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More constructions of semi-bent and plateaued functions in polynomial forms

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Abstract
Plateaued functions and their subclass semi-bent functions have useful applications in cryptography and communications. In this paper we give new constructions of quadratic semi-bent functions in polynomial forms on the finite field $\mathbb{F}_{2^n}$ for both odd and even n. We also present some characterizations of e-plateaued functions with few trace terms when n is even.

Keywords Boolean functions · Semi-bent functions · Plateaued functions

Mathematics Subject Classification 11T06 · 11T71

1 Introduction

In the 1960s and 1990s, two families of m-sequences having low cross correlation were introduced by Gold [1] and Boztas et al. [2] respectively. Each of them has period $2^n - 1$ and a plateaued cross-correlation spectra. That is, for two such m-sequences $u(t) = \mathrm{Tr}_1^n(\alpha^t)$ and $v(t) = \mathrm{Tr}_1^n(\beta^t)$, where α and β have order $2^n - 1$ in the finite field $\mathbb{F}_{2^n}$, we have

$$C_{u,v}(\tau) = \sum_{t=0}^{2^n-2}(-1)^{u(t+\tau)+v(t)} \in \left\{-1, -1 \pm 2^{\frac{n+1}{2}}\right\}.$$

These families of sequences have the trace representations

$$f(x) = \mathrm{Tr}_1^n\left(x^{1+2^i}\right)\ (\gcd(i,n) = 1) \quad \text{and} \quad f(x) = \sum_{i=1}^{\frac{n-1}{2}}\mathrm{Tr}_1^n\left(x^{1+2^i}\right)$$

respectively, where $\mathrm{Tr}_1^n(x) = \sum_{i=0}^{n-1} x^{2^i}$. Such families of maximum-length sequences, whose cross-correlation spectra attain exactly the values above, have a wide range of applications in cryptography and code-division multiple-access communication systems [3, 4]. Such sequences can be represented by Boolean functions which we call *semi-bent* functions, using the terminology of Khoo et al. [5].

In order to construct more sequences having the nice property as the above two sequences, Khoo et al. [6] investigated the problem of determining the function

$$f(x) = \sum_{i=1}^{\frac{n-1}{2}} c_i\mathrm{Tr}_1^n\left(x^{1+2^i}\right), \quad c_i \in \mathbb{F}_2$$

defined on $\mathbb{F}_{2^n}$ with n odd is semi-bent, where this sum has more than one term. To such a function a cyclic code of length $2^n - 1$ was associated, spanned by

$$c(x), xc(x), \ldots, x^{n-1}c(x), \quad \text{where} \quad c(x) = \sum_{i=1}^{\frac{n-1}{2}} c_i\left(x^i + x^{n-i}\right).$$

Then it was proved that f is semi-bent if and only if $\gcd(c(x), x^n + 1) = x + 1$. This gives a very convenient tool for determining whether a function f having certain number of trace terms is semi-bent or not.

Following this work, Charpin et al. studied the following function [7]:

$$f(x) = \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} c_i\mathrm{Tr}_1^n\left(x^{1+2^i}\right), \quad c_i \in \mathbb{F}_2. \tag{1}$$

When n is odd, they provide some semi-bent functions with three or four trace terms. When n is even, they proved that

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$f(x)$ is semi-bent if and only if $\gcd(c(x), x^n+1) = x^2+1$, where $c(x) = \sum_{i=1}^{\frac{n}{2}-1} c_i(x^i + x^{n-i})$. Moreover, they found that the concatenation of two suitably chosen such semi-bent functions will yield a semi-bent function with higher algebraic degrees. After this work, a lot of research has been devoted to finding new families of quadratic semi-bent and bent functions in the form of Eq. (1) [8–16]. In 2013, Dong et al. present some new constructions of quadratic semi-bent functions. For odd $n = pq$ with $p(3 \nmid p), q$ odd, they proved that the function

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^q}\right),$$

is semi-bent. For even $n = 2m$ with odd m, a necessary and sufficient condition for the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i-1}^{\frac{m-1}{2}} c_i \mathrm{Tr}_1^n\left(\beta x^{1+2^{2i}}\right),$$

to be semi-bent is given. For some special cases of $c_i(1 \le i \le \frac{m-1}{2})$, they proved that f is semi-bent.

Motivated by the paper [8], we present new constructions of quadratic semi-bent and e-plateaued functions in polynomial forms. We study the function defined by

$$f(x) = \sum_{i=1}^{s} c_i \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \sum_{j=1}^{t} d_j \mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right), \tag{2}$$

where $c_i, d_j \in \mathbb{F}_2$, $1 \le s \le \frac{q-1}{2}$, $1 \le t \le \frac{p-1}{2}$, $n = pq$, p, q odd, $\gcd(p,q) = 1$, and the function defined by

$$f(x) = \sum_{i=1}^{\frac{m-1}{2}} c_i \mathrm{Tr}_1^n\left(\beta x^{1+2^{ei}}\right), \tag{3}$$

where $n = em$, $e = 2^l$, m is odd, $c_i \in \mathbb{F}_2$ $(1 \le i \le \frac{m-1}{2})$, $\beta \in \mathbb{F}_{2^e}^*$. For odd n, we find five new classes of semi-bent functions of the form Eq. (2) by choosing suitable vectors $(c_1, \ldots, c_s) \in \mathbb{F}_2^s$ and $(d_1, \ldots, d_t) \in \mathbb{F}_2^t$. For even n, we give a necessary and sufficient condition under which $f(x)$ given by Eq. (3) is e-plateaued and provide some new e-plateaued and semi-bent functions with few trace terms.

To the best of our knowledge, we give a list of the quadratic semi-bent functions on $\mathbb{F}_{2^n}$ as follows:

When n is odd, the following functions are semi-bent.

(1) $f(x) = \mathrm{Tr}_1^n(x^{1+2^i})$, $\gcd(i,n) = 1$ [1].

(2) $f(x) = \sum_{i=1}^{\frac{n-1}{2}} \mathrm{Tr}_1^n(x^{1+2^i})$ [2].

(3) $f(x) = \sum_{i=1}^{\frac{n-1}{2}} c_i \mathrm{Tr}_1^n(x^{1+2^i})$, $c_i \in \mathbb{F}_2$ for $1 \le i \le \frac{n-1}{2}$, $\gcd(\sum_{i=1}^{\frac{n-1}{2}} c_i\left(x^i + x^{n-i}\right), x^n+1) = x+1$ [6].

(4) $f(x) = \sum_{i=0}^{r} \mathrm{Tr}_1^n(x^{1+2^{a+id}})$, $\gcd(2a+rd, n) = \gcd((r+1)d, n) = 1$ [5].

(5) $f(x) = \mathrm{Tr}_1^n(x^{1+2^i}) + \mathrm{Tr}_1^n(x^{1+2^j})$, $\gcd(i+j, n) = \gcd(i-j, n) = 1$ [5].

(6) $f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^q}\right)$, $n = pq$, $p(3 \nmid p), q$ are odd positive integers such that $\gcd(p,q) = 1$ [8].

(7) $f(x) = \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n(x^{1+2^{qj}})$, $n = pq$, p, q odd, $\gcd(p,q) = 1$, and i, j are two positive integers such that $\gcd(i,q) = \gcd(j,p) = 1$ (Theorem 3 of this paper).

(8) $f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{pi}}) + \mathrm{Tr}_1^n(x^{1+2^{qj}})$, $n = pq$, $p(3 \nmid p), q$ are odd positive integers such that $\gcd(p,q) = 1$, $\gcd(j,p) = 1$ (Theorem 4 of this paper).

