Graded quasi-Lie algebras of Witt type *)

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Received 16 August 2006

In this article, we introduce a new class of graded algebras called quasi-Lie algebras of Witt type. These algebras can be seen as a generalization of other Witt-type algebras like Lie algebras of Witt type and their colored version, Lie color algebras of Witt type.

PACS: 02.10.Hh
Key words: graded quasi-Lie algebra, Lie color algebra, Witt-type algebra

1 Introduction

Since 1980's, when research on quantum deformations (or q-deformations) of Lie algebras began a period of rapid expansion in connection with the introduction of quantum groups motivated by applications to the Yang–Baxter equation and quantum inverse scattering methods, several other versions of (q-)deformed Lie algebras have appeared, especially in physical contexts such as the string theory. The main objects for these deformations were infinite-dimensional algebras, primarily the Heisenberg algebras (oscillator algebras) and the Virasoro algebra, see [1–12] and the references therein. An important common feature for these algebras is appearance of some deformed (twisted) versions of skew-symmetry or Jacobi identity. At the same time, in a well-known direct generalization of Lie algebras and Lie superalgebras to general commutative grading groups, the class of Lie color algebras, generalized skew-symmetry and Jacobi-type identities, graded by a commutative group and twisted by a scalar bicharacter, hold. A remarkable and not yet fully exploited and understood feature of Lie color algebras and Lie superalgebras is that they often appear simultaneously with usual Lie algebras in various deformation families of algebras for initial, final or other important special values of the deformation parameters in the deformed algebras.

We let $\mathbb{F}$ be a field of characteristic zero, $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ and $(\Gamma, \hat{+})$ an Abelian group. Recall that a *Lie color algebra* or *Γ-graded ε-Lie algebra* (see [13, 14]) is a Γ-graded linear space L with a bilinear multiplication $\langle \cdot, \cdot \rangle$ satisfying

*) Presented at the 15^{th} International Colloquium on "Integrable Systems and Quantum Symmetries", Prague, 15–17 June 2006.

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- $\langle x, y\rangle = -\varepsilon(\gamma_x, \gamma_y)\langle y, x\rangle$,
- $\varepsilon(\gamma_z, \gamma_x)\langle x, \langle y, z\rangle\rangle + \varepsilon(\gamma_x, \gamma_y)\langle y, \langle z, x\rangle\rangle + \varepsilon(\gamma_y, \gamma_z)\langle z, \langle x, y\rangle\rangle = 0$

for $x \in L_{\gamma_x}$, $y \in L_{\gamma_y}$ and $z \in L_{\gamma_z}$ where $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$. The map $\varepsilon : \Gamma \times \Gamma \to \mathbb{F}^*$, called a commutation factor, is a bicharacter on Γ with a symmetry property $\varepsilon(\gamma_x, \gamma_y)\varepsilon(\gamma_y, \gamma_x) = 1$. Notice that Lie color algebras include both Γ-graded Lie algebras (with $\varepsilon(\gamma_x, \gamma_y) = 1$ for all $\gamma_x, \gamma_y \in \Gamma$) and Lie superalgebras (with $\Gamma = \mathbb{Z}_2$ and $\varepsilon(\gamma_x, \gamma_y) = (-1)^{\gamma_x\gamma_y}$).

Let $C : \Gamma \to \mathbb{F}$ be a function defined over Γ and consider the Γ-graded vector space $L = \bigoplus_{g\in\Gamma} \mathbb{F}\mathbf{e}_g$ over $\mathbb{F}$ with the basis $\{\mathbf{e}_g \mid g \in \Gamma\}$. Assume that $\varepsilon : \Gamma\times\Gamma \to \mathbb{F}^*$ is a commutation factor on Γ. The linear space L endowed with the bracket product

$$[\mathbf{e}_g, \mathbf{e}_h] = \big(C(h) - \varepsilon(g, h)C(g)\big)\,\mathbf{e}_{g+h} \tag{1}$$

is a Lie color algebra $L(\Gamma, C)$ under certain conditions on ε, Γ and the function C. $L(\Gamma, C)$ is then said to be a *Lie color algebra of Witt type.* For the full details see [15]. The special case given by $\varepsilon(g, h) = 1$ for all g, $h \in \Gamma$ and with C satisfying the condition

$$\big(C(g) - C(h)\big)\big(C(g\hat{+}h) - C(g) - C(h) + C(0)\big) = 0 \quad \text{for all} \quad g, h \in \Gamma \tag{2}$$

is a Γ-graded Lie algebra and it is called a *Lie algebra of Witt type* (see [16] for instance).

