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ON A CHARACTERIZATION OF k-TREES

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Abstract. A graph G is a k-tree if either G is the complete graph on $k+1$ vertices, or G has a vertex v whose neighborhood is a clique of order k and the graph obtained by removing v from G is also a k-tree. Clearly, a k-tree has at least $k+1$ vertices, and G is a 1-tree (usual tree) if and only if it is a 1-connected graph and has no K_3-minor. In this paper, motivated by some properties of 2-trees, we obtain a characterization of k-trees as follows: if G is a graph with at least $k+1$ vertices, then G is a k-tree if and only if G has no K_{k+2}-minor, G does not contain any chordless cycle of length at least 4 and G is k-connected.

Keywords: characterization; k-tree; K_t-minor

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1. Introduction

Graphs in this paper are finite and simple. Let G be a graph. For $X \subseteq V(G)$ and $v \in V(G)$, the neighborhood of v in X is denoted by $N_X(v)$. Further, for $X \subseteq V(G)$ and $Y \subseteq V(G)$, we denote $N_X(Y) = \bigcup_{v\in Y} N_X(v)$. For $X \subseteq V(G)$, the induced subgraph of G on X is denoted by $G[X]$. Let K_t be a complete graph on t vertices. We say that K_t is a *minor* of G if K_t can be obtained from a subgraph of G by contracting edges (and deleting the resulting multiple edges and loops).

A graph G is a *k-tree* if either G is the complete graph on $k+1$ vertices, or G has a vertex v whose neighborhood is a clique of order k and the graph obtained by removing v from G is a k-tree. Clearly, a k-tree has at least $k+1$ vertices and 1-trees are usual trees. It is also obvious that G is a 1-tree if and only if it is a 1-connected graph and has no K_3-minor. An *edge bonding* of two disjoint graphs G and G' is any

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graph constructed from G and G' by identifying an edge of G with an edge of G'. Cai [3] showed that an edge bonding of two disjoint 2-trees is also a 2-tree. Some properties of 2-trees can be summarized as follows (see [1], [3]): if G is a 2-tree, then G is planar, G is the edge-maximal graph having no K_4-minor, G does not contain any chordless cycle of length at least 4 and G is 2-connected.

From [1], [4], it is known that k-trees are intrinsically related to treewidth, which is an important parameter in the Robertson-Seymour theory of graph minors and in algorithmic complexity. In particular, a graph has *treewidth* k if and only if it is a subgraph of a k-tree. Thus, k-trees are the edge-maximal graphs of treewidth k. Bose et al. [2] gave a characterization of the degree sequences of 2-trees. Motivated by the properties of 2-trees, we can obtain a characterization of k-trees as follows.

Theorem 1.1. *Let G be a graph with at least $k+1$ vertices. Then G is a k-tree if and only if* (a)–(c) *are fulfiled*

(a) *G has no K_{k+2}-minor;*

(b) *G does not contain any chordless cycle of length at least* 4*;*

(c) *G is k-connected.*

2. Proof of Theorem 1.1

We first extend the concept of 'an edge bonding' due to Cai [3] to the concept of 'a K_t-bonding'. Let G and G' be two disjoint graphs and have K_t as a subgraph. A *K_t-bonding* of G and G' is any graph constructed from G and G' by identifying a K_t of G with a K_t of G'. An *ear* in a k-tree is a vertex of degree k whose neighbors are adjacent to each other.

Lemma 2.1. *A K_k-bonding of two disjoint k-trees is also a k-tree.*

Proof. Let G_1 be a k-tree on s vertices and G_2 be a k-tree on t vertices. Then G_1 and G_2 have K_k as a subgraph. Let G be a K_k-bonding of G_1 and G_2. We now use induction on s. If $s = k+1$, then $G_1 = K_{k+1}$, and hence G is the graph obtained from G_2 by adding an ear. Thus G is a k-tree. Assume that $s > k+1$. It is known that the set of all ears of G_1 is an independent set in G_1 and has at least two elements. This implies that there exists an ear v in G_1 with $v \notin V(K_k)$. Then $G-v$ is a K_k-bonding of $G_1 - v$ and G_2. By the induction hypothesis, $G-v$ is a k-tree. Thus G is also a k-tree. □

We now prove Theorem 1.1.