(9) $f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{pi}}) + \mathrm{Tr}_1^n(x^1 + 2^q) + \mathrm{Tr}_1^n(x^{1+2^{qj}})$, $n = pq, p$, q odd and $\gcd(p,q) = 1, j = 2^l$ and l is a positive integer such that $\gcd(l,n) = 1$ (Theorem 5 of this paper).

(10) $f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{pi}}) + \mathrm{Tr}_1^n(x^{1+2^{qj}}) + \mathrm{Tr}_1^n(x^{1+2^{qk}})$, $n = pq$, p, q odd and $\gcd(p,q) = 1$, $j = 2^{u-1} - 2^{v-1}$, $k = 2^{u-1} + 2^{v-1}$, $u > v \ge 1$ (Theorem 6 of this paper).

(11) $f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{pi}}) + \mathrm{Tr}_1^n(x^{1+2^{qj}}) + \mathrm{Tr}_1^n(x^{1+2^{qk}})$, $n = pq, p$, q odd and $\gcd(p,q) = 1, j = 2^u$, $k = 2^v$, $u > v \ge 1$, $\gcd(u-v, n) = 1$ (Theorem 7 of this paper).

When $n = 2m$ is even, the following functions are semi-bent:

(1) $f(x) = \sum_{i=1}^{\frac{n}{2}-1} c_i \mathrm{Tr}_1^n(x^{1+2^i})$, $c_i \in \mathbb{F}_2$, $\gcd(\sum_{i=1}^{\frac{n}{2}-1} c_i(x^i + x^{n-i}), x^n+1) = x^2+1$ [7].

(2) $f(x) = \mathrm{Tr}_1^n(\alpha x^{1+2^i})$, $\alpha \in \mathbb{F}_{2^n}^*$, i even, m odd [10].

(3) $f(x) = \mathrm{Tr}_1^n(\alpha x^{1+2^i})$, $\alpha \in \{x^3 \mid x \in \mathbb{F}_{2^n}^*\}$, i odd, m even [10].

(4) $f(x) = \mathrm{Tr}_1^n(\alpha x^{1+2^i})$, $\alpha \in \{x^3 \mid x \in \mathbb{F}_{2^n}^*\}$, i odd, m odd and $\gcd(i,m) = 1$ [10].

(5) $f(x) = \mathrm{Tr}_1^n(x^{1+2^i}) + \mathrm{Tr}_1^n(x^{1+2^j})$, m odd, $1 \le i < j \le m$, $\gcd(i+j, n) = \gcd(j-i, n) = 1$ or $\gcd(i+j, n) = \gcd(j-i, n) = 2$ [10].

(6) $f(x)=\sum_{i=1}^{\frac{m-1}{2}}\mathrm{Tr}_1^n(\beta x^{1+4^i})$, m odd, $\beta\in\mathbb{F}_4^*$ [8].

(7) $f(x)=\sum_{i=1}^{\frac{m-1}{2}}c_i\mathrm{Tr}_1^n(\beta x^{1+4^i})$, $c_i\in\mathbb{F}_2$, $\beta\in\mathbb{F}_4^*$, m odd, $\gcd(\sum_{i=1}^{\frac{m-1}{2}}c_i(x^i+x^{m-i}),x^m+1)=x+1$ [8].

(8) $f(x)=\sum_{i=1}^{k}\mathrm{Tr}_1^n(\beta x^{1+4^{di}})$, $\beta\in\mathbb{F}_4^*$, m odd, $d\geq 1$, $1\leq k\leq\frac{m-1}{2}$, $\gcd(k+1,m)=\gcd(k,m)=\gcd(d,m)=1$ [8].

(9) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+4^i}+\beta x^{1+4^j})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j\leq\lfloor\frac{n}{4}\rfloor$, $\gcd(i+j,m)=\gcd(j-i,m)=1$ [8].

(10) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+4^i}+\beta x^{1+4^j}+\beta x^{1+4^t})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j<t\leq\lfloor\frac{n}{4}\rfloor$, $i+j=t$, $\gcd(i,m)=\gcd(j,m)=\gcd(t,m)=1$ [8].

(11) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+4^i}+\beta x^{1+4^j}+\beta x^{1+4^t})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j<t\leq\lfloor\frac{n}{4}\rfloor$, $i+j=2t$, $j-i=3^hp$, $3\not| p$, $n=3^kq$, $3\not| q$, $\gcd(2t,m)=1$, $h\geq k$ [8].

(12) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+4^i}+\beta x^{1+4^j}+\beta x^{1+4^t})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j<t\leq\lfloor\frac{n}{4}\rfloor$, $j-i=2t$, $t\neq i$, $j+i=3^up$, $3\not| p$, $n=3^vq$, $3\not| q$, $\gcd(2t,m)=1$, $u\geq v$ [8].

(13) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+4^i}+\beta x^{1+4^j}+\beta x^{1+4^t}+\beta x^{1+4^s})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i,j,t,s\leq\lfloor\frac{n}{4}\rfloor$, $i<j$, $t<s$, $i+j=t+s=r$, $t\neq i$, $\gcd(r,m)=\gcd(s-i,m)=\gcd(s-j,m)=1$ [8].

(14) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{2i}})$, $1\leq i\leq m-1$, $\gcd(i,m)=1$ and m odd (Corollary 5 of this paper).

(15) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{2i}}+\beta x^{1+2^{2j}}+\beta x^{1+2^{2t}})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j<t\leq m-1$, $i+j=2t$, $\gcd(t,m)=1$ (Corollary 9 of this paper).

(16) $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{2i}}+\beta x^{1+2^{2j}}+\beta x^{1+2^{2t}})$, $\beta\in\mathbb{F}_4^*$, m odd, $1\leq i<j<t\leq m-1$, $j-i=2t$, $\gcd(t,m)=1$ (Corollary 10 of this paper).

2 Preliminaries

Let $\mathbb{F}_{2^n}$ be the finite field with 2^n elements, and we use $\mathcal{B}_n$ to denote the set of Boolean functions from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$. In this paper, we mainly investigate the Boolean function of the form

$$f(x)=\sum_{i=1}^{s}c_i\mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right)+\sum_{j=1}^{t}d_j\mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right),$$

where $c_i,d_j\in\mathbb{F}_2$, $1\leq s\leq\frac{q-1}{2}$, $1\leq t\leq\frac{p-1}{2}$, $n=pq$, p, q odd, $\gcd(p,q)=1$, and the function defined by

$$f(x)=\sum_{i=1}^{\frac{m-1}{2}}c_i\mathrm{Tr}_1^n\left(\beta x^{1+2^{ei}}\right),$$

where $n=em$, $e=2^l$, m is odd, $c_i\in\mathbb{F}_2$ $(1\leq i\leq\frac{m-1}{2})$, $\beta\in\mathbb{F}_{2^e}^*$. The Walsh transform of f at $\lambda\in\mathbb{F}_{2^n}$ is given by

$$W_f(\lambda)=\sum_{x\in\mathbb{F}_{2^n}}(-1)^{f(x)+\mathrm{Tr}_1^n(\lambda x)}.$$

Definition 1 ([17]) Let $f(x)\in\mathcal{B}_n$. For any $\lambda\in\mathbb{F}_{2^n}$, if $W_f(\lambda)\in\{0,\pm 2^{\frac{n+r}{2}}\}$, for some fixed r, $r=0,1,\ldots,n$, then $f(x)$ is called r-plateaued. 0-plateaued (when n is even) functions are called bent. 1-plateaued (when n is odd) and 2-plateaued (when n is even) functions are called semi-bent.

The r-plateaued functions exist only when $n-r$ is even, or equivalently, if n and r have the same parity [18]. It is well-known that all the quadratic functions are plateaued [19].

