



Milankovitch equations with spinors

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Abstract

We investigate the use of spinors to describe the secular evolution of quasi-Keplerian systems. Evaluating their Poisson brackets, we show that the components of a properly chosen spinor are canonical variables. We illustrate this formalism using the classical problems of the orbital motion about an oblate planet and the Kozai–Lidov mechanism. Although the present formalism breaks down for circular or radial orbits, it may be of some practical utility in numerical recipes.

Keywords Milankovitch equations · Spinors · Secular dynamics

1 Introduction

The three-body problem is one of the oldest open problems in astronomy (Valtonen and Karttunen 2006). Although it can be applied as a good model to many astrophysical phenomena ranging from the dynamics of minor planets (Kozai 1962) and satellites (Lidov 1962) to that of supermassive black holes (Naoz et al. 2019; Alexander 2017), it cannot be generically solved.¹ It is non-integrable, i.e. chaotic (Masoliver and Ros 2011).

There exist a couple of approaches to predict the future of triple systems: (i) one directly integrates the equations of motion numerically (Heggie and Hut 2003); (ii) one derives the probability distributions of the orbital elements as a result of the chaotic three-body evolution (e.g. Stone and Leigh 2019); (iii) one uses approximations (Valtonen and Karttunen 2006). A frequently used approximation is the *restricted* three-body problem in which one of the three bodies has zero mass (Szebehely 1967). Another possibility is the *hierarchical* problem in which a tight binary is perturbed by a more massive object (Naoz 2016). The hierarchical three-body problem puts the focus on the system's evolution on secular timescales,

¹ There exists a solution in the form of an infinite series (Sundman 1912), but it is of no practical use due to its slow convergence (Diacu 1997).

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i.e. timescales much longer than the orbital period. It is achieved by averaging the system’s Hamiltonian, for example, using the von Zeipel transformation (Ito and Ohtsuka 2019). The secular evolution equations for the orbit can then be obtained from the averaged Hamiltonian in different ways.

A common method is Lagrange’s (non-canonical) planetary equations of motion which use the standard orbital elements (e.g. Murray and Dermott 1999). However, the associated equations suffer from vanishing denominators if the eccentricity is either zero or unity, or if the inclination is zero (Valtonen and Karttunen 2006).

In order to cure the aforementioned problems, one can introduce another set of equations, named after Milankovitch (Milankovitch 1939). More precisely, one defines \mathbf{L} as the angular momentum vector and \mathbf{K} (the pericenter vector) as the vector which points to the orbit’s pericenter and whose magnitude is equal to the angular momentum. The (non-canonical) evolution equations then read

$$\dot{L}_a = \{L_a, \mathcal{H}\} = \{L_a, L_b\} \frac{\partial \mathcal{H}}{\partial L_b} + \{L_a, K_b\} \frac{\partial \mathcal{H}}{\partial K_b}, \tag{1a}$$

$$\dot{K}_a = \{K_a, \mathcal{H}\} = \{K_a, L_b\} \frac{\partial \mathcal{H}}{\partial L_b} + \{K_a, K_b\} \frac{\partial \mathcal{H}}{\partial K_b}, \tag{1b}$$

where Einstein summation is implied for the Latin indices, which run from 1 to 3. Here, \mathcal{H} is the system’s Hamiltonian and $\{ \cdot, \cdot \}$ is the Poisson bracket

$$\{f(q, p), g(q, p)\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}, \tag{2}$$

with f and g arbitrary functions of (q, p) , the canonical coordinates and momenta (Goldstein et al. 2002). These equations are not limited to the three-body problem, i.e. \mathcal{H} is arbitrary. The standard version of Milankovitch equations uses the Runge–Lenz–Laplace or eccentricity vector instead of our pericenter vector (e.g. Tremaine et al. 2009; Rosengren and Scheeres 2014). Some other works (e.g. Fouvry et al. 2022) use the Klein variables (Klein 1924). Via the use of Pfaffian forms, Reference Musen (1964) found a canonical set of vectors which reformulates the Milankovitch equations as Hamilton’s equations of motion. The reason for our present choice is clarified in Sect. 2.

