QUASITRIANGULAR HOPF GROUP ALGEBRAS AND BRAIDED MONOIDAL CATEGORIES

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(Received March 31, 2013)

Abstract. Let π be a group, and H be a semi-Hopf π-algebra. We first show that the category ${}_H\mathcal{M}$ of left π-modules over H is a monoidal category with a suitably defined tensor product and each element α in π induces a strict monoidal functor F_α from ${}_H\mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf π-algebra, and show that a semi-Hopf π-algebra H is quasitriangular if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf π-algebras and describe the categories of modules over them.

Keywords: Hopf π-algebra; H-π-modules; braided monoidal category; braided monoidal functor

MSC 2010: 16T05, 08C05

1. Introduction

The notion of a quasitriangular Hopf algebra was introduced by Drinfel'd [4], when he studied the Yang-Baxter equation. The category of modules over a quasitriangular Hopf algebra is a braided monoidal category. Moreover, the braiding structure of a braided monoidal category can supply solutions to the quantum Yang-Baxter equation. Recently, Turaev [9] introduced Hopf π-coalgebra, which generalizes the notion of Hopf algebra. Virelizier also studied algebraic properties of Hopf group-coalgebras and generalized the main properties of quasitriangular Hopf algebras to the setting of quasitriangular Hopf π-coalgebras in [10]. Wang introduced the concept of semi-Hopf group algebra and investigated properties of coquasitriangular Hopf group algebras

This work is supported by NSF of China, No. 11171291, by Doctorate United Foundation, No. 20123250110005, of Ministry of China and Jiangsu Province, and by Qing Lan Project of Jiangsu Province.

in [11]. Zhu, Chen and Li studied the categories of modules and comodules over a Hopf group coalgebra in [13] and [14], respectively.

In this paper, we first investigate the category ${}_H\mathcal{M}$ of left modules over a semi-Hopf π-algebra H, where π is a group. We define a tensor product module of two modules over H, and show that ${}_H\mathcal{M}$ is a monoidal category with respect to such a tensor product, and each element α in π induces a strict monoidal functor F_α from ${}_H\mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf π-algebra, and show that a semi-Hopf π-algebra H is quasitriangular if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf π-algebras and discuss the categories of modules over them.

2. Preliminaries

Throughout the paper, let π be a discrete group (with neutral element 1) and k be a fixed field. All algebras and coalgebras, π-algebras and Hopf π-algebras are defined over k. The definitions and properties of an algebra, coalgebra, Hopf algebra, category and monoidal category can be found in [5]–[7], [12]. We use the standard Sweedler notation for comultiplication. The tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-spaces, $\tau_{U,V}\colon U \otimes V \to V \otimes U$ will denote the twist map defined by $\tau_{U,V}(u \otimes v) = v \otimes u$. The following definitions and notations can be found in [1], [8]–[11].

Definition 2.1. A π-algebra (over k) is a family $A = \{A_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $m = \{m_{\alpha,\beta}\colon A_\alpha \otimes A_\beta \to A_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of k-linear maps (the multiplication) and a k-linear map $u\colon k \to A_1$ (the unit) such that m is associative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{A_\gamma}) = m_{\alpha,\beta\gamma}(\mathrm{id}_{A_\alpha} \otimes m_{\beta,\gamma}),$$
$$m_{\alpha,1}(\mathrm{id}_{A_\alpha} \otimes u) = \mathrm{id}_{A_\alpha} = m_{1,\alpha}(u \otimes \mathrm{id}_{A_\alpha}).$$

Note that $(A_1, m_{1,1}, u)$ is an algebra in the usual sense.

Definition 2.2. Let $A = (\{A_\alpha\}_{\alpha\in\pi}, m, u)$ be a π-algebra. A left π-module over A is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $\eta = \{\eta^M_{\alpha,\beta}\colon A_\alpha \otimes M_\beta \to M_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of k-linear maps such that for any $\alpha, \beta, \gamma \in \pi$,

(1) $\eta^M_{\alpha,\beta\gamma}(\mathrm{id}_{A_\alpha} \otimes \eta^M_{\beta,\gamma}) = \eta^M_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{M_\gamma})$;
(2) $\eta^M_{1,\alpha}(u \otimes \mathrm{id}_{M_\alpha}) = \mathrm{id}_{M_\alpha}$.

Definition 2.3. Assume that $A = (\{A_\alpha\}_{\alpha\in\pi}, m, u)$ is a π-algebra. Let $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ be two left π-modules over A. A left A-π-module map from M to N is a family $f = \{f_\alpha\colon M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ of k-linear maps such that

$$\eta^N_{\alpha,\beta}(\mathrm{id}_{A_\alpha} \otimes f_\beta) = f_{\alpha\beta}\eta^M_{\alpha,\beta}, \quad \alpha, \beta \in \pi.$$

Definition 2.4. A semi-Hopf π-algebra is a π-algebra $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ such that:

(1) Each H_α is a k-coalgebra with comultiplication Δ_α and counit ε_α, $\alpha \in \pi$.

(2) $u\colon k \to H_1$ and $m_{\alpha,\beta}\colon H_\alpha \otimes H_\beta \to H_{\alpha\beta}$ are coalgebra maps, $\alpha, \beta \in \pi$.
Furthermore, if there is a family $S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of k-linear maps (the antipode) such that the following condition (3) is satisfied, then $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is called a Hopf π-algebra.

(3) $m_{\alpha^{-1},\alpha}(S_\alpha \otimes \mathrm{id}_{H_\alpha})\Delta_\alpha = u\varepsilon_\alpha = m_{\alpha,\alpha^{-1}}(\mathrm{id}_{H_\alpha} \otimes S_\alpha)\Delta_\alpha$, $\alpha \in \pi$.

3. Category of modules over a semi-Hopf π-algebra

Throughout this section, assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra. Denote by ${}_H\mathcal{M}$ the category of all left π-modules over H, whose morphisms are left H-π-module maps.

Lemma 3.1. *Suppose that (M, η^M) and (N, η^N) are left π-modules over H. Then the tensor product $M \otimes N = \{(M \otimes N)_\alpha\}_{\alpha\in\pi}$ is also a left π-module over H, where $(M \otimes N)_\alpha = M_\alpha \otimes N_\alpha$, the structure maps $\eta^{M\otimes N} = \{\eta^{M\otimes N}_{\alpha,\beta}\colon H_\alpha \otimes M_\beta \otimes N_\beta \to M_{\alpha\beta} \otimes N_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ are given by*

$$\eta^{M\otimes N}_{\alpha,\beta} := (\eta^M_{\alpha,\beta} \otimes \eta^N_{\alpha,\beta})(\mathrm{id}_{H_\alpha} \otimes \tau_{H_\alpha,M_\beta} \otimes \mathrm{id}_{N_\beta})(\Delta_\alpha \otimes \mathrm{id}_{M_\beta} \otimes \mathrm{id}_{N_\beta}).$$

Proof. On the one hand, for any $h \in H_\alpha$, $l \in H_\beta$, $m \in M_\gamma$ and $n \in N_\gamma$, we have

$$\begin{aligned}
&\eta^{M\otimes N}_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha} \otimes \eta^{M\otimes N}_{\beta,\gamma})(h \otimes l \otimes m \otimes n)\\
&\quad= \eta^{M\otimes N}_{\alpha,\beta\gamma}\Big(\sum h \otimes l_1 \cdot m \otimes l_2 \cdot n\Big)\\
&\quad= \sum h_1 \cdot (l_1 \cdot m) \otimes h_2 \cdot (l_2 \cdot n)\\
&\quad= \sum (h_1 l_1) \cdot m \otimes (h_2 l_2) \cdot n\\
&\quad= \sum (hl)_1 \cdot m \otimes (hl)_2 \cdot n\\
&\quad= \eta^{M\otimes N}_{\alpha\beta,\gamma}(hl \otimes m \otimes n)\\
&\quad= \eta^{M\otimes N}_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{(M\otimes N)_\gamma})(h \otimes l \otimes m \otimes n).
\end{aligned}$$

