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ORIGINAL ARTICLE

Relativistic gravitational force

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Abstract
What would the universe be like if we would give up the *principle of equivalence* and as a consequence incorporate gravity on a more traditional interaction basis within the theory of special relativity? The current work proposes such a gravitational model where a new law of gravitation is expressed in a Lorentz invariant form and is simultaneously in great agreement with many observations (including general relativity). We find the following predictions so far to be consistent with observations: (1) precession formulas explaining, for example, the Mercury apsidal precession, (2) gravitational bending of light near the limb of a massive object, (3) gravitational redshift, (4) escape velocity near a black hole, (5) several relativistic orbit details are derived in the strong field regime and found to be consistent with GR and also the observed shadow of the M87 black hole. Curved space effects such as gravitational time dilation or Shapiro delay cannot be explained within the realms of special relativity (Sect. 3.11).

Keywords Relativistic celestial mechanics · Gravitation · Special relativity

1 Introduction

Today's consensus is that the topology of the universe is flat, at least on cosmological scales. For example, the large-scale curvature of spacetime was measured by the Wilkinson Microwave Anisotropy Probe (WMAP) to be flat within 0.4% error (Wilkinson Microwave Anisotropy Probe 2020). The locally curved spacetime seems to be well indicated by Einstein's theory of general relativity (GR). However, the traditional GR geometrical approach is not the only way to describe general relativity. It is known that GR can also be cast from the point of view of the *classical limit* of a massless spin-2 approach within quantum field theory (i.e., in flat space). Weinberg described this well in his book, and the purpose was to try to narrow the gap between quantum theories in particle physics and general relativity, see, e.g., preface in Weinberg (1972) and Weinberg (1964a, b). The same spirit can also be seen in Feynman's book (Feynman 1995) and many others, e.g., Kraichnan (1955), Deser (1970) and Bronstein (2012), where attempts were made to construct a consistent theory of quantum

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gravity which in the classical limit would approach general relativity. As put by Weinberg: "the passage of time has shown us that a quantum field theory of gravity which is renormalizable and without divergences is unlikely to succeed" (Weinberg 1972). Freeman Dyson has even argued that gravitation might be a purely classical field without any correspondence to quantum field theory (Dyson 2020).

Dealing with gravity within *classical* physics, from the point of view of special relativity, was abandoned many years ago by the bulk of scientists in related fields such as mathematical physics, astrophysics, astronomy and cosmology. There are several reasons for this. One is that Einstein himself thought that in any scalar theory there are unavoidable violations with regard to (1) universality of free fall and (2) energy conservation. However, both of these objections were refuted in considerable detail by Giulini (2008) who reviewed these historical details from a modern perspective. He discussed the intricate problems that arise (and those that do not) and came to the conclusion that the real problem with various proposed scalar theories is poor agreement with experiment. The incompatibility between special relativity and gravitation is not strictly proven according to Vankov (2008). Also in Sattinger (2015), the objections to a linear Lorentz invariant field theory of gravity are refuted. One of the early promising models in flat spacetime was due to Birkhoff where several observational facts were correctly reproduced (Birkhoff 1943). However, the model met difficulties when it turned out that photons did not follow the postulated classical solution and also attempts to quantize the theory failed, see, e.g., Feynman p. 81 (Feynman 1995). Yet another well-known issue is that Newton's law of gravitation is not invariant under Lorentz transformation (violation of the *principle of relativity*). Of course, the same problem exists for Coulomb's law which therefore also is invalid (except in a static problem where the Lorentz force reduces to Coulomb's law). One of the earliest, but unsuccessful, attempts to study the required form of a relativistic gravitational force was due to Poincaré in 1905, see paragraph 9 in Poincare (1905). Multiple authors have also attempted to create a theory of gravitation analogous to electromagnetism, see, e.g., Gravitoelectromagnetism (2022) (also see references within). Unfortunately, these models are not invariant under Lorentz transformations. Also in common with previous studies, these models do not pass the three classical tests in the solar system that a competing gravitational model must do (bending of light, Mercury apsidal precession and gravitational redshift). Another study dealt with a gravitational force in special relativity where the proper mass was proposed to depend on the gravitational field strength (Vankov 2008). There are also other models of including gravity in flat space, but it would seem that they all are in a continuous developing process of proposals/modifications in order to adjust to various observational facts (Friedman 2016; Biswas 1994). The difficulty of creating a consistent flat space theory/model that is in accord with observations has been well known for a long time (Giulini 2008). For example, if one naively just plug in Newton's law of gravitation into special relativity, one would find that the Mercury precession of its perihelion only becomes 1/6 or 1/3 of the measured value (Lemmon and Mondragon 2016) (depending on whether rest mass or relativistic mass were used in Newton's law of gravitation). In the case of the bending of light around the Sun, only 1/2 can be attributed to the observed one (the same as in Newtonian mechanics).

The theory of general relativity has so far been very successful in explaining almost all gravity-related phenomena. However, until recent black hole observations (Akiyama et al. 2019; Pounds 2018; Bower and van Langevelde 2022, https://www.mpifr-bonn.mpg.de/pressreleases/2019/1) the investigations have mostly dealt with the weak-field regime of this theory. Despite the common belief that the general theory is correct at also the larger scales (i.e., the ultra-weak regime), there are still unsolved issues. For example, mismatch of rotation curves of galaxies or galaxy clusters that presumably is related to dark matter and/or

MOND (Milgrom 1983), peculiar MOND behavior of wide star binaries (Hernandez et al. 2012a, b), conflicting expansion rates of the universe (Nielsen et al. 2016; Oguri 2019). Other problematic properties also exist like non-removable singularities (Vishwakarma 2016), an expected asymmetry of black hole images, but observed to be symmetrical and also much smaller than predicted (Issaoun 2019, https://www.mpifr-bonn.mpg.de/pressreleases/2019/1) no time-reversal symmetry of a black hole (information paradox), the source of the gravitational field (the stress-energy tensor) is continuous but mass comes in discrete elements (particles), troublesome relations to quantum field theory, etc. A well-known disadvantage with the nonlinear tensorial theory of general relativity is that it is always mathematically tedious to apply, even for the simplest physical problem. Cosmological perturbation theory, linearizations in the weak-field limit, post-Newtonian expansions (Will 2014), numerical relativity, approximate metrics, problematic N-body treatments etc. are common practice. A great deal of constructive criticism is summarized in the relatively recent and excellent review by Vishwakarma (2016).

Because of this complexity in GR, we believe it may still be of interest to consider an alternative gravitational model if it is mathematically straightforward (Occam's razor) and has sufficiently good prediction properties. In the current work we will make an attempt to identify such a model, namely addressing relativistic celestial mechanics from a special relativistic point of view, and at the same time derive corrections so the *principle of relativity* is respected. The purpose of such a model is not to replace GR, but to rather identify a linear theory in flat Minkowski space that a) can serve as an approximation to GR to conveniently deal with relativistic celestial mechanics and b), provide a possible way to enable a transition in to a theory of quantum gravity. We shall see that this model does not only reproduce some standard tests in the weak-field regime, but unexpectedly also reproduce several tests in the strong field regime.

We shall begin by outlining the proposed theory, we will then consider a series of mathematical/numerical experiments and compare those findings with established general relativistic results and also observations. We also provide an "Appendix" with various additional supportive material. Although it is very common in the mathematical community to use the *integral of least action* and the Lagrange equations of motion, the theory presented here will instead be cast in the equivalent language of *special relativistic force* and its law of motion. Given today's developments in computer technology, it is not only very simple, but also convenient to plug in N-body equations and accurately solve them by applying numerical analysis. The model to be presented here, is also especially convenient, because it respects the *principle of superposition* (N-body problems then become computationally efficient and easy). In order to further simplify, we shall usually work and derive results from the perspective of a single inertial frame. Relativistic mechanics can then be performed in three-space plus coordinate time, see, e.g., Equation 7–89 and the subsequent discussion by, e.g., Goldstein (1980), or the end of pp. 26–13 in Feynman's lectures (Feynman 1963). Also, in order to express the mathematics more compactly, we will frequently use relativistic mass m instead of rest mass (i.e., the old notation). With the above simplifications, we feel that the focus is emphasized on the physics and less so on formal mathematical details. This, we hope, will be a straightforward display for the broader astrophysics audience.

2 Inclusion of gravitation into special relativity

In the present work, it will be convenient to call the application of the presented methodology for "*Relativi*stic *Gravitational Force*" (RGF). Throughout, we will use the notation of *relativistic mass* $m = m_0 \left(1 - u^2/c^2\right)^{-\frac{1}{2}}$, as the equations presented here often become more compact. In the presentation, the term "mass" will be used without prefix to mean relativistic mass unless otherwise stated. Also, we will only consider *pure forces* (i.e., the rest mass m_0 is constant). The magnitude of various terms in the derivations is more easily seen by not using units where the speed of light $c = 1$ or the gravitational constant $G = 1$. We shall work mostly in Euclidean 3-space and treat time as in Newtonian mechanics. Unlike GR, where gravity is assumed to be equivalent to an accelerated local frame, we cannot apply such a principle here as it is known that it would lead to an inconsistency with special relativity (Schild 1960). Today, there are several complicated formulations: weak, strong and the Einstein *principle of equivalence* and also critical discussions regarding these (Chae 2020). A possible way to remain within the realms of special relativity, is to replace the *principle of equivalence* by the postulates 2 and 3 below. They are very easily stated and also supported by experimental facts.

The theory outlined in the present work is based on the following three postulates:

1. The mechanics and assumptions within special relativity are assumed to be correct. It is thus assumed that the speed of light in vacuum (c) is a universal constant.
2. Initial acceleration $\mathbf{a}_0$ of horizontal light bending in a locally homogeneous gravitational field $\mathbf{g}$ (i.e., when the velocity $\mathbf{u} \perp \mathbf{g}$) is not given by $\mathbf{a}_0 = \mathbf{g}$, but instead by $\mathbf{a}_0 = 2\mathbf{g}$. Light has been observed to have this peculiar behavior in many experiments ((Bruns 2018) is a recent and excellent example) as light deflects near the limb of a massive object.
3. RGF weak equivalence principle[1]: The relativistic inertial mass m in the *relativistic law of motion* (e.g., Equation (12)) equals the corresponding gravitational mass m in the gravitational force $\mathbf{f}$. The relativistic force $\mathbf{f}$ including propagation delay is listed in Eq. (29). It is there assumed that the distance r is small relative to the cosmological scale (cf. Appendix 4.3).

