Lagrangian fractional mechanics — a noncommutative approach *)

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The extension of coordinate–velocity space with noncommutative algebra structure is proposed. For action of fractional mechanics considered on such a space the respective Euler–Lagrange equations are derived via minimum action principle. It appears that equations of motion in the noncommutative framework do not mix left and right derivatives thus being simple to solve at least in the linear case. As an example, two models of oscillator with fractional derivatives are studied.

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1 Introduction

In recent years fractional differential and integral calculus (see [1] for overview of results) was applied in various fields of physics: for example in different problems of transport theory ([2–5] and references therein), in models of nonconservative mechanical systems [6, 7]; in field theory the Klein–Gordon and wave equations with fractional derivatives were studied in [8] and quantum mechanics with fractional derivatives was developed via path integral method [9]. In classical mechanics with fractional derivatives [6, 7] Euler–Lagrange equations were obtained by minimum action principle. It appears that these equations mix left and right fractional derivatives, even if the initial action depends only on one type of them.

The aim of the present paper is to show that on the changed, noncommutative coordinate–velocity space we can prevent mixing of derivatives and arrive at a new type of equations of motion. Let us start with a brief review of some properties of fractional derivatives and previous results of fractional mechanics.

The left and right fractional Riemann–Liouville derivatives are defined as follows for $m < \mathrm{Re}\,\alpha < m+1$:

$$
{}_bD^{\alpha}_{\mathrm{R}} f(t) = \frac{1}{\Gamma(m+1-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_t^b \frac{f(s)}{(s-t)^{\alpha-m}}\,\mathrm{d}s\,,
$$

$$
{}_aD^{\alpha}_{\mathrm{L}} f(t) = \frac{1}{\Gamma(m+1-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_a^t \frac{f(s)}{(t-s)^{\alpha-m}}\,\mathrm{d}s\,.
$$

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Throughout the paper we shall use notation: ${}_aD^{\alpha}_{\mathrm{L}} = D^{\alpha}_{\mathrm{L}}$ and ${}_bD^{\alpha}_{\mathrm{R}} = D^{\alpha}_{\mathrm{R}}$. Both derivatives are connected via the reflection operator $Qf(t) = f(a+b-t)$:

$$D^{\alpha}_{\mathrm{L}} = QD^{\alpha}_{\mathrm{R}}Q\,. \tag{1}$$

They also obey the formula of integration by parts [1]

$$\int_a^b [D^{\alpha}_{\mathrm{L}} f(t)] \cdot g(t)\mathrm{d}t = \int_a^b f(t) \cdot D^{\alpha}_{\mathrm{R}} g(t)\mathrm{d}t \tag{2}$$

provided the respective boundary conditions are fulfilled for function f or g:

$$f^{(k)}(a) = f^{(k)}(b) = 0\,, \quad g^{(k)}(a) = g^{(k)}(b) = 0\,, \qquad k = 0,\ldots,m\,. \tag{3}$$

We see that in this formula left and right derivatives are mixed contrary to the classical formula of integration by parts, where only classical first order derivative appears.

In the mentioned above papers on fractional mechanics the action depending on fractional derivatives of coordinates was considered:

$$S = \int_a^b L\left(\{q^r_n, Q^r_{n'}\}, t\right) \mathrm{d}t\,,$$

where the generalized coordinates can be defined in sequential form (see [7]) with $q^r_n := (D^{\alpha}_{\mathrm{L}})^n x_r(t)$, $Q^r_{n'} := (D^{\beta}_{\mathrm{R}})^{n'} x_r(t)$ or in non-sequential form proposed by Riewe [6] $q^r_n := D^{\alpha_n}_{\mathrm{L}} x_r(t)$, $Q^r_{n'} := D^{\beta_{n'}}_{\mathrm{R}} x_r(t)$ (where $r = 1,\ldots,R$ denotes the number of fundamental coordinates).

After application of properties of fractional derivatives from the minimum action principle condition: $\delta S(\vec{\eta}) = 0$ the generalized Euler–Lagrange equations were obtained for the sequential case [7]:

$$\frac{\partial L}{\partial q^r_0} + \sum_{n=1}^{N} (D^{\alpha}_{\mathrm{R}})^n \frac{\partial L}{\partial q^r_n} + \sum_{n'=1}^{N'} (D^{\beta}_{\mathrm{L}})^{n'} \frac{\partial L}{\partial Q^r_{n'}} = 0$$

and for the non-sequential case [6]:

$$\frac{\partial L}{\partial q^r_0} + \sum_{n=1}^{N} D^{\alpha_n}_{\mathrm{R}} \frac{\partial L}{\partial q^r_n} + \sum_{n'=1}^{N'} D^{\beta_{n'}}_{\mathrm{L}} \frac{\partial L}{\partial Q^r_{n'}} = 0\,.$$

