

A NOTE ON INFINITE aS -GROUPS

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Abstract. Let G be a group. If every nontrivial subgroup of G has a proper supplement, then G is called an aS -group. We study some properties of aS -groups. For instance, it is shown that a nilpotent group G is an aS -group if and only if G is a subdirect product of cyclic groups of prime orders. We prove that if G is an aS -group which satisfies the descending chain condition on subgroups, then G is finite. Among other results, we characterize all abelian groups for which every nontrivial quotient group is an aS -group. Finally, it is shown that if G is an aS -group and $|G| \neq pq, p$, where p and q are primes, then G has a triple factorization.

Keywords: infinite aS -group; supplemented subgroup; nilpotent group

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1. INTRODUCTION

A subgroup K of a group G is said to be *supplemented* in G if there exists a subgroup L of G such that $G = KL$. In this case, L is called a *supplement* of K in G . If $K \cap L = \{1\}$, then K is *complemented* in G and L is called a *complement* of K in G . If every nontrivial subgroup of G has a proper supplement (complement), then G is called an aS -group (aC -group). Both aS - and aC -groups have been investigated by Kappe and Kirtland in [8]. Since the study of supplementation (complementation) in a group can reveal important properties about the structure of the group, this field has received a good deal of attention from several authors (for example see [3], [6] and [8]). In this paper, we investigate infinite aS -groups. For instance, we investigate nilpotent aS -groups. We also characterize abelian groups such that every nontrivial quotient group is an aS -group.

In this paper, the set of all maximal subgroups of a group G is denoted by $\text{Max}(G)$. The intersection of all maximal subgroups of a group G , called the *Frattini subgroup* of G , is denoted by $\Phi(G)$. If G has no maximal subgroups, then $\Phi(G) = G$. If K

is a subgroup of G , then we write $K \leq G$. By G' we mean the derived subgroup of G . A *subdirect product* is a subgroup G of a direct product $\prod_{i=1}^n G_i$ such that every induced projection is surjective. We say G has a *triple factorization* if there exist three proper and nontrivial subgroups H, K and L such that $HK = HL = KL = G$. For more information on triple factorizations see [1] and [9].

2. NILPOTENT aS -GROUPS

Kappe and Kirtland in [8] showed that the class of finite aS -groups coincides with the class of finite aC -groups, which were characterized by Hall [6]. We start with the following result. It should be noted that it was shown in [8] that every subgroup and homomorphic image of an aS -group is also an aS -group.

Theorem 1. *Let G be a finite group. Then the following statements are equivalent:*

- (i) G is an aS -group.
- (ii) G is an aC -group.
- (iii) Every cyclic subgroup of prime order of G is complemented in G .
- (iv) G is supersolvable and its Sylow subgroups are all elementary abelian.
- (v) G is isomorphic with a subgroup of a direct product of groups of squarefree order.

Proof. The result follows from Corollary 3.7 in [8], Corollary 2 in [3], and Theorems 1 and 2 in [6]. □

Kappe and Kirtland in [8] have shown that there exists a group such that $\Phi(G) = 1$ but G is not an aS -group. In the next theorem, we show that they are equivalent for a nilpotent group.

Theorem 2. *Let G be a nilpotent group. Then the following statements are equivalent:*

- (i) G is an aS -group.
- (ii) $\Phi(G) = 1$.
- (iii) G is the subdirect product of a family of cyclic groups of prime orders.

Proof.

(i) \Rightarrow (ii) It follows from Proposition 3.4 in [8].

(ii) \Rightarrow (iii) First, we claim that G is an abelian group. Since G is nilpotent, we have $G' \subseteq \Phi(G) = \{1\}$ and so G is abelian. Consider the homeomorphism $\psi: G \rightarrow \prod G/\mathfrak{m}_i$, where $\mathfrak{m}_i \in \text{Max}(G)$. For every $\mathfrak{m}_i \in \text{Max}(G)$ the quotient group

G/m_i is cyclic of prime order and $\ker(\psi) = \Phi(G) = \{1\}$. Thus one can see that G is isomorphic to a subgroup of a direct product of cyclic groups of prime orders. The direct product of cyclic groups of prime orders as a \mathbb{Z} -module is semisimple. Therefore, G is isomorphic to a subgroup of direct product of cyclic groups, as every submodule of semisimple module is semisimple. Thus G is a subgroup of the direct product of cyclic groups of prime orders. It is not hard to see that G is a subdirect product of cyclic groups of prime orders.