Let us now proceed by noting that in special relativity the definition of a relativistic 3-force is given by *the law of motion* $\mathbf{f}_{SR} = \dot{\mathbf{p}}$, where the linear momentum is given by $\mathbf{p} = m\mathbf{u}$. After differentiation, the force can be expressed as:

$$\mathbf{f}_{SR} = \frac{m}{c^2 - u^2} \left(\mathbf{u} \cdot \mathbf{a}\right) \mathbf{u} + m\mathbf{a} \tag{1}$$

where $\mathbf{u}$ is the particle velocity, $\mathbf{a}$ is the particle acceleration and c is the speed of light in vacuum. Equation (1) may also be rewritten to read

$$\mathbf{f}_{SR} = \frac{1}{c^2} \left(\mathbf{u} \cdot \mathbf{f}_{SR}\right) \mathbf{u} + m\mathbf{a} \tag{2}$$

The main problem, as mentioned in the Introduction, is that Newton's law of gravitation is not consistent with observations, nor with the requirement of Lorentz invariance in special relativity. Here, we are curious to study if there is a consistent adjustment to the relativistic

[1] There is some complexity related to the inertial character of a *field mass* which is described in the Appendices 4.7–4.9. This effect is completely negligible for ordinary objects and is therefore removed from the main sections of this work.

equation of motion ($\mathbf{f}_{SR} = \dot{\mathbf{p}}$) that we could do to improve the situation. Let us therefore consider the ansatz

$$\mathbf{f} = \alpha(u)\, m\, (\mathbf{u}\cdot\mathbf{a})\, \mathbf{u} + \beta(u)\, m\mathbf{a}, \tag{3}$$

where $\alpha(u)$ and $\beta(u)$ are novel scalar functions in u. Consider the law of the *power* (cf. Equation (1))

$$P \equiv \dot{T} = \dot{E} = \frac{dm}{dt}c^2 = \frac{c^2}{c^2-u^2}m\,(\mathbf{u}\cdot\mathbf{a}) = \mathbf{f}_{SR}\cdot\mathbf{u} \tag{4}$$

so we can find a condition for $\alpha(u)$ and $\beta(u)$ in order for the law of power to survive,

$$\mathbf{f}\cdot\mathbf{u} = \alpha(u)\, u^2 m\,(\mathbf{u}\cdot\mathbf{a}) + \beta(u)\, m\,(\mathbf{u}\cdot\mathbf{a}) = \left(\alpha(u)\, u^2 + \beta(u)\right) m\,(\mathbf{u}\cdot\mathbf{a})\,. \tag{5}$$

By comparing with Eq. (4), we must require that

$$\alpha(u)\, u^2 + \beta(u) = \frac{c^2}{c^2-u^2} \tag{6}$$

We note that $\alpha(u) = 1/\left(c^2-u^2\right)$ and $\beta(u) = 1$ in Eq. (1) is consistent with Eq. (6). We also note that in order to approach $\mathbf{f} = m\mathbf{a}$ as $u \to 0$, we must have that $\beta(u) \to 1$ simultaneously. The dimension of $\alpha(u)$ in Eq. (6) also reveals that $\alpha(u) \propto c^{-2}$ as $u \to 0$. Out of curiosity, it is interesting to consider the relativistic force in 1-D:

$$f = \alpha(u)\, mau^2 + \beta(u)\, ma = \left(\alpha(u)\, u^2 + \beta(u)\right) ma = \frac{c^2}{c^2-u^2}ma \tag{7}$$

This result is already well known in special relativity so the dynamics in 1-D will not change by the ansatz 3. Let us also check the law of the *work*

$$W \equiv \int_1^2 \mathbf{f}\cdot\mathbf{dr} = \int_1^2 \mathbf{f}\cdot\mathbf{u}\,dt = \int_1^2 \left(\alpha(u)\, u^2 + \beta(u)\right) m\,(\mathbf{u}\cdot\mathbf{a})\,dt$$

$$= \int_1^2 \frac{c^2}{c^2-u^2}m\,(\mathbf{u}\cdot\mathbf{a})\,dt = \int_1^2 \frac{c^2}{c^2-u^2}mu\frac{du}{dt}dt = \int_1^2 \frac{mu}{1-u^2/c^2}du.$$

The last integral is just $m(u_2)\,c^2 - m(u_1)\,c^2 = T_2 - T_1$. It is thus quite clear that the physical laws in relativity are compatible with the ansatz Eq. (3) as long as Eq. (6) is respected. Now consider the dot product in Eq. (5)

$$\mathbf{u}\cdot\mathbf{a} = \frac{1}{m\left(\alpha(u)\, u^2 + \beta(u)\right)}\mathbf{f}\cdot\mathbf{u} = \frac{c^2-u^2}{mc^2}\mathbf{f}\cdot\mathbf{u}$$

The analog to Eq. (2) can then be written

$$\mathbf{f} = \tau(u)\,(\mathbf{u}\cdot\mathbf{f})\,\mathbf{u} + \beta(u)\, m\mathbf{a}, \tag{8}$$

where $\tau(u) = \alpha(u)\left(c^2-u^2\right)/c^2$.

We shall now identify the scalar functions $\tau(u)$ and $\beta(u)$ by performing a simple light deflection experiment. Consider a uniform gravitational field, i.e., $\mathbf{f} = -mg\mathbf{e}_z$ and initial conditions $\mathbf{u}(0) = c\mathbf{e}_x, \mathbf{r}(0) = \mathbf{0}$, i.e., we want to follow how light bends (in the lab frame). By applying Eq. (8), velocity $\mathbf{u} = u_x\mathbf{e}_x + u_z\mathbf{e}_z$ and the third postulate, one obtains

$$\begin{cases} a_x = & \frac{\tau}{\beta} g u_z u_x \\ a_y = & 0 \\ a_z = & \frac{\tau}{\beta} g u_z^2 - \frac{g}{\beta} \end{cases} \tag{9}$$

Through integration, the velocity components of a photon are explicitly given by

$$\begin{cases} u_x = & c e^{-\frac{\tau}{\beta} g|z|} \\ u_y = & 0 \\ u_z = & -c\sqrt{1 - e^{-2\frac{\tau}{\beta} g|z|}} \end{cases} \tag{10}$$

In general relativity, a well-known first-order result is that $z = -(1/2)\,2gt^2 = -\left(g/c^2\right)x^2$, i.e., for small t or x (Ferraro 2003). It is seen that the initial acceleration is given by $a_z = -2g$. This acceleration $(-2g)$ is split between one part due to the equivalence principle and the other part as a curved space contribution (Ferraro 2003). The question whether measurements are really performed within a flat space or a curved space is not as important as the actual experimental result itself which for light would be $a_z = -2g$ (cf. deflection near the limb of the Sun). We thus apply the second postulate that an experiment of horizontal light bending in a homogeneous gravitational field would result in exactly the initial acceleration $a_z = -2g$. In our setup $u_z = 0$ initially, so this then implies that $\beta = 1/2$, see Eq. (9). After sufficiently long time, the light ray travels vertically, i.e., $a_x = u_x = 0$ and $a_z = 0,\ u_z = -c$. Equation (9) then gives that

$$0 = \frac{\tau}{\beta} g c^2 - 2g,$$

so $\tau/\beta = 2/c^2$. This ratio is taken to be an universal constant, i.e., $\tau(u)/\beta(u) = 2/c^2$ is true for any particle speed u (more details in Appendix 4.1). By using $\tau/\beta = 2/c^2$, Eq. (6) and $\tau = \alpha\left(c^2 - u^2\right)/c^2$, one can then identify the general scalar functions as (also shown in Appendix 4.1)

$$\tau(u) = \frac{2}{c^2 + u^2},\ \beta(u) = \frac{c^2}{c^2 + u^2} \tag{11}$$

Thus, a new *relativistic equation of motion* is proposed:

$$\mathbf{f} = \frac{2}{c^2 + u^2}(\mathbf{u} \cdot \mathbf{f})\,\mathbf{u} + \frac{c^2}{c^2 + u^2} m\mathbf{a} \tag{12}$$

Further, in terms of acceleration Eq. (12) can be expressed as

$$\mathbf{f} = \frac{2mc^2}{\left(c^2 - u^2\right)\left(c^2 + u^2\right)}(\mathbf{u} \cdot \mathbf{a})\,\mathbf{u} + \frac{c^2}{c^2 + u^2} m\mathbf{a} \tag{13}$$

The easiest way to show that the two above expressions are equivalent is to take the dot product of Eq. (13) with the velocity $\mathbf{u}$. For completeness, with regard to the above photon experiment, we provide an analogous expression of how a point mass falls in a homogeneous gravitational field, see Appendix 4.2. The initial condition is $\mathbf{u}(0) = u_0\mathbf{e}_x, \mathbf{r}(0) = \mathbf{0}$. In Appendix 4.2 we have used Eq. (12), but we also find it convenient to identify a *constant of* *motion* resulting in

$$u_z^2 = c^2\left(1 - e^{-\frac{2g}{c^2}|z|}\right) + u_0^2 e^{-\frac{2g}{c^2}|z|}\left(1 - e^{-\frac{2g}{c^2}|z|}\right).$$

A Taylor expansion reveals that

$$u_z^2 \approx 2g|z|\left(1 + \frac{u_0^2}{c^2}\right)$$

which can be compared with the Newtonian result $u_z^2 = 2\,g|z|$. The initial acceleration ($z = 0$) is given by

$$a_z = -g\left(1 + \frac{u_0^2}{c^2}\right)$$

so for a point mass the initial acceleration is nearly $-g$ (or exactly $-g$ if $u_0 = 0$) as expected from Newtonian mechanics. Later on, a_z will eventually decline as the speed becomes relativistic, see last equations in "Appendix 4.2".

