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n-FLAT AND n-FP-INJECTIVE MODULES

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Abstract. In this paper, we study the existence of the n-flat preenvelope and the n-FP-injective cover. We also characterize n-coherent rings in terms of the n-FP-injective and n-flat modules.

Keywords: n-flat module, n-FP-injective module, n-coherent ring, cotorsion theory

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1. Introduction

We use R-Mod (resp., Mod-R) to denote the category of all left (resp., right) R-modules. For any R-module M, $\mathrm{pd}_R M$ (resp., $\mathrm{id}_R M$, $\mathrm{fd}_R M$) denotes the projective (resp., injective, flat) dimension. The character module $\mathrm{Hom}_Z(M, Q/Z)$ is denoted by M^+.

Coherent rings have been characterized in various ways. The deepest result is the one due to Chase [2] which claims that the ring R is left coherent if and only if products of flat right R-modules are again flat if and only if products of copies of R are flat right R-modules. Lee [6] introduced the notions of left n-coherent and n-coherent rings and characterized them in various ways, using n-flat and n-FP-injective modules. In this paper we continue the study of n-coherent rings.

A ring R is called left n-coherent (for integers $n > 0$ or $n = \infty$) if every finitely generated submodule of a free left R-module whose projective dimension is $\leqslant n-1$ is finitely presented. Accordingly, all rings are left 1-coherent, and the left coherent rings are exactly those which are d-coherent (d denotes the left global dimension of R). In particular, left ∞-coherent rings are left coherent. A right R-module M will

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be called n-flat if $\operatorname{Tor}_1^R(M,N)=0$ holds for all finitely presented left R-modules N with $\operatorname{pd}_R N \leqslant n$. A left R-module A is said to be n-FP-injective if $\operatorname{Ext}_R^1(N,A)=0$ holds for all finitely presented left R-modules N of projective dimension $\leqslant n$.

Given a class $\mathscr{C}$ of R-modules, let ${}^{\perp}\mathscr{C}$ be the class of R-modules F such that $\operatorname{Ext}_R^1(F,C)=0$ for every $C\in\mathscr{C}$ and let $\mathscr{C}^{\perp}$ be the class of R-modules F such that $\operatorname{Ext}_R^1(C,F)=0$ for every $C\in\mathscr{C}$. A pair of classes of R-modules $(\mathscr{F},\mathscr{C})$ is called a cotorsion theory if $\mathscr{F}^{\perp}=\mathscr{C}$ and ${}^{\perp}\mathscr{C}=\mathscr{F}$. A cotorsion theory is said to be complete if for every R-module M there is an exact sequence $0\to C\to F\to M\to 0$ such that $C\in\mathscr{C}$ and $F\in\mathscr{F}$. A cotorsion theory is said to be perfect if every R-module has an $\mathscr{F}$-cover and a $\mathscr{C}$-envelope. A cotorsion theory is said to be hereditary if $0\to F'\to F\to F''\to 0$ is exact with $F,F''\in\mathscr{F}$, then $F'\in\mathscr{F}$.

We recall that given a class of R-modules $\mathscr{T}$, a morphism $\varphi\colon T\to M$ where $T\in\mathscr{T}$ is called a $\mathscr{T}$-cover of M if the following conditions hold:

(1) For any linear map $\varphi'\colon T'\to M$ with $T'\in\mathscr{T}$, there exists a linear map $f\colon T'\to T$ with $\varphi'=\varphi f$, or equivalently, $\operatorname{Hom}_R(T',T)\to\operatorname{Hom}_R(T',M)\to 0$ is exact for any $T'\in\mathscr{T}$.

(2) If f is an endomorphism of T with $\varphi=\varphi f$, then f must be an automorphism.

If (1) holds (and perhaps not (2)), $\varphi\colon T\to M$ is called a $\mathscr{T}$-precover. A $\mathscr{T}$-envelope and $\mathscr{T}$-preenvelope are defined dually.

2. n-flat and n-FP-injective modules

Let n be a non-negative integer. In what follows, $\mathscr{F}_n$ stands for the class of all n-flat right R-modules and $\mathscr{F}\mathscr{I}_n$ denotes the class of all n-FP-injective left R-modules.

