Czechoslovak Mathematical Journal, 55 (130) (2005), 1041–1054

ON HARMONIC MAJORIZATION OF THE MARTIN FUNCTION AT INFINITY IN A CONE

I. Miyamoto, M. Yanagishita, and H. Yoshida, Chiba

(Received April 14, 2003)

Abstract. This paper shows that some characterizations of the harmonic majorization of the Martin function for domains having smooth boundaries also hold for cones.

Keywords: harmonic majorization, cone, minimally thin

MSC 2000: 31B05, 31B20

1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbb{R}^n$ $(n \geqslant 2)$ the n-dimensional Euclidean space. A point in $\mathbb{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbb{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbb{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbb{R}^n$ are denoted by ∂S and $\overline{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbb{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by

$$x_1 = r\biggl(\prod_{j=1}^{n-1} \sin\theta_j\biggr) \quad (n \geqslant 2), \quad y = r\cos\theta_1,$$

and if $n \geqslant 3$, then

$$x_{n+1-k} = r\biggl(\prod_{j=1}^{k-1} \sin\theta_j\biggr)\cos\theta_k \quad (2 \leqslant k \leqslant n-1),$$

where $0 \leqslant r < +\infty$, $-\frac{1}{2}\pi \leqslant \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geqslant 3$, then $0 \leqslant \theta_j \leqslant \pi$ $(1 \leqslant j \leqslant n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbb{S}^{n-1}$ and $\mathbb{S}^{n-1}_+$, respectively. For simplicity, a point $(1,\Theta)$ on $\mathbb{S}^{n-1}$ and the set $\{\Theta\colon (1,\Theta)\in\Omega\}$ for a set Ω, $\Omega\subset\mathbb{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda\subset\mathbb{R}_+$ and $\Omega\subset\mathbb{S}^{n-1}$, the set

$$\{(r,\Theta)\in\mathbb{R}^n\colon\ r\in\Lambda,\ \ (1,\Theta)\in\Omega\}$$

in $\mathbb{R}^n$ is simply denoted by $\Lambda\times\Omega$. In particular, we denote by $C_n(\Omega)$ the set $\mathbb{R}_+\times\Omega$ in $\mathbb{R}^n$ with the domain Ω on $\mathbb{S}^{n-1}$ ($n\geqslant 2$). We call it a cone. Then the half-space $\mathbb{T}_n=\{(X,y)\in\mathbb{R}^n\colon\ y>0\}$ is a cone obtained by putting $\Omega=\mathbb{S}^{n-1}_+$.

To extend a result of Beurling [7] for n=2, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m=(X_m,y_m)\in\mathbb{T}_n$, $|P_m|\to+\infty$ ($m\to+\infty$) *characterizes the positive harmonic majorization of* y, if every positive harmonic function h in $\mathbb{T}_n$ which majorizes the function y on the set $\{P_m\colon\ m=1,2,\ldots\}$ majorizes y everywhere in $\mathbb{T}_n$, i.e.

$$\inf_{P\in\mathbb{T}_n}\frac{h(P)}{y}=\inf_m\frac{h(P_m)}{y_m}\quad (P=(X,y)\in\mathbb{T}_n).$$

They proved

Theorem A (Beurling [7] for $n=2$, Armitage and Kuran [4, Theorem 1] for $n\geqslant 2$)**.** *Let $\{P_m\}$ be a sequence of points,*

$$P_m=(r_m,\Theta_m)\in\mathbb{T}_n,\quad \Theta_m=(\theta_{1,m},\theta_{2,m},\ldots,\theta_{(n-1),m})$$

in $\mathbb{T}_n$ satisfying

$$r_{m+1}\geqslant a\,r_m\quad (m=1,2,\ldots) \tag{1.1}$$

for a certain $a>1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of y if and only if

$$\sum_{m=1}^{\infty}(\cos\theta_{1,m})^n=+\infty. \tag{1.2}$$

Theorem A was also extended by Maz'ya [15] to positive solutions of a second order elliptic differential equation in an n-dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ ($0<\alpha<1$).