Thus even for Lagrangian depending initially on one type of derivatives we obtain Euler–Lagrange equations with mixed left and right fractional derivatives. This feature is connected with mixing derivatives in the formula of integration by parts in fractional calculus (2). The partial solution of this difficulty was proposed in [7] in the form of fractional mechanics with symmetric derivatives:

$$\mathcal{D}^{\alpha} := \tfrac{1}{2}\left(D^{\alpha}_{\mathrm{L}} + (-1)^m D^{\alpha}_{\mathrm{R}}\right).$$

The symmetric derivative obeys the following rule of integration by parts for $m < \operatorname{Re}\alpha < m+1$:

$$\int_a^b \left[\mathcal{D}^\alpha f(t)\right] \cdot g(t)\mathrm{d}t = (-1)^m \int_a^b f(t) \cdot \mathcal{D}^\alpha g(t)\mathrm{d}t$$

provided the boundary conditions (3) are fulfilled. When the action depends only on the symmetric derivatives (with generalized coordinates defined as follows: $\tilde{q}_n^r = (\mathcal{D}^\alpha)^n x_r(t)$) by the minimal action principle, we arrive at the set of generalized Euler–Lagrange equations including only the initial symmetric derivatives. Still, the basic equation of models of this type: $\mathcal{D}^\alpha x(t) = f(t)$ is very difficult to solve in general case for arbitrary α and f.

Let us change the algebra of functions on coordinate–velocity space in such a way as to obtain the integration by parts formula without mixing different types of derivatives. The appropriate framework includes extension of the space and the new algebra is noncommutative.

2 Euler−Lagrange equations for models on noncommutative extended algebra of functions

2.1 Noncommutative extended algebra of functions

Let us introduce the new product using the reflection operator from Section 1:

$$f \bullet g = fQg = f(Qg)Q\,.$$

This product has the following properties:

– it is associative

$$(f \bullet g) \bullet h = f \bullet (g \bullet h)\,;$$

– Q is the left and right neutral element

$$f \bullet Q = fQQ = Q \bullet f = QQf = f\,;$$

– $Q\left(\frac{1}{f}\right)$ is the left and right inverse element for $f \neq 0$

$$f \bullet Q\left(\frac{1}{f}\right) = Q\left(\frac{1}{f}\right) \bullet f = Q\,;$$

– it is noncommutative

$$f \bullet g \neq g \bullet f\,;$$

– under the integral the product is commutative (we apply $(Q\mathrm{d}t) = -\mathrm{d}t$ and $Qf \bullet g = (Qf) \bullet (Qg)$):

$$\int_a^b f \bullet g\,\mathrm{d}t = \int_a^b g \bullet f\,\mathrm{d}t\,. \tag{4}$$

Applying the commutation relation for fractional derivatives and the reflection operator (1) we obtain the formula of integration by parts for the new product:

$$\begin{aligned} \int_a^b f \bullet (D^{\alpha}_{\mathrm{L}} g)\,\mathrm{d}t &= \int_a^b (D^{\alpha}_{\mathrm{L}} f) \bullet g\,\mathrm{d}t\,, \\ \int_a^b f \bullet (D^{\beta}_{\mathrm{R}} g)\,\mathrm{d}t &= \int_a^b (D^{\beta}_{\mathrm{R}} f) \bullet g\,\mathrm{d}t \end{aligned} \tag{5}$$

provided the respective boundary conditions (3) are fulfilled. Let us notice that it is analogous to the classical formula and does not mix two types of fractional derivatives. In order to derive the differential calculus on extended algebra of functions we define the derivative on monomials:

$$\frac{\delta}{\delta q}(q^{\bullet n}) := n q^{\bullet (n-1)}\,.$$

This derivative is connected with partial derivative on classical algebra of functions:

$$\frac{\delta}{\delta q_i}\, L_{\bullet}(q_0,\ldots,q_N) = T^{-1} S\, \frac{\partial}{\partial q_i}\, T S\, L_{\bullet}(q_0,\ldots,q_N)\,,$$

where S is the symmetrizer and the discrete operator T is the switch operator acting as follows on symmetric products:

$$T(f \bullet g + g \bullet f) = fg + gf\,, \quad TQ = 1\,, \quad T1 = 1\,.$$

The relation connecting new derivative with partial derivative on commutative algebra of functions yields the corresponding formula for the differential of first order:

$$\Delta L_{\bullet}(q_0,\ldots,q_N)(\eta_0,\ldots,\eta_N) = \left[\sum_{i=0}^{N} \left(S \frac{\delta}{\delta q_i}\, L_{\bullet}\right) \bullet \eta_i\right]_{\mathrm{sym}}\,,$$

when $(\eta_0,\ldots,\eta_N)$ is a variation of the vector $(q_0,\ldots,q_N)$.

