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# MULTIPLE POSITIVE SOLUTIONS TO MULTIPOINT ONE-DIMENSIONAL $p$-LAPLACIAN BOUNDARY VALUE PROBLEM WITH IMPULSIVE EFFECTS

Yuansheng Tian, Chenzhou, Anping Chen, Chenzhou, Weigao Ge, Beijing

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*Abstract.* In this paper, using a fixed point theorem on a convex cone, we consider the existence of positive solutions to the multipoint one-dimensional $p$-Laplacian boundary value problem with impulsive effects, and obtain multiplicity results for positive solutions.

*Keywords*: $p$-Laplacian operator, boundary value problem, impulsive differential equations, fixed-point theorem, positive solutions

*MSC 2010*: 34B15, 34B18, 34B37

## 1. Introduction

In this paper we study the multiplicity of positive solutions to the multipoint one-dimensional $p$-Laplacian boundary value problem with impulsive effects

$$
\begin{cases}
(\varphi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, \quad t \neq t_i,\ 0 < t < 1,\\
\Delta u(t_i) = I_i(u(t_i)), \quad i = 1,2,\dots,n,\\
\Delta \varphi_p(u'(t_i)) = -\overline{I}_i(u(t_i),u'(t_i)), \quad i = 1,2,\dots,n,\\
u(0) = \sum\limits_{j=1}^{m-2} \alpha_j u(\xi_j), \quad \varphi_p(u'(1)) = \sum\limits_{j=1}^{m-2} \beta_j \varphi_p(u'(\eta_j)),
\end{cases}
\tag{1.1}
$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1$, $(\varphi_p)^{-1} = \varphi_q$, $1/p+1/q = 1$ and $\Delta u(t_i) = u(t_i^+)-u(t_i^-)$, $\Delta\varphi_p(u'(t_i)) = \varphi_p(u'(t_i^+)) - \varphi_p(u'(t_i^-))$, $u(t_i^+)$ and $u(t_i^-)$ represent the right-hand

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limit and the left-hand limit of the function $u(t)$ at $t = t_i$. The sequences $\{t_i\}$, $\{\xi_i\}$ and $\{\eta_i\}$ satisfy $0 < t_1 < t_2 < \ldots < t_n < 1$, $n \in \mathbb{N}$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$, and $\xi_j, \eta_j \neq t_i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m-2$. The constants $\alpha_j, \beta_j \in \mathbb{R}^+$ satisfy $0 \leqslant \sum_{j=1}^{m-2} \alpha_j, \sum_{j=1}^{m-2} \beta_j < 1$, where $\mathbb{R}^+ = [0, \infty)$.

In this paper we assume that

($C_1$) $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$,

($C_2$) $q \in C[0,1]$ is nonnegative and there exists an integer $k \geqslant 3$ such that $\int_{1/k}^{1-1/k} q(t)\,\mathrm{d}t > 0$,

($C_3$) $I_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function, $\overline{I}_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2, \ldots, n$.

We set $G_i = \sup_{u \in [0,+\infty)} I_i(u)$, $D_1 = 1/\Big(1 - \sum_{j=1}^{m-2} \alpha_j\Big)$, $D_2 = 1/\Big(1 - \sum_{j=1}^{m-2} \beta_j\Big)$, $h = D_1\Big(1 - \sum_{j=1}^{m-2} \alpha_j (1-\xi_j)^2\Big)$, $b_1 > 0$ is a constant which is given by Theorem 4.1.

($C_4$) $\max\{G_1, G_2, \ldots, G_n\} \leqslant (kb_1(h-1))/n$.

Differential equations involving impulsive effects have been applied in many fields, for example, in population dynamics, biological systems, industrial robotics, optimal control and so on. The boundary value problems for impulsive differential equations have been studied extensively in literature (see [2]–[9], [11], [12], [14] and the references therein). Most of those papers have studied the two-point or periodic boundary value problems for impulsive differential equations. The literature devoted to the multipoint one-dimensional $p$-Laplacian boundary value problem with impulsive effects is not too extensive. Recently there are papers studying some special cases of the problem (1.1). For example, when $\overline{I}_i = 0$ and $\beta_j = 0$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m-2$), Zhang et al. [14] have considered the special case of the problem (1.1) when the nonlinear term $f$ does not involve the first-order derivative, and have obtained the existence of multiple positive solutions to the following multipoint one-dimensional $p$-Laplacian boundary value problem with impulsive effects, by using the classical fixed-point index theorem for compact maps:

$$\begin{cases} -(\varphi_p(u'(t)))' = f(t, u(t)), \quad t \neq t_i,\ 0 < t < 1, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, n, \\ u(0) = \sum_{j=1}^{m-2} \alpha_j u(\xi_j), \quad u'(1) = 0. \end{cases}$$

For the case of $I_i = 0$ and $\overline{I}_i = 0$ ($i = 1, 2, \dots, n$), Wang et al. [13] have researched the multipoint boundary value problem with a one-dimensional $p$-Laplacian

$$\begin{cases} (\varphi_p(u'(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1, \\ \varphi_p(u'(0)) = \sum\limits_{j=1}^{m-2} \alpha_j \varphi_p(u'(\xi_j)), \quad u(1) = \sum\limits_{j=1}^{m-2} \beta_j u(\xi_j). \end{cases}$$

In the paper we consider the more general situation (1.1), we get over some new difficulties such as the construction of the cone and the operator used. We prove that under some conditions the problem (1.1) possesses multiple positive solutions. The detailed statement and proof of our main result are given in Section 4. In Section 5 we give an example to support our main result.

## 2. Preliminaries

In this section we give a brief introduction to the theory of cones in Banach spaces, and to the so called Bai-Ge's fixed point theorem.

**Definition 2.1.** Let $E$ be a Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is called a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and $a \geqslant 0$, $b \geqslant 0$,
(2) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by $x \leqslant y$ if and only if $y - x \in P$.