(iii) \Rightarrow (i) Suppose that G is a subdirect product of cyclic groups of prime orders. Thus G is a subgroup of the direct product of cyclic groups of prime orders and G is semisimple as a \mathbb{Z} -module. Hence G is an aS -group. \square

The classification of $aPNS$ -groups (groups in which every nontrivial subgroup has a proper normal supplement) was done by Kappe and Kirtland in [8]. Now, by Theorem 2, we have the following theorem.

Theorem 3. *Let G be a nilpotent group. Then the following statements are equivalent:*

- (i) G is an aS -group.
- (ii) G is an $aPNS$ -group.
- (iii) G is the subdirect product of a family of cyclic groups of prime orders.
- (iv) G is abelian with $\bigcap_{p \in \pi} G^p = \{1\}$, where π is the set of all primes.
- (iv) $\Phi(G) = 1$.

3. CHARACTERIZATION OF aS -GROUPS

First we show that every aS -group has a maximal subgroup of finite index.

Theorem 4. *Let G be an aS -group. Then every subgroup has a supplement which is a maximal subgroup of finite index.*

Proof. Let $g \in G$. It is enough to show that the subgroup $\langle g \rangle$ has a supplement which is a maximal subgroup of finite index. Since G is an aS -group, there exists a nontrivial subgroup H such that $\langle g \rangle H = G$. Set $\Sigma = \{K < G: H \leq K, g \notin K\}$. By Zorn's Lemma, Σ has a maximal element, say \mathfrak{m} . First, we prove that \mathfrak{m} is a maximal subgroup of G . Assume that there exists a subgroup \mathfrak{n} such that $\mathfrak{m} \subsetneq \mathfrak{n}$. The equality $\langle g \rangle H = G$ implies that $\mathfrak{n} = G$ and hence \mathfrak{m} is a maximal subgroup.

We now show that \mathfrak{m} is of finite index. With no loss of generality, we can assume that $\langle g \rangle$ is a torsion-free group. Now, consider $\langle g^2 \rangle$. Thus, there is a maximal subgroup \mathfrak{m} such that $\langle g^2 \rangle \mathfrak{m} = G$. Since $\langle g^2 \rangle \subseteq \langle g \rangle$, we have $\langle g \rangle \mathfrak{m} = \langle g^2 \rangle \mathfrak{m} = G$.

Hence, there are $n \in \mathbb{Z}$ and $m \in \mathfrak{m}$ such that $g = g^{2n}m$ and so $g^{1-2n} \in \mathfrak{m}$. Since $g^{1-2n} \in \mathfrak{m}$, we deduce that \mathfrak{m} is a maximal subgroup of finite index. \square

The following result shows that no aS -group which is not \mathbb{Z}_p , where p is prime, is simple.

Corollary 5. *Let G be an aS -group and $G \not\cong \mathbb{Z}_p$, where p is prime. Then G is not a simple group.*

Proof. If G is finite, then by Theorem 1, G is a supersolvable group. So G is not simple. Now, let G be infinite. It follows from Theorem 4 that G has a maximal subgroup of finite index. So G has a normal subgroup and G is not simple, as desired. \square

It was shown in [8] that if G is an aS -group which satisfies the descending chain condition on subgroups, then G is an aC -group. Here, we show that G is indeed a finite group.

Theorem 6. *If G is an aS -group which satisfies the descending chain condition on subgroups, then G is finite.*

Proof. It follows from Theorem 4 that G has a (maximal) subgroup H of finite index whose supplement is a maximal subgroup, say \mathfrak{m}_1 , such that $[G : \mathfrak{m}_1] < \infty$. If H is finite, then there is nothing to prove. Hence we may assume that H is infinite. Since $H\mathfrak{m}_1 = G$ and $[G : \mathfrak{m}_1] < \infty$, we conclude that $H \cap \mathfrak{m}_1$ is not trivial. If $H \cap \mathfrak{m}_1$ is a finite subgroup, then the Poincaré inequality (see [10], 1.7.10) implies that $[G : H \cap \mathfrak{m}_1] < \infty$ and so G is finite. Thus, we can assume that $H \cap \mathfrak{m}_1$ is infinite. By Theorem 4, $H \cap \mathfrak{m}_1$ has a supplement \mathfrak{m}_2 of finite index. Again, we should have $|H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2| > 1$. Now, if $H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2$ is finite, then the Poincaré inequality implies that $[G : H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2] < \infty$ and so G is finite. By repeating this procedure, if $H \cap \bigcap_{i=1}^{i=k} \mathfrak{m}_i$ is not a finite subgroup, for every $k \in \mathbb{N}$, we have a non-stopping descending chain of subgroups $\left\{ H \cap \bigcap_{i \in \mathbb{N}} \mathfrak{m}_i \right\}$. Since G satisfies the descending chain condition on subgroups, we get a contradiction. Hence, there is $k \in \mathbb{N}$ such that $H \cap \bigcap_{i=1}^{i=k} \mathfrak{m}_i$ is finite. By the Poincaré inequality, one can see that $\left[G : H \cap \bigcap_{i=1}^{i=k} \mathfrak{m}_i \right] < \infty$ and so G is finite, as desired. \square

Let G be an aS -group which satisfies the descending chain condition on subgroups. Then by Theorem 6, G is finite. Now, it follows from Theorem 1 that G is a supersolvable group. It would be interesting if one could prove the converse of this result.