The quadratic Boolean functions on $\mathbb{F}_{2^n}$ are as follows:

$$f(x)=\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}c_i\mathrm{Tr}_1^n\left(x^{1+2^i}\right),\ c_i\in\mathbb{F}_2.$$

Any such Boolean function with n variables has rank $2t$ with $0\leq t\leq\lfloor\frac{n}{2}\rfloor$ [3], and the rank can be found as follows. Let

$$\Omega_f(x;y)=f(0)+f(x)+f(y)+f(x+y). \tag{4}$$

Then the rank of $f(x)$ is $2t$ if and only if the equation

$$\Omega_f(x;y)=0,\ \text{for any}\ y\in\mathbb{F}_{2^n}$$

in x just has 2^{n-2t} solutions. The rank of quadratic Boolean functions is connected with the distribution of its Walsh transform values. Furthermore, the following theorem holds.

Lemma 1 ([3]) Let $f(x)\in\mathcal{B}_n$ is a quadratic function, and the rank of $f(x)$ is $2t$, $0\leq t\leq\lfloor\frac{n}{2}\rfloor$, then the distribution of its Walsh transform values is given by

$$W_f(\lambda)=\begin{cases}2^{n-t}, & 2^{2t-1}+2^{t-1}\ \text{times},\\ 0, & 2^{n-2t}\ \text{times},\\ -2^{n-t}, & 2^{2t-1}-2^{t-1}\ \text{times}.\end{cases}$$

From the above theorem, it is easy to see that a quadratic Boolean function is semi-bent if and only if the rank of $f(x)$ is $n-2$ when n is even, or the rank of $f(x)$ is $n-1$ when n is odd.

Definition 2 ([20]) The polynomials $l(x) = \sum_{i=0}^{n} a_i x^i$ and $L(x) = \sum_{i=0}^{n} a_i x^{q^i}$ over $\mathbb{F}_{q^m}$ (q is a prime integer) are called q-associates of each other. More specifically, $l(x)$ is the conventional q-associate of $L(x)$ and $L(x)$ is the linearized q-associate of $l(x)$.

Lemma 2 ([20]) Let $L_1(x)$ and $L(x)$ be q-polynomials over $\mathbb{F}_q$ with conventional q-associates $l_1(x)$ and $l(x)$. Then $L_1(x)$ divides $L(x)$ holds if and only if $l_1(x)$ divides $l(x)$.

Lemma 3 Let $\phi(x) = x + \frac{1}{x}$ be a function defined over $\mathbb{F}_{2^n}^*$. Then the following two statements hold.

(i) If $\phi(x) = \phi(y)$ for two elements x and y in $\mathbb{F}_{2^n}^*$, then $x = y$ or $xy = 1$.
(ii) $\phi(x) = 0$ if and only if $x = 1$.

Proof

(i) If $\phi(x) = \phi(y)$ for $x, y \in \mathbb{F}_{2^n}^*$, i.e.,

$$x + \frac{1}{x} = y + \frac{1}{y},$$

then

$$x^2 y + y = xy^2 + x,$$

which implies

$$(x + y)(xy + 1) = 0.$$

Therefore $x = y$ or $xy = 1$.

(ii) $\phi(x) = 0$ if and only if $x + \frac{1}{x} = 0$, which is equivalent to $x^2 = 1$. Since $\gcd(2, 2^n - 1) = 1$, the equation $x^2 = 1$ has only one solution $x = 1$ in $\mathbb{F}_{2^n}^*$.

Lemma 4 Let p, q and i, j be positive integers satisfying that p, q are odd and $\gcd(p, q) = 1$ and $\gcd(i, q) = \gcd(j, p) = 1$. Then $\gcd(pq, pi \pm qj) = 1$.

Proof If $\gcd(pq, pi \pm qj) \neq 1$, then there exists a prime integer t such that $t \mid \gcd(pq, pi \pm qj)$, i.e.,

$$t \mid pq \quad \text{or} \quad t \mid pi \pm qj. \tag{5}$$

Since t is a prime, by Eq. (5) we have $t \mid p$ or $t \mid q$.

If $t \mid p$, then by Eq. (5), we have $t \mid qj$. Therefore, $t \mid q$ or $t \mid j$. If $t \mid q$, then $t \mid \gcd(p, q) = 1$, which is impossible. If $t \mid j$, then $t \mid \gcd(j, p) = 1$, which is also a contradiction with the assumption that t is a prime.

If $t \mid q$, we can similarly deduce that $t = 1$, a contradiction. This completes the proof.

3 New constructions of semi-bent functions on $\mathbb{F}_{2^n}$ with n odd

In this section, several classes of semi-bent functions are constructed on $\mathbb{F}_{2^n}$, where $n = pq$ and p, q are odd integers such that $\gcd(p, q) = 1$.

Theorem 3 Let $n = pq$ with p, q odd and $\gcd(p, q) = 1$. Then the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right) \tag{6}$$

is semi-bent, where i, j are two positive integers such that $\gcd(i, q) = \gcd(j, p) = 1$.

Proof By Lemma 1, in order to prove that $f(x)$ is a semi-bent function, we just need to prove that the rank of $f(x)$ is $n - 1$. By Eq. (4), we have

$$\Omega_f(x; y) = \mathrm{Tr}_1^n\left(y\left(x^{2^{pi}} + x^{2^{pq-pi}} + x^{2^{qj}} + x^{2^{pq-qj}}\right)\right).$$

Let $L(x) = x^{2^{pi}} + x^{2^{pq-pi}} + x^{2^{qj}} + x^{2^{pq-qj}}$, and it is easy to see that $x^2 + x \mid L(x)$. To prove that the rank of $f(x)$ is $n - 1$, we need to show that $L(x)$ has two solutions in $\mathbb{F}_{2^n}$ or equivalently to prove $\gcd(L(x), x^{2^{pq}} + x) = x^2 + x$. By Lemma 2, we need to show

$$\gcd(l(x), x^{pq} + 1) = x + 1,$$

where $l(x) = x^{pi} + x^{pq-pi} + x^{qj} + x^{pq-qj}$. To do this, we divide the remaining proof into three cases.

If $\beta \neq 1$ is root of $x^{pq} + 1$ and $\beta^p = 1$, then $l(\beta) = \beta^{qj} + \beta^{-qj} \neq 0$. Otherwise, if $l(\beta) = 0$, then $\beta^{2qj} = 1$. Since $\gcd(2, 2^n - 1) = 1$, we have $\beta^{qj} = 1$. Recall that $\beta^p = 1$, $\gcd(p, q) = 1$ and $\gcd(p, j) = 1$, we have $\beta = 1$, which is a contradiction with the assumption $\beta \neq 1$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^q = 1$, we can similarly deduce that $l(\beta) = \beta^{pi} + \beta^{-pi} \neq 0$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^p \neq 1$, $\beta^q \neq 1$, then $l(\beta) = \beta^{pi} + \beta^{-pi} + \beta^{qj} + \beta^{-qj}$. If $l(\beta) = 0$, i.e., $\phi(\beta^{pi}) = \phi(\beta^{qj})$, where $\phi(x)$ is the function defined in Lemma 2. By Lemma 2, we have

$$\beta^{pi} = \beta^{qj} \quad \text{or} \quad \beta^{pi+qj} = 1. \tag{7}$$

Since $\gcd(p, q) = 1$ and $\gcd(i, q) = \gcd(j, p) = 1$, by Lemma 4 we have $\gcd(pq, pi \pm qj) = 1$. Recall that $\beta^{pq} = 1$, then by Eq. (7), we have $\beta = \beta^{\gcd(pq, pi+qj)} = 1$ or $\beta = \beta^{\gcd(pq, pi-qj)} = 1$, both of which contradict with the assumption $\beta \neq 1$.