For a bound orbit, the Poisson brackets of these vectors are

$$\{L_a, L_b\} = \epsilon_{abc} L_c, \tag{3a}$$

$$\{K_a, L_b\} = \epsilon_{abc} K_c, \tag{3b}$$

$$\{K_a, K_b\} = -\epsilon_{abc} L_c, \tag{3c}$$

with ϵ_{abc} the Levi–Civita antisymmetric tensor. We check these in “Appendix A” using the canonical Delaunay variables. Inserting these expressions into Eqs. (1) yields

$$\dot{L}_a = \epsilon_{abc} L_c \frac{\partial \mathcal{H}}{\partial L_b} - \epsilon_{abc} K_b \frac{\partial \mathcal{H}}{\partial K_c}, \tag{4a}$$

$$\dot{K}_a = \epsilon_{abc} K_c \frac{\partial \mathcal{H}}{\partial L_b} + \epsilon_{abc} L_b \frac{\partial \mathcal{H}}{\partial K_c}. \tag{4b}$$

Hereafter we restrict ourselves to secular evolution. An essential feature of it is that the semi-major axis is constant due to the orbit-averaging over the (fast) mean anomaly (Morbidelli 2002). For a time-independent averaged Hamiltonian, i.e. when the total energy is constant as well, we have two conserved quantities. These are manifested in the first integrals of motion

of Eqs. (4)

$$\|\mathbf{L}\| - \|\mathbf{K}\| = \text{const.}, \tag{5a}$$

$$\mathbf{L} \cdot \mathbf{K} = \text{const.}, \tag{5b}$$

with “ \cdot ” the usual scalar product. The associated constants are set to zero because \mathbf{L} and \mathbf{K} have the same magnitudes and are perpendicular to each other, by definition. These two constraints reduce the number of degrees of freedom from 6 to 4. Given the geometric meaning of the vectors involved, it is understood that Eqs. (5) hold in the non-averaged case, too.

Contrary to the standard vector algebra just presented, Hestenes (1983, H83 hereafter) suggested an alternative formulation for celestial mechanics, namely geometric algebra. In doing so, H83 derives the equations of motion for a spinor instead of a pair of vectors. We now revisit this result and rewrite the associated equations of motion using the Poisson brackets of spinors. The letter is structured as follows. Section 2 introduces spinors and expresses the equations of motion with them. Section 3 applies the spinorial equations to the pedagogical case of the secular dynamics around an oblate planet. Section 5 summarises the benefits and disadvantages of the spinorial formalism.

2 The spinorial Milankovitch equations

First, let us take three orthonormal unit vectors $\sigma_a \in \mathbb{R}^3$ with $a = 1, 2, 3$. Let us then define the geometric product between two of them as

$$\sigma_a \sigma_b = \sigma_a \cdot \sigma_b + \sigma_a \wedge \sigma_b, \tag{6}$$

where “ \cdot ” is again the usual (symmetric) dot product giving a scalar, while “ \wedge ” is an antisymmetric product giving a bivector. From this, it follows that

$$\sigma_a \sigma_b = -\sigma_b \sigma_a \tag{7}$$

for $a \neq b$ and

$$\sigma_1 \sigma_1 = \sigma_2 \sigma_2 = \sigma_3 \sigma_3 = 1, \tag{8}$$

where 1 is a scalar and the geometric product is not denoted explicitly.² Using the associativity of the geometric product, one can easily show that

$$(\sigma_1 \sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)(\sigma_1 \sigma_2 \sigma_3) = -1, \tag{9}$$

which motivates a notation similar to that of the complex unit, namely $\sigma_1 \sigma_2 \sigma_3 = \mathbf{I}$, a trivector. Now we can define a spinor and its conjugate via

$$\underline{s} = x_0 + \text{I}x_a \sigma_a, \tag{10a}$$

$$\underline{s}^\dagger = x_0 - \text{I}x_a \sigma_a, \tag{10b}$$

where $x_{0 \leq \mu \leq 3}$ are real. As illustrated in “Appendix B,” spinors can be efficiently used to describe rotations.³ Along the same line, “Appendix C” investigates the use of bivectors in the context of Milankovitch equations.

² The notation σ_a is motivated by the fact that the Pauli matrices, which satisfy exactly the same algebra, are usually denoted in this way too.

³ For a detailed introduction into the application of geometric algebra in physics, see Hestenes (1971, 2002).