Theorem 7. *If G is an aS -group and G' satisfies the descending chain condition on its subgroups, then G is solvable.*

Proof. If G' is a finite subgroup, then by Proposition 3.5 of [8] and Theorem 1, G' is supersolvable. Since every finite supersolvable group is solvable, G' is solvable and so G is solvable. Thus, we can suppose that G' is an infinite subgroup. By Theorem 4, G' has a supplement of finite index, say \mathfrak{m}_1 . Since $[G : \mathfrak{m}_1] < \infty$ and G' is infinite, the subgroup $G' \cap \mathfrak{m}_1$ is nontrivial. If $|G' \cap \mathfrak{m}_1| < \infty$, then $[G' : G' \cap \mathfrak{m}_1] = [G : \mathfrak{m}_1] < \infty$. This implies that G' is finite. Hence, we can assume that $G' \cap \mathfrak{m}_1$ is an infinite subgroup. There is a supplement of finite index for $G' \cap \mathfrak{m}_1$. By a similar argument to that of Theorem 6, one concludes that G is solvable. \square

Next, we find a class of aS -groups such that every element of this class contains a normal maximal subgroup.

Theorem 8. *Let G be a torsion aS -group. Then G contains a normal maximal subgroup.*

Proof. Let p be the smallest prime integer such that $a^p = 1$, where a runs over all elements of G . Fix the element a and consider the subgroup $\langle a \rangle$. There exists a subgroup \mathfrak{m} such that $\langle a \rangle \mathfrak{m} = G$. We will show that \mathfrak{m} is a normal maximal subgroup of G . First we prove that $[G : \mathfrak{m}] = p$. Clearly, $[G : \mathfrak{m}] \leq p$. If $a^i \mathfrak{m} = a^j \mathfrak{m}$, then $a^{i-j} \in \langle a \rangle \cap \mathfrak{m}$, a contradiction. Thus $[G : \mathfrak{m}] = p$ and so \mathfrak{m} is maximal. We claim that if $x \notin \mathfrak{m}$, then $x^i \notin \mathfrak{m}$ for $i = 1, \dots, p-1$. Assume that $o(x) = l$ and let t be the smallest integer such that $x^t \in \mathfrak{m}$. Let $l = tq + r$ for integers t and q with $0 \leq r < t$. If $r \neq 0$, then the equality $1 = x^{tq} x^r$ implies that $x^r \in \mathfrak{m}$ which contradicts the minimality of t . Hence $r = 0$ and $l = tq$. Thus $(x^{l/t})^t = 1$, a contradiction and so the claim is proved.

If \mathfrak{m} is not normal in G , then $y = x^{-1}bx \notin \mathfrak{m}$ for some $x \in G$ and $b \in \mathfrak{m}$. Since $x^i \notin \mathfrak{m}$ and $y^i \notin \mathfrak{m}$ for $i = 1, \dots, p-1$, we conclude that $\mathfrak{m}x = \mathfrak{m}y^j$ for some j , $1 \leq j \leq p-1$. Thus $x = mx^{-1}b^jx$ for some $m \in \mathfrak{m}$, and hence $x \in \mathfrak{m}$, which is a contradiction. Therefore, \mathfrak{m} is normal maximal, as desired. \square

Factorizable groups have been intensively studied. Both finite and infinite groups have been the subject of many investigations. An important class of factorizable groups are groups which have a triple factorization. For more information on factorizable groups and triple factorization we refer the reader to [1], [2], [4], [7] and [9].

The next theorem states that every aS -group G for which $|G| \neq pq$ or $|G| \neq p$, where p, q are prime numbers, has a triple factorization.

Theorem 9. *Let G be an aS -group such that $|G| \neq pq$ or $|G| \neq p$ where p, q are prime numbers. Then G has a triple factorization.*

Proof. First, assume that G is finite. We continue the proof in the following two cases:

Case (i). There exist three distinct prime numbers p, q, r such that $p, q, r \mid |G|$. Since G is an aS -group, there exist subgroups $\mathfrak{m}_1, \mathfrak{m}_2$ and \mathfrak{m}_3 of indices p, q, r , respectively. It follows from Theorem 1.3.13 in [11] that $G = \mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_2\mathfrak{m}_3 = \mathfrak{m}_3\mathfrak{m}_1$.