From the analysis of above, we can see that $\gcd(l(x), x^{pq} + 1) = x + 1$. Thus the rank of $f(x)$ is $n - 1$, and this completes the proof.

Corollary 1 Let $n = pq$ with p, q distinct odd prime integers. Then the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right) \tag{8}$$

is semi-bent for any integers i, j.

Theorem 4 Let $n = pq$ with p, q odd, $\gcd(p, q) = 1$ and $3 \nmid p$. Then the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right) \tag{9}$$

is semi-bent, where j is a positive integer such that $\gcd(j, p) = 1$.

Proof From Lemma 1, in order to prove that $f(x)$ is a semi-bent function, we just need to prove that the rank of $f(x)$ is $n - 1$. By Eq. (9), we have

$$\begin{aligned} f(x+y) &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x+y\right)^{1+2^{pi}} + \mathrm{Tr}_1^n(x+y)^{1+2^{qj}}\Big) \\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}} + y^{1+2^{pi}}\right) + \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}}y + y^{2^{pi}}x\right) \\ &\quad + \mathrm{Tr}_1^n\left(x^{1+2^{qj}} + y^{1+2^{qj}} + x^{2^{qj}}y + y^{2^{qj}}x\right). \end{aligned}$$

Hence,

$$\begin{aligned} \Omega_f(x; y) &= f(0) + f(x) + f(y) + f(x+y) \\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}}y + y^{2^{pi}}x\right) + \mathrm{Tr}_1^n\left(x^{2^{qj}}y + y^{2^{qj}}x\right) \\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}}y + yx^{2^{pq-pi}}\right) + \mathrm{Tr}_1^n\left(x^{2^{qj}}y + yx^{2^{pq-qj}}\right) \\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{\frac{q-1}{2}}\left(x^{2^{pi}} + x^{2^{pq-pi}}\right)\right)\right) + \mathrm{Tr}\mathrm{Tr}_1^n\left(y\left(x^{2^{qj}} + x^{2^{pq-qj}}\right)\right) \\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{q-1} x^{2^{pi}} + x^{2^{qj}} + x^{2^{pq-qj}}\right)\right). \end{aligned}$$

Let $L(x) = \sum_{i=1}^{q-1} x^{2^{pi}} + x^{2^{qj}} + x^{2^{pq-qj}}$, and it is easy to see that $x^2 + x \mid L(x)$. To prove that the rank of $f(x)$ is $n - 1$, we need to show that $\gcd(L(x), x^{2^{pq}} + x) = x^2 + x$, which is equivalent to show that

$$\gcd(l(x), x^{pq} + 1) = x + 1$$

from Lemma 2, where $l(x) = \sum_{i=1}^{q-1} x^{pi} + x^{qj} + x^{pq-qj}$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^p = 1$, then $l(\beta) = \beta^{jq} + \beta^{-jq} \neq 0$. Otherwise, we have $\beta^{2jq} = 1$. Since $\gcd(2, 2^n - 1) = 1$, $\beta^{jq} = 1$. From the conditions $\beta^p = 1$ and $\gcd(j, p) = 1$, we have $\beta = 1$, which is a contradiction with the assumption.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^q = 1$, then $l(\beta) = \frac{\beta^{pq}+\beta^p}{\beta^p+1} + \beta^{q(p-j)} + \beta^{qj} = 1 \neq 0$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^p \neq 1$, $\beta^q \neq 1$, thus

$$l(\beta) = \frac{\beta^{pq} + \beta^p}{\beta^p + 1} + \beta^{q(p-j)} + \beta^{qj} = 1 + \beta^{q(p-j)} + \beta^{qj}.$$

If $l(\beta) = 0$, then $\beta^{3jq} = 1$. Since $\beta^{pq} = 1$, $3 \nmid p$ and $\gcd(j, p) = 1$, $\beta^{\gcd(pq,3jq)} = \beta^{\gcd(p,3j)q} = \beta^q = 1$, which contradicts with the assumption $\beta^q \neq 1$. Hence we also have $l(\beta) \neq 0$ in this case.

From the analysis of above, we can see that $\gcd(l(x), x^{pq} + 1) = x + 1$. Thus the rank of $f(x)$ is $n - 1$, and this completes the proof.

For $j = 1$ in Theorem 4, we have the following corollary.

Corollary 2 ([8]) Let $n = pq$ with p, q odd, $\gcd(p, q) = 1$ and $3 \nmid p$. Then the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^q}\right) \tag{10}$$

is semi-bent.

Theorem 5 Let $n = pq$ with p, q odd and $\gcd(p, q) = 1$. Then the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n\left(x^{1+2^q}\right) + \mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right) \tag{11}$$

is semi-bent, where $j = 2^l$ and l is a positive integer such that $\gcd(l, n) = 1$.

Proof From Lemma 1, in order to prove that $f(x)$ is a semi-bent function, we just need to prove that the rank of $f(x)$ is $n - 1$. By Eq. (11), we have

$$\begin{aligned} f(x+y) &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\Big((x+y)^{1+2^{pi}} + \mathrm{Tr}_1^n((x+y)^{1+2^q}) + \mathrm{Tr}_1^n((x+y)^{1+2^{qj}}\Big) \\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{pi}} + y^{1+2^{pi}}) + \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n(x^{2^{pi}}y + y^{2^{pi}}x) \\ &\quad + \mathrm{Tr}_1^n(x^{1+2^q} + y^{1+2^q} + x^{2^q}y + y^{2^q}x) \\ &\quad + \mathrm{Tr}_1^n(x^{1+2^{qj}} + y^{1+2^{qj}} + x^{2^{qj}}y + y^{2^{qj}}x). \end{aligned}$$

Hence,

$$\begin{aligned}\Omega_f(x;y) &= f(0)+f(x)+f(y)+f(x+y)\\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}}y+y^{2^{pi}}x\right)+\mathrm{Tr}_1^n\left(x^{2^q}y+y^{2^q}x\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{2^{qj}}y+y^{2^{qj}}x\right)\\ &= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}}y+yx^{2^{pq-pi}}\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{2^q}y+yx^{2^{pq-q}}\right)+\mathrm{Tr}_1^n\left(x^{2^{qj}}y+yx^{2^{pq-qj}}\right)\\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{\frac{q-1}{2}}\left(x^{2^{pi}}+x^{2^{pq-pi}}\right)\right)\right)+\mathrm{Tr}_1^n\left(y\left(x^{2^q}+x^{2^{pq-q}}\right)\right)\\ &\quad+\mathrm{Tr}_1^n\left(y\left(x^{2^{qj}}+x^{2^{pq-qj}}\right)\right)\\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{q-1}x^{2^{pi}}+x^{2^q}+x^{2^{pq-q}}+x^{2^{qj}}+x^{2^{pq-qj}}\right)\right).\end{aligned}$$

Let $L(x)=\sum_{i=1}^{q-1}x^{2^{pi}}+x^{2^q}+x^{2^{pq-q}}+x^{2^{qj}}+x^{2^{pq-qj}}$, and it is easy to see that $x^2+x\,|\,L(x)$. To prove that the rank of $f(x)$ is $n-1$, we need to show that $\gcd(L(x),x^{2^{pq}}+x)=x^2+x$, which equals

$$\gcd(l(x),x^{pq}+1)=x+1$$

from Lemma 2, where $l(x)=\sum_{i=1}^{q-1}x^{pi}+x^q+x^{pq-q}+x^{qj}+x^{pq-qj}$.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^p=1$, then $l(\beta)=\beta^q+\beta^{-q}+\beta^{jq}+\beta^{-jq}$. Let $w=\beta^q+\beta^{-q}$, then $l(\beta)=w+w^j$, where $j=2^l$ and $w\neq0,1$. We claim that $l(\beta)\neq0$. Otherwise, we have $w=w^{2^l}$. Since $\gcd(2^l-1,2^n-1)=1$, $w=0$ or 1, which is a contradiction.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^q=1$, then $l(\beta)=1\neq0$.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^p\neq1$, $\beta^q\neq1$, thus

$$l(\beta)=1+\beta^q+\beta^{-q}+\beta^{jq}+\beta^{-jq}.$$

If $l(\beta)=0$, then $w+w^{2^l}=1$. Since n is odd, $\mathrm{Tr}_1^n(1)=1$. But $\mathrm{Tr}_1^n(w+w^{2^l})=0$, leading to a contradiction. Hence we also have $l(\beta)\neq0$ in this case.