Let D be a domain in $\mathbb{R}^n$ and $\Delta(D)$ the Martin boundary of D. The Martin function at $Q\in\Delta(D)$ is denoted by $K_Q(P)$ ($P\in D$) (for these definitions see e.g. Helms [14, pp. 243–245], Armitage and Gardiner [5, pp. 235–237]). Following Armitage and Kuran [4], we say that a subset E of D *characterizes the positive harmonic majorization of* $K_Q(P)$, if every positive harmonic function h in D which majorizes $K_Q(P)$ on E majorizes $K_Q(P)$ everywhere in D, i.e.

$$\inf_{P\in D}\frac{h(P)}{K_Q(P)}=\inf_{P\in E}\frac{h(P)}{K_Q(P)}. \tag{1.3}$$

We set

$$B(P,r)=\{P'\in\mathbb{R}^n:\ |P'-P|<r\}\quad (r>0)$$

and

$$d(P)=\inf_{Q\notin D}|P-Q|$$

for any $P\in D$. For a subset E of D and a number ϱ $(0<\varrho<1)$ we put

$$E_\varrho=\bigcup_{P\in E}B(P,\varrho d(P)). \tag{1.4}$$

Dahlberg proved

Theorem B (Dahlberg [10, Theorem 1]). *Let D be a Liapunov-Dini domain in $\mathbb{R}^n$ and $Q\in\partial D$. If $E\subset D$, then the following conditions on E are equivalent:*
(i) *E characterizes the positive harmonic majorization of $K_Q(P)$;*
(ii) *for every ϱ, $0<\varrho<1$*

$$\int_{E_\varrho}|P-Q|^{-n}\,\mathrm{d}P=+\infty;$$

(iii) *for some ϱ, $0<\varrho<1$*

$$\int_{E_\varrho}|P-Q|^{-n}\,\mathrm{d}P=+\infty.$$

Since (1.3) is closely related to the notion of minimal thinness of E_ϱ in (1.4) (see Sjögren [18], Ancona [3] and Zhang [21]), which will be also seen in Theorem 2 of this paper, Aikawa and Essén [2, Corollary 7.4.7] also proved Theorem B in a way different from Dahlberg's.

By using a suitable Kelvin transformation which maps $\mathbb{T}_n$ onto a ball, the following Theorem C follows from Theorem B.

Theorem C (Dahlberg [10, Theorem 3]). *If $E \subset \mathbb{T}_n$, then the following conditions on E are equivalent:*

(i) *E characterizes the positive harmonic majorization of y;*

(ii) *for every ϱ, $0 < \varrho < 1$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty;$$

(iii) *for some ϱ, $0 < \varrho < 1$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty.$$

All proofs of Theorems A and B are based on the smoothness of the boundary having no wedges, e.g. a ball. For a domain having rougher boundary, e.g. a Lipschitz domain, Ancona [3, Theorem 7.4] and Zhang [21, Theorem 3] gave more complicated results which generalize Theorem A.

In this paper we shall prove that Theorems A and C can be still extended in the similar form to a result at a corner point of a wedge, i.e. to a result at ∞ of a cone (Theorem 3). We remark that a half-space is one of cones. To prove this result, we need a result (Theorem 2) which is a specialized version of that due to Aikawa [1, Theorem 1]. Since his proof is too complicated we give a simple proof based on an example of positive harmonic functions (Theorem 1).

For a Lipschitz domain and an NTA domain D, Zhang [21, Corollary 1] and Aikawa [1, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset E of D to characterize the positive harmonic majorization of $K_Q(P)$ by connecting it with minimal thinness of E_ϱ in (1.4). On the other hand, with respect to the quantitative Theorem B Aikawa said in his paper [1] that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. However, if we observe in this paper that a cone has a wedge, at the corner point of which Theorem B still holds, against Aikawa's opinion we may ask whether Theorem B can be extended in the similar form to a result for a Lipschitz domain or an NTA domain.