Proposition 2.1. *$\mathscr{F}_n$ and $\mathscr{F}\mathscr{I}_n$ are closed under pure submodules.*

Proof. Let $B\in\mathscr{F}_n$ and let $A\subseteq B$ be a pure submodule. Then $0\to(B/A)^+\to B^+\to A^+\to 0$ is split and B^+ is n-FP-injective by [6, Lemma 5], and so A is n-flat by [6, Lemma 5].

Let $M\in\mathscr{F}\mathscr{I}_n$, let S be a pure submodule of M and let N be any finitely presented left R-module with $\operatorname{pd}_R N\leqslant n$. Then we can get an induced exact sequence

$$0\longrightarrow\operatorname{Hom}_R(N,S)\longrightarrow\operatorname{Hom}_R(N,M)\longrightarrow\operatorname{Hom}_R(N,M/S)\longrightarrow 0,$$

and so $\operatorname{Ext}_R^1(N,S)=0$ since $\operatorname{Ext}_R^1(N,M)=0$. It follows that $S\in\mathscr{F}\mathscr{I}_n$. □

Lemma 2.1. *The following conditions are equivalent:*

(1) *M is n-FP-injective if and only if $\operatorname{Ext}_R^1(R/I,M)=0$ for any finitely generated left ideal I with $\operatorname{pd}_R I\leqslant n-1$;*

(2) *N is n-flat if and only if* $\operatorname{Tor}_1^R(N, R/I) = 0$ *for any finitely generated left ideal I with* $\operatorname{pd}_R I \leqslant n-1$.

P r o o f. (1) "$\Rightarrow$" is trivial.

"$\Leftarrow$" Let L be any finitely presented left R-module with $\operatorname{pd}_R L \leqslant n$. Then there is an exact sequence $0 \to A \to R^n \to L \to 0$ for some $n \geqslant 0$ and $A \subseteq R^n$ finitely generated with $\operatorname{pd}_R A \leqslant n-1$. Consider the following pullback of $A \to R^n$ and $R \to R^n$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & R^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & R^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $L \cong R/\operatorname{Im} f$ and $\operatorname{Im} f \cong B$ is finitely generated with $\operatorname{pd}_R \operatorname{Im} f \leqslant n-1$. Thus $\operatorname{Ext}_R^1(L, M) \cong \operatorname{Ext}_R^1(R/\operatorname{Im} f, M) = 0$, which gives that M is n-FP-injective.

(2) By analogy with the proof of (1). □

Theorem 2.1. *Let n be a non-negative integer and R a ring. Then*

(1) $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ *is a perfect cotorsion theory;*

(2) $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ *is a complete cotorsion theory.*

P r o o f. (1) Let $\operatorname{Card}(R) \leqslant \aleph_\beta$ and $F \in \mathscr{F}_n$. Then we can write F as a union of a continuous chain $(F_\alpha)_{\alpha<\lambda}$ of pure submodules of F such that $\operatorname{Card}(F_0) \leqslant \aleph_\beta$ and $\operatorname{Card}(F_{\alpha+1}/F_\alpha) \leqslant \aleph_\beta$ whenever $\alpha+1 < \lambda$. If N is a right R-module such that $\operatorname{Ext}_R^1(F_0, N) = 0$ and $\operatorname{Ext}_R^1(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\operatorname{Ext}_R^1(F, N) = 0$ by [5, Theorem 7.3.4]. Since F_α is a pure submodule of F for any $\alpha < \lambda$, we have $F_\alpha \in \mathscr{F}_n$ by Proposition 2.1. On the other hand, F_α is a pure submodule of $F_{\alpha+1}$ whenever $\alpha+1<\lambda$, hence $F_{\alpha+1}/F_\alpha \in \mathscr{F}_n$ by Proposition 2.1. Let X be a set of representatives of all modules $G \in \mathscr{F}_n$ with $\operatorname{Card}(G) \leqslant \aleph_\beta$. Then $\mathscr{F}_n^{\perp} = X^{\perp}$. So $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ is a cotorsion theory by [1, Corollary 2.13]. Since $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ is cogenerated by the set X, $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ is a complete cotorsion theory by [5, Theorem 7.4.1]. Moreover, $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ is a perfect cotorsion theory by [5, Theorem 7.2.6] since $\mathscr{F}_n$ is closed under direct limits.