Case (ii). $|G| = p^nq^m$, where p and q are prime numbers and $n, m \in \mathbb{N} \cup \{0\}$. Let p be the smallest prime number which divides $|G|$, and $o(g) = p$ for some $g \in G$. Since G is an aS -group, there exists a maximal subgroup \mathfrak{m} such that $\langle g \rangle \mathfrak{m} = G$. Note that \mathfrak{m} is a normal subgroup. If there exists $\mathfrak{m}_i \in \text{Max}(G)$ such that $\mathfrak{m} \cap \mathfrak{m}_i \neq 1$, then we are done, as there exists a subgroup L such that $L(\mathfrak{m} \cap \mathfrak{m}_i) = G$. Since $\mathfrak{m} \cap \mathfrak{m}_i \subseteq \mathfrak{m}_i$, we have $L\mathfrak{m}_i = G$ and so we find the factorization $\mathfrak{m}_i\mathfrak{m} = \mathfrak{m}_iL = \mathfrak{m}L = G$. Thus, we can suppose that $\mathfrak{m} \cap \mathfrak{m}_i = 1$ for each $\mathfrak{m}_i \in \text{Max}(G)$. Let \mathfrak{m}_1 be a subgroup of index q . By Theorem 1.3.13 in [11], we have $\mathfrak{m}_1\mathfrak{m} = G$ and so $|G| = pq$.

Now, suppose that G is an infinite group. If $g \in G$ is a torsion free element, then by Theorem 4 there exists a subgroup L of finite index such that $\langle g \rangle L = G$. Therefore, we have $\langle g \rangle \cap L \neq 1$, as g is torsion free and $[G : L] < \infty$. Again, there exists a subgroup K such that $K(\langle g \rangle \cap L) = G$ and we can consider the factorization $\langle g \rangle K = \langle g \rangle L = LK = G$. So we can assume that G is a torsion group. It follows from Corollary 3.4 in [8] that G has a maximal subgroup \mathfrak{m} . By Theorem 4, \mathfrak{m} has a supplement K of finite index. Since \mathfrak{m} is infinite, $\mathfrak{m} \cap K \neq 1$. By a similar argument we can show that G has a triple factorization. \square

We close this paper with the following result which shows that if G is an abelian group with G/H an aS -group for every nontrivial subgroup H of G , then G itself is an aS -group.

Theorem 10. *For every nontrivial subgroup H of an abelian group G , G/H is an aS -group if and only if G is isomorphic to a direct sum of cyclic groups of prime orders.*

Proof. First suppose that for every nontrivial subgroup H of a group G the group G/H is an aS -group. We claim that G is a torsion group. If $G \cong \mathbb{Z}$, we get a contradiction since \mathbb{Z}_{12} is not an aS -group. Thus $G \not\cong \mathbb{Z}$. If $\mathbb{Z} \leq G$, then by the assumption, $G/12\mathbb{Z}$ is an aS -group. Thus Proposition 3.5 in [8] implies that $\mathbb{Z}/12\mathbb{Z}$ is an aS -group, a contradiction and so the claim is proved. Now, by the primary decomposition (see Theorem 8.4 in [5]), $G \cong \bigoplus A_p$, where A_p is a p -group. Suppose that G is a p -group. Let $K = \langle k \rangle$ be a subgroup of G of order p^i , $i \geq 4$ and H a subgroup of K of order p . Since G/H is an aS -group, by Proposition 3.5 in [8]

we deduce that K/H is an aS -group, a contradiction. So, the order of each element of G is at most p^3 . By the Baer-Prüfer Theorem (see Theorem 17.2 in [5]), G is a direct sum of cyclic groups. If \mathbb{Z}_{p^3} is a direct summand of G , then there exists a subgroup $\langle g \rangle$ such that $G/L \cong \mathbb{Z}_{p^3}$, a contradiction. Hence G is a direct sum of cyclic groups of order at most p^2 and so G is isomorphic to one of the following groups: $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$, \mathbb{Z}_{p^2} , \mathbb{Z}_p , or G has at least three summands. If G has at least three summands, then either $G \cong \bigoplus \mathbb{Z}_p$ or there exists a subgroup $\langle g \rangle$ such that either $G/L \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ or $G/L \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$. If $G \cong \bigoplus \mathbb{Z}_p$, then we are done. If not, then we get a contradiction. Now, suppose that $G \cong \bigoplus A_p$, with at least two summands. By a similar argument, one can show that G is a direct sum of cyclic groups of prime orders. The converse is clear. \square

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