2. Statements of results

Let Ω be a domain on $\mathbb{S}^{n-1}$ ($n \geqslant 2$) with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \tau) f = 0 \quad \text{on } \Omega,$$
$$f = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n:

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$; hence

$$\int_\Omega f_\Omega^2(\Theta)\,\mathrm{d}\sigma_\Theta = 1,$$

where $\mathrm{d}\sigma_\Theta$ is the surface element on $\mathbb{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by α_Ω, $-\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbb{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbb{S}^{n-1}$.

To simplify our next consideration, we shall assume that if $n \geqslant 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbb{S}^{n-1}$ (see e.g. Gilbarg and Trudinger [12, pp. 88–89] for the definition of a $C^{2,\alpha}$-domain). It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each point of which is a minimal Martin boundary point, and the Martin kernel at ∞ with respect to a reference point chosen suitably is $K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$ $(P = (r,\Theta) \in C_n(\Omega))$ (see e.g. Yoshida [20, pp. 276–277]). In particular, y is the Martin function at ∞ of $\mathbb{T}_n$.

A subset E of a domain D in $\mathbb{R}^n$ is said to be *minimally thin* at $Q \in \Delta(D)$ (Brelot [8, p. 122], Doob [11, p. 208]), if there exists a point $P \in D$ such that

$$\hat{R}^E_{K_Q(\cdot)}(P) \neq K_Q(P),$$

where $\hat{R}^E_{K_Q(\cdot)}(P)$ is the regularized reduced function of $K_Q(P)$ relative to E (Helms [14, p. 134]).

The following results are conical versions of Dahlberg's results [10, p. 239].

Theorem 1. *Let E be a set in $C_n(\Omega)$ satisfying $\overline{E} \cap \partial C_n(\Omega) = \emptyset$. If E_ϱ with a positive number ϱ $(0 < \varrho < 1)$ is minimally thin at ∞, then there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P\in E} \frac{h(P)}{K_\infty(P)}.$$

Theorem 2. *Let E be a subset of $C_n(\Omega)$. The following conditions on E are equivalent:*

(i) *E characterizes the positive harmonic majorization of $K_\infty(P)$;*

(ii) *for any ϱ, $0 < \varrho < 1$, E_ϱ is not minimally thin at ∞;*

(iii) *for some ϱ, $0 < \varrho < 1$, E_ϱ is not minimally thin at ∞.*

The following Theorem 3 extends Theorem C.

Theorem 3. *Let E be a subset of $C_n(\Omega)$. Then the following conditions on E are equivalent:*

(i) *E characterizes the positive harmonic majorization of $K_\infty(P)$;*

(ii) *for every ϱ $(0 < \varrho < 1)$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty;$$

(iii) *for some ϱ $(0 < \varrho < 1)$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty.$$

A sequence $\{P_m\}$ of points $P_m \in D$ is said to be *separated*, if there exists a positive constant c such that

$$|P_i - P_j| \geqslant c d(P_i) \quad (i, j = 1, 2, \ldots, \ \ i \neq j)$$

(see e.g. Ancona [3, p. 18], Aikawa and Essén [2, p. 156]).

From Theorem 3 we immediately obtain the following Corollary which extends Theorem A.

Corollary. *Let $\{P_m\}$, $P_m \in C_n(\Omega)$ be a separated sequence satisfying*

$$\inf_m |P_m| > 0.$$

The sequence $\{P_m\}$ characterizes the positive harmonic majorization of $K_\infty(P)$ if and only if

$$\sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n = +\infty.$$

Let f and g be two positive real valued functions defined on a set S. Then we shall write $f \approx g$, if there exist two constants A_1, A_2, $0 < A_1 \leqslant A_2$ such that $A_1 g \leqslant f \leqslant A_2 g$ everywhere on S. For a subset S in $\mathbb{R}^n$, the interior of S and the diameter of S are denoted by $\operatorname{int} S$ and $\operatorname{diam} S$, respectively. For two subsets S_1 and S_2 in $\mathbb{R}^n$, the distance between S_1 and S_2 is denoted by $\operatorname{dist}(S_1, S_2)$. A cube $\mathcal{M}_k$ $(k = 0, \pm 1, \pm 2, \ldots)$ is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \ldots \times [l_n 2^{-k}, (l_n+1)2^{-k}]$$

where $l_1, \ldots, l_n$ are integers. Let ϱ be a number satisfying $0 < \varrho \leqslant \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with ϱ is the set of cubes having the following properties:

(i) $\bigcup\limits_i W_i = C_n(\Omega)$,

(ii) $\operatorname{int} W_i \cap \operatorname{int}\ W_j = \emptyset\ (i \neq j)$,

(iii) $[8/(3\varrho)] \operatorname{diam} W_i \leqslant \operatorname{dist}(W_i, \mathbb{R}^n \setminus C_n(\Omega)) \leqslant 2([8/(3\varrho)] + 1) \operatorname{diam} W_i$,

where [a] denotes the integer satisfying $[a] \leqslant a < [a] + 1$ (Stein [19, p. 167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

Lemma 1 (I. Miyamoto, M. Yanagishita and H. Yoshida [16, Theorems 2 and 3])**.** *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞. Then we have*

$$\int_E \frac{\mathrm{d}P}{(1+|P|)^n} < +\infty. \tag{3.1}$$

If E is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with ϱ $(0 < \varrho \leqslant \frac{1}{2})$, then (3.1) is also sufficient for E to be minimally thin at ∞.

For a set $E \subset C_n(\Omega)$ and a number ϱ $(0 < \varrho \leqslant \frac{1}{2})$, define E_ϱ and $E_{\varrho/4}$ as in (1.4).

Lemma 2. *Let $\{W_i\}_{i \geqslant 1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with ϱ. Let E be a subset of $C_n(\Omega)$. Then there exists a subsequence $\{W_{i_j}\}_{j \geqslant 1}$ of $\{W_i\}_{i \geqslant 1}$ such that*

(i) $\bigcup\limits_j W_{i_j} \subset E_\varrho$,

(ii) $W_{i_j} \cap E_{\varrho/4} \neq \emptyset$ $(j = 1, 2, \ldots)$, $E_{\varrho/4} \subset \bigcup\limits_j W_{i_j}$.

Proof. Let k be an integer. Let $c = [8/(3\varrho)] + 1$ and set

$$I_k = \{P \in C_n(\Omega)\colon\ c\sqrt{n}2^{-k} < \operatorname{dist}(P, \partial C_n(\Omega)) \leqslant c\sqrt{n}2^{-k+1}\}.$$

Let $\{W_{i_j}\}_{j\geqslant 1}$ be a subsequence of all Whitney cubes from $\{W_i\}_{i\geqslant 1}$ such that

$$W_{i_j} \cap E_{\varrho/4} \neq \emptyset \quad (j = 1, 2, \ldots).$$

Then it is evident that (ii) holds. We shall also show that this $\{W_{i_j}\}_{j\geqslant 1}$ satisfies (i), i.e. $W_{i_j} \subset E_\varrho$ $(j = 1, 2, \ldots)$.

Take any W_{i_j} $(j = 1, 2, \ldots)$. Since $W_{i_j} \cap E_{\varrho/4} \neq \emptyset$, there exists a point P_j in E such that

$$B(P_j, \frac{\varrho}{4} d(P_j)) \cap W_{i_j} \neq \emptyset. \tag{3.2}$$

We can easily see that $W_{i_j} \in \mathcal{M}_{m+1} \cup \mathcal{M}_m \cup \mathcal{M}_{m-1}$, if there is a point $P \in I_m$ such that $W_{i_j} \cap B(P, \frac{\varrho}{4} d(P)) \neq \emptyset$. Hence, for an integer k satisfying $W_{i_j} \in \mathcal{M}_k$, P_j taken above satisfies $P_j \in I_{k+1} \cup I_k \cup I_{k-1}$. So, if $P_j \in I_{k+1}$, then

$$\varrho d(P_j) - \frac{\varrho}{4} d(P_j) = \frac{3}{4} \varrho d(P_j) > \frac{3}{4} \varrho \Big(\Big[\frac{8}{3\varrho} \Big] + 1 \Big) \sqrt{n} 2^{-(k+1)} > \sqrt{n} 2^{-k}.$$

Since the diameter of W_{i_j} is $\sqrt{n} 2^{-k}$, we have from (3.2) that $W_{i_j} \subset B(P_j, \varrho d(P_j))$ and hence $W_{i_j} \subset E_\varrho$. If $P_j \in I_k$ or $P_j \in I_{k-1}$, then we similarly have $W_{i_j} \subset E_\varrho$. □

P r o o f of Theorem 1. If E is a bounded subset of $C_n(\Omega)$, then let h be a constant function. When E is unbounded, we shall follow Dahlberg [10, p. 240] to make the required function.