2.1 The superposition principle

With regard to the validity of the *superposition principle* in RGF, it can be noted that Eq. (12) is linear in $\mathbf{f}$ and $\mathbf{a}$, so this principle is perfectly legitimate in the case of a N-body problem. This is easily seen by considering two forces $\mathbf{f}_1$ and $\mathbf{f}_2$ acting on a single particle moving with the velocity $\mathbf{u}$ at a certain time t:

$$\mathbf{f}_1 = \frac{2}{c^2 + u^2}\left(\mathbf{u} \cdot \mathbf{f}_1\right)\mathbf{u} + \frac{c^2}{c^2 + u^2} m\mathbf{a}_1$$
$$\mathbf{f}_2 = \frac{2}{c^2 + u^2}\left(\mathbf{u} \cdot \mathbf{f}_2\right)\mathbf{u} + \frac{c^2}{c^2 + u^2} m\mathbf{a}_2$$

The sum of the above equations yields:

$$\mathbf{f}_1 + \mathbf{f}_2 = \frac{2}{c^2 + u^2}\left(\mathbf{u} \cdot \mathbf{f}_1 + \mathbf{u} \cdot \mathbf{f}_2\right)\mathbf{u} + \frac{c^2}{c^2 + u^2} m\left(\mathbf{a}_1 + \mathbf{a}_2\right)$$

This of course is just Eq. (12) again with $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ and $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ is the total acceleration of the particle. It is well known that the superposition principle is not a valid principle in the nonlinear GR theory. This is the reason why it is so cumbersome to apply GR in dealing with N-body problems.

2.2 Relativistic gravitational force

In Eqs. (12, 13), it may appear that we have sacrificed the original definition of relativistic force, i.e., $\mathbf{f}=\dot{\mathbf{p}}$. However, it is really just a matter of interpretation. What we have here is actually $\mathbf{f}=\dot{\mathbf{p}} + \mathbf{q}$, where $\mathbf{q}$ is a correction that has been introduced. One could just as well say that a new special relativistic gravitational force $\mathbf{f} - \mathbf{q}=\dot{\mathbf{p}} \equiv \mathbf{f}_{SR}$ has been discovered and that special relativity is left unchanged. The correction $\mathbf{q}$ is explicitly given by

$$\mathbf{q} = \frac{1}{c^2}\left\{\left(\mathbf{u} \cdot \mathbf{f}\right)\mathbf{u} - u^2\mathbf{f}\right\} = \frac{1}{c^2}\mathbf{u} \times \left(\mathbf{u} \times \mathbf{f}\right) \tag{14}$$

It is obvious that $\mathbf{q} = 0$ in 1-D problems (i.e., in perfect agreement with Eq. (7)). Also, the vectors $\mathbf{u}$ and $\mathbf{q}$ are always orthogonal ($\mathbf{u} \cdot \mathbf{q} = 0$) so there is no *work* performed by this correction force $\mathbf{q}$ (i.e., $\mathbf{q}$ itself cannot change the kinetic energy of the particle). This is in analogy with the magnetic Lorentz force which also is always perpendicular to the velocity. By considering $\mathbf{u} \times \mathbf{f}$ and either of Eqs. (12–13), it is immediately clear that the acceleration form of $\mathbf{q}$ is given by

$$\mathbf{q} = \frac{m}{\left(c^2 + u^2\right)} \mathbf{u} \times (\mathbf{u} \times \mathbf{a}) \tag{15}$$

We can note that as the particle speed u becomes nonrelativistic, $\mathbf{q}$ will be negligible and the Newtonian $\mathbf{f}=\dot{\mathbf{p}}$ is restored. According to our third postulate, the gravitational mass within Newton's gravitational law ($\mathbf{f}$) is the same as the relativistic inertial mass m, see Eq. (12). The implication is that the mass m becomes irrelevant and plays no role in the dynamical solution (i.e., for a purely gravitational problem).

The above relativistic gravitational force $\mathbf{f} - \mathbf{q}$ may be inappropriate for very large scales since it is still today not clear if Newtonian gravity of type $1/r^2$ is correct at very large scales, or if MOND behavior is more appropriate (Milgrom 1983). For the Sun we derive in Appendix 4.3 that at a distance of about 7000 AU it is possible that the MOND effect could become relevant. As all our examples in the present work are concerned with distances much less than 7000 AU, one can safely neglect the MOND effect.

2.3 Field formulation

In terms of fields, the *relativistic equation of motion* of a test particle m in a two-body problem can be expressed as (cf. Equation (12))

$$\begin{cases} \mathbf{F} = \mathbf{g} + \mathbf{u} \times \mathbf{h} = \frac{1}{c^2} (\mathbf{u} \cdot \mathbf{F})\, \mathbf{u} + \mathbf{a} \\ \mathbf{g} = -\frac{GM}{r^2} \mathbf{e}_r \\ \mathbf{h} = \frac{1}{c^2} \mathbf{g} \times \mathbf{u} \end{cases} \tag{16}$$

Note that the relativistic gravitational force is actually $m\mathbf{F} = \mathbf{f}_{SR}$ but Eq. (2) clearly shows that the relativistic mass m plays no role in determining the motion and can thus be dropped. In order to keep this simple it is assumed that the second body M is essentially at rest (e.g., the Sun; $M \approx M_0 \gg m$) so the effect on the $\mathbf{g}$-field from retarded time due to a finite propagation speed of gravity is unimportant (see Sect. 2.7 for a more general $\mathbf{g}$). Equation (16) is still exact if one understands that the computation of the $\mathbf{g}$-field may be nontrivial and not simply given by $\mathbf{g} = -GM/r^2 \mathbf{e}_r$. Even though there is no explicit dependence on m in Eq. (16), there is an exception in the case of an non-inertial frame. For example, if the origin is placed on the central body M (heliocentric system), the above expression needs a minor modification since such a frame is subject to a weak acceleration. As we here consider the case that M is moving slowly, i.e., classically, one can then simply apply Coriolis's theorem, i.e., one subtracts the acceleration of the origin $\mathbf{a}_{\odot}$, where $\mathbf{a}_{\odot} = Gm/r^2 \mathbf{e}_r$ (see the Appendix 4.4 for a simple analysis). A sufficiently accurate expression in the solar system for the acceleration of a planet m would then be

$$\mathbf{a} \approx \mathbf{g} + \mathbf{u} \times \mathbf{h} - \frac{1}{c^2} (\mathbf{u} \cdot \mathbf{g})\, \mathbf{u} - \frac{Gm}{r^2} \mathbf{e}_r . \tag{17}$$

In Sect. 2.4, we will see that the form of Eq. (16) is convenient in order to prove that it respects Lorentz invariance. Although Eq. (16) is mathematically equivalent to Eq. (12), it is often more straightforward to apply the latter in the various examples to be presented below.

Also, one may get the impression that Eq. (16) would imply some kind of similarity to the Lorentz force in electrodynamics. In Appendix 4.5, we provide details showing that there is almost no resemblance, i.e., RGF is not at all some kind of gravitomagnetic theory.

2.4 The principle of relativity

According to the *principle of relativity*, any proposed law (such as Eq. (16)) must display the same form in an arbitrary Lorentz frame. One way of looking at this is to consider the 4-acceleration $\mathbb{A}$ defined by

$$\mathbb{A} = \gamma_u^2 \left(\frac{\mathbf{u} \cdot \mathbf{a}}{c} \gamma_u^2, \frac{\mathbf{u} \cdot \mathbf{a}}{c^2} \gamma_u^2 \mathbf{u} + \mathbf{a} \right),$$

see 2.5.2 in Steane (2012). In the rest frame $\mathbf{u} = \mathbf{0}$, so then we have

$$\mathbb{A} = (0, \mathbf{a}_0)$$

where $\mathbf{a}_0 = -\frac{GM}{r^2}\mathbf{e}_r$. The Lorentz scalar $\mathbb{A}^\mu \mathbb{A}_\mu = a_0^2$ should be conserved in all Lorentz frames in 4-space. According to Eq. (16), the 3-acceleration is given by

$$\mathbf{a} = \mathbf{F} - \frac{1}{c^2} (\mathbf{u} \cdot \mathbf{F})\, \mathbf{u} \tag{18}$$

so

$$\mathbf{u} \cdot \mathbf{a} = \frac{\mathbf{u} \cdot \mathbf{F}}{\gamma_u^2}. \tag{19}$$

Thus, the 4-acceleration can be written

$$\mathbb{A} = \gamma_u^2 \left(\frac{\mathbf{u} \cdot \mathbf{F}}{c}, \frac{(\mathbf{u} \cdot \mathbf{F})}{c^2} \mathbf{u} + \mathbf{a} \right)$$

It is sufficient to show that this general formula fulfills $\mathbb{A}^\mu \mathbb{A}_\mu = a_0^2$,

$$\mathbb{A}^\mu \mathbb{A}_\mu = \gamma_u^4 \left(-\frac{1}{c^2} (\mathbf{u} \cdot \mathbf{F})^2 + \frac{u^2}{c^4} (\mathbf{u} \cdot \mathbf{F})^2 + \frac{2}{c^2} (\mathbf{u} \cdot \mathbf{F}) (\mathbf{u} \cdot \mathbf{a}) + \mathbf{a} \cdot \mathbf{a} \right)$$

This expression can be reduced further by using Eq. (18)

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= \left(\mathbf{F} - \frac{1}{c^2} (\mathbf{u} \cdot \mathbf{F})\, \mathbf{u} \right) \cdot \left(\mathbf{F} - \frac{1}{c^2} (\mathbf{u} \cdot \mathbf{F})\, \mathbf{u} \right) \\ &= \mathbf{F} \cdot \mathbf{F} - \frac{2}{c^2} (\mathbf{u} \cdot \mathbf{F})^2 + \frac{1}{c^4} (\mathbf{u} \cdot \mathbf{F})^2 u^2 \end{aligned}$$

By also inserting Eq. (19), the 4-acceleration scalar becomes

$$\mathbb{A}^\mu \mathbb{A}_\mu = \gamma_u^4 \left((\mathbf{u} \cdot \mathbf{F})^2 \underbrace{\left\{ -\frac{1}{c^2} + \frac{u^2}{c^4} + \frac{2}{c^2 \gamma_u^2} - \frac{2}{c^2} + \frac{u^2}{c^4} \right\}}_{-1/c^2} + \mathbf{F} \cdot \mathbf{F} \right)$$

According to Eq. (16),

$$\mathbf{F}\cdot\mathbf{F}=\frac{1}{c^2}\left(\mathbf{u}\cdot\mathbf{F}\right)^2+\mathbf{F}\cdot\mathbf{a}.$$

Thus, we found the following simplification:

$$\mathbb{A}^{\mu}\mathbb{A}_{\mu}=\gamma_u^4\left(\mathbf{F}\cdot\mathbf{a}\right)$$

Through Eq. (19), the relation (18) can be rewritten as

$$\mathbf{F}=\frac{\gamma_u^2}{c^2}\left(\mathbf{u}\cdot\mathbf{a}\right)\mathbf{u}+\mathbf{a}$$

so

$$\mathbf{F}\cdot\mathbf{a}=\frac{\gamma_u^2}{c^2}\left(\mathbf{u}\cdot\mathbf{a}\right)^2+a^2$$

which finally gives that the Lorentz scalar becomes

$$\mathbb{A}^{\mu}\mathbb{A}_{\mu}=\gamma_u^4\left(\mathbf{F}\cdot\mathbf{a}\right)=\frac{\gamma_u^6}{c^2}\left(\mathbf{u}\cdot\mathbf{a}\right)^2+\gamma_u^4a^2.$$

A standard result in special relativity is that

$$a_0^2=\frac{\gamma_u^6}{c^2}\left(\mathbf{u}\cdot\mathbf{a}\right)^2+\gamma_u^4a^2, \tag{20}$$

see Eq. 2.61 in Steane (2012), so it has been established that in fact it is true in general that

$$\mathbb{A}^{\mu}\mathbb{A}_{\mu}=a_0^2.$$

The law of motion as given by Eq. (16) is thus consistent with the *principle of relativity*. In order to complete the proof, it is convenient to study the mathematical structure of the three-vector $\mathbf{F}=\mathbf{g}+\mathbf{u}\times\mathbf{h}$ in 4-space to see if this 4-vector can be derived from a *manifestly covariant* tensor. The mathematics then becomes much more compact, and the *principle of relativity* can be demonstrated in just a few lines, see “Appendix 4.6”.