**Definition 2.2.** A map $\psi$ is called a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ provided that $\psi\colon P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geqslant t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$. Similarly, we say a map $\varphi$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ provided that $\varphi\colon P \to [0, \infty)$ is continuous and

$$\varphi(tx + (1-t)y) \leqslant t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Let $r > a > 0$ and $L > 0$ be constants, $\psi$ is a nonnegative continuous concave functional and $\varphi$ and $\omega$ are nonnegative continuous convex functionals on the cone $P$.

Define the following convex sets:

$$\begin{aligned} P(\varphi, r; \omega, L) &= \{y \in P\colon\ \varphi(y) < r,\ \omega(y) < L\}, \\ \overline{P}(\varphi, r; \omega, L) &= \{y \in P\colon\ \varphi(y) \leqslant r,\ \omega(y) \leqslant L\}, \\ P(\varphi, r; \omega, L; \psi, a) &= \{y \in P\colon\ \varphi(y) < r,\ \omega(y) < L,\ \psi(y) > a\}, \\ \overline{P}(\varphi, r; \omega, L; \psi, a) &= \{y \in P\colon\ \varphi(y) \leqslant r,\ \omega(y) \leqslant L,\ \psi(y) \geqslant a\}. \end{aligned}$$

The following assumptions as regards the nonnegative continuous convex functions $\varphi$, $\omega$ are used:

($H_1$) there exists $M > 0$ such that $\|x\| \leqslant M \max\{\varphi(x), \omega(x)\}$ for all $x \in P$;
($H_2$) $P(\varphi, r; \omega, L) \neq \emptyset$ for any $r > 0$ and $L > 0$.

To prove our result in Section 4, we need the following fixed point theorem due to Bai and Ge in [1].

**Lemma 2.1.** *Let $P$ be a cone in a real Banach space $E$ and let $r_2 \geqslant d > b > r_1 > 0$, $L_2 \geqslant L_1 > 0$. Assume that $\varphi$ and $\omega$ are nonnegative continuous convex functions satisfying ($H_1$) and ($H_2$), $\psi$ is a nonnegative continuous concave function on $P$ such that $\psi(y) \leqslant \varphi(y)$ for all $y \in \overline{P}(\varphi, r_2; \omega, L_2)$ and $T\colon \overline{P}(\varphi, r_2; \omega, L_2) \to \overline{P}(\varphi, r_2; \omega, L_2)$ is a completely continuous operator. Suppose that*

($A_1$) *$\{y \in \overline{P}(\varphi, d; \omega, L_2; \psi, b)\colon\ \psi(y) > b\} \neq \emptyset$, $\psi(Ty) > b$ for $y \in \overline{P}(\varphi, d; \omega, L_2; \psi, b)$,*
($A_2$) *$\varphi(Ty) < r_1$, $\omega(Ty) < L_1$ for all $y \in \overline{P}(\varphi, r_1; \omega, L_1)$,*
($A_3$) *$\psi(Ty) > b$ for all $y \in \overline{P}(\varphi, r_2; \omega, L_2; \psi, b)$ with $\varphi(Ty) > d$.*

*Then $T$ has at least three fixed points $y_1, y_2$ and $y_3 \in \overline{P}(\varphi, r_2; \omega, L_2)$ with*

$$\begin{aligned} &y_1 \in P(\varphi, r_1; \omega, L_1), \\ &y_2 \in \{\overline{P}(\varphi, r_2; \omega, L_2; \psi, b)\colon\ \psi(y) > b\}, \\ &y_3 \in \overline{P}(\varphi, r_2; \omega, L_2) \setminus (\overline{P}(\varphi, r_2; \omega, L_2; \psi, b) \cup \overline{P}(\varphi, r_1; \omega, L_1)). \end{aligned}$$

## 3. Some lemmas

In order to get the solutions of problem (1.1), we introduce the following notation.

Let $J = [0, 1]$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\ldots$, $J_{n-1} = (t_{n-1}, t_n]$, $J_n = (t_n, 1]$, $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$.

Set $PC(J) = \{u\colon [0, 1] \to \mathbb{R}\colon\ u \in C(J'),\ u(t_i^+)$ and $u(t_i^-)$ exist, and $u(t_i^-) = u(t_i),\ 1 \leqslant i \leqslant n\}$,

$PC^1(J) = \{u \in PC(J)\colon\ u \in C^1(J'), u'(t_i^-)$ and $u'(t_i^+)$ exist, and $u'(t_i^-) = u'(t_i),\ 1 \leqslant i \leqslant n\}$.

Obviously, $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_{PC} = \max_{0\leqslant t\leqslant 1} |u(t)|, \quad \|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\},$$

respectively. A function $u \in PC^1(J)\cap C^2(J')$ is called a solution to (1.1) if it satisfies all equations of (1.1).

Define the cone $P \subset PC^1(J)$ by $P = \Big\{u \in PC^1(J)\colon\ u(t) \geqslant 0,\ u$ is concave on $J_i$ $(1 \leqslant i \leqslant n)$ and $u'(t) \geqslant 0,\ u'(t)$ is non-increasing on $[0,1],\ u(0) = \sum\limits_{j=1}^{m-2} \alpha_j u(\xi_j)\Big\}$.

Define nonnegative continuous functionals $\varphi$, $\omega$ and $\psi$ by

$$\varphi(u) = \max_{0\leqslant t\leqslant 1} |u(t)|, \quad \omega(u) = \max_{0\leqslant t\leqslant 1} |u'(t)|, \quad \psi(u) = \min_{1/k\leqslant t\leqslant (k-1)/k} |u(t)|, \quad u \in P.$$

Then on the cone $P$, $\psi$ is a concave functional, $\varphi$ and $\omega$ are convex functionals satisfying $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$.