Let us now demand that (i) the norm of the spinor, \underline{s} , is related to the angular momentum via

$$\underline{s}^\dagger \underline{s} = 4 \|\mathbf{L}\| = 4 \|\mathbf{K}\|, \tag{11}$$

and that (ii) it rotates the basis vectors σ_1 and σ_3 to the direction of \mathbf{K} and \mathbf{L} , respectively (we follow H83 for that convention), i.e.

$$\mathbf{L} = \underline{s}^\dagger \sigma_3 \underline{s}, \tag{12a}$$

$$\mathbf{K} = \underline{s}^\dagger \sigma_1 \underline{s}. \tag{12b}$$

These equations can be recast as

$$4L_a \sigma_a = (x_0 - Ix_b \sigma_b) \sigma_3 (x_0 + Ix_c \sigma_c), \tag{13a}$$

$$4K_a \sigma_a = (x_0 - Ix_b \sigma_b) \sigma_1 (x_0 + Ix_c \sigma_c). \tag{13b}$$

Matching the prefactors of the $\{\sigma_i\}$ on both sides finally yields

$$L_1 = \frac{1}{2}(x_0 x_2 + x_1 x_3), \tag{14a}$$

$$L_2 = \frac{1}{2}(x_2 x_3 - x_0 x_1), \tag{14b}$$

$$L_3 = \frac{1}{4}(x_0^2 - x_1^2 - x_2^2 + x_3^2), \tag{14c}$$

$$K_1 = \frac{1}{4}(x_0^2 + x_1^2 - x_2^2 - x_3^2), \tag{14d}$$

$$K_2 = \frac{1}{2}(x_0 x_3 + x_1 x_2), \tag{14e}$$

$$K_3 = \frac{1}{2}(x_1 x_3 - x_0 x_2). \tag{14f}$$

The action of the spinor on the basis vectors is to rotate and multiply both vectors by $4\|\mathbf{L}\|$. Such an operation cannot provide us with two vectors of different magnitudes, hence our choice in Sect. 1 of using the pericenter vector rather than the eccentricity one.

We now express the Hamiltonian with the spinor $\mathcal{H}(\underline{s}) = \mathcal{H}(x_0, x_1, x_2, x_3)$. Analogously to the vectorial case in Eqs. (1), we have

$$\dot{x}_\mu = \{x_\mu, x_\nu\} \partial_{x_\nu} \mathcal{H}, \tag{15}$$

where Greek indices run from 0 to 3. The task is to calculate the Poisson brackets of the $\{x_\mu\}$, just like for (\mathbf{L}, \mathbf{K}) in Eqs. (3). Substituting Eqs. (14) into the Poisson brackets from Eqs. (3) gives us

$$\{x_1, x_2\} = \{x_3, x_0\} = 1, \tag{16a}$$

$$\{x_2, x_3\} = \{x_0, x_1\} = 0, \tag{16b}$$

$$\{x_0, x_2\} = \{x_3, x_1\} = 0. \tag{16c}$$

Putting these brackets back into Eq. (15) finally yields the equations of motion

$$\dot{x}_0 = -\partial_{x_3} \mathcal{H}, \tag{17a}$$

$$\dot{x}_1 = \partial_{x_2} \mathcal{H}, \tag{17b}$$

$$\dot{x}_2 = -\partial_{x_1} \mathcal{H}, \tag{17c}$$

$$\dot{x}_3 = \partial_{x_0} \mathcal{H}. \tag{17d}$$

These are the *spinorial* analogues of Milankovitch Eq. (4). We make two remarks about them. First, the transformations in Eqs. (14) have the same structure as the Kustaanheimo–Stiefel (KS) transformation in the regularisation of the 3-dimensional Kepler problem (Stiefel and

Kustaanheimo 1965; Waldvogel 2006). Here, they are applied to both the angular momentum and pericenter vectors. Second, the set $\{x_\mu\}$ is symplectic, i.e. it obeys the canonical Poisson relations. Phrased differently, Eqs. (17) are simply Hamilton’s canonical equations with (x_1, x_3) being coordinates and (x_2, x_0) their respective conjugate momenta.⁴