(2) Let $X \in ({}^{\perp}\mathscr{F}\mathscr{I}_n)^{\perp}$ and let N be finitely presented with $\operatorname{pd}_R N \leqslant n$. Then $N \in {}^{\perp}\mathscr{F}\mathscr{I}_n$. So $\operatorname{Ext}^1_R(N, X) = 0$, which gives that $X \in \mathscr{F}\mathscr{I}_n$ and $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ is a cotorsion theory. By Lemma 2.1, M is n-FP-injective if and only if $\operatorname{Ext}^1_R(R/A, M) = 0$ for any finitely generated $A \subseteq R$ with $\operatorname{pd}_R A \leqslant n-1$. Set $X = \oplus R/A$, where the sum is over all finitely generated left ideals A of R with $\operatorname{pd}_R A \leqslant n-1$. Then $\mathscr{F}\mathscr{I}_n = X^{\perp}$. So $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ is a complete cotorsion theory by [5, Theorem 7.4.1]. □

3. n-coherent rings

In this section we characterize n-coherent rings in terms of the n-FP-injective and n-flat modules. We obtain some characterizations of the situation when every R-module has a monic $\mathscr{F}_n$-preenvelope and an epic $\mathscr{F}_n$-preenvelope.

Theorem 3.1. *For a ring R and any n $(0 < n \leqslant \infty)$, the following conditions are equivalent:*

(1) *R is left n-coherent;*
(2) *every right R-module has an $\mathscr{F}_n$-preenvelope;*
(3) *any direct limit of n-FP-injective left R-modules is n-FP-injective;*
(4) *$\operatorname{Ext}^1_R(N, \varinjlim M_i) \to \varinjlim \operatorname{Ext}^1_R(N, M_i)$ is an isomorphism for any finitely presented left R-module N with $\operatorname{pd}_R N \leqslant n$ and any direct system $(M_i)_{i\in I}$ of left R-modules;*
(5) *$\mathscr{F}\mathscr{I}_n$ is a coresolving subcategory;*
(6) *$({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ is a hereditary cotorsion theory.*

Proof. (1) ⇒ (4) By [3, Lemma 2.9 (2)]; (4) ⇒ (3) and (5) ⇒ (6) are obvious.

(1) ⇒ (2) Let N be any right R-module. Then there is a cardinal number $\aleph_\alpha$ such that for any homomorphism $f\colon N \to L$ with L n-flat, there is a pure submodule Q of L such that $\operatorname{Card}(Q) \leqslant \aleph_\alpha$ and $f(N) \subseteq Q$. Note that Q is n-flat by Proposition 2.1 and $\mathscr{F}_n$ is closed under products by [6, Theorem 5], and so N has an $\mathscr{F}_n$-preenvelope by [5, Proposition 6.2.1].

(2) ⇒ (1) Let $(F_i)_{i\in I}$ be a family of n-flat right R-modules and let $\prod_{i\in I} F_i \to F$ be an $\mathscr{F}_n$-preenvelope. Then there are factorizations $\prod_{i\in I} F_i \to F \to F_j$, where $\prod_{i\in I} F_i \to F_j$ is the canonical projection for each j. This gives rise to a map $F \to \prod_{i\in I} F_i$ with the composition $\prod_{i\in I} F_i \to F \to \prod_{i\in I} F_i$ being the identity. Hence $\prod_{i\in I} F_i$ is isomorphic to a summand of F, and so $\prod_{i\in I} F_i$ is n-flat, which implies that R is left n-coherent.

(3) ⇒ (1) Let K be a finitely generated submodule of a free left R-module F whose projective dimension is $\leqslant n-1$. Consider the exact sequence $0 \to K \to F \to$

$F/K \to 0$. Then F/K is finitely presented and $\mathrm{pd}_R\, F/K \leqslant n$. So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F/K, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(F, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(K, \varinjlim M_i) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\varinjlim \mathrm{Hom}_R(F/K, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(K, M_i) & \longrightarrow & 0
\end{array}$$

Since α and β are isomorphisms, γ is an isomorphism by Five lemma. Thus K is finitely presented.

