# On the Yangian $Y(\mathfrak{gl}_{m|n})$ and its quantum Berezinian *)

Lucy Gow

*School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia*

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Jonathan Brundan and Alexander Kleshchev recently introduced a new family of presentations for the Yangian $Y(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$. In this article, we extend some of their ideas to consider the Yangian $Y(\mathfrak{gl}_{m|n})$ of the Lie superalgebra $\mathfrak{gl}_{m|n}$. In particular, we give a new proof of the result by Nazarov that the quantum Berezinian is central.

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## 1 Introduction

### 1.1 Definition of Yangian

The Yangian $Y(\mathfrak{gl}_{m|n})$ is defined in [1] to be the $\mathbb{Z}_2$-graded associative algebra over $\mathbb{C}$ with generators $t_{ij}^{(r)}$ and certain relations described below. We define the formal power series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots$$

and a matrix

$$T(u) = \sum_{i,j=1}^{m+n} t_{ij}(u) \otimes E_{ij}\,(-1)^{\bar{j}(\bar{i}+1)}\,, \tag{1}$$

where $E_{ij}$ is the standard elementary matrix and $\bar{i}$ is the parity of the index $i$. In analogy with the usual Yangian $Y(\mathfrak{gl}_n)$ (see for example [2–4]), the defining relations are expressed by the matrix product

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)\,,$$

where

$$R(u-v) = 1 - \frac{1}{(u-v)}\,P_{12}$$

and $P_{12}$ is the permutation matrix: $P_{12} = \sum_{i,j=1}^{m+n} E_{ij} \otimes E_{ji}(-1)^{\bar{j}}$. Then we have the following equivalent form of the defining relations:

$$[t_{ij}(u),\ t_{kl}(v)] = \frac{(-1)^{\overline{ij}+\overline{ik}+\overline{jk}}}{(u-v)}\,\big(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)\big)\,.$$

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Throughout this article we observe the following notation for entries of the inverse of the matrix $T(u)$:

$$T(u)^{-1} =: \left(t'_{ij}(u)\right)_{i,j=1}^{n} .$$

A straightforward calculation yields the following relation in $Y(\mathfrak{gl}_{m|n})$:

$$[t_{ij}(u), t'_{kl}(v)] = \frac{(-1)^{\overline{ij}+\overline{ik}+\overline{jk}}}{(u-v)} \left( \delta_{kj} \sum_{s=1}^{m+n} t_{is}(u) t'_{sl}(v) - \delta_{il} \sum_{s=1}^{m+n} t'_{ks}(v) t_{sj}(u) \right) . \quad (2)$$

### 1.2 Gauss decomposition of $T(u)$

In [5], the Drinfeld presentation is described in terms of the quasideterminants of Gelfand and Retakh [6, 7]. We make use of the analogous set of generators for the Yangian $Y(\mathfrak{gl}_{m|n})$. First we recall the definition of quasideterminant.

**Definition 1.1.** *Let $X$ be a square matrix over a ring with identity such that its inverse matrix $X^{-1}$ exists, and such that its $(j,i)$th entry is an invertible element of the ring. Then the $(i,j)$th* quasideterminant *of $X$ is defined by the formula*

$$|X|_{ij} = \left((X^{-1})_{ji}\right)^{-1} .$$

Equivalently, we may define quasideterminants inductively as follows.

If $X = (x_{11})$ is a $(1 \times 1)$-matrix, then there is only one quasideterminant of $X$; this is $|X|_{11} = x_{11}$. For $n > 1$, we have

$$|X|_{ij} = x_{ij} - \sum_{k \neq i, l \neq j} x_{ik} (|X^{ij}|_{lk})^{-1} x_{lj} ,$$

where $X^{ij}$ is the matrix obtained from $X$ by removing both the $i$th row and the $j$th column. We also use the following notation for quasideterminants:

$$|X|_{ij} =: \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ & \cdots & & \cdots & \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{in} \\ & \cdots & & \cdots & \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix} .$$

The matrix $T(u)$ defined in (1) has the following Gauss decomposition in terms of quasideterminants (by Theorem 4.96 in [6]; see §5 in [5]):

$$T(u) = F(u) D(u) E(u)$$

for unique matrices

$$D(u) = \begin{pmatrix} d_1(u) & & \cdots & 0 \\ & d_2(u) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_{m+n}(u) \end{pmatrix} ,$$

$$E(u)=\begin{pmatrix} 1 & e_{12}(u) & \cdots & e_{1,m+n}(u) \\ & \ddots & & e_{2,m+n}(u) \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}, \qquad F(u)=\begin{pmatrix} 1 & & \cdots & 0 \\ f_{21}(u) & & \ddots & \vdots \\ \vdots & & & \ddots \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1 \end{pmatrix},$$

where

$$d_i(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & & \vdots \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ii}(u)} \end{vmatrix},$$

$$e_{ij}(u) = d_i(u)^{-1} \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,i}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ij}(u)} \end{vmatrix},$$

$$f_{ji}(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\ t_{ji}(u) & \cdots & t_{j,i-1}(u) & \boxed{t_{ji}(u)} \end{vmatrix} d_i(u)^{-1}.$$