3 Motion around an oblate planet

As a simple demonstration, we follow H83 and apply the spinorial formalism to the secular dynamics of a satellite around an oblate planet.⁵ The averaged perturbing Hamiltonian is Beletsky (2001)

$$\mathcal{H} = (C/L^3)[1 - 3(L_{\parallel}/L)^2], \tag{18}$$

with C a constant. Here, L is the satellite’s angular momentum and L_{\parallel} its projection on the planet’s axis of rotation. When expressed with the spinor components from Eqs. (14), they read

$$L = \frac{1}{4} (x_0^2 + x_1^2 + x_2^2 + x_3^2), \tag{19a}$$

$$L_{\parallel} = \frac{1}{2}n_1(x_0x_2 + x_1x_3) + \frac{1}{2}n_2(x_2x_3 - x_0x_1) + \frac{1}{4}n_3(x_0^2 - x_1^2 - x_2^2 + x_3^2), \tag{19b}$$

with $\mathbf{n} = [n_1, n_2, n_3]$ the unit vector along the planet’s rotational axis.⁶ After injecting the Hamiltonian from Eq. (18), Eqs. (17) have an exact solution in closed form (see “Appendix D”). Assuming $\mathbf{n} = [0, 0, 1]$, it reads

$$x_0 = A_+ \sin(\omega_+t + \delta_+), \tag{20a}$$

$$x_1 = A_- \sin(\omega_-t + \delta_-), \tag{20b}$$

$$x_2 = A_- \cos(\omega_-t + \delta_-), \tag{20c}$$

$$x_3 = A_+ \cos(\omega_+t + \delta_+), \tag{20d}$$

with δ_{\pm} some given phases, the amplitudes A_{\pm} satisfying the constraints

$$A_+^2 + A_-^2 = 4L, \tag{21a}$$

$$A_+^2 - A_-^2 = 4L_{\parallel}, \tag{21b}$$

and the constant frequencies

$$\omega_{\pm} = \frac{3C}{2} \left(\pm \frac{L^2 - 5L_{\parallel}^2}{L^6} + \frac{2L_{\parallel}}{L^5} \right). \tag{22}$$

When substituting into Eqs. (14), we recover that both \mathbf{L} and \mathbf{K} precess with constant magnitudes (Beletsky 2001). The critical inclination (Lubowe 1969) follows from the resonance condition $\omega_+ = \omega_-$.

⁴ The matching between $\{x_\mu\}$ and the canonical variables can be reordered simply by using a different choice of basis vectors in Eqs. (12).

⁵ See Ref. Rosengren (2014) for another interesting solution based on the use of second-order tensors (dyads).

⁶ Note that the angular momentum in Eq. (19a) is a quadratic function of the spinor, i.e. spinors are the “square roots” of vectors (Coddens 2017).

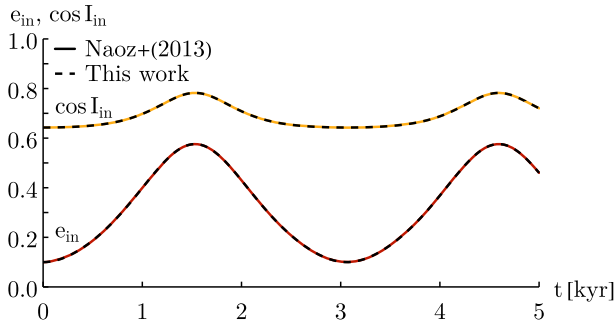


Fig. 1 Numerical solution of Eqs. (17) for the Hamiltonian from Eq. (23). The perturber’s (fixed) orbital parameters are $m_{\text{per}} = 1 M_{\odot}$, $a_{\text{per}} = 5 \text{ au}$, $e_{\text{per}} = 0$ and $g_{\text{per}} = 0$, with g the argument of pericenter, see “Appendix A.” The inner test particle orbits around an object of mass $M = 0.001 M_{\odot}$, with semi-major axis $a_{\text{in}} = 0.1 \text{ au}$. At the initial time, the inner orbit’s parameters are set to $e_{\text{in}} = 0.1$, $g_{\text{in}} = 0$, $I_{\text{in}} = 50^{\circ}$ and $h_{\text{in}} = 0$, with I and h the inclination and argument of node, respectively. The integration of the spinorial equations of motion was performed using the classical fourth-order Runge–Kutta scheme (see, e.g. Hairer et al. 2008) with timestep $\delta t = 0.1 \text{ yr}$, giving a final relative error in the total energy of order 10^{-13} . The time evolution is compared with the use of the independent code from Naoz et al. (2013)