We can assume $\varrho \leqslant \frac{1}{2}$. Let $\{P_j\}$ be a sequence of points P_j which are the central points of cubes W_{i_j} in Lemma 2. Then by our assumption on E, $\{P_j\}$ can not accumulate to any finite boundary point of $C_n(\Omega)$ and hence $|P_j| \to +\infty$, because $P_j \in E_\varrho$ due to (i) of Lemma 2. Since E_ϱ is minimally thin at ∞ and

$$\int_{W_{i_j}} \frac{\mathrm{d}P}{(1 + |P|)^n} \approx \Big(\frac{d(P_j)}{|P_j|} \Big)^n \quad (j = 1, 2, \ldots),$$

Lemma 1 and (i) of Lemma 2 give

$$\sum_{j=1}^{\infty} \Big(\frac{\mathrm{d}(P_j)}{|P_j|} \Big)^n < +\infty. \tag{3.3}$$

Hence we can take a positive integer J such that $d(P_j) \leqslant \frac{1}{2} |P_j|$ for every $j \geqslant J$.

Now, take a point $Q_j = (t_j, \Phi_j) \in \partial C_n(\Omega) \setminus \{O\}$ satisfying

$$|P_j - Q_j| = d(P_j) \quad (j = J, J+1, \ldots).$$

Then we also see $|Q_j| \geqslant \frac{1}{2}|P_j|$ and hence $|Q_j| \to +\infty$ $(j \to +\infty)$. We define $h_1(P)$ by

$$h_1(P) = \sum_{j=J}^{\infty} \mathbb{P}_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}}, \quad \mathbb{P}_{Q_j}(P) = \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} \quad (P \in C_n(\Omega)),$$

where $G(P_1, P_2)$ $(P_1, P_2 \in C_n(\Omega))$ is the Green function of $C_n(\Omega)$ and $\partial/\partial n_Q$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then h_1 is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$ we have

$$\mathbb{P}_{Q_j}(P) \approx r^{\alpha_\Omega} f_\Omega(\Theta) t_j^{-\beta_\Omega - 1} \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

for every Q_j satisfying $t_j \geqslant 2r$ (see Azarin [6, Lemma 1]).

First, to see

$$\inf_{P \in E} \frac{h_1(P)}{K_\infty(P)} > 0, \tag{3.4}$$

denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $\mathbb{P}_j(P, Q)$ $(P \in B_j, Q \in \partial B_j)$. Then we have

$$\mathbb{P}_{Q_j}(P) \geqslant \mathbb{P}_j(P, Q_j) \quad (P \in B_j;\ j = J, J+1, \dots)$$

and hence

$$\mathbb{P}_{Q_j}(P_j) \geqslant \mathbb{P}_j(P_j, Q_j) = s_n^{-1}\{d(P_j)\}^{1-n} \quad (j = J, J+1, \dots).$$

Since

$$f_\Omega(\Theta) \approx d(P') \quad (P' = (1, \Theta),\ \ \Theta \in \Omega),$$

we obtain

$$h_1(P_j) \geqslant \mathbb{P}_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}} \geqslant AK_\infty(P_j) \quad (j = J, J+1, \dots) \tag{3.5}$$

with some positive constant A. Now, take any $P \in E$. Then by (ii) of Lemma 2 there exists a point P_j such that

$$|P - P_j| < \frac{1}{2} \operatorname{diam}(W_{i_j}) \leqslant \delta d(P_j) \quad \Big(\delta = \frac{1}{2}\Big[\frac{8}{3\varrho}\Big]^{-1}\Big).$$

From Harnack's inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]) we have

$$h_1(P) \geqslant \frac{1-\delta}{(1+\delta)^{n-1}} h_1(P_j), \quad K_\infty(P) \leqslant \frac{1+\delta}{(1-\delta)^{n-1}} K_\infty(P_j).$$

These inequalities and (3.5) immediately give (3.4).