Lie color algebras are examples of Γ-graded quasi-Lie algebras (see [12]). The main purpose of this article is to describe a general class of Γ-graded quasi-Lie algebras of Witt type.

2 Γ-graded quasi-Lie algebras of Witt type

We form the Γ-graded $\mathbb{F}$-vector space $W := \bigoplus_{g\in\Gamma} \mathbb{F}\mathbf{e}_g$, where $\mathbb{F}$ is our field. In addition, let $A, B : \Gamma \times \Gamma \to \mathbb{F}$ be two functions. Note, however, that we do not assume that A and B are group morphisms in any of their arguments. Let $\mathrm{L}_{\mathbb{F}}(L)$ be the set of linear maps of the linear space L over the field $\mathbb{F}$. By $\circlearrowleft_{x,y,z}$ we denote cyclic summation with respect to x, y, z.

Definition 1. A *Γ-graded (color) quasi-Lie algebra* is a tuple $\big(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega, \theta\big)$, where

- $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ is a Γ-graded linear space over $\mathbb{F}$,
- $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map called a product or bracket in L,
- α, $\beta : L \to L$ are linear maps mapping $\cup_{\gamma\in\Gamma} L_\gamma$ to $\cup_{\gamma\in\Gamma} L_\gamma$,
- $\omega : D_\omega \to \mathrm{L}_{\mathbb{F}}(L)$ and $\theta : D_\theta \to \mathrm{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq \cup_{\gamma\in\Gamma} L_\gamma \times \cup_{\gamma\in\Gamma} L_\gamma$,

such that the following conditions hold:

- (Γ-grading axiom) $\langle L_{\gamma_1}, L_{\gamma_2}\rangle_L \subseteq L_{\gamma_1 \hat{+} \gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$,
- (ω-symmetry) $\langle x, y\rangle_L = \omega(x,y)\langle y, x\rangle_L$ for all $(x,y) \in D_\omega$,
- (quasi-Jacobi identity) $\circlearrowleft_{x,y,z} \left\{ \theta(z,x)\big(\langle \alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\big)\right\} = 0$, whenever $(z,x) \in D_\theta$, $(x,y) \in D_\theta$, $(y,z) \in D_\theta$.

Equip W with the bracket product

$$\langle \mathbf{e}_g, \mathbf{e}_h\rangle := \big(A(g,h) - B(g,h)\big)\mathbf{e}_{g+h} = \mathrm{S}^g_h \mathbf{e}_{g+h}\,, \tag{3}$$

where we have put $\mathrm{S}^g_h := A(g,h) - B(g,h)$ in order to simplify calculations. When $\big(W, \langle\cdot,\cdot\rangle\big)$ is a quasi-Lie algebra, then we call it a quasi-Lie algebra of Witt type.

Let

$$\alpha(\mathbf{e}_k) = \sum_{i\in\Gamma} \alpha^i_k \mathbf{e}_i\,, \quad \beta(\mathbf{e}_k) = \sum_{i\in\Gamma} \beta^i_k \mathbf{e}_i\,, \quad \theta(\mathbf{e}_g, \mathbf{e}_h)(\mathbf{e}_k) = \sum_{i\in\Gamma} \theta^{k,i}_{gh} \mathbf{e}_i\,, \tag{4}$$

where $\alpha^i_k, \beta^i_k, \theta^{k,i}_{gh} \in \mathbb{F}$ for all $i,k,g,h \in \Gamma$. Also, we assume that there are only finitely many non-zero terms in either sum in (4).