4 Kozai–Lidov oscillations

To test our formalism against a numerical example, we now investigate the secular dynamics of a tight binary with a test particle, perturbed by a distant third object. Truncating the Hamiltonian at third order in the semi-major axes ratio, the dynamics of the inner binary follows from the Hamiltonian [see Eqs. (2) and (3) in Katz et al. 2011]

$$\mathcal{H} = C(\mathcal{H}_q + \epsilon_o \mathcal{H}_o), \tag{23}$$

where \mathcal{H}_q (resp. \mathcal{H}_o) is the quadrupolar (resp. octupolar) contribution. They read

$$\mathcal{H}_q = \frac{3}{4} \left(\frac{1}{2} L_z^2 + D^2 - \frac{5}{2} D_z^2 - \frac{1}{6} \right), \tag{24a}$$

$$\mathcal{H}_o = \frac{75}{64} \left(D_x \left(\frac{1}{5} - \frac{8}{5} D^2 + 7 D_z^2 - L_z^2 \right) - 2 D_z L_x L_z \right), \tag{24b}$$

where \mathbf{L} is the inner binary’s specific angular momentum and $\mathbf{D} = \sqrt{1 - L^2} \mathbf{K} / L$ its eccentricity vector (see Eq. (9.133) of Goldstein et al. 2002). The constants are

$$C = \frac{G m_{\text{per}} a_{\text{in}}^2}{a_{\text{per}}^3 (1 - e_{\text{per}}^2)^{3/2}}, \tag{25a}$$

$$\epsilon_o = \frac{a_{\text{in}}}{a_{\text{per}}} \frac{e_{\text{per}}}{1 - e_{\text{per}}^2}, \tag{25b}$$

where G is the gravitational constant, m_{per} , a_{per} and e_{per} are, respectively, the mass, semi-major axis and eccentricity of the distant perturber, and a_{in} the semi-major axis of the tight binary.

Using the Hamiltonian from Eq. (23), we integrated Eqs. (17) numerically. The solution is shown in Fig. 1. It exhibits the well-known Kozai–Lidov oscillations and perfectly agrees with the solution obtained from an independent integration using the code from Naoz et al. (2013).

5 Discussion and summary

We followed H83 in using spinors to describe quasi-Keplerian systems on secular timescales. In that case, an orbit-averaged Keplerian orbit is represented by a single spinor rather than two vectors. The Poisson brackets of these spinor components turn out to be remarkably simple: they are canonically conjugate variables. As such, the spinorial counterpart of the vectorial Milankovitch equations is thus a set of standard canonical Hamiltonian equations.

The spinorial formalism has difficulties if the orbit is either circular or radial, which limits its applicability. In the circular case, it has an extra degree of freedom associated with the orientation of the pericenter vector, which is unphysical at circular orbits. In the radial case, the spinor is identically zero (see Eq. 19a), and one loses the information about the orbit's orientation.

Future work will be devoted to testing alternative normalisations other than Eq. (11) that could help at extreme eccentricities, as well as using the bivector formulation from "Appendix C." We will also explore if the formalism above could be used efficiently in numerical integrations (see, e.g. McLachlan et al. 2014).

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Declarations

Conflict of interest The authors declare no competing interests.

Appendix A: The canonical basis of Delaunay variables

Following the notation from Murray and Dermott (1999), the angular momentum and pericenter vectors can be expressed on the canonical basis of Delaunay variables via

$$L_1 = \sqrt{G^2 - H^2} \sin h, \quad (\text{A1a})$$

$$L_2 = -\sqrt{G^2 - H^2} \cos h, \quad (\text{A1b})$$

$$L_3 = H = G \cos i, \quad (\text{A1c})$$

$$K_1 = G \cos g \cos h - H \sin g \sin h, \quad (\text{A1d})$$

$$K_2 = G \cos g \sin h + H \sin g \cos h, \quad (\text{A1e})$$

$$K_3 = \sqrt{G^2 - H^2} \sin g, \quad (\text{A1f})$$

with g the argument of pericenter, G the magnitude of the angular momentum, h the argument of node, H the third component of the angular momentum vector, and i the inclination.⁷ The Poisson brackets of the Delaunay variables are

$$\{g, G\} = \{h, H\} = 1, \quad (\text{A2})$$

while all the others are zero.

