(f_1(p),\dots,f_n(p),g_1(q),\dots,g_m(q)) := \int_0^1 \frac{\partial\omega}{\partial x_i}(t(f_1(p) - f_1(p_0)) + f_1(p_0),\dots,t(f_n(p) - f_n(p_0)) + f_n(p_0), g_1(q),\dots,g_m(q))\,\mathrm{d}t.$$

It follows from Corollary 2.1 that $f \in \bigcap\limits_{q\in N} \mathfrak{m}^{k+1}_{(p_0,q)}$. It is also obvious that $\bigcap\limits_{q\in N} \mathfrak{m}^{k+1}_{(p_0,q)} = \Big(\bigcap\limits_{q\in N} \mathfrak{m}_{(p_0,q)}\Big)^{k+1}$. Now, equation (2.5) can be proved by induction. □

We introduce now the notion of *smoothness* for tangent vector fields of an arbitrary order.

A map $X\colon M \to \bigcup\limits_{p\in M} T^k_p M$, where $k \in \mathbb{N}$, is called a k^{th} *order tangent vector field to* (M,C), if $X(p) \in T^k_p M$ for every $p \in M$. Additionally, X is called *smooth*, if Xf, defined by $(Xf)(p) := X(p)(f)$, belongs to C for any $f \in C$. The C-module of all k^{th} order smooth tangent vector fields to (M,C) will be denoted by $\mathfrak{X}^k(M)$.

Let (M,C) be a differential space and let $A \subset M$. (Obviously, we assume $A \neq \emptyset$.) Then (A,C_A) is called a *differential subspace of* (M,C). (See Definition 2.1.) The embedding of (A,C_A) into (M,C) is denoted by ι_A. We say that $X \in \mathfrak{X}^k(M)$ is *tangent to* A, if for every $p \in A$ there exists $v \in T^k_p A$ such that $X(p) = (\iota_A)_* v$. The set of all elements of $\mathfrak{X}^k(M)$ which are smooth k^{th} order vector fields tangent to A will be denoted by $\mathfrak{X}^k_A(M)$. Obviously, $\mathfrak{X}^k_A(M)$ is a C-submodule of the C-module $\mathfrak{X}^k(M)$.

Notice that we defined vectors (let us stay with 1^{st} order ones for a moment) in an algebraic way. Indeed, for smooth manifolds it is well known that there are various ways of introducing tangent vectors, for example, as derivations, velocities of curves, via the cotangent bundle (see, for example, [26]). In the case of a smooth manifold these notions are equivalent. However, it can be not so for a differential space. Our aim is to introduce a tangent vector and a smooth vector field in such a way that for smooth manifolds the classical differential geometry is recovered, and simultaneously as much as possible the correspondence known for the classical differential geometry would pass to the differential spaces category. For example, we defined a smooth vector field in such a way that for 1st order tangent vector fields there is a 1-1 correspondence with derivations of the algebra C (see, for example, [6]). Such a way guarantees us also that the definition via the cotangent bundle is equivalent [26]. However, denote by $\varphi_t(x)$ the point on the maximal integral curve through x of a given derivation, corresponding to the value of t. For smooth manifolds $\varphi_t(x)$ is a diffeomorphism of some open neighbourhood of x onto some open neighbourhood of $\varphi_t(x)$. In the case of a differential space $\varphi_t(x)$ can fail to be a local diffeomorphism [39].

Theorem 2.2. *Let (M, C) be a differential space, $\emptyset \neq A \subset M$, $k \in \mathbb{N}$, and $X \in \mathfrak{X}^k(M)$. Then the following conditions are equivalent:*

(1) $X \in \mathfrak{X}^k_A(M)$;

(2) *for all $p \in A$ exists just one $Y \in \mathfrak{X}^k(A)$ that $X(p) = (\iota_A)_{*p} Y(p)$;*

(3) $f|_A = 0 \Rightarrow (Xf)|_A = 0$.

Proof. (1) $\Rightarrow$ (2): Let $X \in \mathfrak{X}^k_A(M)$ and define the tangent vector field $Y \colon A \to \bigcup_{p \in A} T^k_p A$ by $Y(p) := (\iota_A)^{-1}_{*p}(X(p))$, $p \in A$. Of course, $Y \in \mathfrak{X}^k(A)$ and it is the unique tangent vector field satisfying condition (2).

(2) $\Rightarrow$ (3): Let $f \in C$ be such that $f|_A = 0$. Because of (2), there exists $Y \in \mathfrak{X}^k(A)$ such that $(Xf)|_A = Y(f|_A)$, and, hence, $(Xf)|_A = 0$.