2.5 Conservation of mechanical energy in RGF

In the derivation just below Eq. (7)), it was proved that the work in RGF is still given by the law of the kinetic energy in special relativity, i.e.,

$$W_{12}=T_2-T_1=m\left(u_2\right)c^2-m\left(u_1\right)c^2. \tag{21}$$

We would now like to consider the law of gravitation with the modification that the masses are relativistic, i.e., in agreement with our postulate 3 in Sect. 2. As shown in Sect. 2.2, the special relativistic gravitational force is $\mathbf{f}-\mathbf{q}$ but $\mathbf{q}$ makes no contribution to the work so it is sufficient to deal with $\mathbf{f}$. Also, we here consider the simplified case where the central body M is at rest, but the test particle m may move at relativistic speeds. The effect of propagation delays are then absent (Sect. 2.7 and also Eq. (29) can be discarded). The gravitational force is then essentially Newton’s law of gravitation but with m being the relativistic mass. We thus keep it very simple here. An interested reader can consult a more general discussion of the two-particle problem in Appendix 4.11. The work is for the current situation simply given by

$$W_{12} = \int_1^2 \mathbf{f} \cdot \mathbf{dr} = \int_{r_1}^{r_2} -\frac{GMm}{r^2} dr = -GM \int_{r_1}^{r_2} \frac{m}{r^2} dr$$

and in this case m is viewed as varying as a function of r (instead of u) as it changes from r_1 to r_2. This dependence will be studied later in Sect. 3.6 (see Eq. (53)). We then have that

$$c^2 \ln \frac{m}{m_1} = -GM \left(-\frac{1}{r} + \frac{1}{r_1} \right) = \phi_1 - \phi,$$

where ϕ is the classical gravitational potential. We thus find that

$$m\left(r\right) = m\left(r_1\right) e^{(\phi_1 - \phi)/c^2} = A_1 e^{-\phi/c^2},$$

where the constant A_1 is determined by the initial condition. The work is therefore given by

$$W_{12} = -GMA_1 \int_{r_1}^{r_2} \frac{e^{-\phi/c^2}}{r^2} dr.$$

This integral is solved by the variable substitution $t = 1/r$. One then finds that

$$W_{12} = c^2 A_1 e^{-\phi_2/c^2} - c^2 A_1 e^{-\phi_1/c^2}, \tag{22}$$

and Eq. (21) shows that

$$T_2 - T_1 = c^2 A_1 e^{-\phi_2/c^2} - c^2 A_1 e^{-\phi_1/c^2}.$$

Thus, the following conservation law has been identified

$$T - c^2 A_1 e^{-\phi/c^2} = const.$$

Now, it would be nice if this relativistic expression would coincide with the Newtonian mechanical energy as $c \to \infty$. Through Taylor expansions, it turns out that the appropriate relativistic mechanical energy then can be written

$$E = T + m_1 c^2 \left(1 - e^{(\phi_1 - \phi)/c^2}\right) + m_1 \phi_1, \tag{23}$$

where $T = mc^2 - m_0 c^2$ (m_0 being the rest mass). In Appendix 4.11, we provide a treatment of the general two-particle problem.

2.6 Lagrangian formulation

In Eq. (16), it was shown that the special relativistic gravitational force acting on m due to body M at rest is given by $\mathbf{f} - \mathbf{q} = m\mathbf{g} + m\mathbf{u} \times \mathbf{h}$. If a corresponding generalized potential U can be identified, the relativistic Lagrangian can be written as follows:

$$L = -m_0 c^2 \sqrt{1 - u^2/c^2} - U.$$

The Lagrange equations of motion are then still valid and given by

$$\frac{d}{dt} \frac{\partial L}{\partial u_x} = \frac{\partial L}{\partial x}.$$

It is easy to derive that

$$\frac{\partial L}{\partial u_x} = p_x - \frac{\partial U}{\partial u_x}, \ \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$

so clearly

$$-\frac{\partial U}{\partial x} = \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial u_x} = \dot{p}_x - \frac{d}{dt}\frac{\partial U}{\partial u_x},$$

and one must therefore require that

$$F_x \equiv \dot{p}_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial u_x}.$$

Thus, for the Lagrange equations to be valid for the gravitational force we need to prove that

$$mg_x + m\left(\mathbf{u} \times \mathbf{h}\right)_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial u_x}. \tag{24}$$

This exercise is thus to identify an appropriate generalized potential U. We suggest that such an appropriate candidate is given by

$$U = U_0 + U_1 + U_2 = U_0 - A_1 c^2 e^{-\phi/c^2} - m\mathbf{i} \cdot \mathbf{u} \tag{25}$$

and we will show that Eq. (24) is indeed fulfilled for this choice. Here, A_1 is a constant and ϕ is the classical gravitational potential, see these details in Sect. 2.5. Since M is at rest the **g**-field is stationary so a treatment related to propagation delays can be discarded. The relativistic mass m is usually expressed as being speed-dependent, but Sect. 3.6 will show that it can be viewed as being distance-dependent, i.e.,

$$m\left(r\right) = A_1 e^{-\phi/c^2},$$

where $\phi = \phi\left(x, y, z\right)$. The vector potential $\mathbf{i} = \mathbf{i}\left(x, y, z, t\right)$ is defined by $m\mathbf{h} = \nabla \times m\mathbf{i}$. This relation can be written in this way because

$$\nabla \cdot m\mathbf{h} = \underbrace{\frac{m}{c^2}\mathbf{u} \cdot (\nabla \times \mathbf{g})}_{static\,central\,field \Rightarrow 0} - \frac{1}{c^2}\underbrace{\mathbf{g} \cdot (\nabla \times m\mathbf{u})}_{\nabla \times m\mathbf{u} = \frac{m}{c^2}\mathbf{g} \times \mathbf{u} \Rightarrow 0} = 0,$$

where we used for example $\partial m/\partial x = mg_x/c^2$.

Let us now investigate the derivatives of the generalized potential. First, we define U_0 by requiring that

$$-\frac{\partial U_0}{\partial x} = m\frac{\partial i_x}{\partial t}.$$

This term will be canceled by another term below. Further, we have that

$$-\frac{\partial U_1}{\partial x} = -m\frac{\partial \phi}{\partial x} = mg_x$$
$$-\frac{\partial U_2}{\partial x} = \frac{\partial m}{\partial x}\mathbf{i} \cdot \mathbf{u} + m\frac{\partial \mathbf{i}}{\partial x} \cdot \mathbf{u},$$

and we also have that

$$\frac{\partial U}{\partial u_x} = \frac{\partial U_2}{\partial u_x} = -mi_x \equiv f\left(x, y, z, t\right)$$
$$\frac{d}{dt}\frac{\partial U}{\partial u_x} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}u_x + \frac{\partial f}{\partial y}u_y + \frac{\partial f}{\partial z}u_z$$

where

$$\frac{\partial f}{\partial t} = -m\frac{\partial i_x}{\partial t}$$

which is canceled by the U_0 contribution as mentioned before. Further, we have, for example, that

$$\frac{\partial f}{\partial x} = -\frac{\partial (mi_x)}{\partial x} = -\frac{\partial m}{\partial x} i_x - m\frac{\partial i_x}{\partial x}.$$

One finds that

$$-\frac{\partial U_0}{\partial x} - \frac{\partial U_2}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial u_x} = \frac{\partial m}{\partial x} i_y u_y + \frac{\partial m}{\partial x} i_z u_z + m\frac{\partial i_y}{\partial x} u_y + m\frac{\partial i_z}{\partial x} u_z$$
$$-\frac{\partial m}{\partial y} i_x u_y - \frac{\partial m}{\partial z} i_x u_z - m\frac{\partial i_x}{\partial y} u_y - m\frac{\partial i_x}{\partial z} u_z.$$

This expression is exactly equal to

$$(\mathbf{u} \times (\nabla \times m\mathbf{i}))_x = (\mathbf{u} \times m\mathbf{h})_x = m\,(\mathbf{u} \times \mathbf{h})_x\,.$$

We have thus successfully proved that

$$-\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial u_x} = mg_x + m\,(\mathbf{u} \times \mathbf{h})_x$$

and U in Eq. (25) is therefore an appropriate generalized potential for which the Lagrange equations are fulfilled.