From the analysis of above, we can see that $\gcd(l(x),x^{pq}+1)=x+1$. Thus the rank of $f(x)$ is $n-1$, and this completes the proof.

Theorem 6 Let $n=pq$, with p, q odd and $\gcd(p,q)=1$. Let $j=2^{u-1}-2^{v-1}$ and $k=2^{u-1}+2^{v-1}$, where u, v are positive integers such that $u>v$. Then

$$f(x)=\sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right)+\mathrm{Tr}_1^n\left(x^{1+2^{qj}}\right)+\mathrm{Tr}_1^n\left(x^{1+2^{qk}}\right) \tag{12}$$

is semi-bent on $\mathbb{F}_{2^n}$.

Proof From Lemma 1, in order to prove that $f(x)$ is a semi-bent function, we just need to prove that the rank of $f(x)$ is $n-1$. By Eq. (12), we have

$$\begin{aligned}f(x+y) &= \sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left((x+y)^{1+2^{pi}}\right)+\mathrm{Tr}_1^n\left((x+y)^{1+2^{qj}}\right)\\ &\quad+\mathrm{Tr}_1^n\left((x+y)^{1+2^{qk}}\right)\\ &= \sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left(x^{1+2^{pi}}+y^{1+2^{pi}}\right)+\sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left(x^{2^{pi}}y+y^{2^{pi}}x\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{1+2^{qj}}+y^{1+2^{qj}}+x^{2^{qj}}y+y^{2^{qj}}x\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{1+2^{qk}}+y^{1+2^{qk}}+x^{2^{qk}}y+y^{2^{qk}}x\right).\end{aligned}$$

Hence,

$$\begin{aligned}\Omega_f(x;y) &= f(0)+f(x)+f(y)+f(x+y)\\ &= \sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left(x^{2^{pi}}y+y^{2^{pi}}x\right)+\mathrm{Tr}_1^n\left(x^{2^{qj}}y+y^{2^{qj}}x\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{2^{qk}}y+y^{2^{qk}}x\right)\\ &= \sum_{i=1}^{\frac{q-1}{2}}\mathrm{Tr}_1^n\left(x^{2^{pi}}y+yx^{2^{pq-pi}}\right)+\mathrm{Tr}_1^n\left(x^{2^{qj}}y+yx^{2^{pq-qj}}\right)\\ &\quad+\mathrm{Tr}_1^n\left(x^{2^{qk}}y+yx^{2^{pq-qk}}\right)\\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{\frac{q-1}{2}}\left(x^{2^{pi}}+x^{2^{pq-pi}}\right)\right)\right)+\mathrm{Tr}_1^n\left(y\left(x^{2^{qj}}+x^{2^{pq-qj}}\right)\right)\\ &\quad+\mathrm{Tr}_1^n\left(y\left(x^{2^{qk}}+x^{2^{pq-qk}}\right)\right)\\ &= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{q-1}x^{2^{pi}}+x^{2^{qj}}+x^{2^{pq-qj}}+x^{2^{qk}}+x^{2^{pq-qk}}\right)\right).\end{aligned}$$

Let $L(x)=\sum_{i=1}^{q-1}x^{2^{pi}}+x^{2^{qj}}+x^{2^{pq-qj}}+x^{2^{qk}}+x^{2^{pq-qk}}$, and it is easy to see that $x^2+x\,|\,L(x)$. To prove that the rank of $f(x)$ is $n-1$, we need to show that $\gcd(L(x),x^{2^{pq}}+x)=x^2+x$, which equals

$$\gcd(l(x),x^{pq}+1)=x+1$$

from Lemma 2, where $l(x)=\sum_{i=1}^{q-1}x^{pi}+x^{qj}+x^{pq-qj}+x^{qk}+x^{pq-qk}$.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^p=1$, then $l(\beta)=\beta^{qj}+\beta^{-qj}+\beta^{qk}+\beta^{-qk}$. If $l(\beta)=0$, then $\phi(\beta^{qj})=\phi(\beta^{qk})$, where ϕ is the function defined in Lemma 2. By Lemma 2, we have $\beta^{qj}=\beta^{qk}$ or $\beta^{q(j+k)}=1$. Since $\gcd(j\pm k,p)=1$, $\beta^p=1$ and $\gcd(p,q)=1$, we have $\beta=1$, which is a contradiction with the assumption $\beta\neq1$.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^q=1$, then $l(\beta)=1\neq0$.

If $\beta\neq1$ is a root of $x^{pq}+1$ and $\beta^p\neq1$, $\beta^q\neq1$, thus

$l(\beta) = 1 + \beta^{qj} + \beta^{-qj} + \beta^{qk} + \beta^{-qk}.$

If $l(\beta) = 0$, then

$$\beta^{qj} + \beta^{-qj} + \beta^{qk} + \beta^{-qk} = 1 \quad (13)$$

By Eq. (13), we have

$$\beta^{q(j+k)} + \beta^{q(k-j)} + \beta^{2qk} + \beta^{qk} = 1. \quad (14)$$

Since $k + j = 2^u$ and $k - j = 2^v$, Eq. (14) can be rewritten as

$$(\beta^q)^{2^u} + (\beta^q)^{2^v} + (\beta^{qk})^2 + \beta^{qk} = 1. \quad (15)$$

It is clear that $\mathrm{Tr}_1^n((\beta^q)^{2^u} + (\beta^q)^{2^v} + (\beta^{qk})^2 + \beta^{qk}) = 0$. But $\mathrm{Tr}_1^n(1) = 1$, leading to a contradiction. Hence we also have $l(\beta) \neq 0$ in this case.

From the analysis of above, we can see that $\gcd(l(x), x^{pq} + 1) = x + 1$. Thus the rank of $f(x)$ is $n - 1$, and this completes the proof.

Theorem 7 Let $n = pq$, with p, q odd and $\gcd(p, q) = 1$. Let $j = 2^u$ and $k = 2^v$, where u, v are positive integers such that $u > v$ and $\gcd(u - v, n) = 1$. Then

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \mathrm{Tr}_1^n(x^{1+2^{qj}}) + \mathrm{Tr}_1^n\left(x^{1+2^{qk}}\right) \quad (16)$$

is semi-bent on $\mathbb{F}_{2^n}$.

