Properties of a pseudo-Hermitian Hamiltonian for harmonic oscillator decorated with Dirac delta interactions *)

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We have investigated bound state solutions of the Schrödinger equation for one-dimensional harmonic oscillator potential together with even number of Dirac delta functions. These point interactions are located at symmetric points $x = x_i$ and $x = -x_i$ $(i = 1, 2, \ldots, N)$ and they have complex conjugate strengths $\tilde{\sigma_i}$ and $\tilde{\sigma_i}^*$, respectively.We present explicit forms of eigenfunctions and an algebraic eigenvalue equation and numerical solutions for this $\mathcal{PT}$-symmetric Hamiltonian.

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1 Introduction

By investigating a conjecture of Bessis, Bender and Boettcher showed that eigenvalues of a non-Hermitian Hamiltonian with a complex potential are entirely real [1]. This result initiated an extensive research activity on the Hamiltonians which have $\mathcal{PT}$-symmetry and hundreds of papers were published on such Hamiltonians. Several new $\mathcal{PT}$-symmetric Hamiltonians with complex potentials are designed and their spectral properties have been investigated by many researchers. Mostafazadeh has studied the general properties of pseudo-Hermitian Hamiltonians and showed that exact $\mathcal{PT}$-symmetry is equivalent to Hermicity [2]. Breakdown of $\mathcal{PT}$-symmetry leads to complex eigenvalues in the spectrum. Mostafazadeh proved that the nonreal eigenvalues of a pseudo-Hermitian Hamitonian with a complete biorthonormal eigenbasis will be complex conjugate pairs [3].

Solutions of the Schrödinger equation with Dirac delta functions can be useful for the description of extremely short-range interactions [4]. Albeverio et al. have studied properties of $\mathcal{PT}$-symmetric Hamiltonians with some point interactions [5]. Znojil has used two Dirac delta interactions with complex conjugate strengths at $x = \pm a$ in a one-dimensional box to study double-well type interaction [6]. By using pure imaginary strengths for two delta interactions with $x = \pm a$, Znojil and Jakubský have investigated breakdown of $\mathcal{PT}$-symmetry [7].

Spectral properties of the Schrödinger equation for harmonic oscillator potential together with delta interactions which have real strengths have been studied by several authors [8–17]. In this paper, we investigate bound state solutions of the

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Schrödinger equation for one-dimensional harmonic oscillator potential together with even number of Dirac delta functions which have complex strengths. These point interactions are located at symmetric points $x = x_i$ and $x = -x_i$ ($i = 1, 2, \ldots, N$) and they have complex conjugate strengths $\tilde{\sigma_i}$ and $\tilde{\sigma_i}^*$, respectively. These choices make the Hamiltonian we explore $\mathcal{PT}$-symmetric. By performing numerical calculations for two delta interactions with pure imaginary strengths, we show that $\mathcal{PT}$-symmetry is broken for sufficiently large absolute strengths of delta interactions and complex conjugate eigenvalues were obtained for this case.

2 Harmonic oscillator decorated with Dirac delta interactions

We will investigate bound state solutions of the Schrödinger equation for the Hamiltonian,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 - \frac{\hbar^2}{2m}\sum_{i=1}^{N}\sigma_i\delta(x - x_i) - \frac{\hbar^2}{2m}\sum_{i=1}^{N}\sigma_i^*\delta(x + x_i)\,, \tag{1}$$

where $\omega > 0$ and the strengths of delta interactions, $\tilde{\sigma}_i = -(\hbar^2/2m)\sigma_i$, $\tilde{\sigma}_i^* = -(\hbar^2/2m)\sigma_i^*$, are complex numbers. Here x_is are real numbers and $x_1 < x_2 < \ldots \ldots < x_N$ with $x_i \in (0, +\infty)$.