2.7 Interaction due to retarded time

The effect on the **g**-field from retarded time due to the finite propagation speed of gravity may become difficult to neglect in the case of a highly relativistic N-body system. Provided that the speed of gravity is the same as the speed of light, one can make an analogous treatment as that in relativistic electrodynamics (Section 8.2.4 in Steane (2012)) which then leads to the generalization of the **g**-field in Eq. (16) for a point mass m,

$$\mathbf{g} = -\frac{GM}{r^3\gamma^3\eta\left(\gamma^2\cos^2\theta + \sin^2\theta\right)^{3/2}}\left[\mathbf{r} + \frac{\gamma^2}{c^2}(\mathbf{R}\cdot\mathbf{a})\,\mathbf{r} - \frac{\gamma^2}{c^2}(\mathbf{R}\cdot\mathbf{r})\,\mathbf{a}\right] \tag{26}$$

where $M = M_0\gamma$, $\mathbf{r} = \mathbf{r}_m(t) - \mathbf{r}_M(t)$, $\gamma = \left(1 - u_M^2/c^2\right)^{-1/2}$, $\eta = 1 + u_M^2/c^2$, $\cos^2\theta = (\mathbf{r}\cdot\mathbf{u}_M)^2/(ru_M)^2$, $\mathbf{R} = \mathbf{r} + \mathbf{u}_M R/c$, $\mathbf{a}$ is the acceleration of the retarded position $\mathbf{r}_M(t - r/c)$ and $R \approx r/(1 - \mathbf{r}\cdot\mathbf{u}_M/cr)$. With this updated **g**-field, one may compute the acceleration of the point mass m according to the law given by Eq. (16), i.e.,

$$\mathbf{a}_m = \mathbf{g} + \mathbf{u}_m \times \mathbf{h} - \frac{1}{c^2}(\mathbf{u}_m\cdot\mathbf{g})\,\mathbf{u}_m.$$

At a sufficiently large distance r, only the radiation part of Eq. (26) is significant (as long as $\mathbf{a} \neq \mathbf{0}$)

$$\mathbf{g}_{rad} = -\frac{GM_0}{r^3\eta c^2\left(\gamma^2\cos^2\theta + \sin^2\theta\right)^{3/2}}\mathbf{R} \times (\mathbf{r} \times \mathbf{a}) \tag{27}$$

This is the expression that provides a prediction for gravitational waves in RGF. The above retardation effects could become significant but for most examples to be presented in the present work it turns out that the retardation effect is very small as the usual speed in orbital problems is largely nonrelativistic. Also, in the case of a static **g**-field, i.e., $u_M = 0$, the retardation effect is obviously absent (e.g., the black hole problem in Sect. 3.7). We apply the full equations provided here in Sect. 2.8 related to linear momentum and also in Sect. 3.9–3.10. In Sect. 3.8 we study the case $m << M$ of periastron precession and found that the retardation effect is indeed negligible since $u_M \approx 0$. The propagation effect for a pulsar and its companion can be studied where a slow orbital decay is occurring due to radiation. This orbital decay is due to the recoil acceleration of, for example, a point mass m due to its emission of gravitational radiation (Appendix 4.10)

$$\mathbf{a}_{rad} = \frac{2}{3}\frac{u_m^2}{c^2}\frac{Gm_0}{c^3}\dot{\mathbf{a}}_m.$$

This small acceleration was derived for the relevant case of nonrelativistic speeds and should be added to $\mathbf{a}_m$ (see the above expression) after $\dot{\mathbf{a}}_m$ has been determined. This is a simple matter in a numerical computer solution.

2.8 Linear momentum in RGF

It is worthwhile to retrieve some information about momentum laws in RGF. We shall here consider systems where also effects due to propagation delays are taken into account (see Sect. 2.7). Let us start with the simplest case, i.e., the situation for a single particle. The linear momentum given by $\mathbf{p} = m\mathbf{u}$ is clearly conserved if $\mathbf{f}=\mathbf{q}$ since $\mathbf{f}-\mathbf{q} = \dot{\mathbf{p}}$ (Sect. 2.2). Equation (14) gives that $\mathbf{q} = -\left(u^2/c^2\right)\mathbf{q}$, since $\mathbf{u}\cdot\mathbf{q} = 0$, and Eq. (12) gives that $\mathbf{q} = c^2 m\mathbf{a}/\left(c^2 + u^2\right)$. This can only hold if $\mathbf{q} = \mathbf{0}$ so $\mathbf{f} = \mathbf{0}$ and also $\mathbf{a} = \mathbf{0}$. Notice that $\mathbf{f} = \mathbf{0}$ directly leads to $\mathbf{a} = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$, see Eqs. (12, 14). Thus, it can be concluded that the linear momentum for a single particle is conserved if $\mathbf{f} = \mathbf{0}$, i.e., it is behaving in the usual way.

Now let us consider the much more interesting case of a two-body problem where the masses are given by m and M. According to Eq. (14), the gravitational forces and relativistic corrections acting on body m and body M are given by

$$\mathbf{F}_m = \mathbf{f}_m - \mathbf{q}_m = \mathbf{f}_m - \frac{1}{c^2}\mathbf{u}_m \times (\mathbf{u}_m \times \mathbf{f}_m) = \dot{\mathbf{p}}_m$$
$$\mathbf{F}_M = \mathbf{f}_M - \mathbf{q}_M = \mathbf{f}_M - \frac{1}{c^2}\mathbf{u}_M \times (\mathbf{u}_M \times \mathbf{f}_M) = \dot{\mathbf{p}}_M \qquad (28)$$

The gravitational force is $\mathbf{f}_m = -GMm/r^2\mathbf{e}_r$ if propagation is assumed to occur instantly (i.e., $c \to \infty$ or if M is static). Newton's third law would then be correct, i.e., $\mathbf{f}_M = -\mathbf{f}_m$. However, let us consider Sect. 2.7 and study this assumption. Then we have that

$$\mathbf{f}_m = -\frac{GMm}{r^3\gamma_M^3\eta_M\left(\gamma_M^2\cos^2\theta_M + \sin^2\theta_M\right)^{3/2}}\left[\mathbf{r} + \frac{\gamma_M^2}{c^2}\left(\mathbf{R}_M\cdot\mathbf{a}_M\right)\mathbf{r} - \frac{\gamma_M^2}{c^2}\left(\mathbf{R}_M\cdot\mathbf{r}\right)\mathbf{a}_M\right]$$
$$\mathbf{f}_M = \frac{GMm}{r^3\gamma_m^3\eta_m\left(\gamma_m^2\cos^2\theta_m + \sin^2\theta_m\right)^{3/2}}\left[\mathbf{r} + \frac{\gamma_m^2}{c^2}\left(\mathbf{R}_m\cdot\mathbf{a}_m\right)\mathbf{r} - \frac{\gamma_m^2}{c^2}\left(\mathbf{R}_m\cdot\mathbf{r}\right)\mathbf{a}_m\right] \qquad (29)$$

Although one can certainly see that $\mathbf{f}_M = -\mathbf{f}_m$ for various symmetrical situations, it is not true in general for relativistically moving particles. One way to realize this is to consider the radiation terms

$$\frac{\gamma^2}{c^2}\left(\mathbf{R}\cdot\mathbf{a}\right)\mathbf{r} - \frac{\gamma^2}{c^2}\left(\mathbf{R}\cdot\mathbf{r}\right)\mathbf{a} = \frac{\gamma^2}{c^2}\mathbf{R}\times(\mathbf{r}\times\mathbf{a}) \approx 0.$$

For a gravitational two-particle problem, it is nearly true that $\mathbf{r} \parallel \mathbf{a}$. The smallness of this cross-product is further lessened by the factor γ^2/c^2. By neglecting the radiation terms, we are left with the factors:

$$\gamma_m^3 \eta_m \left(\gamma_m^2 \cos^2\theta_m + \sin^2\theta_m\right)^{3/2}$$
$$\gamma_M^3 \eta_M \left(\gamma_M^2 \cos^2\theta_M + \sin^2\theta_M\right)^{3/2}$$

One can see that these are not necessarily equal. Thus, Newton's third law cannot hold exactly in general. From symmetry, however, one can see that if $\mathbf{u}_m = \pm\mathbf{u}_M$ then it is true that $\mathbf{f}_M = -\mathbf{f}_m$. Such a situation is also compatible with

$$\mathbf{F}_m + \mathbf{F}_M = \frac{d}{dt}\left(\mathbf{p}_m + \mathbf{p}_M\right) = -\frac{1}{c^2}\mathbf{u}_m \times (\mathbf{u}_m \times \mathbf{f}_m) + \frac{1}{c^2}\mathbf{u}_m \times (\mathbf{u}_m \times \mathbf{f}_m) = 0$$

so symmetry dictates that the total *mechanical* momentum $\mathbf{P}_{mech} = \mathbf{p}_m + \mathbf{p}_M$ would then be conserved. These features are analogous to what is found in electrodynamics for the Lorentz force, see 8.2 in Griffiths (1999). Given that the mechanical momentum is not conserved in general, we may adopt the very general principle in physics that the total momentum $\mathbf{P}$ must be conserved. The situation can be rectified by introducing the concept of *field momentum* (compare with, e.g., Chapt. 27 in Feynman (1963)). Its rate would then be given by

$$\dot{\mathbf{p}}_{field} = -\mathbf{F}_m - \mathbf{F}_M,$$

in which case the desired result becomes

$$\frac{d}{dt}\left(\mathbf{p}_m + \mathbf{p}_M + \mathbf{p}_{field}\right) = \frac{d\mathbf{P}}{dt} = 0.$$

So in this way the total momentum $\mathbf{P}$ of the system is always conserved. We note that the symmetry principle $\mathbf{u}_m = \pm\mathbf{u}_M$ makes $\dot{\mathbf{p}}_{field} = 0$ in which case $\mathbf{p}_{field} = const.$ (the total mechanical momentum $\mathbf{P}_{mech} = \mathbf{p}_m + \mathbf{p}_M$ is also conserved). Let us exemplify with $\mathbf{u}_m = -\mathbf{u}_M$. In order for this to be maintained at all times, the masses must be equal $m = M$. By comparing with Eq. (26) and the exact expressions for $\mathbf{f}_m$ and $\mathbf{f}_M$ above, we see that $\gamma_m = \gamma_M$, $\theta_m = \theta_M$, $\mathbf{R}_m = -\mathbf{R}_M$. This means that $(\mathbf{R}_M \cdot \mathbf{a}_M)\,\mathbf{r} = (\mathbf{R}_m \cdot \mathbf{a}_m)\,\mathbf{r}$ and $(\mathbf{R}_M \cdot \mathbf{r})\,\mathbf{a}_M = (\mathbf{R}_m \cdot \mathbf{r})\,\mathbf{a}_m$ so as expected it is then exactly true that $\mathbf{f}_m = -\mathbf{f}_M$. Thus, when there is symmetry in the problem, both the mechanical momentum $\mathbf{p}_m + \mathbf{p}_M$ and $\mathbf{p}_{field}$ are *exactly* conserved (separately). In all other situations, Newton's third law is only approximately true and only the total momentum $\mathbf{P}$ is conserved.