(3) $\Rightarrow$ (1): Suppose that $X \in \mathfrak{X}^k(M)$ fulfils (3). For every $p \in A$, let $v_p \colon C_A \to \mathbb{R}$ be the map $v_p(f) := X(p)(g)$, with $f \in C_A$ and $g \in C$, where $f|_{U \cap A} = g|_{U \cap A}$ for some open neighbourhood $U \in \tau_C$ of p. It is obvious that $v_p \in T^k_p A$ and $X(p) = (\iota_A)_{*p} v_p$. □

Theorem 2.3. *Let (M, C) be a differential space, $A \subset M$ and $X \in \mathfrak{X}(M)$. If $X^k := X \circ \ldots \circ X \in \mathfrak{X}^k_A(M)$ for some $k \in \mathbb{N}$, then $X \in \mathfrak{X}^k_A(M)$ and, moreover, $X^l \in \mathfrak{X}^l_A(M)$ for every $l \in \mathbb{N}$.*

Proof. Let $f \in C$ be such that $f|_A = 0$. Then it is easy to notice that $X^k(f^k)|_A = k!(X(f))^k|_A$. Because of Theorem 2.2, $X^k(f^k)|_A = 0$. Therefore, $Xf|_A = 0$ for an arbitrary $f \in C$ such that $f|_A = 0$. Consequently, $X \in \mathfrak{X}^k_A(M)$.

It is not hard to check that $X^l(f) = 0$ for an arbitrary $l \in \mathbb{N}$. As a result, $X^l \in \mathfrak{X}^l_A(M)$. $\square$

The next two lemmas (see [25] for their proofs) will also be useful in the sequel.

Lemma 2.4. *Let (M, C) be a differential space and $k \in \mathbb{N}$. If $A, B \subset M$ satisfy the condition $(\iota_{A\cap B})_{*p} T^k_p(A \cap B) = (\iota_A)_{*p} T^k_p A \cap (\iota_B)_{*p} T^k_p B$ for every $p \in A \cap B$, then $\mathfrak{X}^k_A(M) \cap \mathfrak{X}^k_B(M) \subset \mathfrak{X}^k_{A\cap B}(M)$.*

Lemma 2.5. *Let $A \subset M$. Then $\mathfrak{X}^k_A(M) = \mathfrak{X}^k_{\bar{A}}(M)$, where $\bar{A}$ denotes the (topological) closure of A in τ_C.*

Finally, let $X \in \mathfrak{X}(M)$ be a smooth vector field tangent to (M, C). For any $k \in \mathbb{N}$ we define $\partial^k X \colon T^k M \to T^{k+1} M$ by $(\partial^k X)(V) := V \circ X$. Then $\partial^k X$ turns out to be a smooth mapping between the differential spaces $(T^k M, T^k C)$ and $(T^{k+1} M, T^{k+1} C)$. Indeed, $d^{k+1} f \circ \partial^k X = d^k(X(f))$ for every $f \in C$ and $\pi^{k+1} \circ \partial^k X = \pi^k$.

3. The main results

Let (M, C) and (N, D) be two differential spaces. Let $H\colon (M, C) \to (N, D)$ be a diffeomorphism, in the category-theoretic sense (morphisms are just smooth mappings as defined in Section 2). Consider a nonempty subset $\Delta_M \subset M$ and its image $\Delta_N \colon = H(\Delta_M)$ under the diffeomorphism H. Let $h := H|_{\Delta_M}$ and let ϱ_h be the equivalence relation on the disjoint sum $(M \sqcup N, C \sqcup D)$ identifying $p \in \Delta_M$ with $h(p) \in \Delta_N$, with $C \sqcup D := \{\alpha\colon M \sqcup N \to \mathbb{R};\ \alpha|_M \in C, \alpha|_N \in D\}$. For $f \in C$ and $g \in D$ we define $f \sqcup g\colon M \sqcup N \to \mathbb{R}$ by $(f \sqcup g)|_M = f$ and $(f \sqcup g)|_N = g$. (See [32] for more details.)

For our purposes, it suffices to mention that there are at least two known methods of gluing differential spaces. One is the "global" technique, which works solely with generators [6]. Another is "local", i.e., we first introduce a diffeomorphism between some subspaces, then we identify points with their images, and, finally, we make the glued space a Hausdorff one with help of the equivalence relation [31], [30]. Here, we will exploit the latter, but with the extra assumption that the gluing is of an arbitrary order.

Definition 3.1. The quotient space $((M \sqcup N)/\varrho_h, (C \sqcup D)/\varrho_h)$, with $(C \sqcup D)/\varrho_h) := \{\alpha \in C \sqcup D;\ \alpha|_{\Delta(M)} = \alpha \circ h\}$, is called the *gluing of the differential spaces* (M, C) *and* (N, D). It is denoted by $(M \cup_h N, C \cup_h D)$.