Hence $\eta^{M\otimes N}_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^{M\otimes N}_{\beta,\gamma}) = \eta^{M\otimes N}_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{(M\otimes N)_\gamma})$. On the other hand, for any $\lambda\in k$, $m\in M_\alpha$ and $n\in N_\alpha$, we have

$$\eta^{M\otimes N}_{1,\alpha}(u\otimes\mathrm{id}_{(M\otimes N)_\alpha})(\lambda\otimes m\otimes n) = \eta^{M\otimes N}_{1,\alpha}(\lambda 1_H\otimes m\otimes n) = \lambda(m\otimes n).$$

Hence $\eta^{M\otimes N}_{1,\alpha}(u\otimes\mathrm{id}_{(M\otimes N)_\alpha}) = \mathrm{id}_{(M\otimes N)_\alpha}$. Thus, $M\otimes N = \{(M\otimes N)_\alpha\}_{\alpha\in\pi}$ is a left π-module over H. □

Let $M,N,P\in {}_H\mathcal{M}$. Define $a_{M,N,P} = \{a_\alpha\}_{\alpha\in\pi}\colon (M\otimes N)\otimes P\to M\otimes(N\otimes P)$ by $a_\alpha\colon (M_\alpha\otimes N_\alpha)\otimes P_\alpha\to M_\alpha\otimes(N_\alpha\otimes P_\alpha)$, $(m\otimes n)\otimes p\mapsto m\otimes(n\otimes p)$, where $m\in M_\alpha$, $n\in N_\alpha$, $p\in P_\alpha$. Then we have the following lemma.

Lemma 3.2. *The family $a_{M,N,P}$ is a family of left H-π-module natural isomorphisms, where $M,N,P\in {}_H\mathcal{M}$.*

P r o o f. For any $\alpha,\beta\in\pi$, $h\in H_\alpha$, $m\in M_\beta$, $n\in N_\beta$ and $p\in P_\beta$, we have

$$\begin{aligned}\eta^{M\otimes(N\otimes P)}_{\alpha,\beta}&(\mathrm{id}_{H_\alpha}\otimes a_\beta)(h\otimes((m\otimes n)\otimes p))\\ &= \eta^{M\otimes(N\otimes P)}_{\alpha,\beta}(h\otimes(m\otimes(n\otimes p)))\\ &= \sum h_1\cdot m\otimes h_2\cdot(n\otimes p) = \sum h_1\cdot m\otimes(h_2\cdot n\otimes h_3\cdot p)\\ &= a_{\alpha\beta}\Big(\sum (h_1\cdot m\otimes h_2\cdot n)\otimes h_3\cdot p\Big)\\ &= a_{\alpha\beta}\Big(\sum h_1\cdot(m\otimes n)\otimes h_2\cdot p\Big)\\ &= a_{\alpha\beta}\eta^{(M\otimes N)\otimes P}_{\alpha,\beta}(h\otimes((m\otimes n)\otimes p)).\end{aligned}$$

This shows that $\eta^{M\otimes(N\otimes P)}_{\alpha,\beta}(\mathrm{id}_{H_\alpha}\otimes a_\beta) = a_{\alpha\beta}\eta^{(M\otimes N)\otimes P}_{\alpha,\beta}$, and so $a_{M,N,P}$ is a left H-π-module morphism. Consequently, $a_{M,N,P}$ is a left H-π-module isomorphism. Obviously, it is a family of natural isomorphisms of H-π-modules. □

Lemma 3.3. *Let $K = \{K_\alpha\}_{\alpha\in\pi}$ with $K_\alpha = k$. Define $\eta^K_{\alpha,\beta}\colon H_\alpha\otimes K_\beta\to K_{\alpha\beta}$ by $\eta^K_{\alpha,\beta}(h\otimes\lambda) = h\cdot\lambda := \varepsilon_\alpha(h)\lambda$. Then K is a left π-module over H.*

P r o o f. For any $h\in H_\alpha$, $l\in H_\beta$, $m\in K_\gamma = k$, $\lambda\in k$, $n\in K_\alpha = k$, we have

$$\begin{aligned}\eta^K_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^K_{\beta,\gamma})(h\otimes l\otimes m) &= \eta^K_{\alpha,\beta\gamma}(h\otimes\varepsilon_\beta(l)m)\\ &= \varepsilon_\alpha(h)(\varepsilon_\beta(l)m) = \varepsilon_{\alpha\beta}(hl)m = \eta^K_{\alpha\beta,\gamma}(hl\otimes m)\\ &= \eta^K_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{K_\gamma})(h\otimes l\otimes m)\end{aligned}$$

and

$$\eta^K_{1,\alpha}(u\otimes\mathrm{id}_{K_\alpha})(\lambda\otimes n) = \eta^K_{1,\alpha}(\lambda 1_H\otimes n) = \varepsilon_1(\lambda 1_H)n = \lambda n.$$

This shows that $\eta^K_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^K_{\beta,\gamma}) = \eta^K_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{K_\gamma})$ and $\eta^K_{1,\alpha}(u\otimes\mathrm{id}_{K_\alpha}) = \mathrm{id}_{K_\alpha}$. Thus, K is a left π-module over H. □

For any left π-module M over H, we have $(K \otimes M)_\alpha = K_\alpha \otimes M_\alpha = k \otimes M_\alpha$ and $(M \otimes K)_\alpha = M_\alpha \otimes K_\alpha = M_\alpha \otimes k$, $\alpha \in \pi$. Define $l_M\colon K \otimes M \to M$ and $r_M\colon M \otimes K \to M$ by

$$(l_M)_\alpha\colon k \otimes M_\alpha \to M_\alpha, \quad \lambda \otimes m \mapsto \lambda m,$$
$$(r_M)_\alpha\colon M_\alpha \otimes k \to M_\alpha, \quad m \otimes \lambda \mapsto \lambda m.$$

Then it is easy to see that $l = \{l_M\}$ and $r = \{r_M\}$ are two families of natural isomorphisms of left H-π-modules.

Summarizing the above discussion, one gets the the following theorem.

Theorem 3.4. *$({}_H\mathcal{M}, \otimes, K, a, l, r)$ is a monoidal category, where K is the unit object.*

For any $\alpha \in \pi$, define a functor $F_\alpha\colon {}_H\mathcal{M} \to {}_H\mathcal{M}$ by

$$F_\alpha(M)_\beta = M_{\beta\alpha}, \quad \eta^{F_\alpha(M)}_{\beta,\gamma} = \eta^M_{\beta,\gamma\alpha}, \quad F_\alpha(f)_\beta = f_{\beta\alpha},$$

where M is a left π-module over H and f is an H-π-module map. Obviously, $F_\alpha(K) = K$ and $(F_\alpha(M) \otimes F_\alpha(N))_\beta = F_\alpha(M)_\beta \otimes F_\alpha(N)_\beta = M_{\beta\alpha} \otimes N_{\beta\alpha} = (M \otimes N)_{\beta\alpha} = F_\alpha(M \otimes N)_\beta$, where M and N are left π-modules over H. Then by a straightforward verification, one can check the following theorem.

Theorem 3.5. *F_α is a strict monoidal functor from $({}_H\mathcal{M}, \otimes, K, a, l, r)$ to itself, where $\alpha \in \pi$.*

4. Quasitriangular semi-Hopf π-algebras

Throughout this section, assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra, and that ${}_H\mathcal{M}$ is the category of left π-modules over H, which is a monoidal category as stated in the last section.