We compute

$$\langle \alpha(\mathbf{e}_g), \langle \mathbf{e}_h, \mathbf{e}_k\rangle\rangle = \mathrm{S}^h_k \left\langle \sum_{i\in\Gamma} \alpha^i_g \mathbf{e}_i, \mathbf{e}_{h+k}\right\rangle = \mathrm{S}^h_k \sum_{i\in\Gamma} \alpha^i_g \mathrm{S}^i_{h+k} \mathbf{e}_{i+h+k}\,,$$

and

$$\beta\langle \mathbf{e}_g, \langle \mathbf{e}_h, \mathbf{e}_k\rangle\rangle = \mathrm{S}^h_k \mathrm{S}^g_{h+k} \sum_{j\in\Gamma} \beta^j_{g+h+k} \mathbf{e}_j\,.$$

Adding up these expressions and letting θ act on the sum yields

$$\begin{aligned}
&\theta(\mathbf{e}_k, \mathbf{e}_g)\big\{\langle \alpha(\mathbf{e}_g), \langle \mathbf{e}_h, \mathbf{e}_k\rangle\rangle + \beta(\langle \mathbf{e}_g, \langle \mathbf{e}_h, \mathbf{e}_k\rangle\rangle)\big\} \\
&\qquad = \theta(\mathbf{e}_k, \mathbf{e}_g)\left\{\mathrm{S}^h_k \left(\sum_{i\in\Gamma} \alpha^i_g \mathrm{S}^i_{h+k} \mathbf{e}_{i+h+k} + \mathrm{S}^g_{h+k} \sum_{j\in\Gamma} \beta^j_{g+h+k} \mathbf{e}_j\right)\right\} \\
&\qquad = \mathrm{S}^h_k \sum_{j\in\Gamma} \left(\alpha^{j-(h+k)}_g \mathrm{S}^{j-(h+k)}_{h+k} + \mathrm{S}^g_{h+k} \beta^j_{g+h+k}\right) \theta(\mathbf{e}_k, \mathbf{e}_g)(\mathbf{e}_j) \\
&\qquad = \mathrm{S}^h_k \sum_{j\in\Gamma} \left(\alpha^{j-(h+k)}_g \mathrm{S}^{j-(h+k)}_{h+k} + \mathrm{S}^g_{h+k} \beta^j_{g+h+k}\right) \sum_{l\in\Gamma} \theta^{j,l}_{kg} \mathbf{e}_l \\
&\qquad = \mathrm{S}^h_k \sum_{l\in\Gamma} \left(\sum_{j\in\Gamma} \theta^{j,l}_{kg} \left(\alpha^{j-(h+k)}_g \mathrm{S}^{j-(h+k)}_{h+k} + \mathrm{S}^g_{h+k} \beta^j_{g+h+k}\right)\right) \mathbf{e}_l\,.
\end{aligned}$$

Permuting g, h, k cyclically and adding up we get

$$\begin{aligned}
\sum_{l\in\Gamma} \Big\{ &\mathrm{S}^h_k \sum_{j\in\Gamma} \theta^{j,l}_{kg} \big(\alpha^{j-(h+k)}_g \mathrm{S}^{j-(h+k)}_{h+k} + \mathrm{S}^g_{h+k} \beta^j_{g+h+k}\big) \\
&\quad + \mathrm{S}^k_g \sum_{j\in\Gamma} \theta^{j,l}_{gh} \big(\alpha^{j-(k+g)}_h \mathrm{S}^{j-(k+g)}_{k+g} + \mathrm{S}^h_{k+g} \beta^j_{g+h+k}\big) \\
&\qquad + \mathrm{S}^g_h \sum_{j\in\Gamma} \theta^{j,l}_{hk} \big(\alpha^{j-(g+h)}_k \mathrm{S}^{j-(g+h)}_{g+h} + \mathrm{S}^k_{g+h} \beta^j_{g+h+k}\big) \Big\} \mathbf{e}_l\,.
\end{aligned}$$