In other words, for arbitrary $f \in C$ and $g \in D$ such that $f|_{\Delta_M} = g \circ h$, the function from $C \cup_h D$ corresponding to the function $f \sqcup g \in C \sqcup D$ is denoted by $f \cup_h g$. Then $C \cup_h D := \{f \cup_h g;\ f \in C, g \in D, f|_{\Delta_M} = g \circ h\}$.

Definition 3.2. Let $k \in \mathbb{N}$. Then $C \cup_h^k D := \{f \cup_h g \in C \cup_h D;\ d_p^k f = d_p^k (g \circ H), p \in \Delta_M\}$.

It is clear that $C \cup_h^k D$ is just one of the various possible differential structures on $M \cup_h N$ (in the sense of Definition 2.2). The reason why we consider this particular structure is that it is the most workable one for our purposes.

Of course,

$$C \cup_h D \supset C \cup_h^1 D \supset C \cup_h^2 D \supset \ldots \supset C \cup_h^k D \supset \ldots$$

Definition 3.3. The differential space $(M \cup_h N, C \cup_h^k D)$ is called the k^{th} *order gluing of the differential spaces* (M, C) *and* (N, D). Shortly, we will denote it by $M \cup_h^k N$.

Let $\pi_{\varrho_h}\colon M \sqcup N \to M \cup_h N$ be the natural projection and define $\hat{\imath}_M := \pi_{\varrho_h}|_M$, $\hat{\imath}_N := \pi_{\varrho_h}|_N$, $\widehat{M} := \hat{\imath}_M(M)$, $\widehat{N} := \hat{\imath}_N(N)$, $\widehat{C}^k := (C \cup_h^k D)_{\widehat{M}}$, $\widehat{D}^k := (C \cup_h^k D)_{\widehat{N}}$ and $\Delta := \pi_{\varrho_h} \Delta_M$.

Now, we would like to check the relation between the topologies induced by the differential structures from Definition 3.1 and Definition 3.2. Also, we want to study the relation between the differential structures before and after the gluing. It is also interesting to check the behaviour of the tangent vectors of higher-order.

Theorem 3.1. *Let* (M, C) *and* (N, D) *be two differential spaces,* $H : (M, C) \to (N, D)$ *a diffeomorphism, and let* $\Delta_M \subset M$, $\Delta_N := H(\Delta_M)$ *be closed. Then, for any* $k \in \mathbb{N}$, *the following conditions hold:*

(1) $\tau_{C \cup_h^k D} = \tau_{C \cup_h D}$;
(2) $\hat{\imath}_M\colon (M, C) \to (\widehat{M}, \widehat{C}^k)$ *and* $\hat{\imath}_N\colon (N, D) \to (\widehat{N}, \widehat{D}^k)$ *are diffeomorphisms;*
(3) $(\hat{\imath}_M)_{*p} T_p^i M = (\hat{\imath}_N)_{*q} T_q^i N$ *for* $i = 1, \ldots, k$, $p \in \Delta_M$ *and* $q = h(p)$;
(4) $(\hat{\imath}_M)_{*p} T_p^{k+1} M \cap (\hat{\imath}_N)_{*q} T_q^{k+1} N = (\hat{\imath}_M|_{\Delta_M})_{*p} T_p^{k+1} \Delta_M = (\hat{\imath}_N|_{\Delta_N})_{*q} T_q^{k+1} \Delta_N$.

P r o o f. (1): Of course, $\tau_{C \cup_h^k D} \subset \tau_{C \cup_h D}$. So it is enough to prove that $\tau_{C \cup_h^k D} \supset \tau_{C \cup_h D}$. For any open set $W \in \tau_{C \cup_h D}$ there exists $V = \pi_{\varrho_h}(\pi_{\varrho_h}^{-1}(U)) \subset W$, where $U \in \tau_C$. Let $f \in C$ be such that $\operatorname{supp} f \subset U$. It is clear that $f \cup_h (f \circ H^{-1}) \in C \cup_h^k D$ and $\operatorname{supp} f \cup_h (f \circ H^{-1}) \subset V \subset W$. So W is also open in the topology of $C \cup_h^k D$.

(2): It is clear that $\pi_{\varrho_h}\colon (M \sqcup N, C \sqcup D) \to (M \cup_h N, C \cup_h^k D)$ is smooth for any $k \in \mathbb{N}$. As a result, $\hat{\imath}_M$ and $\hat{\imath}_N$ are smooth as restrictions of π_{ϱ_h}. It is also true that

$f \circ \hat{\imath}_M^{-1} = f \cup_h (f \circ H^{-1})|_{\widehat{M}}$ and $g \circ \hat{\imath}_N^{-1} = (g \circ H) \cup_h (g|_{\widehat{N}})$ for any $f \in C$ and $g \in D$. Consequently, $\hat{\imath}_M^{-1}$ and $\hat{\imath}_N^{-1}$ are smooth.