Definition 4.1. H is called a quasitriangular semi-Hopf π-algebra, if there exists an invertible element $R \in H_1 \otimes H_1$ such that the following conditions are satisfied:

(1) $\Delta^{\rm cop}_\alpha(h)R = R\Delta_\alpha(h)$;
(2) $(\Delta_1 \otimes \mathrm{id})(R) = R_{13}R_{23}$;
(3) $(\mathrm{id} \otimes \Delta_1)(R) = R_{13}R_{12}$,

where $\alpha \in \pi$, $h \in H_\alpha$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\tau_{H_1,H_1} \otimes \mathrm{id})(1 \otimes R) \in H_1 \otimes H_1 \otimes H_1$ and $\Delta^{\rm cop}_\alpha = \tau_{H_\alpha,H_\alpha} \circ \Delta_\alpha$. In this case, R is called a quasitriangular structure of H.

Remark 4.2. We remark that H_1 is a usual quasitriangular bialgebra if H is quasitriangular, and that H is called an almost cocommutative semi-Hopf π-algebra if only (1) is satisfied.

Let $R = \sum\limits_i s_i \otimes t_i$. Then the three conditions in Definition 4.1 can be formulated as follows:

(1) $\sum\limits_i h_2 s_i \otimes h_1 t_i = \sum\limits_i s_i h_1 \otimes t_i h_2$;
(2) $\sum\limits_i (s_i)_1 \otimes (s_i)_2 \otimes t_i = \sum\limits_{i,j} s_i \otimes s_j \otimes t_i t_j$;
(3) $\sum\limits_i s_i \otimes (t_i)_1 \otimes (t_i)_2 = \sum\limits_{i,j} s_i s_j \otimes t_j \otimes t_i$,

where $\alpha \in \pi$, $h \in H_\alpha$ and $\Delta_\alpha(h) = \sum h_1 \otimes h_2$ as usual.

Lemma 4.3. *If H is almost cocommutative, then there exists a left H-π-module isomorphism $M \otimes N \cong N \otimes M$ for any left π-modules M and N over H.*

Proof. Assume that $R = \sum\limits_i s_i \otimes t_i \in H_1 \otimes H_1$ is an invertible element satisfying condition (1) of Definition 4.1. Let M and N be two left π-modules over H. For any $\alpha \in \pi$, define $(c_{M,N})_\alpha\colon\ M_\alpha \otimes N_\alpha \to N_\alpha \otimes M_\alpha$ by

$$(c_{M,N})_\alpha(m \otimes n) := \tau_{M_\alpha,N_\alpha}(R \cdot (m \otimes n)) = \sum_i t_i \cdot n \otimes s_i \cdot m,$$

where $m \in M_\alpha$ and $n \in N_\alpha$. Since R is invertible, $(c_{M,N})_\alpha$ is a k-linear isomorphism. Now for any $\alpha, \beta \in \pi$, $m \in M_\beta$, $n \in N_\beta$ and $h \in H_\alpha$, we have

$$\begin{aligned}
&\eta^{N\otimes M}_{\alpha,\beta}(\mathrm{id}_{H_\alpha} \otimes (c_{M,N})_\beta)(h \otimes m \otimes n)\\
&\quad = \eta^{N\otimes M}_{\alpha,\beta}\Big(\sum_i h \otimes t_i \cdot n \otimes s_i \cdot m\Big)\\
&\quad = \sum_i h_1 \cdot (t_i \cdot n) \otimes h_2 \cdot (s_i \cdot m) = \sum_i (h_1 t_i) \cdot n \otimes (h_2 s_i) \cdot m\\
&\quad = \sum_i (t_i h_2) \cdot n \otimes (s_i h_1) \cdot m = \sum_i t_i \cdot (h_2 \cdot n) \otimes s_i \cdot (h_1 \cdot m)\\
&\quad = (c_{M,N})_{\alpha\beta}\Big(\sum h_1 \cdot m \otimes h_2 \cdot n\Big) = (c_{M,N})_{\alpha\beta}\eta^{M\otimes N}_{\alpha,\beta}(h \otimes m \otimes n).
\end{aligned}$$

Hence $\eta^{N\otimes M}_{\alpha,\beta}(\mathrm{id}_{H_\alpha} \otimes (c_{M,N})_\beta) = (c_{M,N})_{\alpha\beta}\eta^{M\otimes N}_{\alpha,\beta}$. This shows that $c_{M,N}$ is a left H-π-module map, and so

$$c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha\in\pi}\colon\ M \otimes N \to N \otimes M$$

is a left H-π-module isomorphism. □

Theorem 4.4. *Assume that H is quasitriangular with a quasitriangular structure R. Then the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$.*

Proof. By Theorems 3.4 and 3.5, it follows that ${}_H\mathcal{M}$ is a monoidal category and F_α is a strict monoidal functor for any $\alpha \in \pi$.

For any $M, N \in {}_H\mathcal{M}$, let

$$c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha\in\pi}\colon\ M \otimes N \to N \otimes M$$

be defined as in Lemma 4.3. Then $c_{M,N}$ is a left H-π-module isomorphism. Let $f = \{f_\alpha\}_{\alpha\in\pi}\colon\ M \to M'$ and $g = \{g_\alpha\}_{\alpha\in\pi}\colon\ N \to N'$ be two left H-π-module maps. Then for any $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$, we have

$$\begin{aligned}(g_\alpha \otimes f_\alpha)(c_{M,N})_\alpha(m \otimes n) &= (g_\alpha \otimes f_\alpha)\Big(\sum_i t_i \cdot n \otimes s_i \cdot m\Big)\\ &= \sum_i g_\alpha(t_i \cdot n) \otimes f_\alpha(s_i \cdot m) = \sum_i t_i \cdot g_\alpha(n) \otimes s_i \cdot f_\alpha(m)\\ &= (c_{M',N'})_\alpha(f_\alpha(m) \otimes g_\alpha(n)) = (c_{M',N'})_\alpha(f_\alpha \otimes g_\alpha)(m \otimes n).\end{aligned}$$

Hence $(g \otimes f)c_{M,N} = c_{M',N'}(f \otimes g)$, which shows that $c_{M,N}$ is a family of natural isomorphisms of left H-π-modules.

Now let $M, N, P \in {}_H\mathcal{M}$ and $\alpha \in \pi$. Then for any $m \in M_\alpha$, $n \in N_\alpha$ and $p \in P_\alpha$, we have

$$\begin{aligned}(c_{M,N\otimes P})_\alpha(m \otimes n \otimes p) &= \sum_i t_i \cdot (n \otimes p) \otimes s_i \cdot m = \sum_i (t_i)_1 \cdot n \otimes (t_i)_2 \cdot p \otimes s_i \cdot m\\ &= \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes (s_j s_i) \cdot m = \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes s_j \cdot (s_i \cdot m)\\ &= (\mathrm{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)\Big(\sum_i t_i \cdot n \otimes s_i \cdot m \otimes p\Big)\\ &= (\mathrm{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)((c_{M,N})_\alpha \otimes \mathrm{id}_{P_\alpha})(m \otimes n \otimes p)\end{aligned}$$

and

$$\begin{aligned}(c_{M\otimes N,P})_\alpha(m \otimes n \otimes p) &= \sum_i t_i \cdot p \otimes s_i \cdot (m \otimes n) = \sum_i t_i \cdot p \otimes (s_i)_1 \cdot m \otimes (s_i)_2 \cdot n\\ &= \sum_{i,j} (t_j t_i) \cdot p \otimes s_j \cdot m \otimes s_i \cdot n = \sum_{i,j} t_j \cdot (t_i \cdot p) \otimes s_j \cdot m \otimes s_i \cdot n\\ &= ((c_{M,P})_\alpha \otimes \mathrm{id}_{N_\alpha})\Big(\sum_i m \otimes t_i \cdot p \otimes s_i \cdot n\Big)\\ &= ((c_{M,P})_\alpha \otimes \mathrm{id}_{N_\alpha})(\mathrm{id}_{M_\alpha} \otimes (c_{N,P})_\alpha)(m \otimes n \otimes p).\end{aligned}$$

This shows that $c_{M,N\otimes P} = (\mathrm{id}_N \otimes c_{M,P})(c_{M,N} \otimes \mathrm{id}_P)$ and $c_{M\otimes N,P} = (c_{M,P} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes c_{N,P})$. Therefore, ${}_H\mathcal{M}$ is a braided monoidal category with the braiding c.