Next, for a fixed ray L which is inside $C_n(\Omega)$ and starts from O, we shall show

$$\lim_{|P|\to+\infty,\, P\in L} \frac{h_1(P)}{K_\infty(P)} = 0. \tag{3.6}$$

Put

$$g_j(P) = \frac{\mathbb{P}_{Q_j}(P)}{K_\infty(P)} |P_j|^{\beta_\Omega+1} \quad (P \in C_n(\Omega);\ j = J, J+1, \ldots).$$

Then we have

$$\frac{h_1(P)}{K_\infty(P)} = \sum_{j=J}^{\infty} g_j(P) \left(\frac{d(P_j)}{|P_j|} \right)^n.$$

Since

$$\mathbb{P}_{Q_j}(P) \approx t_j^{\alpha_\Omega-1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j) \quad (P = (r,\Theta) \in C_n(\Omega),\ r \geqslant 2t_j) \tag{3.7}$$

(see Azarin [6, Lemma 1]), we see that

$$\lim_{|P|\to+\infty,\, P\in L} g_j(P) = 0$$

for any fixed $j \geqslant J$. Hence if we can show that

$$|g_j(P)| \leqslant M \quad (P \in L;\ j = J, J+1, \ldots) \tag{3.8}$$

for some constant M independent of j, then we shall have (3.6) from (3.3) and Lebesgue's dominated convergence theorem.

Now we shall prove (3.8) by dividing the proof into three cases. If $r \leqslant \frac{t_j}{2}$, then we have

$$\mathbb{P}_{Q_j}(P) \approx r^{\alpha_\Omega} t_j^{-\beta_\Omega-1} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

and hence

$$|g_j(P)| \leqslant M \quad (P = (r,\Theta) \in C_n(\Omega);\ j = J, J+1, \ldots).$$

If $r \geqslant 2t_j$, then we have

$$|g_j(P)| \leqslant M \quad (P = (r,\Theta) \in C_n(\Omega);\ j = J, J+1, \ldots)$$

from (3.7). Finally, put $R_1 = r/t_j$, $u = t_j$ and $\Theta_1 = \Theta$ in

$$u^{n-2}G((uR_1,\Theta_1),(uR_2,\Theta_2)) = G((R_1,\Theta_1),(R_2,\Theta_2)),$$
$$((R_1,\Theta_1),(R_2,\Theta_2) \in C_n(\Omega)).$$

When (R_2,Θ_2) approaches $(1,\Phi_j)$ along the inward normal, we obtain

$$\frac{\partial G(P,Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}}\frac{\partial G}{\partial n_{Q_j'}}\Bigl(\Bigl(\frac{r}{t_j},\Theta\Bigr),(1,\Phi_j)\Bigr).$$

If $\frac{1}{2}t_j \leqslant r \leqslant 2t_j$, then

$$t_j^{n-1}\mathbb{P}_{Q_j}(P) \leqslant M' \quad (P=(r,\Theta)\in L;\ j=J,J+1,\ldots)$$

for some constant M' and hence

$$|g_j(P)| \leqslant M \quad (P\in L;\ j=J,J+1,\ldots).$$

Finally, put $\gamma = \max\limits_{1\leqslant j<J} K_\infty(P_j)$ and $h(P)=h_1(P)+\gamma$ for any $P\in C_n(\Omega)$. Then we easily see from (3.4) and (3.6) that $h(P)$ is also a positive harmonic function on $C_n(\Omega)$ required in Theorem 1.