2.9 Coordinates in flat versus curved space

In the following sections, the GR results are frequently expressed in Schwarzschild coordinates, whereas the RGF results are derived and presented in the simpler Euclidean coordinates. There are difficulties in comparing flat spacetime results with those obtained in a curved spacetime. Are experiments really conducted in a curved space or a flat space is a key question. Results derived in an Euclidean frame are obviously not meaningful to compare with ditto Schwarzschild results (unless the radial coordinate is large). In order to better facilitate comparisons between RGF and GR, we shall here follow Feynman's suggestion to use *isotropic coordinates* that are those that conformally are most similar to spatial Euclidean coordinates, see p. 157 in Feynman (1995). The usage of isotropic coordinates makes the coordinate speed of light the same in all directions at a certain location.

A good description is provided in Vincent (2015). The isotropic metric is given by

$$ds^2 = \frac{\left(1-\frac{r_s}{4r'}\right)^2}{\left(1+\frac{r_s}{4r'}\right)^2}c^2dt^2 - \left(1+\frac{r_s}{4r'}\right)^4\left(dx'^2+dy'^2+dz'^2\right).$$

In RGF, we will denote Euclidean final results related to the radial distance as r'. The notation for a GR result will be expressed as r, i.e., the Schwarzschild radial coordinate. The transformation between r' (RGF) and r (GR) is then approximately given by (p. 157 in Feynman (1995))

$$r' = \frac{1}{2}\left(r - \frac{r_s}{2}\right) + \sqrt{\frac{r}{4}\left(r - r_s\right)} \tag{30}$$

where the constant r_s is the Schwarzschild radius. It is seen that the RGF and GR radial coordinates become, as expected, very close if $r \gg r_s$ ($r' \sim r - \frac{r_s}{2}$). However, as r and r' get closer to r_s, the above formula is appropriate to apply. The inverse of Eq. 30,

$$r = r'\left(1+\frac{r_s}{4r'}\right)^2, \tag{31}$$

provides a mean to translate a RGF (r') result into the GR Schwarzschild coordinate (r). In RGF, one typically gets results expressed as

$$RGF = e^{-\frac{r_s}{2r'}} \sim 1 - \frac{r_s}{2r'}$$

in the various results, whereas in GR one instead find terms like

$$GR = \sqrt{1-\frac{r_s}{r}} \sim 1 - \frac{r_s}{2r}.$$

Since $r' = r$ for large radial coordinates, it is immediately clear that these functions then are essentially the same. However, a remarkable fact is that when r' is translated into the Schwarzschild picture by using Eq. (30) one finds that the equality

$$e^{-\frac{r_s}{2r'}} = \sqrt{1-\frac{r_s}{r}} \tag{32}$$

is almost perfectly true over the whole range $r \in [r_s, \infty]$. This is the type of comparison that will occur frequently in the following sections and we will know that the agreement between GR and RGF is in fact very close. A large r and r' is not required as the below Taylor expansion suggests. There is therefore an expected agreement even into the strong field regime. In Fig. 1 we plot $\sqrt{1-\frac{r_s}{r}}/e^{-\frac{r_s}{2r'}}$ and $\sqrt{1-\frac{r_s}{r}}/e^{-\frac{r_s}{2r}}$. Ideally both ratios should be one. Note the dramatic improvement occurring for the first ratio when it is consistently expressed in the "same coordinate system" (i.e., the first ratio with r' replaced according to Eq. (30)), see Fig. 1. To further understand this, it is convenient to take the square of the ratio. A Taylor expansion then reveals that

$$1 - \frac{r_s}{r} = 1 - x + x^2/2 - 3/16x^3 + \mathcal{O}\left(x^4\right)$$
$$e^{-\frac{r_s}{r'}} = 1 - x + x^2/2 - 3/18x^3 + \mathcal{O}\left(x^4\right)$$

where $x = r_s/r'$. The above similarity explains why the ratio is close to one over such a wide range (as also shown in Fig. 1).

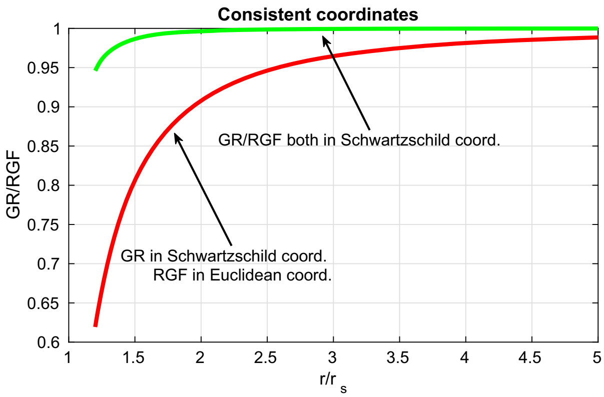

Fig. 1 The strong field regime. Comparisons between the ratio GR/RGF in the "same coordinate system" (top graph). The bottom graph shows the case where GR is derived in one set of coordinates (Schwarzschild) and RGF in another set (Euclidean), i.e., not a meaningful comparison unless r is large

3 Tests and examples of RGF

In the subsections below, we will present a series of examples showing that the RGF approach reproduces many observational facts. The results are often very close to the predictions of general relativity. Although there are deviations from GR in the strong field regime, those seem to a high degree be related to remaining difficulties in the different coordinate representations (i.e., Schwarzschild coordinates versus Euclidean coordinates). RGF is valid for all field strengths and speeds under the assumptions of the three postulates. We shall see that RGF has the great advantage that it can solve many difficult relativistic problems quickly and in a much less mathematically intensive way than GR. Given that the theory is linear and has been cast in the form of relativistic gravitational forces it is straightforward to apply in N-body calculations.

3.1 Two identical point charges

We shall first consider a simple introductory example to see how gravity transforms between frames. Consider the two point charges in Fig. 2 that sense forces from both electromagnetic and gravitational interactions. Such a situation has previously not been possible to reconcile within the framework of special relativity. We will show here that in RGF the simultaneous treatment is straightforward and provides consistent results.

The Lorentz forces acting on the top charge are in S (dynamics) and S' (rest) given, respectively, by

$$\mathbf{F}_{em} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$
$$\mathbf{F}'_{em} = q\mathbf{E}'.$$

The equations of motion are according to Eq. (2) given by

$$\mathbf{F}_{em} = \frac{1}{c^2}\left(\mathbf{u} \cdot \mathbf{F}_{em}\right)\mathbf{u} + m\mathbf{a}_{em} = m\mathbf{a}_{em}$$
$$\mathbf{F}'_{em} = m_0\mathbf{a}'_{em} \tag{33}$$

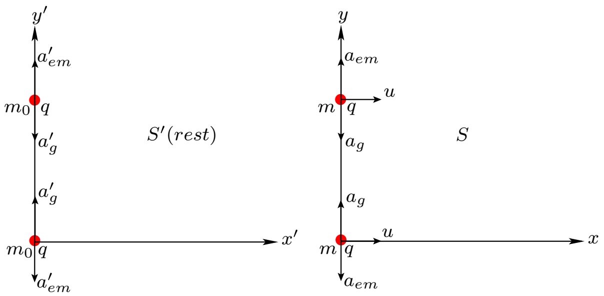

Fig. 2 At $t = 0$ two identical point charges are moving to the right at speed u in the S-system. In the S'-system, the charges are at rest and therefore interacting according to the laws of gravity (Newton's law) and electrostatics (Coulomb's law). In S' the physical system is setup such that the accelerations $a'_{em} = a'_g$ which is fulfilled for $Gm_0^2 = q^2/4\pi\varepsilon_0$. As the point charges are initially at rest, they will remain in rest in S'. The task at hand is to investigate in detail the same physical system but now relative to S where dynamics is occurring

since in S we have that $\mathbf{u}\cdot\mathbf{F}_{em} = 0$ and in S', $\mathbf{u}' = 0$. The magnetic field in S is (c.f. Appendix 4.5)

$$\mathbf{B} = -\frac{1}{c^2}\mathbf{E}\times\mathbf{u} = -\frac{1}{c^2}E\mathbf{e}_y \times u\mathbf{e}_x = \frac{1}{c^2}Eu\mathbf{e}_z$$

so then

$$\mathbf{u}\times\mathbf{B} = u\mathbf{e}_x \times \frac{1}{c^2}Eu\mathbf{e}_z = -\frac{u^2}{c^2}E\mathbf{e}_y.$$

Thus the Lorentz force becomes

$$\mathbf{F}_{em} = q\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}\right) = q\left(E\mathbf{e}_y - \frac{u^2}{c^2}E\mathbf{e}_y\right) = qE\mathbf{e}_y\left(1-\frac{u^2}{c^2}\right)$$

and the acceleration of the top charge in the y-direction is

$$a_{em} = \frac{q}{m}E\left(1-\frac{u^2}{c^2}\right).$$

In S' the acceleration is instead given by

$$a'_{em} = \frac{q}{m_0}E'.$$

From Eq. 2.61 in Steane (2012), a general rule is given that connect accelerations between different frames. This rule is also listed here, see Eq. (20). In our example here $\mathbf{u}\cdot\mathbf{a}_{em} = 0$ so then

$$a'_{em} = \gamma_u^2 a_{em} \tag{34}$$

which gives that

$$\frac{q}{m_0}E' = \frac{q}{m}E = \frac{q}{m_0}E\sqrt{1-\frac{u^2}{c^2}}.$$

We have thus found that

$$E = \frac{E'}{\sqrt{1-\frac{u^2}{c^2}}},$$

where Coulomb's law states that $E' = q/4\pi\varepsilon_0 r'^2$. This agrees with the general law for transformation of fields in electrodynamics (here the y-direction)

$$E_y = \frac{E'_y + vB'_z}{\sqrt{1-\frac{v^2}{c^2}}}.$$

In our example $v = u$, so the charges are stationary in S' and this is the reason why $B'_z = 0$. Furthermore, it is seen that the electric field strength E would become very large as $u \to c$. However, this does not have any dramatic effect on the acceleration because

$$a_{em} = \frac{q}{m}E\left(1-\frac{u^2}{c^2}\right) = \frac{q}{m_0}\sqrt{1-\frac{u^2}{c^2}}\frac{E'}{\sqrt{1-\frac{u^2}{c^2}}}\left(1-\frac{u^2}{c^2}\right)$$
$$= \frac{qE'}{m_0}\left(1-\frac{u^2}{c^2}\right) \to 0.$$