**Lemma 3.1.** *If $u \in P$, then*

$$\max_{0\leqslant t\leqslant 1} |u'(t)| = u'(0), \quad \max_{0\leqslant t\leqslant 1} |u(t)| = u(1), \quad \min_{1/k\leqslant t\leqslant 1-1/k} |u(t)| = u\Big(\frac{1}{k}\Big).$$

Proof. By the definition of the cone $P$, the proof is very easy, so we omit it here. □

**Lemma 3.2.** *If $u \in P$, $k \geqslant 3$, then*

$$\min_{1/k\leqslant t\leqslant 1-1/k} |u(t)| \geqslant \frac{1}{k} \max_{0\leqslant t\leqslant 1} |u(t)| - \frac{1}{k}\sum_{i=1}^{n} |\Delta u(t_i)|.$$

Proof. Let

$$v(t) = \begin{cases} u(t), & t \in J_0,\\ u(t) - |\Delta u(t_1)|, & t \in J_1,\\ \vdots & \\ u(t) - \sum\limits_{i=1}^{n-1} |\Delta u(t_i)|, & t \in J_{n-1},\\ u(t) - \sum\limits_{i=1}^{n} |\Delta u(t_i)|, & t \in J_n. \end{cases}$$

Note that $u'(t)$ is non-increasing on [0,1], hence $v \in C[0,1]$ and $v$ is concave on [0,1]. By Lemma 2.2 in [10] we have

$$\min_{1/k\leqslant t\leqslant 1-1/k} |v(t)| \geqslant \frac{1}{k} \max_{0\leqslant t\leqslant 1} |v(t)|.$$

Moreover, $u(t)$ is non-decreasing on [0,1], and we have

$$\max_{0\leqslant t\leqslant 1}|v(t)| = v(1) = u(1) - \sum_{i=1}^{n}|\Delta u(t_i)| = \max_{0\leqslant t\leqslant 1}|u(t)| - \sum_{i=1}^{n}|\Delta u(t_i)|,$$

$$\begin{aligned}\min_{1/k\leqslant t\leqslant 1-1/k}|v(t)| &= v\Big(\frac{1}{k}\Big) = u\Big(\frac{1}{k}\Big) - \sum_{0<t_i<1/k}|\Delta u(t_i)| \\ &= \min_{1/k\leqslant t\leqslant 1-1/k}|u(t)| - \sum_{0<t_i<1/k}|\Delta u(t_i)|.\end{aligned}$$

Hence,

$$\begin{aligned}\min_{1/k\leqslant t\leqslant 1-1/k}|u(t)| &= \min_{1/k\leqslant t\leqslant 1-1/k}|v(t)| + \sum_{0<t_i<1/k}|\Delta u(t_i)| \\ &\geqslant \frac{1}{k}\max_{0\leqslant t\leqslant 1}|v(t)| \geqslant \frac{1}{k}\max_{0\leqslant t\leqslant 1}|u(t)| - \frac{1}{k}\sum_{i=1}^{n}|\Delta u(t_i)|.\end{aligned}$$

□

**Lemma 3.3.** *Assume that* ($C_1$)–($C_3$) *hold. Then* $u \in PC^1(J)\cap C^2(J')$ *is a solution to problem* (1.1) *if and only if* $u \in PC^1(J)$ *is a solution to the integral equation*

$$\begin{aligned}(3.1)\qquad u(t) &= \int_0^t \varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s \\ &+ D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{\xi_j}\varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s \\ &+ D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<\xi_j}I_i(u(t_i)) + \sum_{0<t_i<t}I_i(u(t_i)),\end{aligned}$$

*where*

$$Y = D_2\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + D_2\sum_{j=1}^{m-2}\beta_j\sum_{\eta_j<t_i<1}\overline{I}_i(u(t_i),u'(t_i)).$$

Proof. First, suppose that $u \in PC^1(J)\cap C^2(J')$ is a solution to problem (1.1). Then

$$(\varphi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0,\quad t\neq t_i,\ i=1,2,\ldots,n.$$

So,

$$\varphi_p(u'(t_n^+)) - \varphi_p(u'(1)) = \int_{t_n}^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s,$$
$$\varphi_p(u'(t)) - \varphi_p(u'(t_n^-)) = \int_t^{t_n} q(s)f(s,u(s),u'(s))\,\mathrm{d}s, \quad t \in J_{n-1}.$$

Thus,

$$\varphi_p(u'(t)) = \varphi_p(u'(1)) + \int_t^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \overline{I}_n(u(t_n),u'(t_n)), \quad t \in J_{n-1}.$$

Repeating the above process, for $t \in [0,1]$ we have

$$\varphi_p(u'(t)) = \varphi_p(u'(1)) + \int_t^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{t<t_i<1} \overline{I}_i(u(t_i),u'(t_i)), \tag{3.2}$$

and taking $t = \eta_j$ in (3.2), we obtain

$$\varphi_p(u'(\eta_j)) = \varphi_p(u'(1)) + \int_{\eta_j}^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{\eta_j<t_i<1} \overline{I}_i(u(t_i),u'(t_i)).$$

So, we have

$$\sum_{j=1}^{m-2} \beta_j \varphi_p(u'(\eta_j)) = \varphi_p(u'(1)) \sum_{j=1}^{m-2} \beta_j + \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s$$
$$+ \sum_{j=1}^{m-2} \beta_j \sum_{\eta_j<t_i<1} \overline{I}_i(u(t_i),u'(t_i)).$$

Since $\varphi_p(u'(1)) = \sum_{j=1}^{m-2} \beta_j \varphi_p(u'(\eta_j))$, we have

$$\varphi_p(u'(1)) = D_2 \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s \tag{3.3}$$
$$+ D_2 \sum_{j=1}^{m-2} \beta_j \sum_{\eta_j<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) := Y.$$