(1) $\Rightarrow$ (5) Let N be a finitely presented left R-module with $\mathrm{pd}_R\, N \leqslant n$ and let $0 \to A \to B \to C \to 0$ be exact in R-Mod with A, $B \in \mathscr{F}\mathscr{I}_n$. Then

$$0 = \mathrm{Ext}^1_R(N, B) \longrightarrow \mathrm{Ext}^1_R(N, C) \longrightarrow \mathrm{Ext}^2_R(N, A) = 0$$

by [6, Theorem 1], and so $C \in \mathscr{F}\mathscr{I}_n$. Thus $\mathscr{F}\mathscr{I}_n$ is a coresolving subcategory.

(6) $\Rightarrow$ (1) Let S be a finitely generated submodule of a free left R-module F whose projective dimension is $\leqslant n-1$. We need to prove that S is finitely presented. Let M be FP-injective and let $0 \to M \to E \to C \to 0$ be exact with E injective. Then $M \in \mathscr{F}\mathscr{I}_n$ and $C \in \mathscr{F}\mathscr{I}_n$, and so

$$\mathrm{Ext}^1_R(S, M) \cong \mathrm{Ext}^2_R(F/S, M) \cong \mathrm{Ext}^1_R(F/S, C) = 0.$$

Thus S is finitely presented, which means that R is left n-coherent. □

Proposition 3.1. *The following conditions are equivalent:*

(1) *R is a left n-coherent ring;*

(2) *$\mathrm{Ext}^1_R(I, N) = 0$ for any FP-injective left R-module N and any finitely generated left ideal I with $pd_R I \leqslant n-1$;*

(3) *$\mathrm{Ext}^2_R(R/I, N) = 0$ for any FP-injective left R-module N and any finitely generated left ideal I with $\mathrm{pd}_R\, I \leqslant n-1$;*

(4) *if $0 \to N \to M \to L \to 0$ is an exact sequence of left R-modules with N FP-injective and M n-FP-injective, then L is n-FP-injective.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let N be an FP-injective left R-module and I a finitely generated left ideal with $\mathrm{pd}_R\, I \leqslant n-1$. Then the exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to the exact sequence

$$0 = \mathrm{Ext}^1_R(I, N) \longrightarrow \mathrm{Ext}^2_R(R/I, N) \longrightarrow \mathrm{Ext}^2_R(R, N) = 0$$

by (2). Thus $\mathrm{Ext}^2_R(R/I, N) = 0$.

(3) $\Rightarrow$ (4) Let I be a finitely generated left ideal of R with $\operatorname{pd}_R I \leqslant n-1$. The exact sequence $0 \to N \to M \to L \to 0$ induces the exactness of

$$0 = \operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, L) \longrightarrow \operatorname{Ext}^2_R(R/I, N) = 0$$

by (3), and hence $\operatorname{Ext}^1_R(R/I, L) = 0$. That is, L is n-FP-injective by Lemma 2.1.

(4) $\Rightarrow$ (1) Let I be a finitely generated left ideal with $\operatorname{pd}_R I \leqslant n-1$. For any FP-injective left R-module N, there is an exact sequence $0 \to N \to E \to E/N \to 0$ with E injective. Note that E/N is n-FP-injective by (4). Hence we get the exact sequence

$$0 = \operatorname{Ext}^1_R(R/I, E/N) \longrightarrow \operatorname{Ext}^2_R(R/I, N) \longrightarrow \operatorname{Ext}^2_R(R/I, E) = 0,$$

and so $\operatorname{Ext}^1_R(I, N) \cong \operatorname{Ext}^2_R(R/I, N) = 0$. It follows that I is finitely presented. Therefore R is left n-coherent. $\square$

Lemma 3.1. *Let R be a left n-coherent ring and let $|M| = \lambda$ for a left R-module M. Let k be as in El Bashir's result. Then any map $A \to M$ with A n-FP-injective can be factored through an n-FP-injective left R-module B with $|B| < k$.*