After integration and application of the property (4) we obtain the following variation of the action:

$$\delta S(\vec{\eta}) = \int_a^b \Delta L_{\bullet}(q_0,\ldots,q_N)(\eta_0,\ldots,\eta_N)\,\mathrm{d}t = \int_a^b \left[\sum_{i=0}^{N} \left(\frac{\delta}{\delta q_i} L_{\bullet}\right) \bullet \eta_i\right] \mathrm{d}t\,,$$

which shall yield the required equations of motion.

2.2 Euler−Lagrange equation for a simple model

Let us apply the differential calculus on the proposed noncommutative algebra to the case of Lagrangian depending only on first order of the fractional derivative of coordinate x:

$$S = \int_a^b L_{\bullet}(q_0, q_1)\,\mathrm{d}t\,,$$

where for $m < \operatorname{Re}\alpha < m+1$; we define $q_0 = x$ and $q_1 = D_{\mathrm{L}}^{\alpha} x$.

The variation of the action looks as follows:

$$\delta S = \int_a^b \Delta L_\bullet(q_0, q_1)(\eta_0, \eta_1)\,\mathrm{d}t = \int_a^b \left(\frac{\delta L_\bullet}{\delta q_0} \bullet \eta_0 + \frac{\delta L_\bullet}{\delta q_1} \bullet \eta_1 \right) \mathrm{d}t\,.$$

As $\eta_1 = D_{\mathrm{L}}^{\alpha}\eta_0$, we can rewrite the variation using the formula of integration by parts (5) for variation η_0 fulfilling the respective boundary conditions (3):

$$\delta S = \int_a^b \left(\frac{\delta L_\bullet}{\delta q_0} + D_{\mathrm{L}}^{\alpha} \frac{\delta L_\bullet}{\delta q_1} \right) \bullet \eta_0 \,\mathrm{d}t$$

and obtain Euler–Lagrange equations for the model containing only left fractional derivative:

$$\frac{\delta L_\bullet}{\delta q_0} + D_{\mathrm{L}}^{\alpha} \frac{\delta L_\bullet}{\delta q_1} = 0\,.$$

2.3 Euler−Lagrange equations: a general case

Let us now pass to the general case and consider the action depending on left fractional derivatives:

$$S = \int_a^b L_\bullet(q_0^r, \ldots, q_N^r)\,\mathrm{d}t\,,$$

where the generalized coordinates look as follows for sequential and nonsequential case respectively:

$$\begin{aligned} q_n^r &= (D_{\mathrm{L}}^{\alpha})^n x^r\,, \qquad r = 1,\ldots,R\,, \quad n = 0,\ldots,N\,,\\ q_n^r &= D_{\mathrm{L}}^{\alpha_n} x^r\,, \qquad r = 1,\ldots,R\,, \quad n = 0,\ldots,N\,. \end{aligned}$$

Euler–Lagrange equations for both models contain only left fractional derivatives ($r = 1,\ldots,R$) for sequential model

$$\frac{\delta L_\bullet}{\delta q_0^r} + \sum_{n=1}^{N} (D_{\mathrm{L}}^{\alpha})^n \frac{\delta L_\bullet}{\delta q_n^r} = 0\,,$$

as well as for non-sequential model

$$\frac{\delta L_\bullet}{\delta q_0^r} + \sum_{n=1}^{N} D_{\mathrm{L}}^{\alpha_n} \frac{\delta L_\bullet}{\delta q_n^r} = 0\,.$$

Let us finally notice that in the noncommutative framework Euler–Lagrange equations depend on both types of derivatives only in the case, when they both appear in the initial action [10].

3 Applications

3.1 Example: Fractional harmonic oscillator

The fractional analog of harmonic oscillator for $m < \mathrm{Re}\,\alpha < m+1$ has the following action:

$$S = \int_a^b \left(\tfrac{1}{2}\, D_{\mathrm{L}}^{\alpha} x \bullet D_{\mathrm{L}}^{\alpha} x + \tfrac{1}{2}\,\omega^2 x \bullet x\right) \mathrm{d}t\,.$$

Using the developed method we derive Euler–Lagrange equation

$$\left[(D_{\mathrm{L}}^{\alpha})^2 + \omega^2\right] x = 0\,. \tag{6}$$

When the solution x is known, the function QxQ solves the advanced version of the obtained equation of motion

$$\left[(D_{\mathrm{R}}^{\alpha})^2 + \omega^2\right] QxQ = 0\,.$$

The equation (6) can be solved by factorization to the following fractional equations:

$$D_{\mathrm{L}}^{\alpha}\, x = \mathrm{i}\omega x\,, \qquad D_{\mathrm{L}}^{\alpha}\, x = -\mathrm{i}\omega x\,.$$