Substituting (3.3) into (3.2), we get

$$\varphi_p(u'(t)) = \int_t^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{t<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y,$$

which implies that

$$u'(t) = \varphi_q\Big(\int_t^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{t<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\Big). \tag{3.4}$$

On the other hand, note that

$$\begin{aligned} u(t_1^-) - u(0) &= \int_0^{t_1} u'(s)\,\mathrm{d}s, \\ u(t) - u(t_1^+) &= \int_{t_1}^t u'(s))\,\mathrm{d}s, \quad t \in J_1, \end{aligned}$$

so that we have

$$u(t) = u(0) + \int_0^t u'(s)\,\mathrm{d}s + I_1(u(t_1)), \quad t \in J_1.$$

Repeating the above process again for $t \in [0,1]$, we obtain

$$u(t) = u(0) + \int_0^t u'(s)\,\mathrm{d}s + \sum_{0<t_i<t} I_i(u(t_i)). \tag{3.5}$$

Substituting (3.4) into (3.5), we get

$$\begin{aligned} u(t) = u(0) &+ \sum_{0<t_i<t} I_i(u(t_i)) \\ &+ \int_0^t \varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s, \end{aligned} \tag{3.6}$$

and taking $t = \xi_j$ in (3.6), we get

$$\begin{aligned} u(\xi_j) = u(0) &+ \sum_{0<t_i<\xi_j} I_i(u(t_i)) \\ &+ \int_0^{\xi_j} \varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s. \end{aligned}$$

So,

$$\begin{aligned} &\sum_{j=1}^{m-2} \alpha_j u(\xi_j) \\ &= u(0)\sum_{j=1}^{m-2} \alpha_j + \sum_{j=1}^{m-2} \alpha_j \sum_{0<t_i<\xi_j} I_i(u(t_i)) \\ &\quad + \sum_{j=1}^{m-2} \alpha_j \int_0^{\xi_j} \varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s. \end{aligned}$$

Since $u(0)=\sum\limits_{j=1}^{m-2}\alpha_j u(\xi_j)$, we have

$$
\begin{aligned}
u(0) = {} & D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<\xi_j} I_i(u(t_i)) \\
& + D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{\xi_j}\varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau \\
& \qquad + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i))+Y\biggr)\,\mathrm{d}s.
\end{aligned}
\tag{3.7}
$$

Substituting (3.7) into (3.6), we get (3.1), which completes the proof of sufficiency.

Conversely, if $u(t)\in PC^1(J)$ is a solution to (3.1), apparently

$$\Delta u(t_i)=I_i(u(t_i)),\quad i=1,2,\ldots,n.$$

The differentiation of (3.1) implies that for $t\neq t_i$

$$
\begin{gathered}
u'(t)=\varphi_q\biggl(\int_t^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s+\sum_{t<t_i<1}\overline{I}_i(u(t_i),u'(t_i))+Y\biggr),\\
(\varphi_p(u'(t)))'=-q(t)f(t,u(t),u'(t)).
\end{gathered}
$$

Hence $u\in C^2(J')$, and

$$
\begin{gathered}
\Delta\varphi_p(u'(t_i))=-\overline{I}_i(u(t_i),u'(t_i)),\quad i=1,2,\ldots,n,\\
u(0)=\sum_{j=1}^{m-2}\alpha_j u(\xi_j),\quad \varphi_p(u'(1))=\sum_{j=1}^{m-2}\beta_j\varphi_p(u'(\eta_j)).
\end{gathered}
$$

The proof is complete. □

Now, define an operator $T\colon P\longrightarrow PC^1(J)$ by

$$
\begin{aligned}
Tu(t) = {} & \int_0^t\varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i))+Y\biggr)\,\mathrm{d}s \\
& + D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{\xi_j}\varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau \\
& \qquad + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i))+Y\biggr)\,\mathrm{d}s \\
& + D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<\xi_j}I_i(u(t_i))+\sum_{0<t_i<t}I_i(u(t_i)).
\end{aligned}
\tag{3.8}
$$

**Lemma 3.4.** *Assume that* $(C_1)$–$(C_3)$ *hold. Then* $T\colon P\to P$ *is a completely continuous operator.*

Proof. From the definition of $T$ we deduce that for each $u(t) \in P$, $Tu$ is nonnegative and

$$\Delta(Tu)(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, n. \tag{3.9}$$

By the differentiation of (3.8), for $t \neq t_i$ we have

$$(Tu)'(t) = \varphi_q\Big(\int_t^1 q(s)f(s, u(s), u'(s))\,\mathrm{d}s + \sum_{t<t_i<1} \overline{I}_i(u(t_i), u'(t_i)) + Y\Big) \geqslant 0,$$

$$(\varphi_p((Tu)'(t)))' = -q(t)f(t, u(t), u'(t)) \leqslant 0,$$

and

$$\Delta\varphi_p((Tu)'(t_i)) = -\overline{I}_i(u(t_i), u'(t_i)), \quad i = 1, 2, \ldots, n.$$

So, $Tu \in PC^1(J)$, $Tu$ is concave on $J_i$ for $0 \leqslant i \leqslant n$ and $(Tu)'(t) \geqslant 0$, $(Tu)'(t)$ is non-increasing on [0,1], $(Tu)(0) = \sum_{j=1}^{m-2} \alpha_j (Tu)(\xi_j)$. Thus $T(P) \subset P$.