(3): Let $w \in (\hat{\imath}_N)_{*q} T_q^i N$ for some $i = 1, \ldots, k$. Then there exists $v \in T_q^i N$ such that $w = (\hat{\imath}_N)_{*q} v$. Since H is a diffeomorphism, there exists $u \in T_p^i M$ such that $H_{*p} u = v$. It is easy to notice that $(\hat{\imath}_M)_{*p} u = (\hat{\imath}_N)_{*q} v$.

(4): It is enough to prove that

$$(\hat{\imath}_M)_{*p} T_p^{k+1} M \cap (\hat{\imath}_N)_{*q} T_q^{k+1} N \subset (\hat{\imath}_M|_{\Delta_M})_{*p} T_p^{k+1} \Delta_M.$$

Let $v \in (\hat{\imath}_M)_{*p} T_p^{k+1} M \cap (\hat{\imath}_N)_{*q} T_q^{k+1} N$. Then there exists a unique $(u_M, u_N) \in T_p^{k+1} M \oplus T_q^{k+1} N$ such that $v = (\hat{\imath}_M)_{*p} u_M = (\hat{\imath}_N)_{*q} u_N$. Moreover, the following condition is satisfied: $f \cup_h g \in C \cup_h^k D \Rightarrow u_M(f) = u_N(g)$. Let $w\colon C_{\Delta_M} \to \mathbb{R}$ be defined by $w(g) := u_M(\widetilde{g}|_M)$ for $g \in C_{\Delta_M}$, where $\widetilde{g} \in C$ is such that $\widetilde{g}|_{\Delta_M \cap U} = g|_{\Delta_M \cap U}$ for some open neighbourhood $U \in \tau_C$ of p. Since $(\hat{\imath}_M|_{\Delta_M})_{*p} w = (\hat{\imath}_M)_{*p} u_M$, it turns out that $v = (\hat{\imath}_M)_{*p} u_M = (\hat{\imath}_M|_{\Delta_M})_{*p} w \in (\hat{\imath}_M|_{\Delta_M})_{*p} T_p^{k+1} \Delta_M$. □

From condition (2) of Theorem 3.1 above it follows that $\widehat{C}^k = \widehat{C}$ and $\widehat{D}^k = \widehat{D}$ for any $k \in \mathbb{N}$.

Theorem 3.2. *If Δ_M is a closed boundary set (see, e.g., [17]) in M, then $\mathfrak{X}^{k+1}(M \cup_h^k N) = \mathfrak{X}_\Delta^{k+1}(M \cup_h^k N)$.*

Proof. Assume that Δ_M is a closed boundary set in M. Then $\widehat{M} \setminus \Delta$ and $\widehat{N} \setminus \Delta$ are open in $M \cup_h^k N$ and dense in $\widehat{M}$ and $\widehat{N}$, respectively. Let $X \in \mathfrak{X}^{k+1}(M \cup_h^k N)$. Since $\widehat{M} \setminus \Delta$ is open and $\widehat{N} \setminus \Delta$ is open, then $X \in \mathfrak{X}_{\widehat{M}}^{k+1}(M \cup_h^k N)$ and $X \in \mathfrak{X}_{\widehat{N}}^{k+1}(M \cup_h^k N)$ and, by Lemma 2.5, $X \in \mathfrak{X}_{\widehat{M} \setminus \Delta}^{k+1}(M \cup_h^k N)$ and $X \in \mathfrak{X}_{\widehat{N} \setminus \Delta}^{k+1}(M \cup_h^k N)$. Theorem 3.1 and Theorem 2.3 imply $X \in \mathfrak{X}_{\widehat{M} \cap \widehat{N}}^{k+1}(M \cup_h^k N) = \mathfrak{X}_\Delta^{k+1}(M \cup_h^k N)$. □

Corollary 3.1. *If Δ_M is a closed boundary set in M, then we have $\mathfrak{X}(M \cup_h^k N) = \mathfrak{X}_\Delta(M \cup_h^k N)$ and $\mathfrak{X}(M \cup_h^k N)$ is contained both in $\mathfrak{X}_{\widehat{M}}(M \cup_h^k N)$ and $\mathfrak{X}_{\widehat{N}}(M \cup_h^k N)$.*

Proof. It is enough to prove that $\mathfrak{X}(M \cup_h^k N) \subset \mathfrak{X}_\Delta(M \cup_h^k N)$. To this end, let $X \in \mathfrak{X}(M \cup_h^k N)$ and consider the k^{th} order vector field given by $Y := X^{k+1} = X \circ \ldots \circ X$ (see Theorem 2.3). Theorem 3.2 guarantees that $Y \in \mathfrak{X}_\Delta^{k+1}(M \cup_h^k N)$, where $X \in \mathfrak{X}_\Delta^{k+1}(M \cup_h^k N)$ in view of Theorem 2.3.