Proof. Consider any homomorphism $A \to M$ with A n-FP-injective. If $|A| < k$, let $B = A$. So suppose $|A| \geqslant k$. Consider a submodule $S \subseteq A$ maximal with respect to the two properties that S is pure in A and that $S \subseteq \operatorname{Ker}(A \to M)$. Let $B = A/S$. Then B is n-FP-injective by Theorem 3.1. We wish to argue that $|B| < k$. Let K be the kernel of $B \to M$. Then $|B/K| \leqslant |M| = \lambda$. So if $|B| \geqslant k$, there is a nonzero pure submodule T/S of B contained in K. But then T is pure in A and is contained in the kernel of $A \to M$. This contradicts the choice of S. $\square$

Theorem 3.2. *Let R be a left n-coherent ring. Then every left R-module has an $\mathscr{F}\mathscr{I}_n$-cover.*

Proof. By Lemma 3.1 and [5, Proposition 5.2.2 and Corollary 5.2.7]. $\square$

Proposition 3.2. *Let R be left n-coherent. Then the following conditions are equivalent:*

(1) *every left R-module has an n-FP-injective cover with the unique mapping property (see [4]);*
(2) *for every left R-modules exact sequence $A \to B \to C \to 0$ with A and B n-FP-injective, C is n-FP-injective.*

P r o o f. (1) $\Rightarrow$ (2) Let $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be exact in R-Mod with A, B n-FP-injective and $\theta\colon H \to C$ an n-FP-injective cover with the unique mapping property. Then there exists a map $\delta\colon B \to H$ such that $g = \theta\delta$. Thus $\theta\delta f = gf = 0 = \theta 0$, and hence $\delta f = 0$, which implies that $\operatorname{Ker} g = \operatorname{Im} f \subseteq \operatorname{Ker} \delta$. Therefore there is a morphism $\gamma\colon C \to H$ such that $\gamma g = \delta$, and so $\theta\gamma g = \theta\delta = g$, which gives that $\theta\gamma = 1_C$. Thus C is isomorphic to a direct summand of H, and so C is n-FP-injective.

(2) $\Rightarrow$ (1) Let M be any left R-module. Then M has an n-FP-injective cover $f\colon L \to M$ by Theorem 3.2. It is enough to show that for any n-FP-injective left R-module G and any homomorphism $g\colon G \to L$ such that $fg = 0$, we have $g = 0$. In fact, there is a homomorphism $\beta\colon L/\operatorname{Im} g \to M$ such that $\beta\pi = f$, where $\pi\colon L \to L/\operatorname{Im} g$ is the natural map. Since $L/\operatorname{Im} g$ is n-FP-injective, there is a map $\alpha\colon L/\operatorname{Im} g \to L$ such that $\beta = f\alpha$, and so $f\alpha\pi = f$. Hence $\alpha\pi$ is an isomorphism. Therefore π is monic and $g = 0$. □

Proposition 3.3. *The following conditions are equivalent:*

(1) $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ *is a hereditary cotorsion theory;*

(2) *R is left n-coherent and* $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ *is a hereditary cotorsion theory;*

(3) $\operatorname{Ext}^2_R(R/I, M) = 0$ *for any finitely generated left ideal I with* $\operatorname{pd}_R I \leqslant n-1$ *and any n-FP-injective left R-module M;*

(4) *R is left n-coherent and* $\operatorname{Tor}_2^R(N, R/I) = 0$ *for any finitely generated left ideal I with* $\operatorname{pd}_R I \leqslant n-1$ *and any n-flat right R-module N.*

P r o o f. (1) $\Rightarrow$ (2) Since $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ is hereditary, R is left n-coherent by Theorem 3.1. On the other hand, let $0 \to A \to B \to C \to 0$ be exact with B, $C \in \mathscr{F}_n$. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is exact with $B^+, C^+ \in \mathscr{F}\mathscr{I}_n$ by [6, Theorem 3], and so $A^+ \in \mathscr{F}\mathscr{I}_n$ by (1), which implies that $A \in \mathscr{F}_n$. That is, $(\mathscr{F}_n, \mathscr{F}_n^{\perp})$ is hereditary.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) By [6, Theorem 1], (4) $\Rightarrow$ (2). It is easy.