On the other hand, by the conditions ($C_1$)–($C_3$), from the definition of $Tu(t)$, it is clear that $T\colon P \to P$ is continuous. Let $\Omega \subset P$ be bounded, i.e., there exists a positive constant $R$ such that $\Omega \subset \{u \in P\colon \|u\|_{PC^1} \leqslant R\}$. Let

$$\begin{aligned}
B_1 &= \max_{(t,u,v)\in[0,1]\times[0,R]\times[0,R]} f(t, u, v) + 1,\\
B_2 &= \max_{1\leqslant i\leqslant n}\Big\{\max_{u\in[0,R]} I_i(u)\Big\},\\
B_3 &= \max_{1\leqslant i\leqslant n}\Big\{\max_{(u,v)\in[0,R]\times[0,R]} \overline{I}_i(u, v)\Big\},\\
R_1 &= \max_{0\leqslant t\leqslant 1} q(t).
\end{aligned}$$

For all $u \in \Omega$ we have

$$\begin{aligned}
Y &\leqslant D_2 \sum_{j=1}^{m-2} \beta_j \int_0^1 q(s)f(s, u(s), u'(s))\,\mathrm{d}s + D_2 \sum_{j=1}^{m-2} \beta_j \sum_{0<t_i<1} \overline{I}_i(u(t_i), u'(t_i))\\
&\leqslant D_2(B_1R_1 + nB_3) \sum_{j=1}^{m-2} \beta_j.
\end{aligned}$$

Hence,

$$
\begin{aligned}
|Tu(t)| \leqslant{} & \int_0^1 \varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
& + D_1\sum_{j=1}^{m-2}\alpha_j \int_0^1 \varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau \\
& \qquad\qquad + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
& + D_1\sum_{j=1}^{m-2}\alpha_j \sum_{0<t_i<1} I_i(u(t_i)) + \sum_{0<t_i<1} I_i(u(t_i)) \\
\leqslant{} & D_1\biggl\{\varphi_q\biggl(B_1R_1 + nB_3 + D_2(B_1R_1+nB_3)\sum_{j=1}^{m-2}\beta_j\biggr) + nB_2\biggr\}, \\
|(Tu)'(t)| \leqslant{} & \varphi_q\biggl(\int_0^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{0<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\biggr) \\
\leqslant{} & \varphi_q\biggl(B_1R_1 + nB_3 + D_2(B_1R_1+nB_3)\sum_{j=1}^{m-2}\beta_j\biggr), \\
& |(\varphi_p((Tu)'(t)))'| \leqslant R_1B_1, \quad t\neq t_i,\ i=1,2,\ldots,n.
\end{aligned}
$$

So, $Tu$ and $(Tu)'$ are bounded on $J$ and equi-continuous on each $J_i$ $(i=0,1,2,\ldots,n)$. This implies that $T\Omega$ is relatively compact. Therefore, the operator $T\colon P\to P$ is completely continuous. □

## 4. Main results

We are now ready to apply the fixed point theorem due to Bai and Ge to the operator $T$ in order to get sufficient conditions for the existence of multiple positive solutions to the problem (1.1).

For the sake of convenience, we introduce the following notation:

$$
\begin{aligned}
H &= D_1\varphi_q\biggl(R_1 + n + D_2(R_1+n)\sum_{j=1}^{m-2}\beta_j\biggr) + nD_1, \\
N &= \varphi_q\biggl(\int_{1/k}^{1-1/k} q(\tau)\,\mathrm{d}\tau\biggr), \\
L &= \varphi_q\biggl(R_1 + n + D_2(R_1+n)\sum_{j=1}^{m-2}\beta_j\biggr).
\end{aligned}
$$

We recall that the integer $k \geqslant 3$ is given in ($C_2$) and $D_1 = 1/\Big(1 - \sum\limits_{j=1}^{m-2} \alpha_j\Big)$, $D_2 = 1/\Big(1 - \sum\limits_{j=1}^{m-2} \beta_j\Big)$, $h = D_1\Big(1 - \sum\limits_{j=1}^{m-2} \alpha_j (1-\xi_j)^2\Big)$, $R_1 = \max\limits_{0\leqslant t\leqslant 1} q(t)$.

**Theorem 4.1.** *Assume* ($C_1$)–($C_4$) *hold and there exist constants* $r_2 \geqslant hkb_1 > b_1 > r_1 > 0$, $L_2 \geqslant L_1 > 0$ *such that* $L_2 \geqslant 2kb_1$, $\varphi_p(kb_1/N) \leqslant \min\{\varphi_p(r_2/H), \varphi_p(L_2/L)\}$. *Let the following conditions be satisfied:*

($B_1$) $\max\{f(t,u,v), \overline{I}_i(u,v)\} < \min\{\varphi_p(r_1/H), \varphi_p(L_1/L)\}$, $I_i(u) \leqslant r_1/H$ *for* $(t,u,v) \in [0,1]\times[0,r_1]\times[0,L_1]$, $1 \leqslant i \leqslant n$;

($B_2$) $f(t,u,v) > \varphi_p(kb_1/N)$ *for* $(t,u,v) \in [1/k, 1-1/k]\times[b_1, hkb_1]\times[0,L_2]$;

($B_3$) $\max\{f(t,u,v), \overline{I}_i(u,v)\} \leqslant \min\{\varphi_p(r_2/H), \varphi_p(L_2/L)\}$, $I_i(u) \leqslant r_2/H$ *for* $(t,u,v) \in [0,1]\times[0,r_2]\times[0,L_2]$, $1 \leqslant i \leqslant n$.

*Then the problem* (1.1) *possesses at least three positive solutions* $u_1$, $u_2$ *and* $u_3$ *such that*

$$\max_{0\leqslant t\leqslant 1} u_1(t) < r_1, \quad \max_{0\leqslant t\leqslant 1} |u_1'(t)| < L_1; \tag{4.1}$$

$$b_1 < \min_{1/k\leqslant t\leqslant (k-1)/k} u_2(t) \leqslant \max_{0\leqslant t\leqslant 1} u_2(t) \leqslant r_2, \quad \max_{0\leqslant t\leqslant 1} |u_2'(t)| \leqslant L_2; \tag{4.2}$$

$$r_1 < \max_{0\leqslant t\leqslant 1} u_3(t) \leqslant r_2, \quad \min_{1/k\leqslant t\leqslant (k-1)/k} u_3(t) < b_1, \quad \max_{0\leqslant t\leqslant 1} |u_3'(t)| \leqslant L_2. \tag{4.3}$$

P r o o f. The problem (1.1) has a solution $u = u(t)$ if and only if $u$ satisfies the operator equation $u = Tu$. Thus we set out to verify that the operator $T$ satisfies all conditions of Lemma 2.1. The proof is divided into four steps.