Delaunay variables are also suitable for expressing the spinor components defined in Eqs. (14), via

$$x_0 = 2\sqrt{L} [\cos(g/2) \cos(i/2) \cos(h/2) - \sin(g/2) \cos(i/2) \sin(h/2)], \quad (\text{A3})$$

⁷ These expressions follow from Eqs. (2.119–120) in Murray and Dermott (1999).

$$x_1 = 2\sqrt{L} \left[\cos(g/2) \sin(i/2) \cos(h/2) + \sin(g/2) \sin(i/2) \sin(h/2) \right], \tag{A4}$$

$$x_2 = 2\sqrt{L} \left[\cos(g/2) \sin(i/2) \sin(h/2) - \sin(g/2) \sin(i/2) \cos(h/2) \right], \tag{A5}$$

$$x_3 = 2\sqrt{L} \left[\cos(g/2) \cos(i/2) \sin(h/2) + \sin(g/2) \cos(i/2) \cos(h/2) \right]. \tag{A6}$$

Appendix B: Spinors and rotations

Spinors are powerful tools to treat rotations. Any rotation of a vector \mathbf{v} can be executed as (see Eq. 2.1 in H83)

$$\mathbf{v} \mapsto \underline{s}^\dagger \mathbf{v} \underline{s}. \tag{B7}$$

In order to illustrate it, let us consider the unit vector σ_1 . The spinor that rotates it by $\pi/2$ around σ_3 is

$$\underline{s} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \mathbf{I} \sigma_3. \tag{B8}$$

Indeed, some easy algebra using Eqs. (7)–(8) leads to

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \mathbf{I} \sigma_3 \right) \sigma_1 \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \mathbf{I} \sigma_3 \right) = \sigma_2. \tag{B9}$$

This matches with the geometric intuition since the $\{\sigma_a\}$ are orthogonal to one another. A general rotation by an angle $\|\theta\|$ around an axis θ is given by (see Eq. 2.3 in H83)

$$\underline{s} = \sum_n \frac{\left(\frac{1}{2} \mathbf{I} \theta\right)^n}{n!} = e^{\frac{1}{2} \theta}, \tag{B10}$$

where H83 normalises the spinor to unity in Eq. (2.2) therein.

H83 derives an evolution equation for the spinor (see Eqs. 3.1–3.3 therein), namely

$$\dot{\underline{s}} = \frac{1}{2} \mathbf{I} \underline{s} \omega, \tag{B11}$$

with ω the angular velocity of the rotation (see Eq. 3.7 in H83). It has the simple formal solution

$$\underline{s} = e^{\frac{1}{2} \omega t}; \tag{B12}$$

i.e. a general rotation as in Eq. (B10).

We finally point out that the spinors of 3D rotations are formally identical to Hamilton quaternions (Hestenes 1983). Indeed, $-\mathbf{I} \sigma_1$, $-\mathbf{I} \sigma_2$ and $-\mathbf{I} \sigma_3$ obey the same algebra as Hamilton’s $\{i, j, k\}$, respectively (Hamilton 1844).

Appendix C: Milankovitch equations with bivectors

In Sect. 2, we introduced the basic concepts of geometric algebra. We now apply it to bivectors. Similarly to vectors, we consider two arbitrary bivectors u and v , and we define the dot and wedge product between them as the symmetric and antisymmetric part of the geometric product, namely

$$u \cdot v = \frac{1}{2} (uv + vu), \tag{C13a}$$

$$u \wedge v = \frac{1}{2} (uv - vu). \tag{C13b}$$

Let us now consider four orthonormal unit vectors $\sigma_a \in \mathbb{R}^4$ with $a = 1, 2, 3, 4$. We can construct a total of 6 unit bivectors out of them.⁸ The key point is to note that their respective wedge products follow a structure similar to the one of the Poisson brackets of the angular momentum and eccentricity vectors (\mathbf{L}, \mathbf{D}) . For example, one has

$$\{L_1, L_2\} = L_3 \iff \sigma_2\sigma_3 \wedge \sigma_1\sigma_3 = \sigma_1\sigma_2. \tag{C14}$$