It is easy to recover each generating series $t_{ij}(u)$ by multiplying together and taking commutators of the series $d_i(u)$, $e_j(u) := e_{j,j+1}(u)$, and $f_j(v) := f_{i+1,i}(u)$ for $1 \le i \le m+n$, $1 \le j \le m+n-1$ (see §5 of [5]). Thus the Yangian $Y(\mathfrak{gl}_{m|n})$ is generated by the coefficients of the latter.

### 1.3 Some useful maps

Here we define some automorphisms of the Yangian $Y(\mathfrak{gl}_{m|n})$ and homomorphisms between Yangians, so that we may refer to them in the next section.

Let $\omega_{m|n} : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n})$ be the automorphism defined by

$$\omega : T(u) \mapsto T(-u)^{-1}\,.$$

Let $\tau : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n})$ be the automorphism defined by

$$\tau(t_{ij}(u)) = t_{ji}(-u) \times (-1)^{\bar{i}(\bar{j}+1)}\,.$$

Let $\rho_{m|n} : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{n|m})$ be the isomorphism defined by

$$\rho_{m|n}(t_{ij}(u)) = t_{m+n+1-i,m+n+1-j}(-u)\,.$$

Let $\varphi_{m|n} : Y(\mathfrak{gl}_{m|n}) \hookrightarrow Y(\mathfrak{gl}_{m+k|n})$ be the inclusion which sends each generator $t_{ij}^{(r)} \in Y(\mathfrak{gl}_{m|n})$ to the generator $t_{k+i,k+j}^{(r)}$ in $Y(\mathfrak{gl}_{m+k|n})$.

Finally, let $\psi_k : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m+k|n})$ be the injective homomorphism defined by

$$\psi_k = \omega_{m+k|n} \circ \varphi_{m|n} \circ \omega_{m|n}\,. \tag{3}$$

This last homomorphism is useful for studying quasideterminants, so we discuss it in some detail with the following remarks.

**Remark 1.1.** *We can calculate $\psi_k(t_{ij}(u))$ explicitly for any $1 \le i, j \le m+n$ (see Lemma 4.2 of [5]):*

$$\psi_k(t_{ij}(u)) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1k}(u) & t_{1,k+j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{k1}(u) & \cdots & t_{kk}(u) & t_{k,k+j}(u) \\ t_{k+i,1}(u) & \cdots & t_{k+i,k}(u) & \boxed{t_{k+i,k+j}(u)} \end{vmatrix}.$$

*In particular, this means that for $k \ge 1$, we have $\psi_k(d_1(u)) = d_{k+1}(u)$, $\psi_k(e_1(u)) = e_{k+1}(u)$, and $\psi_k(f_1(u)) = f_{k+1}(u)$. Furthermore, by (3), we have for any $k, l \ge 1$ that $\psi_k \circ \psi_l = \psi_{k+l}$, so we may generalize this observation to give for instance $\psi_k(d_l(u)) = d_{k+l}(u)$.*

**Remark 1.2.** *Notice that the map $\psi_k$ sends $t'^{(r)}_{ij} \in Y(\mathfrak{gl}_{m|n})$ to the element $t'^{(r)}_{k+i,k+j}$ in $Y(\mathfrak{gl}_{m+k|n})$. Thus the subalgebra $\psi_k(Y(\mathfrak{gl}_{m|n}))$ is generated by the elements $\{t'^{(r)}_{k+s,k+t}\}_{s,t=1}^{n}$. Then, by (2), all elements of this subalgebra commute with those of the subalgebra generated by $\{t^{(r)}_{ij}\}_{i,j=1}^{k}$. By Remark 1.1, this implies in particular that for any $i, j \ge 1$, the quasideterminants $d_i(u)$ and $d_j(v)$ commute.*

## 2 The quantum Berezinian

The quantum Berezinian was defined by Nazarov [1] and plays a similar role in the study of the Yangian $Y(\mathfrak{gl}_{m|n})$ as the quantum determinant does in the case of the Yangian $Y(\mathfrak{gl}_n)$ (see [3]).