Since this Hamiltonian is invariant under the transformations $x \to -x$ and $\mathrm{i} \to -\mathrm{i}$, it is $\mathcal{PT}$-symmetric. Furthermore, $PHP^{-1} = H^\dagger$, hence, this Hamiltonian is a P-pseudo-Hermitian.

By inserting $E = (\xi + \frac{1}{2})\hbar\omega$ in the Schrödinger equation $H\Psi = E\Psi$ and using dimensionless parameter $z = \sqrt{2m\omega/\hbar x}$, we obtain

$$\frac{\mathrm{d}^2\Psi(z)}{\mathrm{d}z^2} + \left[\left(\xi + \tfrac{1}{2}\right) - \tfrac{1}{4}z^2\right]\Psi(z) + \left\{\sum_{i=1}^{N}\mu_i\delta(z - z_i) + \sum_{i=1}^{N}\mu_i^*\delta(z + z_i)\right\}\Psi(z) = 0\,, \tag{2}$$

where $z_i = \sqrt{2m\omega/\hbar}\,x_i$ and $\mu_i = \sigma_i/\sqrt{2m\omega/\hbar}$ for $i = 1, 2, \ldots, N$.

For $z \neq \pm z_i$, the equation (2) has two linearly independent solutions. For $\xi \neq 0, 1, 2, \ldots$, these linearly independent solutions are parabolic cylinder functions $D_\xi(z)$ and $D_\xi(-z)$ where

$$D_\xi(z) = 2^{\xi/2}\mathrm{e}^{-z^2/4}\left\{\frac{\sqrt{\pi}\,\Phi\left(-\frac{1}{2}\xi, \frac{1}{2}; \frac{1}{2}z^2\right)}{\Gamma\left(\frac{1}{2}(1-\xi)\right)} - \frac{\sqrt{2\pi}z\,\Phi\left(\frac{1}{2}(1-\xi), \frac{3}{2}; \frac{1}{2}z^2\right)}{\Gamma\left(-\frac{1}{2}\xi\right)}\right\}, \tag{3}$$

$$D_\xi(-z) = 2^{\xi/2}\mathrm{e}^{-z^2/4}\left\{\frac{\sqrt{\pi}\,\Phi\left(-\frac{1}{2}\xi, \frac{1}{2}; \frac{1}{2}z^2\right)}{\Gamma\left(\frac{1}{2}(1-\xi)\right)} - \frac{\sqrt{2\pi}z\,\Phi\left(\frac{1}{2}(1-\xi), \frac{3}{2}; \frac{1}{2}z^2\right)}{\Gamma\left(-\frac{1}{2}\xi\right)}\right\}, \tag{4}$$

Here $\Phi(\alpha, \gamma; u)$ is the confluent hypergeometric function of the first kind.

Then, the general solution of the equation (2) in a region between two delta interactions is

$$\Psi(z) = a_i D_\xi(z) + b_i D_\xi(-z)\,. \tag{5}$$

The continuity of the wave function at the positions of delta interactions and the jump in the derivative of the wavefunction at $z = z_i$ lead to the recursive relation between the coefficients (a_i,b_i) and (a_{i+1}, b_{i+1}).

Thus, if (a_1, b_1) and (a_{2N+1}, b_{2N+1}) are the coefficients for the wavefunction in the left most and the right most regions, we have

$$\begin{pmatrix} a_{2N+1} \\ b_{2N+1} \end{pmatrix} = M_N M_{N-1} \cdots M_1 M_{-1} \cdots M_{-(N-1)} M_{-N} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \mathbf{X} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \qquad (6)$$

where the matrix $\mathbf{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = M_N M_{N-1} \cdots M_1 M_{-1} \cdots M_{-(N-1)} M_{-N}$ is a function of ξ for given ω, μ_i, μ_i^*,and z_i values. Here, M_J and M_{-J} are the matrices calculated at the locations $z = z_J$ and $z = -z_J$, respectively, and

$$M_i = \begin{pmatrix} 1 + \frac{\mu_i D_\xi(z_i) D_\xi(-z_i)}{W} & \frac{\mu_i (D_\xi(-z_i))^2}{W} \\ -\frac{\mu_i (D_\xi(z_i))^2}{W} & 1 - \frac{\mu_i D_\xi(z_i) D_\xi(-z_i)}{W} \end{pmatrix}. \qquad (7)$$