To have a quasi-Jacobi identity, we need that the coefficients in curly brackets above be zero for all $l \in \Gamma$, i.e.,

$$\begin{aligned} \mathrm{S}_k^h \sum_{j\in\Gamma} \theta_{kg}^{j,l}\big(\alpha_g^{j-(h+k)}\mathrm{S}_{h+k}^{j-(h+k)} + \mathrm{S}_{h+k}^g \beta_{g+h+k}^j\big) \\ + \mathrm{S}_g^k \sum_{j\in\Gamma} \theta_{gh}^{j,l}\big(\alpha_h^{j-(k+g)}\mathrm{S}_{k+g}^{j-(k+g)} + \mathrm{S}_{k+g}^h \beta_{g+h+k}^j\big) \\ + \mathrm{S}_h^g \sum_{j\in\Gamma} \theta_{hk}^{j,l}\big(\alpha_k^{j-(g+h)}\mathrm{S}_{g+h}^{j-(g+h)} + \mathrm{S}_{g+h}^k \beta_{g+h+k}^j\big) = 0 \end{aligned} \tag{5}$$

for all $l \in \Gamma$.

By definition, we assume that α, β are linear maps mapping $\cup_{\gamma\in\Gamma} L_\gamma$ to $\cup_{\gamma\in\Gamma} L_\gamma$. Since every homogeneous subspace L_γ is one-dimensional, there exist functions $p, q : \Gamma \to \Gamma$ such that $\alpha(\mathbf{e}_k) = \alpha_k^{p(k)}\mathbf{e}_{p(k)}, \beta(\mathbf{e}_k) = \beta_k^{q(k)}\mathbf{e}_{q(k)}$ for all $k \in \Gamma$. It follows from (5) that

$$\circlearrowleft_{g,h,k} \mathrm{S}_k^h \big\{ \mathrm{S}_{h+k}^{p(g)} \alpha_g^{p(g)} \theta_{kg}^{p(g)+h+k,l} + \mathrm{S}_{h+k}^g \beta_{g+h+k}^{q(g+h+k)} \theta_{kg}^{q(g+h+k),l} \big\} = 0\,, \tag{6}$$

for all $l \in \Gamma$.

Suppose $\omega(\mathbf{e}_g, \mathbf{e}_h)(\mathbf{e}_k) = \sum_{i\in\Gamma} \omega_{gh}^{k,i}\mathbf{e}_i$. We have by definition (3), $\langle \mathbf{e}_g, \mathbf{e}_h\rangle = \mathrm{S}_h^g \mathbf{e}_{g+h}$. Exchanging g and h we get $\langle \mathbf{e}_h, \mathbf{e}_g\rangle = \mathrm{S}_g^h \mathbf{e}_{g+h}$. By the ω-skew-symmetry axiom these two are thus related by

$$\mathrm{S}_h^g \mathbf{e}_{g+h} = \mathrm{S}_g^h \omega(\mathbf{e}_g, \mathbf{e}_h)(\mathbf{e}_{g+h}) = \sum_{i\in\Gamma} \mathrm{S}_g^h \omega_{gh}^{g+h,i}\mathbf{e}_i\,.$$

Unless $i = g+h$ we must have $\mathrm{S}_g^h \omega_{gh}^{g+h,i} = 0$. Hence

$$\mathrm{S}_g^h \omega_{gh}^{g+h,i} = \mathrm{S}_h^g \delta_{g+h,i} \tag{7}$$

for all $i \in \Gamma$.

Theorem 1. *Suppose α, β and θ are given by (4) for all $g, h \in \Gamma$. Then $\big(W, \langle\cdot,\cdot\rangle\big)$, where $\langle \mathbf{e}_g, \mathbf{e}_h\rangle = \big(A(g,h) - B(g,h)\big)\mathbf{e}_{g+h} = \mathrm{S}_h^g \mathbf{e}_{g+h}$ is a quasi-Lie algebra if and only if (5) and (7) are satisfied.*

Proof. Necessity has already been proved. Sufficiency is clear by simply going backwards in the above calculations. □