Proof From Lemma 1, in order to prove that $f(x)$ is a semi-bent function, we just need to prove that the rank of $f(x)$ is $n - 1$. Through similar calculations as Theorem 5, we have

$$\begin{aligned}
\Omega_f(x; y) &= f(0) + f(x) + f(y) + f(x + y)\\
&= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}} y + y^{2^{pi}} x\right) + \mathrm{Tr}_1^n\left(x^{2^{qj}} y + y^{2^{qj}} x\right)\\
&\qquad + \mathrm{Tr}_1^n\left(x^{2^{qk}} y + y^{2^{qk}} x\right)\\
&= \sum_{i=1}^{\frac{q-1}{2}} \mathrm{Tr}_1^n\left(x^{2^{pi}} y + yx^{2^{pq-pi}}\right) + \mathrm{Tr}_1^n\left(x^{2^{qj}} y + yx^{2^{pq-qj}}\right)\\
&\qquad + \mathrm{Tr}_1^n\left(x^{2^{qk}} y + yx^{2^{pq-qk}}\right)\\
&= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{\frac{q-1}{2}}\left(x^{2^{pi}} + x^{2^{pq-pi}}\right)\right)\right) + \mathrm{Tr}_1^n\left(y(x^{2^{qj}} + x^{2^{pq-qj}})\right)\\
&\qquad + \mathrm{Tr}_1^n\left(y\left(x^{2^{qk}} + x^{2^{pq-qk}}\right)\right)\\
&= \mathrm{Tr}_1^n\left(y\left(\sum_{i=1}^{q-1} x^{2^{pi}} + x^{2^{qj}} + x^{2^{pq-qj}} + x^{2^{qk}} + x^{2^{pq-qk}}\right)\right).
\end{aligned}$$

Let $L(x) = \sum_{i=1}^{q-1} x^{2^{pi}} + x^{2^{qj}} + x^{2^{pq-qj}} + x^{2^{qk}} + x^{2^{pq-qk}}$, and it is easy to see that $x^2 + x \mid L(x)$. To prove that the rank of $f(x)$ is $n - 1$, we need to show that $\gcd(L(x), x^{2^{pq}} + x) = x^2 + x$, which equals

$$\gcd(l(x), x^{pq} + 1) = x + 1$$

from Lemma 2, where $l(x) = \sum_{i=1}^{q-1} x^{pi} + x^{qj} + x^{pq-qj} + x^{qk} + x^{pq-qk}$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^p = 1$, then $l(\beta) = \beta^{qj} + \beta^{-qj} + \beta^{qk} + \beta^{-qk}$. Let $w = \beta^q + \beta^{-q}$, then $l(\beta) = w^{2^u} + w^{2^v}$. If $l(\beta) = 0$, then $w^{2^u} = w^{2^v}$. Note that $w \neq 0$, then we have $w^{2^u - 2^v} = w^{2^v(2^{u-v}-1)} = 1$. Since $\gcd(2^v(2^{u-v} - 1), 2^n - 1) = \gcd(2^{u-v} - 1, 2^n - 1) = 2^{\gcd(u-v,n)} - 1 = 1$, we have $w = 1$. This is impossible, because the equality $w = 1$ will lead to $\beta = 1$, which is a contradiction with the assumption $\beta \neq 1$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^q = 1$, then $l(\beta) = 1 \neq 0$.

If $\beta \neq 1$ is a root of $x^{pq} + 1$ and $\beta^p \neq 1$, $\beta^q \neq 1$, thus

$$l(\beta) = 1 + \beta^{qj} + \beta^{-qj} + \beta^{qk} + \beta^{-qk} = w^{2^u} + w^{2^v}.$$

If $l(\beta) = 0$, then $w^{2^u} + w^{2^v} = 1$. Note that $\mathrm{Tr}_1^n(w^{2^u} + w^{2^v}) = 0$. But $\mathrm{Tr}_1^n(1) = 1$, leading to a contradiction. Hence we also have $l(\beta) \neq 0$ in this case.

From the analysis of above, we can see that $\gcd(l(x), x^{pq} + 1) = x + 1$. Thus the rank of $f(x)$ is $n - 1$, and this completes the proof.

4 New constructions of *e*-plateaued and semi-bent functions on $\mathbb{F}_{2^n}$ with *n* even

In this section, we give some new constructions of quadratic *e*-plateaued and semi-bent functions in polynomial forms with even n. We suppose $n = em$, where e and m are even and odd positive integers respectively in this section.

Theorem 8 For any $\beta \in \mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{\frac{m-1}{2}} \mathrm{Tr}_1^n\left(\beta x^{1+2^{ei}}\right) \quad (17)$$

is e-plateaued.

Proof By Lemma 1, in order to prove that $f(x)$ is an e-plateaued function, we just need to prove that the rank of $f(x)$ is $n - e$. By Eq. (17), we have

$$\begin{aligned}\Omega_f(x;y) &= f(0)+f(x)+f(y)+f(x+y)\\ &= \sum_{i=1}^{\frac{m-1}{2}} \mathrm{Tr}_1^n\left(\beta x y^{2^{ei}} + \beta x^{2^{ei}} y\right)\\ &= \sum_{i=1}^{\frac{m-1}{2}} \mathrm{Tr}_1^n\left(\beta x^{2^{em-ei}} y + \beta x^{2^{ei}} y\right)\\ &= \sum_{i=1}^{\frac{m-1}{2}} \mathrm{Tr}_1^n\left(\beta x^{2^{em-ei}} y + \beta x^{2^{ei}} y\right)\\ &= \sum_{i=1}^{m-1} \mathrm{Tr}_1^n\left(\beta y x^{2^{ei}}\right)\\ &= \mathrm{Tr}_1^n\left(\beta y\left(\mathrm{Tr}_e^n(x)+x\right)\right).\end{aligned}$$

It follows that

$$\Omega_f(x;y) = 0, \quad \text{for any } \; y \in \mathbb{F}_{2^n} \tag{18}$$

holds if and only if

$$\mathrm{Tr}_e^n(x) + x = 0.$$

So $x = \mathrm{Tr}_e^n(x) \in \mathbb{F}_{2^e}$, and for any $x \in \mathbb{F}_{2^e}$, $\mathrm{Tr}_e^n(x) = x\mathrm{Tr}_e^n(1) = x$. This implies that $\mathrm{Tr}_e^n(x) + x = 0$ holds if and only if $x \in \mathbb{F}_{2^e}$. Hence Eq. (18) has only 2^e solutions, so the rank of $f(x)$ is $n-e$. By Lemma 1 and Definition 1, $f(x)$ is an e-plateaued function.

For $e = 2$, we have the following corollary.

Corollary 3 ([8]) For any $\beta \in \mathbb{F}_{2^2}^*$, the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{\frac{m-1}{2}} \mathrm{Tr}_1^n\left(\beta x^{1+2^{2i}}\right)$$

is a semi-bent function.

Now we consider the general case. We study the function defined by

$$f(x) = \sum_{i=1}^{\frac{m-1}{2}} c_i \mathrm{Tr}_1^n\left(\beta x^{1+2^{ei}}\right), \tag{19}$$

where $c_i \in \mathbb{F}_2$, $(1 \le i \le \frac{m-1}{2})$, $\beta \in \mathbb{F}_{2^e}^*$.