Let $\alpha \in \pi$. Then for any $M, N \in {}_H\mathcal{M}$ and $\beta \in \pi$, it is obvious that $F_\alpha(c_{M,N})_\beta = (c_{M,N})_{\beta\alpha} = (c_{F_\alpha(M),F_\alpha(N)})_\beta$. Hence $F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)}$, and consequently, F_α is a strict braided monoidal functor for any $\alpha \in \pi$. □

Theorem 4.5. *Suppose that ${}_H\mathcal{M}$ is a braided monoidal category, and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Then H is quasitriangular.*

Proof. Suppose that ${}_H\mathcal{M}$ is a braided monoidal category with a braiding c, and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Then $c_{H,H}\colon H\otimes H \to H\otimes H$ is a left H-π-module isomorphism, and hence $(c_{H,H})_1\colon H_1\otimes H_1 \to H_1\otimes H_1$ is a k-linear isomorphism. Let $R = \tau_{H_1,H_1}((c_{H,H})_1(1\otimes 1)) \in H_1 \otimes H_1$. Then Lemmas 4.8–4.10 below show that R is a quasitriangular structure of H. □

Throughout the following Lemma 4.6, Corollary 4.7 and Lemmas 4.8–4.10, assume that ${}_H\mathcal{M}$ is a braided monoidal category with a braiding c, F_α is a strict braided monoidal functor for any $\alpha \in \pi$, and let $R = \tau_{H_1,H_1}((c_{H,H})_1(1\otimes 1)) = \sum_i s_i \otimes t_i \in H_1 \otimes H_1$ be given as above. In this case, we have $(c_{H,H})_1(1\otimes 1) = \tau_{H_1,H_1}(R) = \sum_i t_i \otimes s_i$.

Lemma 4.6. *Let $M, N \in {}_H\mathcal{M}$. Then we have*

$$(c_{M,N})_\alpha(m\otimes n) = \tau_{M_\alpha,N_\alpha}(R\cdot(m\otimes n)) = \sum_i t_i\cdot n \otimes s_i \cdot m,$$

where $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$.

Proof. Let $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$. Then one can easily check that the two maps $\overline{m} = \{\overline{m}_\beta\}_{\beta\in\pi}\colon H \to F_\alpha(M)$ and $\overline{n} = \{\overline{n}_\beta\}_{\beta\in\pi}\colon H \to F_\alpha(N)$ defined by $\overline{m}_\beta(h) = h\cdot m$ and $\overline{n}_\beta(h) = h\cdot n$, $\beta \in \pi$, $h \in H_\beta$, are left H-π-module maps. In this case, $\overline{m}_1(1) = m$ and $\overline{n}_1(1) = n$.

Since $c_{M,N}$ is a family of natural isomorphisms of left H-π-modules, we have $c_{F_\alpha(M),F_\alpha(N)}(\overline{m}\otimes\overline{n}) = (\overline{n}\otimes\overline{m})c_{H,H}$. Since F_α is a strict braided monoidal functor, $F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)}$, and hence $(c_{M,N})_\alpha = F_\alpha(c_{M,N})_1 = (c_{F_\alpha(M),F_\alpha(N)})_1$. Thus, we have

$$\begin{aligned}(c_{M,N})_\alpha(m\otimes n) &= (c_{M,N})_\alpha(\overline{m}_1\otimes\overline{n}_1)(1\otimes 1) = (c_{F_\alpha(M),F_\alpha(N)})_1(\overline{m}_1\otimes\overline{n}_1)(1\otimes 1)\\ &= (c_{F_\alpha(H),F_\alpha(H)}(\overline{m}\otimes\overline{n}))_1(1\otimes 1) = ((\overline{n}\otimes\overline{m})c_{H,H})_1(1\otimes 1)\\ &= (\overline{n}_1\otimes\overline{m}_1)(c_{H,H})_1(1\otimes 1) = (\overline{n}_1\otimes\overline{m}_1)\Big(\sum_i t_i\otimes s_i\Big)\\ &= \sum_i t_i\cdot n\otimes s_i\cdot m = \tau_{M_\alpha,N_\alpha}(R\cdot(m\otimes n)).\end{aligned}$$

□

Corollary 4.7. *For any $\alpha \in \pi$ and $x, y \in H_\alpha$, we have*

$$(c_{H,H})_\alpha(x \otimes y) = \tau_{H_\alpha,H_\alpha}(R(x \otimes y)) = \sum_i t_i y \otimes s_i x.$$

Proof. It follows by putting $M = N = H$ in Lemma 4.6. □

Lemma 4.8. *R is an invertible element in $H_1 \otimes H_1$.*

Proof. Since $(c_{H,H})_1 \colon H_1 \otimes H_1 \to H_1 \otimes H_1$ is a k-linear isomorphism, there exists an element $a \in H_1 \otimes H_1$ such that $(c_{H,H})_1(a) = 1 \otimes 1$. From Corollary 4.7, it follows that $\tau_{H_1,H_1}(Ra) = 1 \otimes 1$, and so $Ra = 1 \otimes 1$. Then $(c_{H,H})_1(aR - 1 \otimes 1) = \tau_{H_1,H_1}(R(aR - 1 \otimes 1)) = \tau_{H_1,H_1}(RaR - R) = \tau_{H_1,H_1}(R - R) = 0$, which implies that $aR - 1 \otimes 1 = 0$, since $(c_{H,H})_1$ is a k-linear automorphism of $H_1 \otimes H_1$, and so $aR = 1 \otimes 1$. Thus, R is an invertible element in $H_1 \otimes H_1$ with $R^{-1} = a$. □

Lemma 4.9. *The following equations hold in $H_1 \otimes H_1 \otimes H_1$:*

(1) $(\mathrm{id} \otimes \Delta_1)(R) = R_{13}R_{12}$;
(2) $(\Delta_1 \otimes \mathrm{id})(R) = R_{13}R_{23}$.

Proof. Since c is a braiding and $H \in {}_H\mathcal{M}$, we have

$$c_{H,H\otimes H} = (\mathrm{id}_H \otimes c_{H,H})(c_{H,H} \otimes \mathrm{id}_H), \quad c_{H\otimes H,H} = (c_{H,H} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes c_{H,H}),$$

and hence

$$(c_{H,H\otimes H})_1 = (\mathrm{id}_{H_1} \otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \mathrm{id}_{H_1}),$$
$$(c_{H\otimes H,H})_1 = ((c_{H,H})_1 \otimes \mathrm{id}_{H_1})(\mathrm{id}_{H_1} \otimes (c_{H,H})_1).$$

By Lemma 4.6 (and Corollary 4.7), we have

$$(c_{H,H\otimes H})_1(1 \otimes 1 \otimes 1) = \sum_i t_i \cdot (1 \otimes 1) \otimes s_i = \sum_i \Delta(t_i) \otimes s_i$$

and

$$\begin{aligned}(\mathrm{id}_{H_1} &\otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \mathrm{id}_{H_1})(1 \otimes 1 \otimes 1) \\ &= (\mathrm{id}_{H_1} \otimes (c_{H,H})_1)\Big(\sum_i t_i \otimes s_i \otimes 1\Big) = \sum_{i,j} t_i \otimes t_j \otimes s_j s_i.\end{aligned}$$