**Definition 2.1.** *The quantum Berezinian is the following power series with coefficients in the Yangian $Y(\mathfrak{gl}_{m|n})$:*

$$\begin{aligned} b_{m|n}(u) := &\sum_{\tau \in S_m} \operatorname{sgn}(\tau)\, t_{\tau(1)1}(u) t_{\tau(2)2}(u-1) \cdots t_{\tau(m)m}(u-m+1) \\ &\times \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, t'_{m+1,m+\sigma(1)}(u-m+1) \cdots t'_{m+n,m+\sigma(n)}(u-m+n)\,. \end{aligned}$$

For convenience, let us write:

$$C_m(u) := \sum_{\tau \in S_m} \operatorname{sgn}(\tau) t_{\tau(1)1}(u) t_{\tau(2)2}(u-1) \cdots t_{\tau(m)m}(u-m+1)\,.$$

It is clear that $C_m(u)$ is an element of the subalgebra of $Y(\mathfrak{gl}_{m|n})$ generated by the set $\left\{t^{(r)}_{ij}\right\}_{1 \le i,j \le m; r \ge 0}$. This subalgebra is isomorphic to the Yangian $Y(\mathfrak{gl}_\mathfrak{m})$ of the Lie algebra $\mathfrak{gl}_m$ by the inclusion $Y(\mathfrak{gl}_m) \to Y(\mathfrak{gl}_{m|n})$ which sends each generator $t^{(r)}_{ij}$ in $Y(\mathfrak{gl}_\mathfrak{m})$ to the generator of the same name in $Y(\mathfrak{gl}_{m|n})$. Moreover, $C_m(u)$ is the image under this map of the *quantum determinant* of $Y(\mathfrak{gl}_\mathfrak{m})$ (see [3, 5]). Then it is well known (see Theorem 2.32 in [4]) that we have:

$$C_m(u) = d_1(u)d_2(u-1)\cdots d_m(u-m+1)\,.$$

We can extend this observation as follows:

**Theorem 1.** *We can write the quantum Berezinian as follows:*

$$\begin{aligned} b_{m|n}(u) &= d_1(u)\,d_2(u-1)\cdots d_m(u-m+1) \\ &\quad \times d_{m+1}(u-m+1)^{-1}\cdots d_{m+n}(u-m+n)^{-1}\,. \end{aligned}$$

*Proof.* Note that the second part of the expression for $b_{m|n}(u)$ in Definition 2.1 is the image under the isomorphism $\rho_{n|m}\circ\omega_{n|m} : Y(\mathfrak{gl}_{n|m}) \to Y(\mathfrak{gl}_{m|n})$ of

$$\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) t_{n,\sigma(n)}(u-m+1)\cdots t_{2,\sigma(2)}(u-m+n-1)\, t_{1,\sigma(1)}(u+m-n)\,, \quad (4)$$

where we follow in this expression (4) the convention for denoting generators *in the Yangian* $Y(\mathfrak{gl}_{n|m})$. We recognise (by comparing with (8.3) of [5]) that this is $C_n(u-m+n)$, the image of the quantum determinant of $Y(\mathfrak{gl}_n)$ under the natural inclusion $Y(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_{n|m})$. So to verify the claim we calculate the image of $C_n(u-m+n)$ under this map explicitly in terms of the quasideterminants $d_i(v)$. Applying Proposition 1.6 of [7], we find that the image of $d_i(v)$ in $Y(\mathfrak{gl}_{n|m})$ is $(d_{m+n+1-i}(v))^{-1}$ in $Y(\mathfrak{gl}_{m|n})$. This gives the desired result. □

The following theorem is a result of Nazarov [1]. We give a new proof.

**Theorem 2.** *The coefficients of the quantum Berezinian (2.1) are central in the algebra* $Y(\mathfrak{gl}_{m|n})$.

*Proof.* By Remark 1.2, the quantum Berezinian $b_{m|n}(u)$ commutes with $d_i(v)$ for $1 \le i \le m+n$. In addition, if we know that the quantum Berezinian commutes with $e_i(v)$, then by applying the automorphism $\tau$, we find that it also commutes with $f_i(-v)$. So we need to show that $b_{m|n}(u)$ commutes with $e_i(v)$ for each $i$ between 1 and $m+n-1$. We break this problem into three cases:

*Case 1:* $1 \le i \le m-1$. By Theorem 7.2 in [5], $e_i(v)$ commutes with $C_m(u)$. On the other hand, $e_i(v)$ is an element of the subalgebra generated by $\{t_{jk}^{(r)}\}_{1\le j,k\le m}$ and so by Remark 1.2 it commutes with $d_{m+s}(u-m+s)^{-1} = t'_{m+s,m+s}(u-m+s)$ for $1 \le s \le n$.