*Step 1.* First we show that

$$T\colon \overline{P}(\varphi, r_2; \omega, L_2) \to \overline{P}(\varphi, r_2; \omega, L_2). \tag{4.4}$$

In fact, for $u \in \overline{P}(\varphi, r_2; \omega, L_2)$ we have $\varphi(u) \leqslant r_2$, $\omega(u) \leqslant L_2$, by the condition ($B_3$) we get

$$\begin{aligned} Y &\leqslant D_2 \sum_{j=1}^{m-2} \beta_j \int_0^1 q(s) f(s, u(s), u'(s))\,\mathrm{d}s + D_2 \sum_{j=1}^{m-2} \beta_j \sum_{0<t_i<1} \overline{I}_i(u(t_i), u'(t_i)) \\ &\leqslant \min\{\varphi_p(r_2/H), \varphi_p(L_2/L)\} \times D_2(R_1 + n) \sum_{j=1}^{m-2} \beta_j. \end{aligned}$$

Hence,

$$
\begin{aligned}
\varphi(Tu) = &\max_{0\leqslant t\leqslant 1}|(Tu)(t)| = |(Tu)(1)| \\
= &\int_0^1 \varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
&+ D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{\xi_j}\varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau \\
&\qquad + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
&+ D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<\xi_j}I_i(u(t_i)) + \sum_{0<t_i<1}I_i(u(t_i)) \\
\leqslant &\int_0^1 \varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{0<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
&+ D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{1}\varphi_q\biggl(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau \\
&\qquad + \sum_{0<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
&+ D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<1}I_i(u(t_i)) + \sum_{0<t_i<1}I_i(u(t_i)) \\
\leqslant &D_1\int_0^1 \varphi_q\biggl(\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{0<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr)\,\mathrm{d}s \\
&+ D_1\sum_{0<t_i<1}I_i(u(t_i)) \\
\leqslant &\frac{r_2}{H}\biggl\{D_1\varphi_q\biggl(R_1+n+D_2(R_1+n)\sum_{j=1}^{m-2}\beta_j\biggr) + nD_1\biggr\} = r_2.
\end{aligned}
$$

On the other hand, for $u\in P$ we have $Tu\in P$. Thus $(Tu)'(t)\geqslant 0$, $(Tu)'(t)$ is non-increasing on [0,1], and $\max_{0\leqslant t\leqslant 1}|(Tu)'(t)| = (Tu)'(0)$. Therefore,

$$
\begin{aligned}
\omega(Tu) = &\max_{0\leqslant t\leqslant 1}|(Tu)'(t)| = (Tu)'(0) \\
= &\varphi_q\biggl(\int_0^1 q(s)f(s,u(s),u'(s))\,\mathrm{d}s + \sum_{0<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\biggr) \\
\leqslant &\frac{L_2}{L}\varphi_q\biggl(R_1+n+D_2(R_1+n)\sum_{j=1}^{m-2}\beta_j\biggr) = L_2.
\end{aligned}
$$

So, (4.4) holds.

*Step 2.* We show that condition ($A_1$) in Lemma 2.1 holds.

We take $u(t) = kb_1[h-(1-t)^2]$ for $t \in [0,1]$. Obviously, $h > 1$. It is easy to see that $u(t) \in \overline{P}(\varphi, hkb_1; \omega, L_2; \psi, b_1)$, $\psi(u) = u(1/k) > b_1$ and consequently $\{u \in \overline{P}(\varphi, hkb_1; \omega, L_2; \psi, b_1)\colon\ \psi(u) > b_1\} \neq \emptyset$. Thus for $u \in \overline{P}(\varphi, hkb_1; \omega, L_2; \psi, b_1)$ there is $b_1 \leqslant u(t) \leqslant hkb_1$ for $t \in [1/k, (k-1)/k]$. By condition ($B_2$) we have

$$\begin{aligned}
\psi(Tu) &= \min_{1/k\leqslant t\leqslant (k-1)/k} |(Tu)(t)| = (Tu)\Big(\frac{1}{k}\Big)\\
&= \int_0^{1/k} \varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{s<t_i<1} \overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s\\
&\quad + D_1\sum_{j=1}^{m-2}\alpha_j\int_0^{\xi_j}\varphi_q\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\\
&\qquad\qquad + \sum_{s<t_i<1}\overline{I}_i(u(t_i),u'(t_i)) + Y\Big)\,\mathrm{d}s\\
&\quad + D_1\sum_{j=1}^{m-2}\alpha_j\sum_{0<t_i<\xi_j} I_i(u(t_i)) + \sum_{0<t_i<1/k} I_i(u(t_i))\\
&\geqslant \int_0^{1/k}\varphi_q\Big(\int_{1/k}^{1-1/k} q(\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&> \frac{b_1}{N}\varphi_q\Big(\int_{1/k}^{1-1/k} q(\tau)\,\mathrm{d}\tau\Big) = b_1.
\end{aligned}$$

Therefore,

$$\psi(Tu) > b_1, \quad \forall\, u \in \overline{P}(\varphi, hkb_1; \omega, L_2; \psi, b_1).$$

Consequently, condition ($A_1$) in Lemma 2.1 is satisfied.

*Step 3.* We now show ($A_2$) in Lemma 2.1 is satisfied. If $u \in \overline{P}(\varphi, r_1; \omega, L_1)$, by condition ($B_1$), in the same way as in Step 1, we can obtain that $T\colon\ \overline{P}(\varphi, r_1; \omega, L_1) \to P(\varphi, r_1; \omega, L_1)$. Hence, condition ($A_2$) in Lemma 2.1 is satisfied.