Hence $\sum\limits_i \Delta(t_i) \otimes s_i = \sum\limits_{i,j} t_i \otimes t_j \otimes s_j s_i$, and so $\sum\limits_i s_i \otimes \Delta(t_i) = \sum\limits_{i,j} s_j s_i \otimes t_i \otimes t_j$. This shows equation (1). Equation (2) can be proved similarly. □

Lemma 4.10. *Let $\alpha \in \pi$ and $h \in H_\alpha$. Then we have*

$$\Delta_\alpha^{\mathrm{cop}}(h)R = R\Delta_\alpha(h).$$

Proof. Since $c_{H,H}$ is a left H-π-module map, we have

$$\eta_{\alpha,1}^{H\otimes H}(\mathrm{id}_{H_\alpha} \otimes (c_{H,H})_1) = (c_{H,H})_\alpha \eta_{\alpha,1}^{H\otimes H}, \quad \forall \alpha \in \pi.$$

Let $\alpha \in \pi$ and $h \in H_\alpha$. By Lemma 4.6 or Corollary 4.7, we have

$$\eta_{\alpha,1}^{H\otimes H}(\mathrm{id}_{H_\alpha} \otimes (c_{H,H})_1)(h \otimes 1 \otimes 1) = \eta_{\alpha,1}^{H\otimes H}\Big(h \otimes \sum_i t_i \otimes s_i\Big) = \sum_i h_1 t_i \otimes h_2 s_i$$

and

$$(c_{H,H})_\alpha \eta_{\alpha,1}^{H\otimes H}(h \otimes 1 \otimes 1) = (c_{H,H})_\alpha\Big(\sum h_1 \otimes h_2\Big) = \sum_i t_i h_2 \otimes s_i h_1.$$

Hence $\sum\limits_i h_1 t_i \otimes h_2 s_i = \sum\limits_i t_i h_2 \otimes s_i h_1$, and so $\sum\limits_i h_2 s_i \otimes h_1 t_i = \sum\limits_i s_i h_1 \otimes t_i h_2$. That is, $\Delta_\alpha^{\mathrm{cop}}(h)R = R\Delta_\alpha(h)$. □

Combining Theorems 4.4 and 4.5, one gets the following theorem.

Theorem 4.11. *Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. Then H is a quasitriangular semi-Hopf π-algebra if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$.*

5. Examples

In this section, we will give two examples of Hopf π-algebras, and consider the category of modules over them.

Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. Then H_1 is a usual bialgebra, and hence the category ${}_{H_1}\mathcal{M}$ of the left H_1-modules is a monoidal category as usual. Let $V \in {}_{H_1}\mathcal{M}$. For any $\alpha, \beta \in \pi$, let $M_\alpha = H_\alpha \otimes_{H_1} V$ and $\eta_{\alpha,\beta}^M = m_{\alpha,\beta} \otimes_{H_1} \mathrm{id}_V \colon H_\alpha \otimes H_\beta \otimes_{H_1} V \to H_{\alpha\beta} \otimes_{H_1} V$. Then it is easy to see that $M = \{M_\alpha\}_{\alpha\in\pi}$ is a left π-module over H with the module structure map $\eta = \{\eta_{\alpha,\beta}^M\}_{\alpha,\beta\in\pi}$. Denote M by $H \otimes_{H_1} V$. Let $f \colon U \to V$ be a left H_1-module map. Then $\mathrm{id}_H \otimes_{H_1} f = \{\mathrm{id}_{H_\alpha} \otimes_{H_1} f \colon H_\alpha \otimes_{H_1} U \to H_\alpha \otimes_{H_1} V\}_{\alpha\in\pi}$ is a left H-π-module map. Thus, we have a functor F from ${}_{H_1}\mathcal{M}$ to ${}_H\mathcal{M}$ as follows:

$$F\colon {}_{H_1}\mathcal{M} \to {}_H\mathcal{M}, \quad F(V) = H \otimes_{H_1} V, \quad F(f) = \mathrm{id}_H \otimes_{H_1} f,$$

where V is an object of ${}_{H_1}\mathcal{M}$ and f is a morphism of ${}_{H_1}\mathcal{M}$. We have another functor G from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$ as follows:

$$G\colon {}_H\mathcal{M} \to {}_{H_1}\mathcal{M}, \quad G(M) = M_1, \quad F(f) = f_1,$$

where $M = \{M_\alpha\}_{\alpha\in\pi}$ is an object of ${}_H\mathcal{M}$ and $f = \{f_\alpha\}_{\alpha\in\pi}$ is a morphism of ${}_H\mathcal{M}$. For the unit object K of the monoidal category ${}_H\mathcal{M}$ as stated in the last two sections, $G(K) = K_1 = k$ is exactly the unit object k of the monoidal category ${}_{H_1}\mathcal{M}$. For any $M, N \in {}_H\mathcal{M}$, $G(M \otimes N) = (M \otimes N)_1 = M_1 \otimes N_1 = G(M) \otimes G(N)$. Then one can easily check that G is a strict monoidal functor from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$.

For any H_1-module V, let $\theta(V)\colon GF(V) \to V$ be the canonical H_1-module isomorphism $H_1 \otimes_{H_1} V \to V$, $h\otimes v \mapsto h\cdot v$. Then one can easily check that θ is a natural isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

Example 5.1. Let π be a cyclic group of order 2 generated by α. Then, $\pi = \{1, \alpha\}$ with $\alpha^2 = 1$. Let H_1 be a 2-dimensional k-space with a k-basis $\{h_0, h_2\}$, and H_α a 2-dimensional k-space with a k-basis $\{h_1, h_3\}$. Define k-linear maps $m_{1,1}\colon H_1 \otimes H_1 \to H_1$ by $m_{1,1}(h_0 \otimes h_0) = m_{1,1}(h_2 \otimes h_2) = h_0$ and $m_{1,1}(h_0 \otimes h_2) = m_{1,1}(h_2 \otimes h_0) = h_2$; $m_{\alpha,\alpha}\colon H_\alpha \otimes H_\alpha \to H_1$ by $m_{\alpha,\alpha}(h_1 \otimes h_3) = m_{\alpha,\alpha}(h_3 \otimes h_1) = h_0$ and $m_{\alpha,\alpha}(h_1 \otimes h_1) = m_{\alpha,\alpha}(h_3 \otimes h_3) = h_2$; $m_{1,\alpha}\colon H_1 \otimes H_\alpha \to H_\alpha$ by $m_{1,\alpha}(h_0 \otimes h_1) = m_{1,\alpha}(h_2 \otimes h_3) = h_1$ and $m_{1,\alpha}(h_0 \otimes h_3) = m_{1,\alpha}(h_2 \otimes h_1) = h_3$; and $m_{\alpha,1}\colon H_\alpha \otimes H_1 \to H_\alpha$ by $m_{\alpha,1} = m_{1,\alpha}\tau_{H_\alpha,H_1}$. Define a k-linear map $u \to H_1$ by $u(\lambda) = \lambda h_0$, $\lambda \in k$. Then one can check that $H = (\{H_1, H_\alpha\}, m, u)$ is a π-algebra with $h_0 = 1$.

Define k-linear maps $\Delta_1\colon H_1 \to H_1 \otimes H_1$ by $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_1\colon H_1 \to k$ by $\varepsilon_1(h_i) = 1$, $i = 0, 2$. Then one can see that H_1 is a coalgebra. Similarly, H_α is also a coalgebra with comultiplication and counit given by $\Delta_\alpha\colon H_\alpha \to H_\alpha \otimes H_\alpha$, $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_\alpha\colon H_\alpha \to k$, $\varepsilon_\alpha(h_i) = 1$, $i = 1, 3$.

