AN IDENTITY BETWEEN THE DETERMINANT AND THE PERMANENT OF HESSENBERG-TYPE MATRICES

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Abstract. In this short note we provide an extension of the notion of Hessenberg matrix and observe an identity between the determinant and the permanent of such matrices. The celebrated identity due to Gibson involving Hessenberg matrices is consequently generalized.

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1. Introduction

It is standard that for an $n \times n$ matrix $A = (a_{ij})$, the determinant of A may be given by

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n represents the symmetric group of degree n. Analogously, the permanent of A is

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Due to the apparent similarity of the two definitions, in 1913, George Pólya [14] asked whether it was possible to calculate permanents "using" determinants. To be more precise, whether it was possible to change the signs of some of the entries of a matrix A, getting a new matrix B, in such a way that $\text{per}(A) = \det(B)$. Soon, Gábor Szegö [16] gave a negative answer to Pólya's question and, six decades later, Reich [15] provided a new and concise proof for that fact.

It was clear since the beginning that the calculation of the permanent of a matrix was a more intricated problem than the computation of the determinant, although it seems simpler [1], [13]. Maybe the most celebrated achievement concerning Pólya's Problem is the classical result due to Marcus and Minc [12] stating that the determinant is not "convertible" into the permanent. Other extensions, solutions, and results related to this problem are known [2], [3], [7], [8], [9], [10], [11], [13], [15], [17]. However, there are matrices whose determinant is sign-converted into the permanent [6]. Those matrices are, unsurprisingly, called *convertible*.

It is worth mentioning the long refereed paper [13], where McCuaig gave structural characterizations of sign-nonsingular matrices, digraphs with no even length dicycles, and square non-negative real matrices whose permanent and determinant are equal.

The thrust of this brief note is to extend the famous identity due to Gibson [6] to a broader class of convertible matrices: Hessenberg-type matrices.
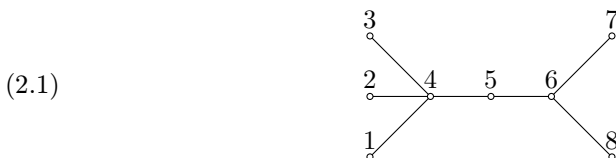
2. G-LOWER HESSENBERG MATRICES

In 1969, Gibson [6] proved that if A is a lower Hessenberg matrix and B is related to A, then $\operatorname{per} A = \det B$. Gibson denoted such a matrix as *semitriangular* and, according to [6], $B = (b_{ij})$ is said *to be related* to A if $b_{ij} = a_{ij}$ when $i \geqslant j$, and $b_{ij} = -a_{ij}$ when $i < j$.

Definition 2.1. For a given (simple) graph G with the vertex set $\{1, \ldots, n\}$, a *G-lower Hessenberg matrix* $A = (a_{ij})$ is an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i < j$ and ij is not an edge of G.

If G is a tree, we say that A is an *acyclic lower Hessenberg matrix*. In particular, if G is a path (whose vertices are ordered in the natural way), i.e., $a_{ij} = 0$ for all $j > i + 1$, A is the classical lower Hessenberg matrix. An extension of this notion, known as the *generalized Hessenberg matrix*, is due to Fiedler and Vavřín [5] and to Elsner [4].

Example 2.1. Let us consider the tree T

(2.1)

Then a T-lower Hessenberg matrix is of the form

(2.2)
$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix}.$$

A *generalized double star* is a tree resulting from joining the central vertices of two stars by a path. In this regard, a path and a cycle can be seen as particular cases of generalized double stars. We assume that all the indices are distributed as in the tree T defined by (2.1), which is itself an example of a generalized double star.

Definition 2.2. For a given generalized double star T, let V_1 and V_2 be the two pendant sets of end vertices. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two T-lower Hessenberg matrices. We say that B is T-*related* to A if

$$b_{ij} = \begin{cases} -a_{ij} & \text{if } i < j \\ a_{ij} & \text{if } i = j \\ -a_{ij} & \text{if } i > j \text{ and } i, j \in V_1 \\ -a_{ij} & \text{if } i > j \text{ and } i, j \in V_2 \\ a_{ij} & \text{otherwise.} \end{cases}$$

Example 2.2. The matrix

$$B = \begin{pmatrix} a_{11} & 0 & 0 & -a_{14} & 0 & 0 & 0 & 0 \\ -a_{21} & a_{22} & 0 & -a_{24} & 0 & 0 & 0 & 0 \\ -a_{31} & -a_{32} & a_{33} & -a_{34} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & -a_{45} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & -a_{56} & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & -a_{67} & -a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & -a_{87} & a_{88} \end{pmatrix}$$

is T-related to A defined in (2.2).

Remark 2.1. Observe that, when T is a path, we get the definition proposed by Gibson in [6].

3. The main result

We are ready now to present our main result.

Theorem 3.1. *Let T be a generalized double star, and let A and B be two related T-lower Hessenberg matrices. Then*

$$\operatorname{per} A = \det B.$$

Proof. The statement is clearly true for generalized double stars of orders 2, 3, or 4. Let us assume that the result is true for any generalized double star of order $< n$. Suppose that the two pendant stars have n_1 and n_2 terminal vertices, incident on the vertices $n_1 + 1$ and $n - n_2$, respectively. Setting $k = n_1 + 1$, we have

$$(3.1) \qquad \det B = a_{11} \det B_1 + (-1)^k a_{1k} \det B_{1k},$$

where B_1 is the matrix obtained from B by deleting both the row and column indexed by 1 and B_{1k} is the matrix obtained from B by deleting the row indexed by 1 and the column indexed by k.

Note that B_{1k} is an $(n-1) \times (n-1)$ acyclic lower Hessenberg matrix. In fact, the underlying tree is still a generalized double star, say $\tilde{T}$: a path with a pendant star with n_2 end vertices at the vertex $n - n_2 - 1$. Since the matrix

$$\begin{pmatrix} -I_{n_1-1} & \\ & I_{n-n_1} \end{pmatrix} B_{1k}$$

is $\tilde{T}$-related to A_{1k}, we conclude, using induction, that $(-1)^{n_1-1} \det B_{1k} = \operatorname{per} A_{1k}$.

Clearly, B_1 is an acyclic lower Hessenberg matrix as well: the tree here is obtained from the original generalized double star by deleting the vertex 1. Therefore, from the inductive assumption, $\det B_1 = \operatorname{per} A_1$. Finally, we get from (3.1),

$$\det B = a_{11} \operatorname{per} A_1 + a_{1k} \operatorname{per} A_{1k} = \operatorname{per} A.$$

$\square$

Remark 3.1. Observe that if $n_2 = 0$, then T is a star.

Corollary 3.2 ([6])**.** *If A and B are two related lower Hessenberg matrices, then*

$$\operatorname{per} A = \det B.$$

We remark the fact that, under the conditions of the theorem, the related T-lower Hessenberg matrices have always at least $\Omega_n = \frac{1}{2}(n^2 - 3n + 2)$ zeros. In [7], Gibson confines his attention to convertible $(0,1)$-matrices.

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