Here $W_i = W[D_\xi(z), D_\xi(-z)]|_{z=z_i} = \{D_\xi(z)(\mathrm{d}D_\xi(-z)/\mathrm{d}z) - (\mathrm{d}D_\xi(z)/\mathrm{d}z) D_\xi(-z)\}|_{z=z_i} = 2^{(\xi+3/2)}\pi / \left[\Gamma\left(-\frac{1}{2}\xi\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\xi\right)\right]$ is the Wronskian. Since W_i does not depend on z_i, we write $W_i = W$.

$D_\xi(-z)$ is regular as $z \to -\infty$, but $|D_\xi(-z)| \to +\infty$ as $z \to +\infty$, and $D_\xi(z)$ is regular as $z \to +\infty$, but $|D_\xi(z)| \to +\infty$ as $z \to -\infty$ [18]. Then, we have to take $a_1 = 0$ and $b_{2N+1} = 0$ which lead to $b_1 D_\xi(-z)$ for the first interval and $a_{2N+1} D_\xi(z)$ for the last interval as the regular solutions of equation (2). Since we demand $a_1 = 0$ and $b_{2N+1} = 0$ for regular solutions, then, we obtain $\mathbf{X}_{22}(\xi) = 0$ for the eigenvalues ξ.

3 Numerical calculations

The equation $\mathbf{X}_{22}(\xi) = 0$ will be a very complicated equation to solve for $2N$ delta interactions for any large N value and arbitrary complex strengths. For investigating properties of eigenvalues, we solve the simplest possible case, that is, we take $2N = 2$ and pure imaginary $\mu_1 = \mathrm{i}V$ for the strengths of delta interactions.Then, one gets the following equation for the eigenvalues ξ:

$$1 + \frac{|V|^2}{W^2} D_\xi^2(z_1)[D_\xi^2(-z_1) - D_\xi^2(z_1)] = 0, \qquad (8)$$

For specific values of $z_1 = 2.0$ and $V = 1.0$, we sought for the roots of this equation. For this case, only real values for $E/(\hbar\omega) - 1/2 = \xi$ were obtained in our searches. These values are 0.056, 1.28, 1.92, 2.98, 4.02, 4.98, 6.00, 7.04, 7.98,... which are very close to the values without delta interactions. However, for $z_1 = 2.0$ and $V = 10.0$, $\mathcal{PT}$-symmetry breaks down and some of the roots are complex numbers. These are $\xi = 0.236, 2.26, 3.42 \pm \mathrm{i}\,0.174, 5.46, 6.14 \pm \mathrm{i}\,0.224, 9.00 \pm \mathrm{i}\,0.298, 9.38, 15.5 \pm \mathrm{i}\,0.776, \ldots$. Thus, for V values larger than a critical value, i.e., $V_{\mathrm{cr}} < |\mu_1|$ where $V_{\mathrm{cr}} < 10.0$, $\mathcal{PT}$-symmetry is broken and we have complex eigenvalues.

4 Conclusion

We have investigated solutions of the Schrödinger equation for one-dimensional harmonic oscillator potential together with even number of Dirac delta functions with complex strengths. We present explicit forms of eigenfunctions in terms of parabolic cylinder functions and an algebraic eigenvalue equation for this $\mathcal{PT}$-symmetric Hamiltonian. Numerical calculations for two delta interactions with pure imaginary strengths demonstrate that $\mathcal{PT}$-symmetry breaks down whenever the strength of the coupling exceeds certain critical value.

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