With regard to gravity, we have according to Eq. (16) and the fact that $\mathbf{u}\cdot\mathbf{g} = 0$ the following relation for the acceleration

$$\mathbf{a}_g = \mathbf{g} + \mathbf{u} \times \mathbf{h}.$$

Through an analogous calculation as above, one then finds that

$$\mathbf{a}_g = -g\mathbf{e}_y - \frac{u^2}{c^2}g\mathbf{e}_y = -g\mathbf{e}_y\left(1+\frac{u^2}{c^2}\right).$$

Thus, for the upper charge we have in S and S', respectively, that

$$a_g = -g\left(1+\frac{u^2}{c^2}\right)$$
$$a'_g = -g' \tag{35}$$

If we assume that the point charges initially are at rest in S' and that $|a'_{em}| = |a'_g|$ then they will remain at rest in S'. The balancing condition for this in S' is given by

$$Gm_0^2 = \frac{q^2}{4\pi\varepsilon_0}.$$

The total acceleration is $a' = 0$, and because of Eq. (34), i.e., $a' = \gamma_u^2 a$, it is clear that also $a = 0$ (the *superposition principle* for 3-accelerations is valid in special relativity, see Sect. 2.1). Thus the charges will not move relative to each other in either system. Now given the field $g' = Gm_0/r'^2$ what relation should we have between g' and g? According to Eq. (34), $a = a'\left(1-u^2/c^2\right)$. Also we found above that $a_g = -g\left(1+u^2/c^2\right)$ and $a'_g = -g'$ so

$$-g\left(1+u^2/c^2\right) = -g'\left(1-u^2/c^2\right) \iff$$
$$g = g'\frac{1-u^2/c^2}{1+u^2/c^2} \tag{36}$$

Again, it is interesting to see if something dramatic happens as $u \to c$. In contrast to the very high field strength in the electric field E as we saw above, we note that here $g \to 0$ as $u \to c$. The acceleration behaves according to

$$a_g = -g\left(1 + \frac{u^2}{c^2}\right) \to 0$$

since $g \to 0$. Thus, for an extreme relativistic system (relative to S) both a_{em} and a_g would just approach zero and the charges would continue in a rectilinear motion (as before). This is consistent with the total acceleration $a' = \gamma_u^2 a$. If $a'_{em} + a'_g = 0$ then also $a_{em} + a_g = 0$, i.e.,

$$a_{em} + a_g = \frac{qE'}{m_0}\left(1 - u^2/c^2\right) - g'\left(1 - u^2/c^2\right) = 0$$

which is valid for any speed u (including $u \to c$). In S' we recognize the condition for balance

$$a'_{em} + a'_g = \frac{qE'}{m_0} - g' = 0.$$

Furthermore, there is a simple transformation formula for the three-force between S and S', see Section 4.1.1 in Steane (2012)

$$f_y = f'_y\sqrt{1 - u^2/c^2} \tag{37}$$

This is of course fulfilled for the Lorentz force, so let us instead check that it is fulfilled also for the case of gravity. The forces in the y-direction are given by (see Eqs. (33, 35))

$$f_y = ma_g = -mg\left(1 + \frac{u^2}{c^2}\right)$$
$$f'_y = m_0 a'_g = -m_0 g'.$$

The connection between g and g' was derived in Eq. (36) so

$$f_y = -mg\left(1 + \frac{u^2}{c^2}\right) = -mg'\left(1 - \frac{u^2}{c^2}\right)$$
$$= -\frac{m_0 g'}{\sqrt{1 - u^2/c^2}}\left(1 - \frac{u^2}{c^2}\right) = f'_y\sqrt{1 - u^2/c^2},$$

which indeed is consistent with Eq. (37).

3.2 Light deflection near a massive object

Let us continue with yet another introductory example. In this case, we will show that bending of light in RGF is consistent with experiment and also GR. With the origin at the center of a massive body M which is assumed to be unaffected by a light particle such as a photon, its acceleration is according to Eq. (12)

$$\mathbf{a} = \frac{c^2 + u^2}{mc^2}\mathbf{f} - \frac{2}{mc^2}\left(\mathbf{u} \cdot \mathbf{f}\right)\mathbf{u};\ \mathbf{f} = -\frac{GMm}{r^2}\mathbf{e}_r \tag{38}$$

Table 1 Light deflection angle in milli-arcsec; r is the Schwarzschild radial coordinate

Object	M	Mean r (AU)	RGF	GR ($4GM/rc^2$)
Sun ($M_\odot$)	$2.9591397{\cdot}10^{-4}$	$4.650467{\cdot}10^{-3}$	1751.209	1751.201[a]
Mercury	$4.9125098{\cdot}10^{-11}$	$1.6306382{\cdot}10^{-5}$	0.0829	0.0829
Venus	$7.2434956{\cdot}10^{-10}$	$4.0453784{\cdot}10^{-5}$	0.493	0.493
Earth	$8.9970652{\cdot}10^{-10}$	$4.2587561{\cdot}10^{-5}$	0.581	0.581
Mars	$9.5496058{\cdot}10^{-11}$	$2.2657408{\cdot}10^{-5}$	0.116	0.116
Jupiter	$2.8247779{\cdot}10^{-7}$	$4.6732617{\cdot}10^{-4}$	16.635	16.635
Saturn	$8.4576657{\cdot}10^{-8}$	$3.8925688{\cdot}10^{-4}$	5.980	5.980
Uranus	$1.2918994{\cdot}10^{-8}$	$1.6953450{\cdot}10^{-4}$	2.097	2.097
Neptune	$1.5240481{\cdot}10^{-8}$	$1.6458790{\cdot}10^{-4}$	2.548	2.548

In the case of the Sun, a recent 2018 exp. reported 1751.2 mas (with 3% error) Bruns (2018)
[a] 1751.208, Eq. (34)

Take the dot product $\mathbf{u}\cdot\mathbf{a}$ on the above acceleration and let $u \to c$

$$\mathbf{c}\cdot\mathbf{a} = \frac{2}{m}\left(\mathbf{c}\cdot\mathbf{f}\right) - \frac{2}{m}\left(\mathbf{c}\cdot\mathbf{f}\right) = 0,$$

so either $\mathbf{a} = 0$ or $\mathbf{a}$ and $\mathbf{c}$ are orthogonal. As expected, there cannot be any acceleration along the propagation, i.e., the *speed* of light is kept constant. However, in the orthogonal direction an acceleration is allowed, so the *velocity* $\mathbf{c}$ is allowed to change (i.e., its direction). The deflection could be studied mathematically of course and has been studied many times before in the literature. However, only approximate results can usually be derived, e.g., expansions valid in the weak-field regime, etc. In order to circumvent this, one can make a short cut and instead apply a N-body computer method that immediately can solve for $\mathbf{u}(t)$ and $\mathbf{r}(t)$ once given an expression for the acceleration, i.e., Eq. (38). The results are then exact to numerical accuracy. What we need, to start a N-body calculation, is to place the central object M at the origin and the initial conditions of the photon: $\mathbf{r}(0) = (0, r)$ and $\mathbf{u}(0) = (c, 0)$, where r is the radius of the massive body M. This radius is usually given in the Schwarzschild coordinate r in the GR results. In order to facilitate comparisons, we need to use Eq. (30) in RGF, i.e., apply the appropriate $\mathbf{r}(0) = \left(0, r'\right)$ as initial condition corresponding to $(0, r)$ in the Schwarzschild picture. Then, N-body computations are carried out until $|\mathbf{r}(t)| > L$ (far away) where we record the velocity components. Then, the total deflection angle in the usual meaning becomes

$$\delta\phi_{RGF} = 2\arctan\left(\frac{u_y}{u_x}\right).$$

Some relevant results in the solar system are given in Table 1. The physical data in Table 1 were obtained from Physical planet data (2021). Also see the footnote[2] for the applied units. It is seen that there are no essential differences between RGF and GR in the case of weak fields, which is the case in the solar system.

It is therefore interesting to also study the behavior of light when the fields are much stronger than in the solar system. One then needs a much more accurate formula for the GR

[2] Units used: c=173.144632684657 AU/day, nominal Solar radius=695,700 km (IAU), 1 AU =149597870.700 km (IAU), G=1.

Table 2 Light deflection near a object of mass $M_{\odot}$ in arcsec; r is the Schwarzschild radial coordinate

r/r_s	RGF (exact)	GR (exact)	GR ($4GM_{\odot}/rc^2$)
300,000	1.375104	1.375104	1.375099
30,000	13.75148	13.75143	13.75099
3000	137.5590	137.5545	137.5099
300	1380.026	1379.573	1375.099
30	14,261.58	14,213.80	13,750.99
3	219,107.0	209,333.1	137,509.9

Column 1 shows the closest distance to $M_{\odot}$, column 2 gives exact results from RGF N-body computations, column 3 shows exact GR Schwarzschild results and column 4 lists approximate GR results

deflection than was applied in Table 1. The references (Gerard and Pireaux 1999; Misner et al. 1973) provide an exact Schwarzschild treatment with regard to the light bending near a massive object. The computation recipe goes as follows

$$\begin{aligned} r_s &= \frac{2GM}{c^2} \\ q &= \sqrt{\left(1-\frac{r_s}{r}\right)\left(1+\frac{3r_s}{r}\right)}r;\ k = \sqrt{\frac{q-r+3r_s}{2q}} \\ \sigma_0 &= \arcsin\left\{\sqrt{\frac{q-r+r_s}{q-r+3r_s}}\right\} \\ F(\sigma,k) &= \int_0^{\sigma}\frac{dy}{\sqrt{1-k^2\sin^2 y}} \\ \delta\phi_{GR} &= 4\sqrt{\frac{r}{q}}\left\{F\left(\frac{\pi}{2},k\right)-F(\sigma_0,k)\right\}-\pi \end{aligned} \tag{39}$$

where r is the closest distance of approach to the object M. The result of this is shown in Table 2 where several examples are provided for RGF and GR in the strong field regime. It is seen that the GR approximation in column 4 soon breaks down. RGF and the exact GR treatment are in excellent agreement. Tiny differences may be observed in the most extreme situations where the distances approach the Schwarzschild radius r_s. It is interesting to note that gravitational bending of light in Table 2 is slightly stronger in RGF compared with GR as given by Eq. (39). However, this could simply be related to remaining coordinate difficulties as described in Sect. 2.9.