Since Y is tangent to M (or N), hence X is tangent to M (or N, respectively). □

Dual to the sequence of inclusions

$$C \cup_h D \supset C \cup_h^1 D \supset C \cup_h^2 D \supset \ldots \supset C \cup_h^k D \supset \ldots$$

is the sequence of smooth mappings

$$(M \cup_h N, C \cup_h D) \xrightarrow{\mathrm{id}} (M \cup_h^1 N, C \cup_h^1 D) \xrightarrow{\mathrm{id}} (M \cup_h^2 N, C \cup_h^2 D) \xrightarrow{\mathrm{id}} \dots$$
$$\xrightarrow{\mathrm{id}} (M \cup_h^k N, C \cup_h^k D) \xrightarrow{\mathrm{id}} \dots$$

Lemma 3.1. *If Δ_M is a closed boundary set in M and $X \in \mathfrak{X}(M \cup_h^k N)$, then there exists a unique $\widetilde{X} \in \mathfrak{X}(M \cup_h N)$ such that $\widetilde{X}|_{C \cup_h^k D} = X$.*

Proof. Observe that $X \in \mathfrak{X}_{\widehat{M}}(M \cup_h^k N)$ and $X \in \mathfrak{X}_{\widehat{N}}(M \cup_h^k N)$, and $X \in \mathfrak{X}_\Delta(M \cup_h^k N)$ as well. Put $\widetilde{X}_M := X|_M$ and $\widetilde{X}_N := X|_N$ and notice that $\widetilde{X}_M|_\Delta = \widetilde{X}_N|_\Delta$. Choose $X_M \in \mathfrak{X}(M)$ such that $(\hat{\imath}_M)_* X_M = \widetilde{X}_M$ and $X_N \in \mathfrak{X}(N)$ such that $(\hat{\imath}_N)_* X_N = \widetilde{X}_N$, and set $\widetilde{X} := X_M \cup_h X_N$. It can be easily proved that $X_M(f) = X(f \cup_h f \circ H^{-1})$ for any $f \in C$ and $X_N(g) = X(g \circ H \cup_h g)$ for any $g \in D$. It follows that $\widetilde{X}|_{C \cup_h^k D} = X$. □

Let $\mathfrak{X}_h^k(M,N) := \{(X_M, X_N) \in \mathfrak{X}_{\Delta_M}(M) \times \mathfrak{X}_{\Delta_N}(N);\ h_*(X_M|_{\Delta_M}) = X_N|_{\Delta_N}, \partial^k X_M|_{\Delta_M^k} = \partial^k X_N \circ h_*|_{\Delta_M^k}\}$, where $\Delta_M^k := \bigsqcup_{p \in \Delta_M} T_p^k M$.

Theorem 3.3. *If Δ_M is a closed boundary set in M, then the following conditions are satisfied:*

(1) $\tau_{C \cup_h^k D} = \tau_{C \cup_h D} = \tau_{C \sqcup D / \varrho_h}$;
(2) *$\mathfrak{X}(M \cup_h^k N)$ is contained both in $\mathfrak{X}_{\widehat{M}}(M \cup_h^k N)$ and $\mathfrak{X}_{\widehat{N}}(M \cup_h^k N)$;*
(3) $\mathfrak{X}(M \cup_h^k N) = \mathfrak{X}_\Delta(M \cup_h^k N)$;
(4) *the $C \cup_h^k D$-module $\mathfrak{X}_\Delta(M \cup_h^k N)$ is isomorphic to the $C \cup_h D$-module $\mathfrak{X}_h^k(M,N)$.*

Proof. In view of the previously stated results (Theorem 3.1 and Corollary 3.1), it is enough to prove (4). Let $X \in \mathfrak{X}(M \cup_h^k N)$ and set $X_M := (\hat{\imath}_M^{-1})_*(X|_{\widehat{M}})$, $X_N := (\hat{\imath}_N^{-1})_*(X|_{\widehat{N}})$. It is easy to notice that $(X_M, X_N) \in \mathfrak{X}_h^k(M,N)$. Now, for any $(Y_M, Y_N) \in \mathfrak{X}_h^k(M,N)$, define $Y_M \cup_h^k Y_N := Y_M \cup_h Y_N|_{C \cup_h^k D}$. Since it is clear that $Y_M \cup_h^k Y_N \in \mathfrak{X}(M \cup_h^k N)$, the sought-for $I\colon \mathfrak{X}_h^k(M,N) \to \mathfrak{X}(M \cup_h^k N)$ can be defined as $I(Y_M, Y_N) := Y_M \cup_h^k Y_N$. □

