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THE AXIOMS FOR IMPLICATION IN ORTHOLOGIC

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Abstract. We set up axioms characterizing logical connective implication in a logic derived by an ortholattice. It is a natural generalization of an orthoimplication algebra given by J. C. Abbott for a logic derived by an orthomodular lattice.

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The classical propositional logic has its algebraic counterpart in a Boolean algebra. The clone of functions generated by the logical connective implication is not the clone of all Boolean functions. An algebraic counterpart of the case mentioned is the so-called implication algebra introduced by J. C. Abbott [1]. Let us recall that by an *implication algebra* we mean a groupoid $\mathscr{A} = (A; \circ)$ satisfying the axioms
(I1) $(x \circ y) \circ x = x$ (contraction)
(I2) $(x \circ y) \circ y = (y \circ x) \circ x$ (quasi-commutativity)
(I3) $x \circ (y \circ z) = y \circ (x \circ z)$ (exchange).

Let us note that, for the sake of brevity, $x \circ y$ is a formal expression for "x implies y" (the symbol $x \Rightarrow y$ is not usually used for possible confusion with implication in a meta-language).

It was proved by J. C. Abbott that every implication algebra contains an algebraic constant 1 (its meaning being the logical value TRUE) such that $x \circ x = 1$ is an identity of $\mathscr{A}$. When introducing a binary relation $\leqslant$ by the rule

$$x \leqslant y \text{ if and only if } x \circ y = 1,$$

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the ordered set $(A; \leqslant)$ becomes a join-semilattice with the greatest element 1 where

$$x \vee y = (x \circ y) \circ y$$

and for each $p \in A$ the principal filter $[p)$ is a Boolean algebra where $x^p = x \circ p$ is a complement of $x \in [p)$ in this filter; of course, $x \wedge_p y = ((x \circ p) \vee (y \circ p)) \circ p$ by the De Morgan law.

In the logic of quantum mechanics, Boolean algebra is not satisfactory since this logic does not contain the "rule of excluded middle". Hence, an orthomodular lattice is usually considered to be its algebraic counterpart.

The axiomatization of implication in this logic was established by J. C. Abbott [2] under the condition of the so-called compatibility. Without the compatibility condition, it was treated and axiomatized in [3] and [4].

Since the hypothesis of orthomodularity need not be accepted in all cases, we will generalize these concepts to a much more general logic based on an ortholattice only.

By an *ortholattice* we mean an algebra $\mathscr{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $^{\perp}$ is the so-called *orthocomplementation*, i.e., a unary operation on L satisfying

$x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$ ($x^{\perp}$ is a complement of x)

$x \leqslant y \Rightarrow y^{\perp} \leqslant x^{\perp}$ (antitony)

$x^{\perp\perp} = x$ (involution).

A typical example of an ortholattice which is not orthomodular is that in Fig. 1.

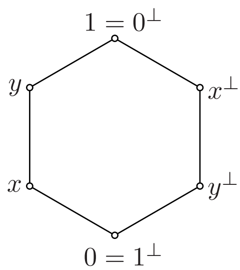

Fig. 1

In fact, an ortholattice is orthomodular if and only if it does not contain a sublattice depicted in Fig. 1.

When describing a connective implication, we do not obtain the whole clone of functions on the corresponding lattice but only those derived by a join-semilattice (for the Boolean case, see [1]). Hence, let us introduce the following crucial concept.

Definition 1. A join-semilattice $\mathscr{S} = (S; \vee, 1)$ with the greatest element 1 is called an *orthosemilattice* if for each $p \in S$ the principal filter $[p)$ is an ortholattice (by $\wedge_p$ the operation meet in $[p)$ is denoted).

Now, we are ready to generalize Abbott's implication algebra for our purposes.