(2) $\Rightarrow$ (4) Let $N \in \mathscr{F}_n$ and let $0 \to K \to P \to N \to 0$ be exact with P projective. Then $K \in \mathscr{F}_n$ by (2), and hence $\operatorname{Tor}_2^R(N, R/I) \cong \operatorname{Tor}_1^R(K, R/I) = 0$ for any finitely generated left ideal I with $\operatorname{pd}_R I \leqslant n-1$. □

Proposition 3.4. *The following conditions are equivalent for a left n-coherent ring R:*

(1) *every n-flat right R-module is flat;*

(2) *every cotorsion right R-module belongs to* $\mathscr{F}_n^{\perp}$*;*

(3) *every n-FP-injective left R-module is FP-injective;*

(4) *every finitely presented left R-module belongs to* $\mathscr{F}\mathscr{I}_n^{\perp}$.

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow from Theorem 2.1.

(1) $\Rightarrow$ (3) Let M be any n-FP-injective left R-module. Then M^+ is n-flat, and so M^+ is flat by (1). On the other hand, for any finitely presented left R-module N, there is an exact sequence

$$\mathrm{Tor}_1^R(M^+, N) \longrightarrow \mathrm{Ext}_R^1(N, M)^+ \longrightarrow 0$$

by [3, Lemma 2.7 (1)]. Thus $\mathrm{Ext}_R^1(N, M) = 0$, and so M is FP-injective.

(3) $\Rightarrow$ (1) Let M be an n-flat right R-module. Then M^+ is n-FP-injective, and so M^+ is FP-injective by (3). Hence M is flat. □

Now we study when every right R-module has a monic $\mathscr{F}_n$-preenvelope and an epic $\mathscr{F}_n$-preenvelope.

Proposition 3.5. *The following conditions are equivalent:*

(1) *every right R-module has a monic $\mathscr{F}_n$-preenvelope;*

(2) *R is left n-coherent and every flat left R-module is n-FP-injective;*

(3) *R is left n-coherent and ${}_RR$ is n-FP-injective;*

(4) *R is left n-coherent and $(\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n^{\perp})$ is a perfect cotorsion theory;*

(5) *R is left n-coherent and every left R-module has an epic $\mathscr{F}\mathscr{I}_n$-cover.*

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2) Let M be a flat left R-module. Then M^+ is injective and M^+ has a monic $\mathscr{F}_n$-preenvelope $\varphi\colon M^+ \to F$. Set $C = \mathrm{Coker}\,\varphi$. Then $0 \to M^+ \to F \to C \to 0$ is split, and so $C \in \mathscr{F}_n$, which gives that $M^+ \in \mathscr{F}_n$ since R is left n-coherent. Thus M is n-FP-injective.

(3) $\Rightarrow$ (4) By analogy with the proof of Theorem 2.1.

(5) $\Rightarrow$ (1) Let M be any right R-module. Then M has an epic $\mathscr{F}\mathscr{I}_n$-cover $E \to M^+$ by (5), and so there is a monomorphism $M \to E^+$. Thus every right R-module has a monic $\mathscr{F}_n$-preenvelope by Theorem 3.1. □

Proposition 3.6. *The following conditions are equivalent:*

(1) *every right R-module has an epic $\mathscr{F}_n$-preenvelope;*

(2) *R is left n-coherent and every submodule of any n-flat right R-module is n-flat;*

(3) *every quotient module of any n-FP-injective left R-module is n-FP-injective;*

(4) *every left R-module has a monic $\mathscr{F}\mathscr{I}_n$-cover.*

Proof. (1) $\Rightarrow$ (2) R is left n-coherent by Theorem 3.1. Now suppose that N is a submodule of an n-flat right R-module L and $\iota\colon N \to L$ is the inclusion.

By (1), N has an epic $\mathscr{F}_n$-preenvelope $f\colon N \to F$. Then there is a homomorphism $g\colon F \to L$ such that the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{\;f\;} & F \\ {\scriptstyle \iota}\big\downarrow & \swarrow {\scriptstyle g} & \\ L & & \end{array}$$

So $gf = \iota$ is monic, and hence f is monic, which gives that f is an isomorphism and $N \cong F$ is n-flat.

(2) $\Rightarrow$ (3) Let M be any n-FP-injective left R-module and let $M \to N \to 0$ be exact. Then $0 \to N^+ \to M^+$ is exact and M^+ is n-flat, and so N^+ is n-flat by (2). Thus N is n-FP-injective by [6, Theorem 3].