With the above structure, a straightforward verification shows that H is a semi-Hopf π-algebra. Moreover, H is a Hopf π-algebra with the antipode $S = \{S_1, S_\alpha\}$ given by

$$S_1\colon H_1 \to H_1, \quad h_0 \mapsto h_0, \quad h_2 \mapsto h_2;$$
$$S_\alpha\colon H_\alpha \to H_\alpha, \quad h_1 \mapsto h_3, \quad h_3 \mapsto h_1.$$

It is easy to see that $R = 1 \otimes 1$ is a (trivial) quasitriangular structure of H. If $\mathrm{Char}(k) \neq 2$, then H has a nontrivial quasitriangular structure as follows:

$$R = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes h_2 + h_2 \otimes 1 - h_2 \otimes h_2).$$

Now we consider the functors $F\colon {}_{H_1}\mathcal{M} \to {}_H\mathcal{M}$ and $G\colon {}_H\mathcal{M} \to {}_{H_1}\mathcal{M}$ given as above. We have already shown that G is a strict monoidal functor. Let $(\varphi_0)_1\colon$

$K_1 = k \to F(k)_1 = H_1 \otimes_{H_1} k$, $\lambda \mapsto \lambda h_0 \otimes_{H_1} 1 = 1 \otimes_{H_1} \lambda$ be the canonical k-linear isomorphism, and let $(\varphi_0)_\alpha\colon K_\alpha = k \to F(k)_\alpha = H_\alpha \otimes_{H_1} k$ be the k-linear map defined by $(\varphi_0)_\alpha(\lambda) = \lambda h_1 \otimes_{H_1} 1 = h_1 \otimes_{H_1} \lambda$. Then one can easily check that $\varphi_0 = \{(\varphi_0)_1, (\varphi_0)_\alpha\}$ is a left H-π-module isomorphism from K to $F(k)$. Let $V, W \in {}_{H_1}\mathcal{M}$. Define $\varphi_2(V,W)_1\colon (F(V) \otimes F(W))_1 \to F(V \otimes W)_1$ by

$$\varphi_2(V,W)_1((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = 1 \otimes_{H_1} (h \cdot v \otimes l \cdot w),$$
$$h, l \in H_1,\ v \in V,\ w \in W;$$

and $\varphi_2(V,W)_\alpha\colon (F(V) \otimes F(W))_\alpha \to F(V \otimes W)_\alpha$ by

$$\varphi_2(V,W)_\alpha((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = h_1 \otimes_{H_1} ((h_3 h) \cdot v \otimes (h_3 l) \cdot w),$$
$$h, l \in H_\alpha,\ v \in V,\ w \in W.$$

Then a straightforward verification shows that $\varphi_2(V,W) = \{\varphi_2(V,W)_1, \varphi_2(V,W)_\alpha\}$ is a left H-π-module isomorphism from $F(V) \otimes F(W)$ to $F(V \otimes W)$. Moreover, one can easily check that $\varphi_2(V,W)$ is a family of natural isomorphisms of left π-modules over H indexed by all couples (V,W) of objects of ${}_{H_1}\mathcal{M}$. Now by a standard verification, one can check that $(F, \varphi_0, \varphi_2)$ is a monoidal functor from ${}_{H_1}\mathcal{M}$ to ${}_H\mathcal{M}$.

We have already seen that there is a natural isomorphism $\theta\colon GF \to \mathrm{id}_{{}_{H_1}\mathcal{M}}$ as given before. It is easy to check that θ is a natural monoidal isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

Let $M = \{M_1, M_\alpha\} \in {}_H\mathcal{M}$. Let $\sigma(M)_1\colon M_1 \to FG(M)_1 = H_1 \otimes_{H_1} M_1$ be the canonical left H_1-module isomorphism, and let $\sigma(M)_\alpha\colon M_\alpha \to FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be the k-linear map defined by $\sigma(M)_\alpha(m) = h_1 \otimes_{H_1} h_3 \cdot m$, $m \in M_\alpha$. Then one can check that $\sigma(M)_\alpha$ is a bijection with the inverse given by $(\sigma(M)_\alpha)^{-1}(h \otimes m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Now by a straightforward verification, one can check that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha \in \pi}$ is a left H-π-module map, and so it is an H-π-module isomorphism. Moreover, σ is a natural isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. Then a standard verification shows that σ is a natural monoidal isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. This shows that ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories.

Finally, since H_1 is the group algebra of the cyclic group $\{1, h_2\}$ of order 2, the category ${}_{H_1}\mathcal{M}$ can be well described. When $\mathrm{Char}(k) \neq 2$, H_1 is semisimple. There are only two simple H_1-modules V_0 and V_1 in this case. V_0 and V_1 are both one-dimensional with the actions given by $h_2 \cdot v = v$ for $v \in V_0$ and $h_2 \cdot v = -v$ for $v \in V_1$. When $\mathrm{Char}(k) = 2$, there is a unique simple H_1-module V_0 as given above, and the regular module H_1 is the unique non-simple indecomposable H_1-module, which is projective and uniserial.

In order to give another example, we first give some properties of a semi-Hopf π-algebra.

Definition 5.1. Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. A family $e = \{e_\alpha\}_{\alpha\in\pi}$ of nonzero elements with $e_\alpha \in H_\alpha$ is called a generalized idempotent if $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in \pi$. Furthermore,

(1) if $e_1 = 1$, then e is called a strong generalized idempotent;

(2) if $\Delta_\alpha(e_\alpha) = e_\alpha \otimes e_\alpha$ for all $\alpha \in \pi$, then e is called a group-like generalized idempotent;

(3) if π is abelian and $e_\alpha h = h e_\alpha$ for all $\alpha, \beta \in \pi$ and $h \in H_\beta$, then e is called a central generalized idempotent.

Remark 5.2. Assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra and $e = \{e_\alpha\}_{\alpha\in\pi}$ is a generalized idempotent in H. Then the set $\{e_\alpha;\ \alpha \in \pi\}$ forms a group, which is isomorphic to π. If e is strong, then $e_\alpha e_{\alpha^{-1}} = e_{\alpha^{-1}} e_\alpha = e_1 = 1$ for all $\alpha \in \pi$. If e is group-like, then $\varepsilon_\alpha(e_\alpha) = 1$ for all $\alpha \in \pi$.

Lemma 5.3. *Assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra and that H has a strong generalized idempotent $e = \{e_\alpha\}_{\alpha\in\pi}$. Then ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent categories.*

P r o o f. We use the functors F and G given before. We have already seen that θ is a natural isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

For any $M = \{M_\alpha\}_{\alpha\in\pi} \in {}_H\mathcal{M}$ and $\alpha \in \pi$, let $\sigma(M)_\alpha\colon M_\alpha \to FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be defined by $\sigma(M)_\alpha(m) = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)$, $m \in M_\alpha$. Then it is obvious that $\sigma(M)_\alpha$ is a k-linear map. Let $\tau(M)_\alpha\colon H_\alpha \otimes_{H_1} M_1 \to M_\alpha$ be the k-linear map defined by $\tau(M)_\alpha(h \otimes_{H_1} m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Then for any $\alpha \in \pi$, $m \in M_\alpha$, $h \in H_\alpha$ and $m' \in M_1$, we have $(\tau(M)_\alpha \sigma(M)_\alpha)(m) = \tau(M)_\alpha(e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)) = e_\alpha \cdot (e_{\alpha^{-1}} \cdot m) = (e_\alpha e_{\alpha^{-1}}) \cdot m = 1 \cdot m = m$ and $(\sigma(M)_\alpha \tau(M)_\alpha)(h \otimes_{H_1} m') = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot (h \cdot m')) = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}} h) \cdot m') = e_\alpha e_{\alpha^{-1}} h \otimes_{H_1} m' = h \otimes_{H_1} m'$. This shows that $\sigma(M)_\alpha$ is a k-linear isomorphism with $(\sigma(M)_\alpha)^{-1} = \tau(M)_\alpha$, $\alpha \in \pi$. Now it is easy to see that $\tau(M) = \{\tau(M)_\alpha\}_{\alpha\in\pi}$ is a left H-π-module map, and so it is an isomorphism. It follows that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha\in\pi}$ is a left H-π-module isomorphism from M to $FG(M)$. Then it is easy to check that $\sigma(M)$ is a family of natural morphisms indexed by all objects M of ${}_H\mathcal{M}$. Therefore, σ is a natural isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. $\square$

Proposition 5.4. *Assume that π is abelian and that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra with a generalized idempotent $e = \{e_\alpha\}_{\alpha\in\pi}$. If e is a central, strong and group-like generalized idempotent, then ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories.*

P r o o f. Suppose that e is a central, strong and group-like generalized idempotent. We use the notations introduced in the proof of Lemma 5.3.