Theorem 9 If $e = 2^l$ for some positive integer l, then for any $\beta \in \mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by Eq. (19) is e-plateaued if and only if

$$\gcd\left(\sum_{i=1}^{\frac{m-1}{2}} c_i(x^i + x^{m-i}), x^m + 1\right) = x+1.$$

Proof By Lemma 1, we only need to prove that the rank of $f(x)$ is $n-e$. Similar to the proof of Theorem 7, the rank of $f(x)$ is $n-e$ if and only if the equation

$$\sum_{i=1}^{\frac{m-1}{2}} c_i\left(x^{2^{ei}} + x^{2^{em-ei}}\right) \tag{20}$$

has only 2^e solutions in $\mathbb{F}_{2^n}$. For any $x \in \mathbb{F}_{2^e}$, it is obvious that $x^{2^{ei}} + x^{2^{em-ei}} = 0\,(1 \le i \le \frac{m-1}{2})$ holds. So

$$x^{2^e} + x \,\Big|\, \sum_{i=1}^{\frac{m-1}{2}} c_i\left(x^{2^{ei}} + x^{2^{em-ei}}\right).$$

In order to show that Eq. (20) has only 2^e solutions in $\mathbb{F}_{2^n}$, we just need to prove that

$$\gcd\left(\sum_{i=1}^{\frac{m-1}{2}} c_i\left(x^{2^{ei}} + x^{2^{em-ei}}\right), x^{2^n} + x\right) = x^{2^e} + x \tag{21}$$

holds. By Lemma 3, Eq. (21) holds if and only if

$$\gcd\left(\sum_{i=1}^{\frac{m-1}{2}} c_i\left(x^{ei} + x^{em-ei}\right), x^{em} + 1\right) = x^e + 1 = (x+1)^e. \tag{22}$$

Eq. (22) holds if and only if

$$\gcd\left(\sum_{i=1}^{\frac{m-1}{2}} c_i\left(x^{i} + x^{m-i}\right), x^{m} + 1\right) = x+1.$$

Theorem 10 If $e = 2^l$ for some positive integer l, then for any $\beta \in \mathbb{F}_{2^e}^*$, $r \ge 1$, $1 \le k \le \frac{m-1}{2}$, the function defined on $\mathbb{F}_{2^n}$ by

$$f(x) = \sum_{i=1}^{k} \mathrm{Tr}_1^n\left(\beta x^{1+2^{eri}}\right)$$

is e-plateaued if and only if

$$\gcd(k+1, m) = \gcd(k, m) = \gcd(r, m) = 1.$$

Proof Similar to the proof of Theorem 8, $f(x)$ is e-plateaued if and only if

$$\gcd(L(x), x^m + 1) = x + 1, \quad \text{where} \;\; L(x) = \sum_{i=1}^{k}\left(x^{ri} + x^{m-ri}\right).$$

We have

$$
\begin{aligned}
L(x) &= \sum_{i=1}^{k}\left(x^{ri}+x^{-ri}\right)\\
&= \sum_{i=0}^{2k}\frac{x^{ri}}{x^{rk}}+1\\
&= \frac{x^{(2k+1)r}+1}{x^{rk}(x^r+1)}+1\\
&= \frac{x^{(2k+1)r}+x^{rk+r}+x^{rk}+1}{x^{rk}(x^r+1)}\\
&= \frac{(x^{rk+r}+1)(x^{rk}+1)}{x^{rk}(x^r+1)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\gcd(L(x),x^m+1) &= \gcd\left(\frac{(x^{rk+r}+1)(x^{rk}+1)}{x^{rk}(x^r+1)},x^m+1\right)\\
&= x+1
\end{aligned}
$$

holds if and only if

$$\gcd(r(k+1),m)=\gcd(rk,m)=\gcd(r,m)=1,$$

which equals

$$\gcd(k+1,m)=\gcd(k,m)=\gcd(r,m)=1.$$

Corollary 4 ([8]) For any $\beta\in\mathbb{F}_{2^2}^*$, $r\geq 1$, $1\leq k\leq\frac{m-1}{2}$, the function defined on $\mathbb{F}_{2^n}$ by

$$f(x)=\sum_{i=1}^{k}\mathrm{Tr}_1^n\left(\beta x^{1+2^{2ri}}\right)$$

is semi-bent if and only if

$$\gcd(k+1,m)=\gcd(k,m)=\gcd(r,m)=1.$$

Theorem 11 If $e=2^l$ for some positive integer l, then for any $\beta\in\mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{ei}})\,(1\leq i\leq m-1)$ is e-plateaued if and only if $\gcd(i,m)=1$ and m is odd.

Proof Let $l(x)=x^i+x^{m-i}$. By Theorem 8, the function is e-plateaued if and only if $\gcd(l(x),x^m+1)=x+1$. As $x^i+x^{m-i}=x^i(x^{m-2i}+1)$ and $\gcd(x^i,x^m+1)=1$. The equality $\gcd(l(x),x^m+1)=x+1$ holds if and only if $\gcd(i,m)=1$ and m is odd.

Corollary 5 For any $\beta\in\mathbb{F}_{2^2}^*$, the function defined by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{2i}})$ $(1\leq i\leq m-1)$, is semi-bent if and only if $\gcd(i,m)=1$ and m odd.

When $k=1$ and $r=1$ in Theorem 10, we have the following corollary.

Corollary 6 If $e=2^l$ for some positive integer l, then for any $\beta\in\mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{ei}}+\beta x^{1+2^{ej}})\,(1\leq i<j\leq\lfloor\frac{n}{4}\rfloor)$ is e-plateaued if and only if $\gcd(m,j+i)=1$, $\gcd(m,j-i)=1$ and m is odd.

Corollary 7 If $e=2^l$ for some positive integer l, then for any $\beta\in\mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{ei}}+\beta x^{1+2^{ej}})$ is e-plateaued for any i,j with $1\leq i<j\leq\lfloor\frac{n}{4}\rfloor$ if and only if m is an odd prime integer.

Corollary 8 ([8]) For any $\beta\in\mathbb{F}_{2^2}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{2i}}+\beta x^{1+2^{2j}})\,(1\leq i<j\leq\lfloor\frac{n}{4}\rfloor)$ is semi-bent if and only if $\gcd(m,j+i)=1$, $\gcd(m,j-i)=1$ and m is odd.

Theorem 12 If $e=2^l$ for some positive integer l, then for any $\beta\in\mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n(\beta x^{1+2^{ei}}+\beta x^{1+2^{ej}}+\beta x^{1+2^{et}})\,(1\leq i<j<t\leq\lfloor\frac{n}{4}\rfloor, i+j=t)$ is e-plateaued if and only if $\gcd(m,i)=1$, $\gcd(m,j)=1$ and $\gcd(m,t)=1$.

Proof Let $l(x)=x^i+x^{m-i}+x^j+x^{m-j}+x^t+x^{m-t}$. By Theorem 8, the function is e-plateaued if and only if $\gcd(l(x),x^m+1)=x+1$. Note that

$$
\begin{aligned}
l(x) &= (1+x^i)(1+x^j)+1+x^m+x^m(1+x^{-i})(1+x^{-j})\\
&= (1+x^i)(1+x^j)(1+x^{m-i-j})+1+x^m.
\end{aligned}
$$

Then the equality $\gcd(l(x),x^m+1)=x+1$ holds if and only if $\gcd(m,i)=1$, $\gcd(m,j)=1$ and $\gcd(m,t)=1$.

Theorem 13 If $e=2^l$ for some positive integer l, then for any $\beta\in\mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x)=\mathrm{Tr}_1^n$ $betax^{1+2^{ei}}+\beta x^{1+2^{ej}}+\beta x^{1+2^{et}})\,(1\leq i<j\leq m-1,\quad i+j=2t)$ is e-plateaued if and only if $\gcd(t,m)=1$.