Definition 2. A groupoid $\mathscr{A} = (A; \circ)$ is called a *pre-implication algebra* if it satisfies the identities (I1) and (I2). A pre-implication algebra is called an *ortho-algebra* if it satisfies the axiom

(A) $$(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1.$$

Lemma 1. *Let $\mathscr{A} = (A, \circ)$ be a pre-implication algebra. Then it satisfies the identity*

(I) $$x \circ x = y \circ y,$$

i.e., there exists an algebraic constant 1 such that $x \circ x = 1$ for each $x \in A$ and $\mathscr{A}$ satisfies the identities

(II) $$1 \circ x = x, \ x \circ 1 = 1 \quad \textit{and} \quad x \circ (x \circ y) = x \circ y.$$

Proof. Applying (I1) twice, we obtain

$$x \circ (x \circ y) = ((x \circ y) \circ x) \circ (x \circ y) = x \circ y;$$

thus, using this and (I2), we infer

$$x \circ x = ((x \circ y) \circ x) \circ x = (x \circ (x \circ y)) \circ (x \circ y) = (x \circ y) \circ (x \circ y).$$

Hence,

$$\begin{aligned} x \circ x &= (x \circ y) \circ (x \circ y) = ((x \circ y) \circ y) \circ ((x \circ y) \circ y) \\ &= ((y \circ x) \circ x) \circ ((y \circ x) \circ x) = (y \circ x) \circ (y \circ x)) = y \circ y, \end{aligned}$$

which proves (I). Denote $x \circ x = 1$. By (I1) we obtain

$$1 \circ x = (x \circ x) \circ x = x$$

and

$$x \circ 1 = x \circ (x \circ x) = x \circ x = 1,$$

thus also (II) is satisfied. □

Lemma 2. *Let* $\mathscr{A} = (A; \circ)$ *be a pre-implication algebra. Define a binary relation* $\leqslant$ *on* A *by the rule*

$$x \leqslant y \quad \text{if and only if} \quad x \circ y = 1. \tag{R}$$

Then $\leqslant$ *is reflexive and antisymmetrical. If* $\mathscr{A}$ *is an orthoalgebra then* $\leqslant$ *is an order on* A *and* 1 *is the greatest element.*

P r o o f. Reflexivity of $\leqslant$ is obvious by Lemma 1. Suppose $x \leqslant y$ and $y \leqslant x$. Then, by (R), $x \circ y = 1$ and $y \circ x = 1$. Applying (I2) we derive

$$x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y,$$

proving the antisymmetry.

Suppose that $\mathscr{A}$ is an ortho-algebra. Let $x \leqslant y$ and $y \leqslant z$. Then $x \circ y = 1$, $y \circ z = 1$ and, by the identity (A),

$$\begin{aligned} 1 &= (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) \\ &= (y \circ z) \circ (x \circ z) = 1 \circ (x \circ z) = x \circ z. \end{aligned}$$

Thus, by (R), $x \leqslant z$, which proves transitivity of $\leqslant$. Hence, $\leqslant$ is an order on A. Due to (II), $x \leqslant 1$ for each $x \in A$. □

From now on, let us call the relation $\leqslant$ defined by (R) the *induced order* of an ortho-algebra $\mathscr{A} = (A; \circ)$.

Lemma 3. *Let* $\mathscr{A} = (A; \circ)$ *be an ortho-algebra and* $\leqslant$ *the induced order. Then*
(a) $x \leqslant y$ *implies* $y \circ z \leqslant x \circ z$ *for all* $x, y, z \in A$;
(b) $x \leqslant (x \circ y) \circ y$ *and* $y \leqslant (x \circ y) \circ y$;
(c) *if* $x, y \leqslant z$ *then* $z \leqslant (x \circ y) \circ y$.

P r o o f. (a) Suppose $x \leqslant y$. Then $x \circ y = 1$, i.e. $(x \circ y) \circ y = 1 \circ y = y$. Using the identity (A), we infer $1 = (((x \circ y) \circ y) \circ z) \circ (x \circ z) = (y \circ z) \circ (x \circ z)$, which yields $y \circ z \leqslant x \circ z$.

(b) Applying (a), we obtain easily $y \leqslant 1 \Rightarrow x = 1 \circ x \leqslant y \circ x$, i.e. $\mathscr{A}$ satisfies the identity

$$x \circ (y \circ x) = 1. \tag{III}$$

Now, by (III) and (I2) we infer

$$x \circ ((x \circ y) \circ y) = x \circ ((y \circ x) \circ x) = 1$$

whence $x \leqslant (x \circ y) \circ y$. Interchanging x and y, we obtain also $y \leqslant (x \circ y) \circ y$.