Note that the unit object of the monoidal category ${}_{H_1}\mathcal{M}$ is the trivial H_1-module k with the action given by $h \cdot 1 = \varepsilon_1(h)$, where $h \in H_1$. Hence $F(k) = H \otimes_{H_1} k = \{H_\alpha \otimes_{H_1} k\}_{\alpha\in\pi}$. For any $\alpha \in \pi$, $H_\alpha = (e_\alpha e_{\alpha^{-1}})H_\alpha = e_\alpha(e_{\alpha^{-1}}H_\alpha) \subseteq e_\alpha H_1 \subseteq H_\alpha$, and hence $H_\alpha = e_\alpha H_1$. It follows that H_α is a free right H_1-module of rank one with an H_1-basis e_α, since $e_{\alpha^{-1}}e_\alpha = 1$. Therefore, $H_\alpha \otimes_{H_1} k$ is a one-dimensional k-vector space with the k-basis $e_\alpha \otimes_{H_1} 1$. Thus, there is a k-linear isomorphism $(\varphi_0)_\alpha\colon K_\alpha = k \to H_\alpha \otimes_{H_1} k$, $\lambda \mapsto \lambda e_\alpha \otimes_{H_1} 1 = e_\alpha \otimes_{H_1} \lambda$ for any $\alpha \in \pi$. Now let $\alpha, \beta \in \pi$, $h \in H_\alpha$ and $\lambda \in K_\beta = k$. Then $h \cdot (\varphi_0)_\beta(\lambda) = h \cdot (e_\beta \otimes_{H_1} \lambda) = (e_\beta h) \otimes_{H_1} \lambda = (e_{\alpha\beta}e_{\alpha^{-1}}h) \otimes_{H_1} \lambda = e_{\alpha\beta} \otimes_{H_1} (e_{\alpha^{-1}}h) \cdot \lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_1(e_{\alpha^{-1}}h)\lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_{\alpha^{-1}}(e_{\alpha^{-1}})\varepsilon_\alpha(h)\lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_\alpha(h)\lambda = (\varphi_0)_{\alpha\beta}(\varepsilon_\alpha(h)\lambda) = (\varphi_0)_{\alpha\beta}(h \cdot \lambda)$. Thus, φ_0 is a left H-π-module isomorphism from K to $F(k)$.

Let $U, V \in {}_{H_1}\mathcal{M}$ and $\alpha \in \pi$. Define $\varphi_2(U,V)_\alpha\colon (F(U) \otimes F(V))_\alpha \to F(U \otimes V)_\alpha$ by

$$\varphi_2(U,V)_\alpha((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v)) = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v),$$

where $h, l \in H_\alpha$, $x \in U$ and $v \in V$. Since H_α is a free right H_1-module of rank one with an H_1-basis e_α as stated before, it is easy to check that $\varphi_2(U,V)_\alpha$ is a k-linear isomorphism. Let $h, l \in H_\alpha$, $y \in H_\beta$ with $\alpha, \beta \in \pi$, $x \in U$ and $v \in V$. Then

$$\begin{aligned}
&y \cdot \varphi_2(U,V)_\alpha((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v))\\
&\quad = ye_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)\\
&\quad = e_{\beta\alpha}e_{\beta^{-1}}y \otimes_{H_1} ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)\\
&\quad = e_{\beta\alpha} \otimes_{H_1} (e_{\beta^{-1}}y) \cdot ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)\\
&\quad = \sum e_{\beta\alpha} \otimes_{H_1} (((e_{\beta^{-1}}y)_1 e_{\alpha^{-1}}h) \cdot x \otimes ((e_{\beta^{-1}}y)_2 e_{\alpha^{-1}}l) \cdot v)\\
&\quad = \sum e_{\beta\alpha} \otimes_{H_1} ((e_{\beta^{-1}}y_1 e_{\alpha^{-1}}h) \cdot x \otimes (e_{\beta^{-1}}y_2 e_{\alpha^{-1}}l) \cdot v)\\
&\quad = \sum e_{\beta\alpha} \otimes_{H_1} ((e_{(\beta\alpha)^{-1}}y_1 h) \cdot x \otimes (e_{(\beta\alpha)^{-1}}y_2 l) \cdot v)\\
&\quad = \varphi_2(U,V)_{\beta\alpha}\Big(\sum (y_1 h \otimes_{H_1} x) \otimes (y_2 l \otimes_{H_1} v)\Big)\\
&\quad = \varphi_2(U,V)_{\beta\alpha}(y \cdot ((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v))).
\end{aligned}$$

It follows that $\varphi_2(U,V)$ is a left H-π-module isomorphism. A straightforward verification shows that $\varphi_2(U,V)$ is a family of natural isomorphisms of left H-π-modules indexed by all couples (U,V) of objects of ${}_{H_1}\mathcal{M}$.

Let $U, V, W \in {}_{H_1}\mathcal{M}$ and $\alpha \in \pi$. For any $h, l, s \in H_\alpha$, $x \in U$, $v \in V$ and $w \in W$, we have

$$\begin{aligned}
&(\varphi_2(U, V \otimes W)_\alpha(\mathrm{id}_{F(U)_\alpha} \otimes \varphi_2(V,W)_\alpha)a_\alpha)(((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v)) \otimes (s \otimes_{H_1} w))\\
&\quad = (\varphi_2(U, V \otimes W)_\alpha(\mathrm{id}_{F(U)_\alpha} \otimes \varphi_2(V,W)_\alpha))((h \otimes_{H_1} x) \otimes ((l \otimes_{H_1} v) \otimes (s \otimes_{H_1} w)))\\
&\quad = \varphi_2(U, V \otimes W)_\alpha((h \otimes_{H_1} x) \otimes (e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}l) \cdot v \otimes (e_{\alpha^{-1}}s) \cdot w)))
\end{aligned}$$

$$= e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h) \cdot x \otimes ((e_{\alpha^{-1}}e_\alpha) \cdot ((e_{\alpha^{-1}}l) \cdot v \otimes (e_{\alpha^{-1}}s) \cdot w)))$$
$$= e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h) \cdot x \otimes ((e_{\alpha^{-1}}l) \cdot v \otimes (e_{\alpha^{-1}}s) \cdot w))$$

and

$$\begin{aligned}
&(F(a)_\alpha\varphi_2(U\otimes V, W)_\alpha(\varphi_2(U,V)_\alpha \otimes \mathrm{id}_{F(W)_\alpha}))(((h\otimes_{H_1} x)\otimes(l\otimes_{H_1} v))\otimes(s\otimes_{H_1} w))\\
&\quad = (F(a)_\alpha\varphi_2(U\otimes V,W)_\alpha)((e_\alpha\otimes_{H_1}((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v))\otimes(s\otimes_{H_1} w))\\
&\quad = F(a)_\alpha(e_\alpha\otimes_{H_1}((e_{\alpha^{-1}}e_\alpha)\cdot((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v)\otimes(e_{\alpha^{-1}}s)\cdot w))\\
&\quad = F(a)_\alpha(e_\alpha\otimes_{H_1}(((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v)\otimes(e_{\alpha^{-1}}s)\cdot w))\\
&\quad = e_\alpha\otimes_{H_1}((e_{\alpha^{-1}}h)\cdot x\otimes((e_{\alpha^{-1}}l)\cdot v\otimes(e_{\alpha^{-1}}s)\cdot w)).
\end{aligned}$$