4. Application and perspectives

In this section we specialize the techniques described earlier to the case when $\Delta_M := \{p\}$ with $p \in M$, $\Delta_N := \{q\}$, where $q := H(p)$. In other words, we study the particular case of the gluing at a point. As before, $h := H|_{\Delta_M}$ and $(M \cup_h N, C \cup_h D)$ is the glued differential space. For any $f \in C$ and $g \in D$, set $\widetilde{f} := f \cup_h f \circ H^{-1}$ and $\widetilde{g} := g \circ H \cup_h g$. Of course, $\widetilde{f}, \widetilde{g} \in C \cup_h^k D$ for an arbitrary $k \in \mathbb{N}$. Consider $\langle f_1, \ldots, f_{k+1} \rangle := (f_1 - f_1(p)) \ldots (f_{k+1} - f_{k+1}(p)) \cup_h 0$ with $f_1, \ldots, f_{k+1} \in C$. It can be easily noticed that $\langle f_1, \ldots, f_{k+1} \rangle \in C \cup_h^k D$. These functions play an important role in generating the differential structure on the glued space, as Theorem 4.1 below shows.

Theorem 4.1. *If C is generated by C_0, then $C \cup_h^k D$ is generated by $\{\widetilde{f};\ f \in C\} \cup \{\langle f_{i_1}, \ldots, f_{i_{k+1}} \rangle;\ f_{i_1}, \ldots, f_{i_{k+1}} \in C_0\}$.*

Proof. If $f \cup_h g \in C \cup_h^k D$, then by Definition 3.2, $d_p^k f = d_p^k (g \circ H)$. As a consequence of Lemma 2.2 there exists an open neighbourhood $U \in \tau_C$ of p and functions $f_1, \ldots, f_n \in C_0$, with $n \in \mathbb{N}$, and $\omega_{i_1 \ldots i_{k+1}} \in C^\infty(\mathbb{R}^n)$, with $i_1, \ldots, i_{k+1} = 1, \ldots, n$, such that $f|_U = \theta \circ (f_1, \ldots, f_n)|_U$, where $\theta \in C^\infty(\mathbb{R}^n)$, and

$$(g \circ H - f)|_U = \left(\sum_{i_1, \ldots, i_{k+1} = 1}^{n} (f_{i_1} - f_{i_1}(p)) \ldots (f_{i_{k+1}} - f_{i_{k+1}}(p)) \omega_{i_1 \ldots i_{k+1}}(f_1, \ldots, f_n) \right) \Bigg|_U .$$

Now, if $V := \pi_{\varrho_h}^{-1}(\pi_{\varrho_h}(U))$, then

$$f \cup_h g|_V = \left(\theta \circ (\widetilde{f}_1, \ldots, \widetilde{f}_n) + \sum_{i_1, \ldots, i_{k+1} = 1}^{n} \langle f_{i_1}, \ldots, f_{i_{k+1}} \rangle \omega_{i_1 \ldots i_{k+1}}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \right) \Bigg|_V .$$

□

Now we introduce the notion of the differential basis (see [12]). A function $f \in C$ is called *differentiably dependent on $g_1, \ldots, g_n \in C$ at $p \in M$*, if there exists an open neighbourhood $U \in \tau_C$ of p and a function $\omega \in C^\infty(\mathbb{R}^n)$ such that $f|_U = \omega \circ (g_1, \ldots, g_n)|_U$. A collection of functions $\{f_1, \ldots, f_n\} \subset C$ is called *differentiably independent at p*, if there is no f_i, $i = 1, \ldots, n$, which is differentiably dependent on the remaining functions at p. A subset $B \subset C$ is said to *reproduce C at p*, if for any $f \in C$ there exist an open neighbourhood $U \in \tau_C$ of p, functions $g_1, \ldots, g_n \in B$ and a function $\omega \in C^\infty(\mathbb{R}^n)$ such that $f|_U = \omega \circ (g_1, \ldots, g_n)|_U$. So, we say that a subset $B \subset C$ is a *differential basis of C at p*, if B is differentiably independent at p and B reproduces C at p.