(c) Let $x, y \leqslant z$. Then, by (a), $z \circ y \leqslant x \circ y$ and hence $(x \circ y) \circ y \leqslant (z \circ y) \circ y = (y \circ z) \circ z = 1 \circ z = z$. □

Theorem 1. *Let $\mathscr{A} = (A; \circ)$ be an ortho-algebra and $\leqslant$ the induced order. Then $\mathscr{A}$ is a join-semilattice with the greatest element 1 where $x \vee y := (x \circ y) \circ y$. For each $p \in A$, the principal filter $[p)$ is an ortholattice where for $x \in [p)$ its orthocomplement in $[p)$ is $x^p = x \circ p$.*

P r o o f. By (b) and (c) of Lemma 3 it follows that $(x \circ y) \circ y$ is the supremum of x, y with respect to $\leqslant$, i.e., $x \vee y = (x \circ y) \circ y$ and $(A; \leqslant)$ is a join-semilattice. Of course, 1 is the greatest element of $(A; \vee)$.

By (a) of Lemma 3, the mapping $x \mapsto x^p = x \circ p$ is antitone in $[p)$. Further,

$$x^{pp} = (x \circ p) \circ p = x \vee p = x$$

for each $x \in [p)$, thus it is an involution. Hence, we can apply the De Morgan law to show that

$$x \wedge_p y = (x^p \vee y^p)^p$$

is the infimum of $x, y \in [p)$ in this filter. Hence, $([p); \vee, \wedge_p)$ is a bounded lattice. Finally, $x \vee x^p = x \vee (x \circ p) = ((x \circ p) \circ x) \circ x = x \circ x = 1$ and thus, due to to the De Morgan law, x^p is an orthocomplement of x in $[p)$. □

Example. A (semi)lattice which is an orthosemilattice but not an orthomodular lattice is depicted in Fig. 2.

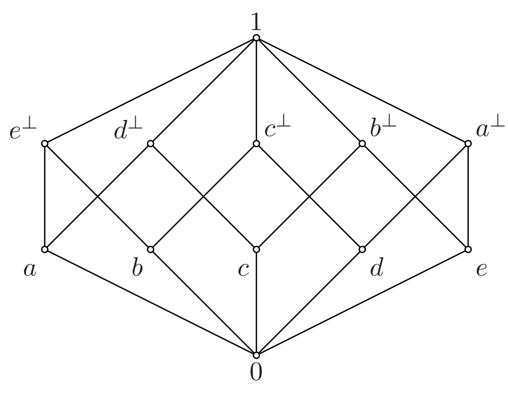

Fig. 2

Really, $a \leqslant e^\perp$ in $[0, 1]$ but

$$a \vee (e^\perp \wedge a^\perp) = a \vee 0 = a \neq e^\perp,$$

which contradicts the orthomodular law.

We are going to prove the converse of Theorem 1. For this, let us mention that $x \circ y$ is an implication in orthologic, i.e., we need to define an implication on a join-semilattice where every principal filter is an ortholattice. In the classical logic we know that $x \Rightarrow y = \neg x \vee y$. However, in the Boolean case, $\neg x \vee y$ can be rewritten as

$$(x \vee y)^y$$

in our notation. Hence, we are going to try the same also for the case of orthosemilattice.

Theorem 2. *Let $\mathscr{S} = (S; \vee, 1)$ be an orthosemilattice. Denote by x^p the orthocomplement of $x \in [p)$ in this filter. Define a binary operation $\circ$ on S by setting*

$$x \circ y = (x \vee y)^y.$$

Then $\circ$ is defined everywhere on S and $(S; \circ)$ is an ortho-algebra.

Proof. Since $x \vee y \in [y)$ for all $x, y \in S$, the operation "$\circ$" is defined everywhere on S. We need only to verify the identities (I1), (I2) and (A). Easily we infer

$$(x \circ y) \circ x = ((x \vee y)^y \vee x)^x = ((x \vee y)^y \vee (x \vee y))^x = 1^x = x$$

since $(x \vee y)^y \geqslant y$ and $(x \vee y)^y$ is the complement of $(x \vee y)$ in $[y)$.