*Step 4.* Finally, we show ($A_3$) in Lemma 2.1 is also satisfied. Suppose that $u \in \overline{P}(\varphi, r_2; \omega, L_2; \psi, b_1)$ with $\varphi(Tu) > hkb_1$. Then, by Lemma 3.2 and condition ($C_4$), we have (see (3.9))

$$\begin{aligned}
\psi(Tu) = \min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}} |(Tu)(t)| &\geqslant \frac{1}{k}\max_{0\leqslant t\leqslant 1}|Tu(t)| - \frac{1}{k}\sum_{i=1}^{n}|\Delta(Tu)(t_i)|\\
&\geqslant \frac{1}{k}\max_{0\leqslant t\leqslant 1}|Tu(t)| - \frac{1}{k}\sum_{i=1}^{n} I_i(u(t_i)) > hb_1 - (h-1)b_1 = b_1.
\end{aligned}$$

Thus, condition ($A_3$) in Lemma 2.1 is satisfied.

Consequently, by Lemma 2.1, the problem (1.1) has at least three positive solutions $u_1, u_2, u_3 \in \overline{P}(\varphi, r_2; \omega, L_2)$ with

$$u_1 \in P(\varphi, r_1; \omega, L_1),$$
$$u_2 \in \{\overline{P}(\varphi, r_2; \omega, L_2; \psi, b_1)\colon\ \psi(u) > b_1\}$$
$$u_3 \in \overline{P}(\varphi, r_2; \omega, L_2) \setminus (\overline{P}(\varphi, r_2; \omega, L_2; \psi, b_1) \cup \overline{P}(\varphi, r_1; \omega, L_1)).$$

The proof is complete. □

From the proof of Theorem 4.1 it is easy to see that, if conditions like $(B_1)$–$(B_3)$ are appropriately combined, we can obtain an arbitrary number of positive solutions of problem (1.1).

**Corollary 4.1.** *Assume* $(C_1)$–$(C_4)$ *hold and there exist constants* $0 < r_1 < b_1 < hkb_1 \leqslant r_2 < b_2 < hkb_2 \leqslant \ldots \leqslant r_l$, $0 < L_1 \leqslant L_2 \leqslant \ldots \leqslant L_l$, $l \in \mathbb{N}$ *such that* $L_{i+1} \geqslant 2kb_i$, $\varphi_p(kb_i/N) \leqslant \min\{\varphi_p(r_{i+1}/H), \varphi_p(L_{i+1}/L)\}$ *for* $1 \leqslant i \leqslant l-1$. *Let the following conditions be satisfied:*

$(D_1)$ $\max\{f(t,u,v), \overline{I}_j(u,v)\} < \min\{\varphi_p(r_i/H), \varphi_p(L_i/L)\}$, $I_j(u) \leqslant r_i/H$ *for* $(t,u,v) \in [0,1] \times [0,r_i] \times [0,L_i]$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant n$;

$(D_2)$ $f(t,u,v) > \varphi_p(kb_i/N)$ *for* $(t,u,v) \in [1/k, 1-1/k] \times [b_i, hkb_i] \times [0, L_{i+1}]$, $1 \leqslant i \leqslant l-1$.

*Then the problem* (1.1) *possesses at least* $2l-1$ *positive solutions.*

Proof. When $l = 1$, it follows from condition $(D_1)$ that

$$T\colon\ \overline{P}(\varphi, r_1; \omega, L_1) \to \overline{P}(\varphi, r_1; \omega, L_1) \subset \overline{P}(\varphi, r_1; \omega, L_1),$$

thus $T$ has at least one fixed point $u_1 \in \overline{P}(\varphi, r_1; \omega, L_1)$ by Schauder's fixed point theorem.

When $l = 2$, it is clear that Theorem 4.1 can be applied to get at least three positive solutions $u_i$ $(i = 1,2,3)$ such that (4.1)–(4.3) hold.

Suppose that for $l = m$ the statement holds, i.e., the problem (1.1) possesses at least $2m-1$ positive solutions $u_1, u_2, \ldots, u_{2m-1}$ such that $\max\limits_{0 \leqslant t \leqslant 1} |u_i(t)| \leqslant r_m$, $i = 1,2,\ldots,2m-1$. When $l = m+1$, by induction hypothesis, in addition to the $2m-1$ positive solutions $u_1, u_2, \ldots, u_{2m-1}$ such that $\max\limits_{0 \leqslant t \leqslant 1} |u_i(t)| \leqslant r_m$, $i = 1,2,\ldots,2m-1$, we can apply Theorem 4.1 to the case

(1) $\max\{f(t,u,v), \overline{I}_j(u,v)\} < \min\{\varphi_p(r_m/H), \varphi_p(L_m/L)\}$, $I_j(u) \leqslant r_m/H$ for $(t,u,v) \in [0,1] \times [0,r_m] \times [0,L_m]$, $1 \leqslant j \leqslant n$;

(2) $f(t,u,v) > \varphi_p(kb_m/N)$ for $(t,u,v) \in [1/k, 1-1/k] \times [b_m, hkb_m] \times [0, L_{m+1}]$;

(3) $\max\{f(t,u,v),\overline{I}_j(u,v)\} \leqslant \min\{\varphi_p(r_{m+1}/H),\varphi_p(L_{m+1}/L)\},\ I_j(u) \leqslant r_{m+1}/H$ for $(t,u,v)\in[0,1]\times[0,r_{m+1}]\times[0,L_{m+1}]$, $1\leqslant j\leqslant n$

to get at least three positive solutions $u_0$, $u_{2m}$ and $u_{2m+1}$ with

$$\max_{0\leqslant t\leqslant 1} u_0(t) < r_m, \quad \max_{0\leqslant t\leqslant 1} |u_0'(t)| < L_m;$$
$$b_m < \min_{1/k\leqslant t\leqslant (k-1)/k} u_{2m}(t) \leqslant \max_{0\leqslant t\leqslant 1} u_2(t) \leqslant r_{m+1}, \quad \max_{0\leqslant t\leqslant 1} |u_{2m}'(t)| \leqslant L_{m+1};$$
$$r_m < \max_{0\leqslant t\leqslant 1} u_{2m+1}(t) \leqslant r_{m+1}, \quad \min_{1/k\leqslant t\leqslant (k-1)/k} u_{2m+1}(t) < b_m,$$
$$\max_{0\leqslant t\leqslant 1} |u_{2m+1}'(t)| \leqslant L_{m+1}.$$