Proof Let $L(x)=x^i+x^{m-i}+x^j+x^{m-j}+x^t+x^{m-t}$. By Theorem 8, the function is e-plateaued if and only if $\gcd(L(x),x^m+1)=x+1$. Note that

$$
\begin{aligned}
L(x) &= x^i+x^j+x^{\frac{i+j}{2}}+x^{m-i}+x^{m-j}+x^{m-\frac{i+j}{2}}\\
&= x^i+x^j+x^{\frac{i+j}{2}}+x^{m-(i+j)}\left(x^i+x^j+x^{\frac{i+j}{2}}\right)\\
&= x^i\left(1+x^{j-i}+x^{\frac{j-i}{2}}\right)\left(1+x^{m-(i+j)}\right),
\end{aligned}
\tag{23}
$$

and $\gcd(x^i,x^m+1)=1$, we have $\gcd(L(x),x^m+1)=\gcd((1+x^{j-i}+x^{\frac{j-i}{2}})(1+x^{m-(i+j)}),x^m+1)$. Since m is odd, $\mathrm{Tr}_1^m(1)=1$. Consequently, $\mathrm{Tr}_1^m(1+x^{j-i}+x^{\frac{j-i}{2}})=1$. That is, for any $a\in\mathbb{F}_{2^m}$, $1+a^{j-i}+a^{\frac{j-i}{2}}\neq 0$. Hence, $\gcd(x+x^{2^{j-i}}+x^{2^{\frac{j-i}{2}}},x^{2^m}+x)=1$. By Lemma 2, we have $\gcd(1+x^{j-i}+x^{\frac{j-i}{2}},x^m+1)=1$. Therefore, $\gcd(L(x),x^m+1)=\gcd\ (x^{m-(i+j)}+1,x^m+1)$. By Theorem 8, $f(x)$ is e-plateaued if and only if $\gcd(x^{m-(i+j)}+1,x^m+1)=x+1$, which is equivalent to the condition $\gcd(i+j,m)=\gcd\ (2t,m)=\gcd(t,m)=1$.

Corollary 9 If $n = 2m$ with $m(>1)$ odd, then for any $\beta \in \mathbb{F}_{2^2}^*$,

$$f(x) = \mathrm{Tr}_1^n\left(\beta x^{1+2^{2i}} + \beta x^{1+2^{2j}} + \beta x^{1+2^{2t}}\right)$$

$(i + j = 2t, 1 \le i < j \le m - 1)$ is semi-bent if and only if $\gcd(t, m) = 1$.

Theorem 14 If $e = 2^l$ for some positive integer l, then for any $\beta \in \mathbb{F}_{2^e}^*$, the function defined on $\mathbb{F}_{2^n}$ by $f(x) = \mathrm{Tr}_1^n(\beta x^1 + 2^{ei} + \beta x^{1+2^{ej}} + \beta x^{1+2^{et}})$ $(1 \le i < j \le m - 1, j - i = 2t)$ is e-plateaued if and only if $\gcd(t, m) = 1$.

Proof Let $L(x) = x^i + x^{m-i} + x^j + x^{m-j} + x^t + x^{m-t}$. By Theorem 8, the function is e-plateaued if and only if $\gcd(L(x), x^m + 1) = x + 1$. Consider

$$\begin{aligned} x^tL(x) &= x^{t+i} + x^{t+j} + x^{2t} + x^{m-i+t} + x^{m-j+t} + x^m \\ &= x^{t+j} + x^{m-i+t} + x^{2t} + x^{t+i} + x^{m-(j-t)} + x^m \\ &= x^{t+j} + x^{m-i+t} + x^{2t} + x^{\frac{i+j}{2}} + x^{m-\frac{i+j}{2}} + x^m \\ &= \left(x^{2t} + 1)(x^{\frac{i+j}{2}} + x^{m-\frac{i+j}{2}} + 1\right) + x^m + 1. \end{aligned} \tag{24}$$

Since $\gcd(x^t, x^m + 1) = 1$, we have

$$\begin{aligned} &\gcd(L(x), x^m + 1) = \gcd(x^tL(x), x^m + 1) \\ &= \gcd\left((x^{2t} + 1)\left(x^{\frac{i+j}{2}} + x^{m-\frac{i+j}{2}} + 1\right), x^m + 1\right). \end{aligned}$$

Suppose that a is a root of $x^tL(x)$, $a \notin \mathbb{F}_2$ and $a^m = 1$, then $a^{\frac{i+j}{2}} + a^{m-\frac{i+j}{2}} + 1 \ne 0$. Otherwise, we have $a^{\frac{i+j}{2}}(a^{\frac{i+j}{2}} + a^{m-\frac{i+j}{2}} + 1) = 0$, i.e.,

$$1 + a^{\frac{i+j}{2}} + a^{i+j} = 0. \tag{25}$$

Note that $\mathrm{Tr}_1^m(1) = 1$, and $\mathrm{Tr}_1^m(a^{\frac{i+j}{2}} + a^{i+j}) = 0$. This induces a contradiction with Eq. (25).

By the analysis above, we have

$$\begin{aligned} \gcd(L(x), x^m + 1) &= \gcd(x^tL(x), x^m + 1) \\ &= \gcd(x^{2t} + 1, x^m + 1). \end{aligned}$$

Consequently, $f(x)$ is e-plateaued if and only if $\gcd(x^{2t} + 1, x^m + 1) = x + 1$, which is equivalent to the condition $\gcd(t, m) = 1$.

Since m is odd, $\mathrm{Tr}_1^m(1) = 1$. Consequently, $\mathrm{Tr}_1^m(1 + x^{j-i} + x^{\frac{j-i}{2}}) = 1$. That is, for any $a \in \mathbb{F}_{2^m}$, $1 + a^{j-i} + a^{\frac{j-i}{2}} \ne 0$. Hence, $\gcd(x + x^{2^{j-i}} + x^{2^{\frac{j-i}{2}}}, x^{2^m} + x) = 1$. By Lemma 2, we have $\gcd(1 + x^{j-i} + x^{\frac{j-i}{2}}, x^m + 1) = 1$. Therefore, $\gcd(L(x), x^m + 1) = \gcd(x^{m-(i+j)} + 1, x^m + 1)$. By Theorem 8, $f(x)$ is e-plateaued if and only if $\gcd(x^{m-(i+j)} + 1, x^m + 1) = x + 1$, which is equivalent to the condition $\gcd(i + j, m) = \gcd(2t, m) = \gcd(t, m) = 1$.

Corollary 10 If $n = 2m$ with $m(>1)$ odd, then for any $\beta \in \mathbb{F}_{2^2}^*$,

$$f(x) = \mathrm{Tr}_1^n(\beta x^{1+2^{2i}} + \beta x^{1+2^{2j}} + \beta x^{1+2^{2t}})$$

$(j - i = 2t, 1 \le i < j \le m - 1)$ is semi-bent if and only if $\gcd(t, m) = 1$.

5 Concluding remarks

In this paper, we study the function defined by

$$f(x) = \sum_{i=1}^{\frac{q-1}{2}} c_i \mathrm{Tr}_1^n\left(x^{1+2^{pi}}\right) + \sum_{i=1}^{\frac{p-1}{2}} d_i \mathrm{Tr}_1^n\left(x^{1+2^{qi}}\right),$$

where $c_i, d_j \in \mathbb{F}_2$, $1 \le i \le \frac{q-1}{2}$, $1 \le j \le \frac{p-1}{2}$, $n = pq$, p, q odd, $\gcd(p, q) = 1$, and the function defined by

$$f(x) = \sum_{i=1}^{\frac{m-1}{2}} c_i \mathrm{Tr}_1^n\left(\beta x^{1+2^{ei}}\right),$$

where $n = em$, $e = 2^l$, m is odd, $c_i \in \mathbb{F}_2$ $(1 \le i \le \frac{m-1}{2})$, $\beta \in \mathbb{F}_{2^e}^*$. We prove that these two kinds of functions contain semi-bent ones in certain cases. Moreover, we present some characterizations of e-plateaued functions with few trace terms when n is even. Furthermore, their are still some problems that need to be studied such as how to obtain semi-bent functions with higher degree by the primary constructions.

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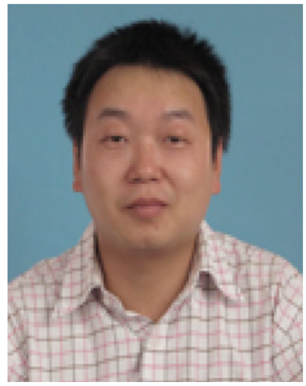

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