Proof of Theorem 2. (i) $\Rightarrow$ (ii). Let c be a positive constant and put $E_1 = \{P\in E\colon K_\infty(P) > c\}$. Then E_1 is a set satisfying $\overline{E_1}\cap\partial C_n(\Omega)=\emptyset$. Since E characterizes the harmonic majorization of $K_\infty(P)$, E_1 also characterizes the harmonic majorization of $K_\infty(P)$. Indeed, otherwise there would exist a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$a = \inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} < \inf_{P\in E_1}\frac{h(P)}{K_\infty(P)} = b.$$

If we put $u(P)=h(P)+bc$ $(P\in C_n(\Omega))$, then $u(P)\geqslant bK_\infty(P)$ for all $P\in E$ and hence

$$\inf_{P\in C_n(\Omega)}\frac{u(P)}{K_\infty(P)} = a < b \leqslant \inf_{P\in E}\frac{u(P)}{K_\infty(P)},$$

which contradicts (i).

If we can show that for any ϱ $(0<\varrho<1)$ $(E_1)_\varrho$ is not minimally thin at ∞, then for any ϱ $(0<\varrho<1)$ E_ϱ is not minimally thin at ∞ either, which is (ii).

So, suppose that for some number ϱ $(0<\varrho<1)$ $(E_1)_\varrho$ is minimally thin at ∞. Then by Theorem 1 there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} < \inf_{P\in E_1}\frac{h(P)}{K_\infty(P)},$$

which contradicts the fact that E_1 characterizes the harmonic majorization of $K_\infty(P)$.

(iii) $\Rightarrow$ (i). Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then there exists a positive harmonic function $h(P)$ in $C_n(\Omega)$ such that

$$a = \inf_{P\in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P\in E} \frac{h(P)}{K_\infty(P)} = b.$$

If we put $v(P) = h(P) - aK_\infty(P)$ $(P \in C_n(\Omega))$, then $v(P)$ is a positive harmonic function on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} = 0. \tag{3.9}$$

Let ϱ be any positive number satisfying $0 < \varrho < 1$. For any $P \in E_\varrho$, there exists a point $P' \in E$ such that $|P - P'| < \varrho d(P')$ and hence

$$\Big(\frac{1-\varrho}{1+\varrho}\Big)^n \frac{v(P')}{K_\infty(P')} \leqslant \frac{v(P)}{K_\infty(P)}$$

by Harnack's inequality. Hence we have

$$\inf_{P\in E_\varrho} \frac{v(P)}{K_\infty(P)} \geqslant \Big(\frac{1-\varrho}{1+\varrho}\Big)^n \inf_{P\in E} \frac{v(P)}{K_\infty(P)} = \Big(\frac{1-\varrho}{1+\varrho}\Big)^n (b-a) > 0. \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} < \inf_{P\in E_\varrho} \frac{v(P)}{K_\infty(P)}$$

for the positive superharmonic function $v(P)$. Hence, from Miyamoto, Yanagishita and Yoshida [16, Theorem 1] it follows that E_ϱ is minimally thin at ∞. This contradicts (iii). $\square$

P r o o f of Theorem 3. (i) $\Rightarrow$ (ii). Suppose that

$$\int_{E_\varrho} (1+|P|)^{-n}\,\mathrm{d}P < +\infty$$

for some ϱ $(0 < \varrho < 1)$. We can assume that this ϱ satisfies $0 < \varrho \leqslant \frac{1}{2}$. Let $\{W_{i_j}\}_{j\geqslant 1}$ be the subsequence of $\{W_i\}_{i\geqslant 1}$ from Lemma 2. Then from (i) of Lemma 2 we also have

$$\int_{\bigcup_j W_{i_j}} \frac{\mathrm{d}P}{(1+|P|)^n} < +\infty.$$

Since $\bigcup_j W_{i_j}$ is a union of cubes from the Whitney cubes of $C_n(\Omega)$ with ϱ, we see from the second part of Lemma 1 that $\bigcup_j W_{i_j}$ is minimally thin at ∞, and hence from (ii) of Lemma 2 that $E_{\varrho/4}$ is minimally thin at ∞.

On the other hand, since E characterizes the positive harmonic majorization of $K_\infty(P)$, it follows from Theorem 2 that $E_{\varrho/4}$ is not minimally thin at ∞, which contradicts the conclusion obtained above.