Thus the full solution of this type oscillator equation is a combination of first order derivatives of Mittag–Leffler functions:

$$x = C_1 x_1 + C_2 x_2\,, \qquad x_{1,2}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \frac{(\pm \mathrm{i}\omega)^k}{\Gamma(1+\alpha k)}\,(t-a)^{\alpha k}\,.$$

When $\alpha \longrightarrow 1^+$, we recover classical equation and solution:

$$x'' + \omega^2 x = 0\,, \qquad x = C_1 x_1 + C_2 x_2\,,$$

$$x_{1,2}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \frac{(\pm \mathrm{i}\omega)^k}{k!}\,(t-a)^k = \pm \mathrm{i}\omega \sum_{k=0}^{\infty} \frac{(\pm \mathrm{i}\omega)^k}{k!}\,(t-a)^k\,.$$

3.2 Example: Fractional version of classical harmonic oscillator

Let us now discuss the following action:

$$S = \int_a^b \left(\tfrac{1}{2}\,(D_{\mathrm{L}}^{1/n})^n x \bullet (D_{\mathrm{L}}^{1/n})^n x + \tfrac{1}{2}\,\omega^2 x \bullet x\right) \mathrm{d}t\,.$$

The corresponding Euler–Lagrange equation:

$$\left[(D_{\mathrm{L}}^{1/n})^{2n} + \omega^2\right] x = 0$$

for part of trajectories, where the composition rule works [1] coincides with classical equation for harmonic oscillator. Let us notice that the operator of this equation is a composition of the following fractional differential operators of order $1/n$:

$$D_{\mathrm{L}}^{1/n} x = \epsilon_k x\,, \quad \epsilon_k = \sqrt[2n]{-\omega^2}\,, \qquad k = 1, \ldots, 2n\,.$$

Thus the general solution of initial equation is a combination of $2n$ basic first order derivatives of Mittag–Leffler functions:

$$x(t) = \sum_{k=1}^{2n} C_k x_k(t)\,,$$

$$x_k(t) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{l=0}^{\infty} \frac{(\epsilon_k)^l}{\Gamma(1+l/n)} (t-a)^{l/n}\,.$$

Basic real solutions look as follows ($k = 1, \dots, n$):

$$\epsilon_{2n-k-1} = \overline{\epsilon_k}\,, \qquad x_{2n-k-1}(t) = \overline{x_k(t)}\,,$$

$$y_k(t) = \tfrac{1}{2}\left[x_k(t) + x_{2n-k-1}(t)\right], \qquad y_{n+k}(t) = \tfrac{1}{2\mathrm{i}}\left[x_k(t) - x_{2n-k-1}(t)\right].$$

4 Final remarks

In the paper the Lagrangian part of mechanics with fractional derivatives was developed on noncommutative extended coordinate–velocity space. Similarly the Hamiltonian mechanics for Hamiltonian in the form

$$H = \sum_{k=0}^{n-1} (p_k \bullet q_{k+1})_{\mathrm{sym}} - L$$

can be derived [10]. It appears that in the proposed formalism it is non-conserved as in other fractional mechanical models [6, 7]. The next step will be Hamilton–Jacobi theory and canonical quantization of fractional mechanical systems. Let us finally notice that classical equations and solutions are recovered via continuous limit, while the connection between classical and noncommutative action is still under investigation.

References

[1] S.G. Samko, A.A. Kilbas and O.I. Marichev: *Fractional Integrals and Derivatives.* Gordon & Breach, Amsterdam, 1993.

[2] R. Hilfer, ed.: *Applications of fractional calculus in physics.* World Scientific, Singapore, 2000.

[3] R. Metzler: Phys. Rev. E **62** (2000) 6233.

[4] R. Metzler and J. Klafter: J. Phys. Chem. B **104** (2000) 3851.

[5] R. Metzler and J. Klafter: Phys. Rep. **339** (2000) 1.

[6] F. Riewe: Phys. Rev. E **53** (1996) 1890; **55** (1997) 3581.

[7] M. Klimek: Czech. J. Phys. **51** (2001) 1348; **52** (2002) 1247.

[8] P. Zavada: Commun. Math. Phys. **192** (1998) 261; `hep-th/0003126`.

[9] N. Laskin: Phys. Lett. A **268** (2000) 298; Phys. Rev. E **66** (2002) 056108; `quant-ph/0504106`.

[10] M. Klimek: *in preparation.*