Assuming that $\alpha(\mathbf{e}_k) = \alpha_k^k \mathbf{e}_k$ and $\beta(\mathbf{e}_k) = \beta_k^k \mathbf{e}_k$ for all $k \in \Gamma$, the condition (5) is simplified to

$$\begin{aligned} \theta_{kg}^{g+h+k,l} \mathrm{S}_k^h \mathrm{S}_{h+k}^g \big(\alpha_g^g + \beta_{g+h+k}^{g+h+k}\big) + \theta_{gh}^{g+h+k,l} \mathrm{S}_g^k \mathrm{S}_{k+g}^h \big(\alpha_h^h + \beta_{g+h+k}^{g+h+k}\big) \\ + \theta_{hk}^{g+h+k,l} \mathrm{S}_h^g \mathrm{S}_{g+h}^k \big(\alpha_k^k + \beta_{g+h+k}^{g+h+k}\big) = 0 \end{aligned}$$

for all $l \in \Gamma$. The special case $\alpha_k^k = \beta_k^k = 1$ for all $k \in \Gamma$ yields

$$\mathrm{S}_k^h \mathrm{S}_{h+k}^g \theta_{kg}^{g+h+k,l} + \mathrm{S}_g^k \mathrm{S}_{k+g}^h \theta_{gh}^{g+h+k,l} + \mathrm{S}_h^g \mathrm{S}_{g+h}^k \theta_{hk}^{g+h+k,l} = 0 \tag{8}$$

for all $l \in \Gamma$. Multiplying the left-hand side by $\mathbf{e}_l$ and summing over l we obtain by (4) for all $g,\ h,\ k \in \Gamma$

$$\big\{\mathrm{S}_k^h \mathrm{S}_{h+k}^g \theta(\mathbf{e}_k, \mathbf{e}_g) + \mathrm{S}_g^k \mathrm{S}_{k+g}^h \theta(\mathbf{e}_g, \mathbf{e}_h) + \mathrm{S}_h^g \mathrm{S}_{g+h}^k \theta(\mathbf{e}_h, \mathbf{e}_k)\big\}(\mathbf{e}_{g+h+k}) = 0\,. \tag{9}$$

This implies that

$$\theta(\mathbf{e}_k, \mathbf{e}_g)\mathrm{S}_k^h \mathrm{S}_{h+k}^g + \theta(\mathbf{e}_g, \mathbf{e}_h)\mathrm{S}_g^k \mathrm{S}_{k+g}^h + \theta(\mathbf{e}_h, \mathbf{e}_k)\mathrm{S}_h^g \mathrm{S}_{g+h}^k = 0 \tag{10}$$

for all $g,\ h,\ k \in \Gamma$. Note that $\mathrm{S}_k^h \mathrm{S}_{h+k}^g \mathbf{e}_{g+h+k} = \big\langle \mathbf{e}_g, \langle \mathbf{e}_h, \mathbf{e}_k\rangle\big\rangle$ and by cyclic permutation of indices $\mathrm{S}_g^k \mathrm{S}_{k+g}^h \mathbf{e}_{g+h+k} = \big\langle \mathbf{e}_h, \langle \mathbf{e}_k, \mathbf{e}_g\rangle\big\rangle$ and $\mathrm{S}_h^g \mathrm{S}_{g+h}^k \mathbf{e}_{g+h+k} = \big\langle \mathbf{e}_k, \langle \mathbf{e}_g, \mathbf{e}_h\rangle\big\rangle$. Since we now have $\alpha = \beta = \mathrm{id}_L$, the choice of θ such that $\theta(x,y)(v) = -\varepsilon(\gamma_x, \gamma_y)v$ for $v \in L$ means that from (10) we recover the Jacobi identity for a colored Lie algebra with commutation factor ε. Recall that $\gamma_x, \gamma_y \in \Gamma$ are the graded degrees of x and y. So taking $\theta(\mathbf{e}_g, \mathbf{e}_h) = -\varepsilon(g,h)\,\mathrm{id}_L$ for $g, h \in \Gamma$ yields by relation (10)

$$\varepsilon(k,g)\mathrm{S}_k^h \mathrm{S}_{h+k}^g + \varepsilon(g,h)\mathrm{S}_g^k \mathrm{S}_{k+g}^h + \varepsilon(h,k)\mathrm{S}_h^g \mathrm{S}_{g+h}^k = 0 \tag{11}$$

for all $g,\ h,\ k \in \Gamma$.

This work was partially supported by the Crafoord Foundation, the Royal Physiographic Society in Lund, the Swedish Royal Academy of Sciences and the Swedish Foundation for International Cooperation in Research and Higher Education (STINT).

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