(iii) ⇒ (i). Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then we see from Theorem 2 that for any ϱ $(0 < \varrho < 1)$ E_ϱ is minimally thin at ∞. Lemma 1 gives that for any ϱ $(0 < \varrho < 1)$

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P < +\infty.$$

This contradicts (iii). □

P r o o f of Corollary. It is easy to see that if $\{P_m\}$ is a separated sequence, then

$$B(P_i, \varrho d(P_i)) \cap B(P_j, \varrho d(P_j)) = \emptyset \quad (i, j = 1, 2, \ldots ;\ i \neq j)$$

for a sufficiently small ϱ $(0 < \varrho < 1)$ and hence

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P \approx \sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n.$$

Hence the corollary immediately follows from (iii) of Theorem 3.

References

[1] *H. Aikawa*: Sets of determination for harmonic functions in an NTA domain. J. Math. Soc. Japan *48* (1996), 299-315.

[2] *H. Aikawa, M. Essén*: Potential Theory-Selected Topics. Lecture Notes in Math. Vol. 1633. Springer-Verlag, 1996.

[3] *A. Ancona*: Positive Harmonic Functions and Hyperbolicity. Lecture Notes in Math. Vol. 1344. Springer-Verlag, 1987, pp. 1–23.

[4] *D. H. Armitage, Ü. Kuran*: On positive harmonic majorization of y in $\mathbb{R}^n \times (0, +\infty)$. J. London Math. Soc. Ser. II *3* (1971), 733–741.

[5] *D. H. Armitage, S. J. Gardiner*: Classical Potential Theory. Springer-Verlag, 2001.

[6] *V. S. Azarin*: Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone Am. Math. Soc. Transl. II. Ser..

[7] *A. Beurling*: A minimum principle for positive harmonic functions. Ann. Acad. Sci. Fenn. Ser. AI. Math. *372* (1965).

[8] *M. Brelot*: On Topologies and Boundaries in Potential Theory. Lect. Notes in Math. Vol. 175. Springer-Verlag, 1971.

[9] *R. Courant, D. Hilbert*: Methods of Mathematical Physics, 1st English edition. Interscience, New York, 1954.

[10] *B. E. J. Dahlberg*: A minimum principle for positive harmonic functions. Proc. London Math. Soc. *33* (1976), 2380–250.

[11] *J. L. Doob*: Classical Potential Theory and its Probabilistic Counterpart. Springer-Verlag, 1984.

[12] *D. S. Jerison, C. E. Kenig*: Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv. Math. *46* (1982), 80–147.
[13] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, 1977.
[14] *L. L. Helms*: Introduction to Potential Theory. Wiley, New York, 1969.
[15] *V. G. Maz'ya*: Beurling's theorem on a minimum principle for positive harmonic functions. Zapiski Nauchnykh Seminarov LOMI *30* (1972); (English translation) J. Soviet. Math. *4* (1975), 367–379.
[16] *I. Miyamoto, M. Yanagishitam, and H. Yoshida*: Beurling-Dahlberg-Sjögren type theorems for minimally thin sets in a cone. Canad. Math. Bull. *46* (2003), 252–264.
[17] *I. Miyamoto, H. Yoshida*: Two criteria of Wiener type for minimally thin sets and rarefied sets in a cone. J. Math. Soc. Japan *54* (2002), 487–512.
[18] *P. Sjögren*: Une propriété des fonctions harmoniques positives d'après Dahlberg, Séminaire de théorie du potentiel. Lecture Notes in Math. Vol. 563. Springer-Verlag, 1976, pp. 275–282.
[19] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 1970.
[20] *H. Yoshida*: Nevanlinna norm of a subharmonic function on a cone or on a cylinder. Proc. London Math. Soc. Ser. III *54* (1987), 267–299.
[21] *Y. Zhang*: Ensembles équivalents a un point frontière dans un domaine lipshitzien, Séminaire de théorie du potentiel. Lecture Note in Math. Vol. 1393. Springer-Verlag, 1989, pp. 256–265.

Authors' address: I. Miyamoto, M. Yanagishita, and H. Yoshida, Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan, e-mails: miyamoto@math.s.chiba-u.ac.jp, myanagis@g.math.s.chiba-u.ac.jp, yoshida@math.s.chiba-u.ac.jp.