Lemma 4.1 ([12]). *The collection $\{f_1,\dots,f_n\}\subset C$ is differentiably independent at p if and only if for any $\omega\in C^\infty(\mathbb{R}^n)$ and for any open neighbourhood $U\in\tau_C$ of p the following condition is satisfied:*

$$\omega\circ(f_1,\dots,f_n)|_U=0\Rightarrow\frac{\partial\omega}{\partial x_i}(f_1(p),\dots,f_n(p))=0,\quad 1\leqslant i\leqslant n.$$

Theorem 4.2 shows how to construct differential bases for glued differential spaces.

Theorem 4.2. *If $\{f_1,\dots,f_n\}$ is a differential basis of C at p, then $\{\widetilde{f}_1,\dots,\widetilde{f}_n\}\cup\{\langle f_{i_1},\dots,f_{i_{k+1}}\rangle;\ 1\leqslant i_1\leqslant\dots\leqslant i_{k+1}\leqslant n\}$ is a differential basis of $C\cup_h^k D$.*

Proof. Let $v_1,\dots,v_n\in T_pM$ be linearly independent and such that $v_i(f_j)=\delta_{ij}$ for $i,j=1,\dots,n$, and set $w_i:=(\hat{\imath}_M)_{*p}v_i$ for $i=1,\dots,n$. Let $\omega\in C^\infty(\mathbb{R}^{n+m})$ and suppose that

$$\omega\circ(\widetilde{f}_1,\dots,\widetilde{f}_n,g_1,\dots,g_m)=0 \tag{4.1}$$

for an open neighbourhood $V\in\tau_{C\cup_h^k D}$ of $p_h:=\pi_{\varrho_h}(p)$, where $g_1,\dots,g_m\in\{f_{i_1},\dots,f_{i_{k+1}};\ 1\leqslant i_1\leqslant\dots\leqslant i_{k+1}\leqslant n\}$. Applying w_i to equation (4.1), we obtain that $(\partial\omega/\partial x_i)(\widetilde{f}_1(p_h),\dots,\widetilde{f}_n(p_h),g_1(p_h),\dots,g_m(p_h))=0$ for $i=1,\dots,n$. Now, define the map $\omega_{i_1\dots i_{k+1}}\colon C\cup_h^k D\to\mathbb{R}$ by $\omega_{i_1\dots i_{k+1}}(\alpha\cup_h\beta):=\theta\circ(f_1,\dots,f_n)(p)$, where $1\leqslant i_1\leqslant\dots\leqslant i_{k+1}\leqslant n$, and $\theta\in C^\infty(\mathbb{R}^n)$ is such that $(\alpha-\beta\circ H)|_U=\theta\circ(f_1,\dots,f_n)|_U$ and $d_p^k(\theta\circ(f_1,\dots,f_n))=0$. It can be easily noticed that $\omega_{i_1\dots i_{k+1}}\in T_{p_h}(M\cup_h^k N)$. Moreover, $\omega_{i_1\dots i_{k+1}}(\langle f_{j_1}-f_{j_1}(p),\dots,f_{j_{k+1}}-f_{j_{k+1}}(p)\rangle)=0$, if $\{i_1,\dots,i_{k+1}\}\neq\{j_1,\dots,j_{k+1}\}$, and $\omega_{i_1\dots i_{k+1}}(\langle f_{j_1}-f_{j_1}(p),\dots,f_{j_{k+1}}-f_{j_{k+1}}(p)\rangle)=1$, if $\{i_1,\dots,i_{k+1}\}=\{j_1,\dots,j_{k+1}\}$. Now, assume that $g_j=\langle f_{j_1}-f_{j_1}(p),\dots,f_{j_{k+1}}-f_{j_{k+1}}(p)\rangle$. Applying $\omega_{i_1\dots i_{k+1}}$ to equation (4.1), we obtain

$$\frac{\partial\omega}{\partial x_{n+j}}(\widetilde{f}_1(p_h),\dots,\widetilde{f}_n(p_h),g_1(p_h),\dots,g_m(p_h))=0$$

and the result follows from Lemma 4.1. □

We conclude this paper with some considerations about the "distributivity" of the Cartesian product with respect to the gluing in the category of differential spaces, collected in Theorem 4.3 below.