*Case 2:* $m+1 \le i \le m+n-1$. Applying Propositions 1.6 and 1.4 of [7] in turn to $f_i(v)$, we find an alternative expression:

$$f_i(v) =$$

$$-\begin{vmatrix} \boxed{t'_{i+1,i+1}(v)} & \cdots & t'_{i+1,m+n}(v) \\ \vdots & & \vdots \\ t'_{m+n,i+1}(v) & \cdots & t'_{m+n,m+n}(v) \end{vmatrix}^{-1} \times \begin{vmatrix} \boxed{t'_{i+1,i}(v)} & t'_{i+1,i+2}(v) & \cdots & t'_{i+1,m+n}(v) \\ t'_{i+2,i}(v) & t'_{i+2,i+2}(v) & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ t'_{m+n,i}(v) & & \cdots & t'_{m+n,m+n}(v) \end{vmatrix}.$$

Then, for $m+1 \le i \le m+n-1$, we have

$$e_i(v) = \rho_{n|m} \circ \omega_{n|m}\big(-f_{m+n-i}(v)\big)\,.$$

Now apply this isomorphism to the results of Case 1 in the Yangian $Y(\mathfrak{gl}_{n|m})$.

*Case 3:* $i = m$. Consider the Yangian $Y(\mathfrak{gl}_{1|1})$ first. For this algebra we have $b_{1|1}(u) = d_1(u)d_2(u)^{-1}$ and we would like to show that it commutes with $e_1(v)$. So it will suffice to show

$$d_1(u)e_1(v)d_2(u) = d_2(u)e_1(v)d_1(u)\,. \tag{5}$$

We have

$$\begin{pmatrix} t_{11}(u)\,t_{12}(u) \\ t_{21}(u)\,t_{22}(u) \end{pmatrix} = \begin{pmatrix} d_1(u) & d_1(u)\,e_1(u) \\ f_1(u)d_1(u) & f_1(u)d_1(u)e_1(u) + d_2(u) \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} t'_{11}(v)\,t'_{12}(v) \\ t'_{21}(v)\,t'_{22}(v) \end{pmatrix} = \begin{pmatrix} d_1(v)^{-1} + e_1(v)d_2(v)^{-1}f_1(v) & -e_1(v)\,d_2(v)^{-1} \\ -d_2(v)^{-1}f_1(v) & d_2(v)^{-1} \end{pmatrix}. \tag{7}$$

An application of (2) gives

$$(u-v)[t_{11}(u), t'_{12}(v)] = t_{11}(u)t'_{12}(v) + t_{12}(u)t'_{22}(v)\,.$$

We substitute in this the expressions from (6) and (7), then cancel $d_2(v)$ and rearrange to find

$$(u-v)e_1(v)d_1(u) = (u-v-1)d_1(u)e_1(v) + d_1(u)e_1(u)\,.$$

Similarly, by considering the commutator $[t_{12}(u), t'_{22}(v)]$, we derive the relation

$$(u-v)e_1(v)d_2(u) = (u-v-1)d_2(u)e_1(v) + d_2(u)e_1(u)\,.$$

From these relations it is clear that (5) holds.

Now we return our attention to the general Yangian $Y(\mathfrak{gl}_{m|n})$. By similar appeals to Remark 1.2 as in the first case, we see that $e_m(v)$ commutes with $d_1(u)\cdots d_{m-1}(u-m+2)$ and with $d_{m+2}(u-m+2)^{-1}\cdots d_{m+n}(u-m+n)^{-1}$. So we need only show that $e_m(v)$ commutes with $d_m(u-m+1)d_{m+1}(u-m+1)^{-1}$. This follows immediately if we apply the homomorphism $\psi_{m-1}$ to the identity (5) in $Y(\mathfrak{gl}_{1|1})$. □

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## References

[1] M. Nazarov: Lett. Math. Phys. **21** (1991) 123.

[2] V. Chari and A. Pressley: *A Guide to Quantum Groups.* CUP, Cambridge, 1994.

[3] A. Molev, M. Nazarov, and G. Olshanskii: Russian Math. Surveys **51** (1996) 205.

[4] A. Molev: in *Handbook of Algebra*, Vol. 3, North-Holland, Amsterdam, 2003, p. 907.

[5] J. Brundan and A. Kleshchev: Commun. Math. Phys. **254** (2005) 191.

[6] I.M. Gelfand, S. Gelfand, V. Retakh, and R.L. Wilson: Adv. Math. **193** (2005) 56.

[7] I.M. Gelfand and V.S. Retakh: Funct. Anal. Appl. **25** (1991) 91.