Further, $(x \circ y) \circ y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ since the orthocomplementation is an involution. Analogously $(y \circ x) \circ x = y \vee x = x \vee y$, whence (I2) is evident.

Finally,

$$\begin{aligned}(((x \circ y) \circ y) \circ z) \circ (x \circ z) &= ((x \vee y) \circ z) \circ (x \circ z) \\ &= ((x \vee y \vee z)^z \vee (x \vee z)^z)^{(x \vee z)^z} = ((x \vee z)^z)^{(x \vee z)^z} = 1\end{aligned}$$

due to antitony of orthocomplementation. □

We close our study by several important congruence properties of ortho-algebras.

Theorem 3. *Ortho-algebras are congruence distributive.*

Proof. Consider the ternary terms $t_0(x, y, z) = x$, $t_1(x, y, z) = (y \circ (z \circ x)) \circ x$, $t_2(x, y, z) = (x \circ y) \circ z$, $t_3(x, y, z) = z$. We can verify that $t_0(x, y, x) = x$, $t_1(x, y, x) = (y \circ 1) \circ x = 1 \circ x = x$, $t_2(x, y, x) = (x \circ y) \circ x = x$ (by (I1)), $t_3(x, y, x) = x$.

For i even we have $t_0(x, x, y) = x = 1 \circ x = (x \circ (y \circ x)) \circ x = t_1(x, x, y)$, by the identity (III) of the proof of Lemma 3, $t_2(x, x, y) = (x \circ x) \circ y = 1 \circ y = y = t_3(x, x, y)$.

For i odd we have $t_1(x, y, y) = (y \circ (y \circ x)) \circ x = (y \circ x) \circ x = (x \circ y) \circ y = t_2(x, y, y)$. Hence, t_0, t_1, t_2, t_3 are Jónsson terms (see [5], [6]), which proving congruence distributivity of the variety of ortho-algebras. □

Let Θ be a congruence on an ortho-algebra $\mathscr{A} = (A; \circ)$. The class $[1]_\Theta$ will be called a *kernel* of Θ. An algebra is called *weakly regular* if every $\Theta \in \operatorname{Con} \mathscr{A}$ is determined by its kernel, i.e., if $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ and $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.

Theorem 4. *The variety of all ortho-algebras is weakly regular.*

Proof. By Theorem 6.4.3 in [5] (Csákány Theorem), a variety $\mathscr{V}$ with a constant 1 is weakly regular if and only if there are binary terms $t_1, \ldots, t_n$ $(n \geqslant 1)$ such that $t_1(x,y) = \ldots = t_n(x,y) = 1$ if and only if $x = y$. Take $n = 2$ and $t_1(x,y) = x \circ y$, $t_2(x,y) = y \circ x$. Of course, $t_1(x,x) = t_2(x,x) = x \circ x = 1$. Conversely, if $t_1(x,y) = t_2(x,y) = 1$ then, by Lemma 2, $x \leqslant y$ and $y \leqslant x$ whence $x = y$. Thus the variety of ortho-algebras is weakly regular. $\square$

Since every $\Theta \in \operatorname{Con} \mathscr{A}$ its determined by its kernel $[1]_\Theta$ it is a natural question to describe the congruence kernels of ortho-algebras. For weakly regular varieties it was already done in [5], thus we can only specify it in our particular case. For this, call a subset D of an ortho-algebra $\mathscr{A} = (A; \circ)$ a *deductive system* if $1 \in D$ and whenever $a \in D$ and $a \circ b \in D$ then also $b \in D$. Let us notice that this is a form of Modus Ponens on the quotient ortho-algebra by the deductive system D (under the condition that D is the kernel of a congruence on $\mathscr{A}$ which will be just shown).

Theorem 5. *A subset D of an ortho-algebra $\mathscr{A} = (A; \circ)$ is a congruence kernel if and only if D is a deductive system. If D is a deductive system then D is a kernel of the congruence*

$$\Theta_D = \{\langle x,y \rangle \in A^2;\ x \circ y \in D,\ \text{and } y \circ x \in D\}.$$

The proof follows immediately by Theorems 9.4.8. and 9.4.9 of [5].

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