Obviously $u_{2m}, u_{2m+1}$ are different from $u_1, u_2, \ldots, u_{2m-1}$. Thus in this way we get at least $2m+1$ positive solutions to the problem (1.1). $\square$

## 5. Example

Let $q(t)\equiv 1$, $p=\frac{3}{2}$, $n=1$. We consider the boundary value problem

$$(5.1)\qquad \begin{cases} (|u'(t))|^{-1/2}u'(t))' + f(t,u(t),u'(t)) = 0, \quad t\neq \frac{1}{2},\ 0<t<1,\\ \Delta u\Big(\dfrac{1}{2}\Big) = I\Big(u\Big(\dfrac{1}{2}\Big)\Big),\\ \Delta\varphi_{3/2}\Big(u'\Big(\dfrac{1}{2}\Big)\Big) = -\overline{I}\Big(u\Big(\dfrac{1}{2}\Big),u'\Big(\dfrac{1}{2}\Big)\Big),\\ u(0) = \dfrac{1}{4}u\Big(\dfrac{1}{3}\Big) + \dfrac{1}{8}u\Big(\dfrac{2}{3}\Big),\\ \varphi_{3/2}(u'(1)) = \dfrac{1}{4}\varphi_{3/2}\Big(u'\Big(\dfrac{1}{3}\Big)\Big) + \dfrac{1}{8}\varphi_{3/2}\Big(u'\Big(\dfrac{2}{3}\Big)\Big), \end{cases}$$

where

$$f(t,u,v) = \begin{cases} \dfrac{1}{18}t^2 + 4u^6 + \dfrac{1}{2}\times\Big(\dfrac{v}{275}\Big)^4, & u<1,\\ \dfrac{1}{18}t^2 + 4 + \dfrac{1}{2}\times\Big(\dfrac{v}{275}\Big)^4, & u\geqslant 1, \end{cases}$$

$$I(u) = \begin{cases} \dfrac{1}{18}u, & 0\leqslant u\leqslant \dfrac{1}{2},\\ \dfrac{1}{36}, & u>\dfrac{1}{2}, \end{cases}$$

$$\overline{I}(u,v) = \frac{1}{150}u + \frac{1}{140}v, \quad u\geqslant 0,\ v\geqslant 0.$$

Choose $k=4$, $r_1=\frac{1}{2}$, $b_1=1$, $r_2=450$, $L_1=1$, $L_2=275$. Then we have

$$D_1=D_2=\frac{8}{5}=1.6,\ \ h=\frac{7}{5}=1.4,\ \ H=17.984,\ \ N=\frac{1}{4}=0.25,\ \ L=10.24.$$

It is easy to verify that $(C_1)$–$(C_4)$ hold and $\varphi_{3/2}(kb/N)\leqslant\min\{\varphi_{3/2}(r_2/H), \varphi_{3/2}(L_2/L)\}$, $L_2\geqslant 2kb$ and the following conditions are satisfied:

$(B_1')$ $\max\{f(t,u,v),\overline{I}(u,v)\}<0.1181<\min\{\varphi_{3/2}(r_1/H),\varphi_{3/2}(L_1/L)\}\approx 0.1667$, $I(u)\leqslant\frac{1}{36}<r_1/H$ for $(t,u,v)\in[0,1]\times[0,\frac{1}{2}]\times[0,1]$;

$(B_2')$ $f(t,u,v)>\varphi_{3/2}(kb/N)=4$ for $(t,u,v)\in[\frac{1}{4},\frac{3}{4}]\times[1,5.6]\times[0,275]$;

$(B_3')$ $\max\{f(t,u,v),\overline{I}(u,v)\}<5<\min\{\varphi_{3/2}(r_2/H),\varphi_{3/2}(L_2/L)\}\approx 5.002$, $I(u)\leqslant\frac{1}{36}<r_2/H$ for $(t,u,v)\in[0,1]\times[0,450]\times[0,275]$.

Thus, all conditions of Theorem 4.1 hold. By Theorem 4.1, the problem (5.1) has at least three positive solutions $u_1$, $u_2$ and $u_3$ such that

$$\max_{0\leqslant t\leqslant 1}u_1(t)<\frac{1}{2},\quad \max_{0\leqslant t\leqslant 1}|u_1'(t)|<1;$$
$$1<\min_{1/4\leqslant t\leqslant 3/4}u_2(t)\leqslant\max_{0\leqslant t\leqslant 1}u_2(t)\leqslant 450,\quad \max_{0\leqslant t\leqslant 1}|u_2'(t)|\leqslant 275;$$
$$\frac{1}{2}<\max_{0\leqslant t\leqslant 1}u_3(t)\leqslant 450,\quad \min_{1/4\leqslant t\leqslant 3/4}u_3(t)<1,\quad \max_{0\leqslant t\leqslant 1}|u_3'(t)|\leqslant 275.$$

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*Authors' addresses*: Y. Tian (corresponding author), Department of Mathematics, Xiangnan University, Chenzhou, Hunan, 423000, P. R. China, e-mail: tys73@163.com A. Chen, Department of Mathematics, Xiangnan University, Chenzhou, Hunan, 423000, P. R. China; W. Ge, Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, P. R. China.