P r o o f of Theorem 1.1. We use induction on n to prove the necessity. Let G be a k-tree on n vertices. Then $n \geqslant k+1$. If $n = k+1$, then $G = K_{k+1}$. Clearly, G satisfies (a)–(c). Assume that $n > k+1$. Let u be an ear of G and denote $G' = G - u$. Let $N_G(u) = \{x_1, \ldots, x_k\}$. Then $\{x_1, \ldots, x_k\}$ is a clique in G.

By the induction hypothesis, G' has no K_{k+2}-minor. If G has K_{k+2}-minor, let H be a subgraph of G so that we can obtain K_{k+2} from H by contracting edges, then $u \in V(H)$. By $d_H(u) \leqslant d_G(u) = k < k+1$, we have that $u \notin V(K_{k+2})$. This implies that some edge ux_j in H will be contracted in the process of forming K_{k+2}. Let H' be the graph obtained from H by contracting ux_j. Since $\{x_1, \ldots, x_k\}$ is a clique in G, it is easy to see that H' is a subgraph of G'. Since we can obtain K_{k+2} from H' by contracting edges, we have that G' has K_{k+2}-minor, a contradiction. Therefore, G has no K_{k+2}-minor.

By the induction hypothesis, G' has no chordless cycle of length at least 4. If G has a chordless cycle C with $|V(C)| \geqslant 4$, then $u \in V(C)$. This is impossible by $G[\{u\} \cup N_G(u)] = K_{k+1}$. Therefore, G has no chordless cycle of length at least 4.

By the induction hypothesis, G' is k-connected. Thus G is also k-connected by $d_{G'}(u) = k$.

We now use induction on n to prove the sufficiency. Let $n \geqslant k+1$ and G be a graph on n vertices satisfying (a)–(c). If $n = k+1$, then $G = K_{k+1}$ by G satisfying (c). Clearly, G is a k-tree. Assume that $n \geqslant k+2$. We first prove the following Claim.

Claim. G contains K_k as a subgraph.

P r o o f of Claim. Since G has no K_{k+2}-minor, G is not a complete graph. Then there exist two vertices $u, v \in V(G)$ with $uv \notin E(G)$. Since G is k-connected, by Menger's theorem, there are at least k internally-disjoint paths between u and v. Let

$$\begin{aligned} P_1 &= ux_{11} \ldots x_{1t_1} v, \\ P_2 &= ux_{21} \ldots x_{2t_2} v, \\ &\vdots \\ P_k &= ux_{k1} \ldots x_{kt_k} v \end{aligned}$$

be the k internally-disjoint paths between u and v so that $|P_1| + |P_2| + \ldots + |P_k|$ is minimal. Let

$$\begin{aligned} X_1 &= \{x_{11}, \ldots, x_{1t_1}\}, \\ X_2 &= \{x_{21}, \ldots, x_{2t_2}\}, \\ &\vdots \\ X_k &= \{x_{k1}, \ldots, x_{kt_k}\}. \end{aligned}$$

Denote $X = X_1 \cup \ldots \cup X_k$. Let s and t be two arbitrary integers with $1 \leqslant s < t \leqslant k$. Since $P_s \cup P_t$ is a cycle of length at least 4, by the minimality of $|P_1| + |P_2| + \ldots + |P_k|$, we have that $N_{X_s}(X_t) \neq \emptyset$ and $N_{X_t}(X_s) \neq \emptyset$. Let $x_{si} \in X_s$ and $x_{tj} \in X_t$ so that $x_{si}x_{tj} \in E(G)$ and $i + j$ is minimal. Since $ux_{s1} \ldots x_{si}x_{tj} \ldots x_{t1}u$ is a chordless cycle of G with length $i + j + 1$, we have that $i + j = 2$. This implies that $i = j = 1$ and $x_{s1}x_{t1} \in E(G)$. Therefore, $G[\{x_{11}, x_{21}, \ldots, x_{k1}\}] = K_k$. The proof of Claim is completed. □