This motivates then the correspondences

$$L_1 \mapsto \sigma_2\sigma_3, \quad D_1 \mapsto \sigma_1\sigma_4, \tag{C15a}$$

$$L_2 \mapsto \sigma_1\sigma_3, \quad D_2 \mapsto \sigma_4\sigma_2, \tag{C15b}$$

$$L_3 \mapsto \sigma_1\sigma_2, \quad D_3 \mapsto \sigma_3\sigma_4, \tag{C15c}$$

Similarly, we are led to defining the bivector

$$v = L_1\sigma_2\sigma_3 + L_2\sigma_1\sigma_3 + L_3\sigma_1\sigma_2 + D_1\sigma_1\sigma_4 + D_2\sigma_4\sigma_2 + D_3\sigma_3\sigma_4, \tag{C16}$$

along with a ∇_v operator following the same pattern

$$\nabla_v = \partial_{L_1}\sigma_2\sigma_3 + \partial_{L_2}\sigma_1\sigma_3 + \partial_{L_3}\sigma_1\sigma_2 + \partial_{D_1}\sigma_1\sigma_4 + \partial_{D_2}\sigma_4\sigma_2 + \partial_{D_3}\sigma_3\sigma_4. \tag{C17}$$

Milankovitch Eqs. (4) then become⁹

$$\dot{v} = \nabla_v \mathcal{H} \wedge v, \tag{C18}$$

i.e. a “rotation in bivector space.” This equation has a first integral, $d(v \cdot v)/dt = 0$, associated with the conserved quantity

$$(L^2 + D^2) + (L_1D_1 + L_2D_2 + L_3D_3) \sigma_1\sigma_2\sigma_3\sigma_4 = \text{const}. \tag{C19}$$

The two terms in this equation are conserved independently, in agreement with, e.g. Eq. (17) of Tremaine et al. (2009).

Appendix D: Solution of the spinorial equations

Substituting Eq. (18) into the spinorial equations of motion (17) yields

$$\dot{x}_0 = -\frac{3C}{2} \left[-\frac{L^2-5L_{\parallel}^2}{L^6} x_3 - \frac{2L_{\parallel}}{L^5} (n_1x_1 + n_2x_2 + n_3x_3) \right], \tag{D20a}$$

$$\dot{x}_1 = \frac{3C}{2} \left[-\frac{L^2-5L_{\parallel}^2}{L^6} x_2 - \frac{2L_{\parallel}}{L^5} (n_1x_0 + n_2x_3 - n_3x_2) \right], \tag{D20b}$$

$$\dot{x}_2 = -\frac{3C}{2} \left[-\frac{L^2-5L_{\parallel}^2}{L^6} x_1 - \frac{2L_{\parallel}}{L^5} (n_1x_3 - n_2x_0 - n_3x_1) \right], \tag{D20c}$$

$$\dot{x}_3 = \frac{3C}{2} \left[-\frac{L^2-5L_{\parallel}^2}{L^6} x_0 - \frac{2L_{\parallel}}{L^5} (n_1x_2 - n_2x_1 + n_3x_0) \right]. \tag{D20d}$$

Applying Eqs. (D20) to Eqs. (19), one gets

$$\dot{L} = 0, \tag{D21a}$$

⁸ For another perspective of the connection between the Kepler problem and four-dimensional geometry, see Oliver (2004) and Ikemori et al. (2023).

⁹ Contrary to Eqs. (4), here we use the eccentricity vector, \mathbf{D} , rather than the pericenter one, \mathbf{K} .

$$\dot{L}_{\parallel} = 0, \quad (\text{D21b})$$

i.e. both the total angular momentum and its projection on the rotation axis are conserved. Using this, Eqs. (D20) become a set of linear differential equations with an antisymmetric matrix. It results in purely imaginary eigenvalues, i.e. the solution is oscillatory. After picking up a particular reference frame like in Sect. 3, the equations of motion simplify to

$$\dot{x}_0 = \omega_+ x_3, \quad (\text{D22a})$$

$$\dot{x}_1 = \omega_- x_2, \quad (\text{D22b})$$

$$\dot{x}_2 = -\omega_- x_1, \quad (\text{D22c})$$

$$\dot{x}_3 = -\omega_+ x_0, \quad (\text{D22d})$$

using the abbreviations from Eq. (22). These equations can be immediately integrated to give Eqs. (20).

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