(3) $\Rightarrow$ (4) By [6, Theorem 2], R is left n-coherent, and hence every left R-module M has an $\mathscr{F}\mathscr{I}_n$-precover $\varphi\colon C \to M$. Note that $\operatorname{Im}\varphi$ is n-FP-injective by (3), so $\operatorname{Im}\varphi \to M$ is a monic $\mathscr{F}\mathscr{I}_n$-cover.

(4) $\Rightarrow$ (1) Let E be an injective left R-module and $S \subseteq E$ a pure submodule. Then E/S has a monic $\mathscr{F}\mathscr{I}_n$-cover $f\colon C \to E/S$. By analogy with the proof (1) $\Rightarrow$ (2), f is an isomorphism and E/S is n-FP-injective, and hence R is left n-coherent by [6, Theorem 2], which means that every right R-module has an $\mathscr{F}_n$-preenvelope by Theorem 3.1. Let M be any right R-module. Then M^+ has a monic $\mathscr{F}\mathscr{I}_n$-cover $E \to M^+$, and hence $M^{++} \to E^+ \to 0$ is exact. Set $K = \operatorname{Ker}(M^{++} \to E^+)$. Consider the following pullback of $M \to M^{++}$ and $K \to M^{++}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & M^{++}/M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & M^{++} & \longrightarrow & M^{++}/M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & E^+ & = & E^+ & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since E^+ is n-flat, M has an epic $\mathscr{F}_n$-preenvelope. □

Proposition 3.7. *The following conditions are equivalent:*

(1) *every left R-module is n-FP-injective;*
(2) *every cotorsion left R-module is n-FP-injective;*
(3) *every right R-module is n-flat;*
(4) *every cotorsion right R-module is n-flat;*
(5) *every right R-module in $\mathscr{F}_n^{\perp}$ is injective;*
(6) *every left R-module in ${}^{\perp}\mathscr{F}\mathscr{I}_n$ is projective;*
(7) *every nonzero right R-module contains a nonzero n-flat submodule;*
(8) $({}^{\perp}\mathscr{F}\mathscr{I}_n, \mathscr{F}\mathscr{I}_n)$ *is a hereditary cotorsion theory and every left R-module in ${}^{\perp}\mathscr{F}\mathscr{I}_n$ is n-FP-injective.*

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (3) $\Rightarrow$ (7) and (1) $\Rightarrow$ (8) are obvious.

(2) $\Rightarrow$ (3) Let M be any right R-module. Then M^+ is n-FP-injective by (2), and so M is n-flat.

(4) $\Rightarrow$ (1) Let M be any left R-module. Then M^+ is n-flat by (4), and so M^{++} is n-FP-injective. Note that M is a pure submodule of M^{++}. So M is n-FP-injective.

(3) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) follow from Theorem 2.1.

(7) $\Rightarrow$ (5) Assume that $0 \to A \to B \to C \to 0$ is any exact sequence. To simplify the notation, we think of A as a submodule of B. Let $M \in \mathscr{F}_n^{\perp}$ and let $f\colon A \to M$ be any homomorphism. By a simple application of Zorn's Lemma, we can find $g\colon D \to M$, where $A \subseteq D \subseteq B$ and $g|_A = f$, such that g cannot be extended to any submodule of B properly containing D. We claim that $D = B$. Indeed, if $D \neq B$, then $B/D \neq 0$. By (7), there is a nonzero submodule N/D of B/D such that N/D is n-flat. Since $M \in \mathscr{F}_n^{\perp}$, there exists $h\colon N \to M$ such that $h|_D = g$. It is obvious that h extends g, thus we get the desired contradiction, and so M is injective.

(8) $\Rightarrow$ (1) Let M be a left R-module. By Theorem 2.1, there is an exact sequence $0 \to K \to F \to M \to 0$ with $F \in {}^{\perp}\mathscr{F}\mathscr{I}_n$, $K \in \mathscr{F}\mathscr{I}_n$. Then $F \in \mathscr{F}\mathscr{I}_n$, and hence $M \in \mathscr{F}\mathscr{I}_n$ by (8). □

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