Denote $F = G[\{x_{11}, x_{21}, \ldots, x_{k1}\}] = K_k$. We now consider the following two cases.

Case 1. $G - V(F)$ is connected.

Let $P = uy_1 \ldots y_l v$ be a path connecting u and v in $G - V(F)$ and denote $Y = \{y_1, \ldots, y_l\}$. If $X \cap Y = \emptyset$, then there exists a subgraph $F \cup P \cup P_1 \cup \ldots \cup P_k$ of G so that we can get a K_{k+2} from this subgraph by contracting edges. In other words, G has K_{k+2}-minor, a contradiction. Thus $X \cap Y \neq \emptyset$. Let $y_{l_0} \in X \cap Y$ so that l_0 is minimal, and denote $P_0 = uy_1 \ldots y_{l_0}$. Then there exists a subgraph $F \cup P_0 \cup P_1 \cup \ldots \cup P_k$ of G so that we can get a K_{k+2} from this subgraph by contracting edges. In other words, G has K_{k+2}-minor, a contradiction.

Case 2. $G - V(F)$ is not connected.

Let $H_1, \ldots, H_m$ be m connected components of $G - V(F)$. If $G[V(H_i) \cup V(F)]$ satisfies (a)–(c) for each i with $1 \leqslant i \leqslant m$, then by the induction hypothesis, $G[V(H_i) \cup V(F)]$ is a k-tree for each i with $1 \leqslant i \leqslant m$. Since G is a K_k-bonding of $G[V(H_1) \cup V(F)], \ldots, G[V(H_m) \cup V(F)]$, we have that G is also a k-tree by Lemma 2.1. We now assume that there exists a r with $1 \leqslant r \leqslant m$ such that $G[V(H_r) \cup V(F)]$ does not satisfy (a)–(c).

If $G[V(H_r) \cup V(F)]$ does not satisfy (a), i.e., $G[V(H_r) \cup V(F)]$ has K_{k+2}-minor, then G also has K_{k+2}-minor as $G[V(H_r) \cup V(H)]$ is a subgraph of G, a contradiction.

If $G[V(H_r) \cup V(F)]$ does not satisfy (b), i.e., $G[V(H_r) \cup V(F)]$ contains a chordless cycle C with $|C| \geqslant 4$, then C is also a chordless cycle in G, a contradiction.

Assume that $G[V(H_r) \cup V(F)]$ does not satisfy (c), i.e., $G[V(H_r) \cup V(F)]$ is not k-connected. If $|V(H_r)| = 1$, then by G satisfying (c), we have that $G[V(H_r) \cup V(F)] = K_{k+1}$, which is a k-connected graph, a contradiction. So $|V(H_r)| \geqslant 2$. Let V' be a vertex-cut of $G[V(H_r) \cup V(F)]$ with $|V'| < k$ and let M_1, M_2 be two connected components of $G[V(H_r) \cup V(F)] - V'$. If $V(M_1) \cap V(F) \neq \emptyset$, then $V(M_2) \cap V(F) = \emptyset$. This implies that $V(M_1) \cap V(F) = \emptyset$ or $V(M_2) \cap V(F) = \emptyset$. Without loss of generality, we let $V(M_1) \cap V(F) = \emptyset$. Then M_1 is also a connected component of $G - V'$. In other words, V' is a vertex-cut of G. Thus G is not k-connected, a contradiction. This completes the proof of Theorem 1.1. □

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