Theorem 4.3. *Let (M,C), (N,D) and (L,B) be differential spaces, $p\in M$, and let $H\colon(M,C)\to(N,D)$ be a diffeomorphism. Let $\mathrm{pr}_L\colon M\times L\to L$, $\mathrm{pr}_M\colon M\times L\to M$ and $\mathrm{pr}_N\colon N\times L\to N$ be the canonical projections and use the notation*

$(p_M, l) \in M \times L$ *and* $(p_N, l) \in N \times L$ *with* $H(p_M) = p_N$ *to indicate that* $p_M \in M$ *and* $p_N \in N$. *If* $H_L := H \times \mathrm{id}_L$ *and* $\Delta_M := \{p\} \times L$, *then:*

(1) *if* C *is generated by* C_0, *then* $(C \times B) \cup_h^k (D \times B)$ *is generated by* $\{\widehat{f};\ f \in C_0\} \cup \{\widehat{g};\ g \in B\} \cup \{\langle \widehat{f}_1, \dots, \widehat{f}_{k+1} \rangle;\ f_1, \dots, f_{k+1} \in C_0\}$, *where* $\widehat{f} := (f \circ \mathrm{pr}_M) \cup_h (f \circ H^{-1} \circ \mathrm{pr}_N)$ *and* $\widehat{g} := (g \circ \mathrm{pr}_L) \cup_h (g \circ \mathrm{pr}_L)$;

(2) $\Phi^*((C \cup_h^k D) \times B) = (C \times B) \cup_h^k (D \times B)$, *where* $\Phi\colon (M \times L) \cup_h (N \times L) \to (M \cup_h N) \times L$ *is the diffeomorphism defined by* $\Phi([(p, l)]) := ([p], l)$, *where* $[(p, l)] \in (M \times L) \cup_h (N \times L)$.

Proof. (1): Let $f \cup_h g \in (C \times B) \cup_h (D \times B)$. It is obvious that $f(p_M, l) = g(p_N, l)$ and $d^k_{(p_M,l)} f = d^k_{(p_M,l)}(g \circ H)$ for any $l \in L$. Moreover, it is true that $g \circ H - f \in \mathfrak{m}_{(p_M,l)}$ for an arbitrary $l \in L$ and $d^k_{(p_M,l)}(g \circ H - f) = 0$. Because of Lemma 2.3, $g \circ H - f$ can be written locally as

$$(g \circ H - f)(p, l) = \sum_{i_1, \dots, i_{k+1}=1}^{n} (f_{i_1}(p) - f_{i_1}(p_M)) \dots (f_{i_{k+1}}(p) - f_{i_{k+1}}(p_M)) \times \omega_{i_1 \dots i_{k+1}}(f_1(p), \dots, f_n(p), g_1(l), \dots, g_m(l)),$$

where $(p, l) \in W$ and $W \in \tau_{C \times B}$. If $U := \pi_{\varrho_h}^{-1}(\pi_{\varrho_h}(W))$, then

$$(f \cup_h g)|_U = \biggl(\omega \circ (\widehat{f}_1, \dots, \widehat{f}_n, \widehat{g}_1, \dots, \widehat{g}_n) + \sum_{i_1, \dots, i_{k+1}=1}^{n} \langle \widehat{f}_{i_1}, \dots, \widehat{f}_{i_{k+1}} \rangle \omega_{i_1 \dots i_{k+1}}(\widehat{f}_1, \dots, \widehat{f}_n, \widehat{g}_1, \dots, \widehat{g}_n)\biggr)\bigg|_W.$$

(2): It can be easily noticed that $\widetilde{f} \circ \mathrm{pr}_{M \cup_h N} \circ \Phi = \widehat{f}$ for any $f \in C$. Moreover, $\langle \widetilde{f}_{i_1}, \dots, \widetilde{f}_{i_{k+1}} \rangle \circ \mathrm{pr}_{M \cup_h N} \circ \Phi = \langle \widehat{f}_{i_1}, \dots, \widehat{f}_{i_{k+1}} \rangle$ for arbitrary $f_{i_1}, \dots, f_{i_{k+1}} \in C$ and $g \circ \mathrm{pr}_L \circ \Phi = \widehat{g}$ for any $g \in B$. Therefore $\Phi^*((C \cup_h^k D) \times B) = (C \times B) \cup_h^k (D \times B)$. □

5. Final remarks

The proposed methods can be useful in the context of jets. For example, consider the differential space $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$, which is simultaneously a smooth manifold. Let $\partial^k_{i_1 \dots i_m}|_p$ be the vectors representing the partial derivatives of k^{th} order, with respect to the variables $x_{i_1}, \dots, x_{i_m}$. It is well known that these vectors are the basis of $T^k_p \mathbb{R}^n$, which (in view of the just stated remark) can be naturally identified with the k^{th} order jet space at p [7], [14], [16]. One would surely like to apply this method to differential equations, as it is well known that jet spaces form a suitable background to study differential equations. Indeed, a "generalized" partial differential equation is a submanifold in the jet space. Therefore, it could be interesting to prolong the theory known for manifolds to differential spaces. Unfortunately, differential equations on differential spaces are not an easy topic and there are serious obstacles against developing a well-working theory. However, further research will be made elsewhere to clarify this situation.

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