Therefore, for any objects U, V, W of ${}_{H_1}\mathcal{M}$, we have

$$\begin{aligned}
&\varphi_2(U, V\otimes W)(\mathrm{id}_{F(U)}\otimes\varphi_2(V,W))a_{F(U),F(V),F(W)}\\
&\qquad\qquad = F(a_{U,V,W})\varphi_2(U\otimes V,W)(\varphi_2(U,V)\otimes\mathrm{id}_{F(W)}).
\end{aligned}$$

For any $h \in H_\alpha$, $v \in V$ and $\lambda \in K_\alpha = k$ with $\alpha \in \pi$, we have

$$\begin{aligned}
(F(l_V)_\alpha\varphi_2(k,V)_\alpha&((\varphi_0)_\alpha\otimes\mathrm{id}_{F(V)_\alpha}))(\lambda\otimes(h\otimes_{H_1} v))\\
&= (F(l_V)_\alpha\varphi_2(k,V)_\alpha)((e_\alpha\otimes_{H_1}\lambda)\otimes(h\otimes_{H_1} v))\\
&= F(l_V)_\alpha(e_\alpha\otimes_{H_1}((e_{\alpha^{-1}}e_\alpha)\cdot\lambda\otimes(e_{\alpha^{-1}}h)\cdot v))\\
&= F(l_V)_\alpha(e_\alpha\otimes_{H_1}(\lambda\otimes(e_{\alpha^{-1}}h)\cdot v))\\
&= e_\alpha\otimes_{H_1}(\lambda(e_{\alpha^{-1}}h)\cdot v)\\
&= e_\alpha\lambda e_{\alpha^{-1}}h\otimes_{H_1} v\\
&= \lambda(h\otimes_{H_1} v)\\
&= (l_{F(V)})_\alpha(\lambda\otimes(h\otimes_{H_1} v)).
\end{aligned}$$

Hence $F(l_V)\varphi_2(k,V)(\varphi_0\otimes\mathrm{id}_{F(V)}) = l_{F(V)}$ for any object V of ${}_{H_1}\mathcal{M}$. Similarly, one can show that $F(r_V)\varphi_2(V,k)(\mathrm{id}_{F(V)}\otimes\varphi_0) = r_{F(V)}$ for any object V of ${}_{H_1}\mathcal{M}$. Thus, we have proved that (F,φ_0,φ_2) is a monoidal functor.

Note that G is a strict monoidal functor from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$ as stated before.

Finally, a straightforward verification shows that θ is a natural monoidal isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$, and σ is a natural monoidal isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. Hence ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories. □

Example 5.2. Assume that $\mathrm{Char}(k) \neq 2$. Let π be any group. For any $\alpha \in \pi$, let H_α be a 4-dimensional vector space with a k-basis $\{e_\alpha, g_\alpha, h_\alpha, x_\alpha\}$. Define k-linear maps $\Delta_\alpha\colon H_\alpha \to H_\alpha \otimes H_\alpha$ and $\varepsilon_\alpha\colon H_\alpha \to k$ by

$$\begin{gathered}
\Delta_\alpha(e_\alpha) = e_\alpha\otimes e_\alpha,\qquad \Delta_\alpha(h_\alpha) = h_\alpha\otimes g_\alpha + e_\alpha\otimes h_\alpha,\\
\Delta_\alpha(g_\alpha) = g_\alpha\otimes g_\alpha,\qquad \Delta_\alpha(x_\alpha) = x_\alpha\otimes e_\alpha + g_\alpha\otimes x_\alpha,\\
\varepsilon_\alpha(e_\alpha) = \varepsilon_\alpha(g_\alpha) = 1,\qquad \varepsilon_\alpha(h_\alpha) = \varepsilon_\alpha(x_\alpha) = 0.
\end{gathered}$$

Then a straightforward verification shows that $(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)$ is a coalgebra over k for any $\alpha \in \pi$.

For any $\alpha, \beta \in \pi$, define a k-linear map $m_{\alpha,\beta}\colon H_\alpha \otimes H_\alpha \to H_{\alpha\beta}$ by

$$\begin{aligned}
&e_\alpha e_\beta = e_{\alpha\beta}, && e_\alpha g_\beta = g_{\alpha\beta}, && e_\alpha h_\beta = h_{\alpha\beta}, && e_\alpha x_\beta = x_{\alpha\beta},\\
&g_\alpha e_\beta = g_{\alpha\beta}, && g_\alpha g_\beta = e_{\alpha\beta}, && g_\alpha h_\beta = x_{\alpha\beta}, && g_\alpha x_\beta = h_{\alpha\beta},\\
&h_\alpha e_\beta = h_{\alpha\beta}, && h_\alpha g_\beta = -x_{\alpha\beta}, && h_\alpha h_\beta = 0, && h_\alpha x_\beta = 0,\\
&x_\alpha e_\beta = x_{\alpha\beta}, && x_\alpha g_\beta = -h_{\alpha\beta}, && x_\alpha h_\beta = 0, && x_\alpha x_\beta = 0,
\end{aligned}$$

where we denote $m_{\alpha,\beta}(y \otimes z)$ by yz for any $y \in H_\alpha$ and $z \in H_\beta$. Then define a k-linear map $u\colon k \to H_1$ by $u(1) = e_1$. A tedious but standard verification shows that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a π-algebra with $e_1 = 1$. Moreover, one can check that H is a semi-Hopf π-algebra.

For any $\alpha \in \pi$, define a k-linear map $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ by $S_\alpha(e_\alpha) = e_{\alpha^{-1}}$, $S_\alpha(g_\alpha) = g_{\alpha^{-1}}$, $S_\alpha(h_\alpha) = x_{\alpha^{-1}}$ and $S_\alpha(x_\alpha) = -h_{\alpha^{-1}}$. Then one can check that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u, S)$ is a Hopf π-algebra.

For any $\lambda \in k$, let

$$\begin{aligned}
R_\lambda = {}&\tfrac{1}{2}(1\otimes 1 + 1\otimes g_1 + g_1\otimes 1 - g_1\otimes g_1)\\
&+ \tfrac{1}{2}\lambda(x_1\otimes x_1 - x_1\otimes h_1 + h_1\otimes x_1 + h_1\otimes h_1).
\end{aligned}$$

Then one can check that R_λ is a quasitriangular structure of H for any $\lambda \in k$.

Let $e = \{e_\alpha\}_{\alpha\in\pi}$. Then e is a strong group-like generalized idempotent. Now assume that π is abelian. Then e is central. It follows from Proposition 5.4 that ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories. Thus, in order to describe the left π-modules over H, we only need to describe the left H_1-modules.

Note that H_1 is a usual Hopf algebra, which is generated, as an algebra, by g_1 and h_1. Algebra H_1 is isomorphic, as a Hopf algebra, to Sweedler's 4-dimensional Hopf algebra. Hence there are only 4 non-isomorphic finite-dimensional indecomposable modules V_0, V_1, U_0 and U_1. Modules V_0 and V_1 are both one-dimensional with the actions given by $g_1 \cdot v = (-1)^i v$ and $h_1 \cdot v = 0$ for all $v \in V_i$, where $i = 0, 1$. Modules U_0 and U_1 are both 2-dimensional. The matrix representation $\varrho_i\colon H_1 \to M_2(k)$ corresponding to U_i is given by

$$\varrho_i(g_1) = \begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^{i-1} \end{pmatrix}, \quad \varrho_i(h_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $i = 0, 1$. Moreover, U_0 and U_1 are both projective and uniserial. For details, one can see [2] and [3].

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