3.3 Light in a region of a uniform field

In this section, we will make further comparisons between RGF and GR. We emphasize the difficulty that it is not feasible to make exact comparisons between a flat 4-space theory and results derived from coordinates in curved 4-space. In general relativity, the definition of flatness is that the Riemann tensor $R_{ijkl} = 0$. From this, it is understood that an arbitrary coordinate transformation leaves the Riemann tensor invariant in flat space, so it is not possible to transform from flat space to a curved space (where $R_{ijkl} \neq 0$). It is well known that in a curved space one can choose coordinates such that near a local point x, $g_{\mu\nu}(x) = \eta_{\mu\nu}$. However, in the following example we are interested in the behavior in a wider range by applying both methods. Ultimately, the question is what exactly is measured experimentally, and that really depends on whether the experimenter's measurements really take place in a flat space or in a curved space.

A well-known first-order GR solution for a horizontal light ray in a weak uniform gravitational field is given by Ferraro (2003),

$$z\,(x) = -\frac{g}{c^2}x^2. \tag{40}$$

We are interested in generalizing this prediction since it facilitates comparisons between present work, GR and experiment into a regime of stronger fields and large x, where Eq. (40) no longer is accurate. Let us therefore derive a more accurate general relativistic prediction for the problem of an uniform gravitational field. We will use the same notation $(t,\ x,\ y,\ z)$ as earlier, see Eq. (10), so in the case of significant curvature, comparisons between the RGF method and GR may not be entirely meaningful. However, the comparison that really should be made is between a theoretical model and experiment. Thus, we show both results expressed in the same coordinate notation.

We shall now proceed from the following static metric ($\Phi \ll c^2$) for the problem at hand

$$ds^2 = \left(1 + 2\Phi\,(\mathbf{r})\,c^{-2}\right)c^2dt^2 - \frac{1}{1 + 2\Phi\,(\mathbf{r})\,c^{-2}}\left(dx^2 + dy^2 + dz^2\right) \tag{41}$$

The behavior of a light ray is described by a null geodesic which can be obtained from:

$$g^{\mu\nu}p_\mu p_\nu = 0, \qquad p_\mu = g_{\mu\delta}\frac{dx^\delta}{d\lambda}, \tag{42}$$

where λ is the length of the trajectory (affine parameter) and p is the momentum. Initially, the ray is at $x = y = z = 0$ and $dz/d\lambda = 0$. For an uniform gravitational potential given by $\Phi = gz$ we have that $p_y^2 = 0$ and $p_0^2 = p_x^2$ for all z. From Eq. (42), one then finds that

$$\left(1 + 2\Phi c^{-2}\right)^{-1}p_x^2 - \left(1 + 2\Phi c^{-2}\right)p_x^2 - \left(1 + 2\Phi c^{-2}\right)^{-1}\left(\frac{dz}{d\lambda}\right)^2 = 0$$

Further, Eq. (42) gives that

$$\frac{dx}{d\lambda} = g^{xx}p_x = -\left(1 + 2\Phi c^{-2}\right)p_x \Rightarrow \left(\frac{dz}{d\lambda}\right)^2 = \left(\frac{dz}{dx}\right)^2\left(1 + 2\Phi c^{-2}\right)^2 p_x^2$$

so

$$\left(1 + 2\Phi c^{-2}\right)^{-1} - \left(1 + 2\Phi c^{-2}\right) - \left(1 + 2\Phi c^{-2}\right)\left(\frac{dz}{dx}\right)^2 = 0$$

which can be written

$$\left(\frac{dz}{dx}\right)^2 = (1 - \alpha|z|)^{-2} - 1 \tag{43}$$

where $\alpha = 2g/c^2$. By dropping $\alpha^2|z|^2$ terms, one finds

$$\left(\frac{dz}{dx}\right)^2 \approx \frac{2\alpha|z|}{1 - 2\alpha|z|} \tag{44}$$

Similarly, we find from Eq. (10) and after expansion that

$$\left(\frac{dz}{dx}\right)^2 = \left(\frac{u_z}{u_x}\right)^2 = \frac{1 - e^{-2\alpha|z|}}{e^{-2\alpha|z|}} \approx \frac{2\alpha|z|}{1 - 2\alpha|z|} \tag{45}$$

Thus, under the assumption of a nearly flat space and a small $\alpha|z|$ we have that both methods must yield nearly the same result.

3.4 Gravitational redshift

A photon is emitted in the radial direction from within a spherically symmetric gravitational field. How will this affect its energy? According to the Planck energy relation for a photon, its energy is given by $E = h\nu$. The change in *the work* dW can by definition then be written as

$$dW = Pdt = \frac{dE}{dt}dt = h\frac{d\nu}{dt}dt = hd\nu,$$

where P is the *power*. We have also from the definition of *work* that

$$dW = (\mathbf{f} - \mathbf{q}) \cdot \mathbf{dr} = \mathbf{f} \cdot \mathbf{dr} = -f dr = -\frac{GMm}{r^2}dr = -\frac{GMmu}{ur^2}dr$$

since $-\mathbf{q} \cdot \mathbf{dr} = -\mathbf{q} \cdot \mathbf{u}dt = 0$, see Eq. (14). In RGF, m is the relativistic mass and the linear momentum is defined by $p = mu$ (in 1-D). For a photon, the linear momentum is replaced by $p = h/\lambda = h\nu/c$ and its speed is constant $u = c$. We then have that

$$dW = -\frac{GMp}{cr^2}dr = -\frac{GMh\nu}{c^2r^2}dr$$

What we have found is thus

$$hd\nu = -\frac{GMh\nu}{c^2r^2}dr \tag{46}$$

Given the usage of a single inertial frame, the frequency ν of the photon is interpreted as continuously changing as it moves through the gravitational field. The cumulative effect is accounted for by integrating

$$\int_{\nu_1}^{\nu_2} \frac{1}{\nu}d\nu = -\frac{GM}{c^2}\int_{r_1}^{r_2} \frac{1}{r^2}dr$$

so

$$\ln\frac{\nu_2}{\nu_1} = -\frac{GM}{c^2}\left(-\frac{1}{r_2} + \frac{1}{r_1}\right) \tag{47}$$

By letting $r_2 \to \infty$ and by renaming the variables: $\nu_1 \to \nu_e$, $\nu_2 \to \nu_\infty$ and $r_1 \to r_e'$ we find that the frequency far away (ν_∞) declines from the emitting source (ν_e) located at r_e' as

$$\nu_\infty = \nu_e\, e^{-\frac{GM}{c^2r_e'}} \tag{48}$$

An analogous calculation for the wavelength leads to

$$\lambda_\infty = \lambda_e\, e^{\frac{GM}{c^2r_e'}} \tag{49}$$

so clearly $c_\infty = \nu_\infty\lambda_\infty = \nu_e\lambda_e = c_e$, i.e., the speed of light is the same everywhere in RGF (as it must be within special relativity). Further, Eq. (48) can be compared with the GR (Schwarzschild metric) prediction (Gravitational redshift 2021)

$$\nu_\infty = \nu_e\sqrt{1 - \frac{2GM}{c^2r_e}} \tag{50}$$

Although these expressions look different, they are the same (Sect. 2.9). For example, at the surface of the Sun $\nu_\infty(GR)/\nu_\infty(RGF) = 1$. Further, the white dwarf Sirius B is expected to have about the same mass as the Sun ($M \approx M_\odot$) but a radius of only $0.0084r_\odot$, so $\nu_\infty(GR)/\nu_\infty(RGF) = 0.999999999998$. Even then, the difference between the two exact expressions would most probably be undetectable. The closest approach of the star S2 to the black hole SgrA* ($M = 4.3 \cdot 10^6 M_\odot$ (Gillessen 2009)) at about $r = 120$ AU was studied in 2018 w.r.t. redshift (Do 2019). Also in this case the difference is insignificant: $\nu_\infty(GR)/\nu_\infty(RGF) = 0.999999999996$. It can be concluded that the RGF prediction is so close to GR, even in quite extreme situations, so it is probably not even meaningful to try to separate the methods observationally.

3.5 Accelerated frame and the equivalence principle

In RGF, it is interesting to check what the results of a photon experiment would be in an accelerated frame S. The behavior of such a thought experiment is in the literature sometimes seen to be derived from the perspective of an external inertial frame. However, if the light experiment is conducted entirely within an accelerated frame, it is in this lab frame that all measurements are taken and should be related to. Formally, in special relativity, an accelerated frame can be dealt with by using a Fermi–Walker tetrad that is extended to a local frame S of the accelerated observer, see p. 172 (Misner et al. 1973). The metric for a uniformly properly accelerated frame S (Rindler metric) is then given by

$$ds^2 = -\left(1 + \frac{a'}{c^2}z\right)^2 c^2dt^2 + dz^2,$$

where a' is the proper acceleration relative to a rest frame S' coinciding with S momentarily and z is the position of the emitted photon within the accelerated frame S. Note that a' is a constant in the Rindler metric. In the accelerated frame, the observer is assumed to be at $z = 0$ so the emitter is at $z = -h$. Next, we assume that the proper acceleration at $z = 0$ is given by the constant

$$a'_O = \frac{GM}{r_e^2}.$$

This is the correct proper acceleration at the observer position O. From the Rindler metric, one can see that

$$\frac{d\tau}{dt} = 1 + \frac{a'_O}{c^2}z.$$

This results in the well-known fact that different points have different proper accelerations

$$a'_e = \frac{a'_O}{\left(1 + \frac{a'_O}{c^2}z\right)} = \frac{a'_O}{\left(1 - \frac{a'_O}{c^2}h\right)}.$$

A simple relation for the Doppler shift in an uniformly accelerated frame was derived in Cochran (1989)

$$\frac{\nu_O}{\nu_e} = \frac{a'_O}{a'_e}$$

One thus finds that

$$\nu_O = \nu_e \left(1 - \frac{a'_O}{c^2} h\right) = \nu_e \left(1 - \frac{GMh}{c^2 r_e^2}\right) \tag{51}$$

which also is in agreement with (Formiga and Romero 2007). A simple re-derivation for the case of an uniform field $\mathbf{g} = -GM/r_e^2 \mathbf{e}_z$ in Sect. 3.4 yields similarly

$$\nu_O = \nu_e e^{-\frac{GMh}{c^2 r_e^2}} \sim \nu_e \left(1 - \frac{GMh}{c^2 r_e^2}\right)$$

which is close to Eq. (51) if h is small. However, due to the exponential dependence the results are not really similar for a somewhat larger h. The Rindler metric itself is not without limitations. As can be seen at p. 172 (Misner et al. 1973), it is not possible to generalize the results to an extended region since the validity of the accelerated coordinates eventually breaks down for $|h| \geq c^2/a'_O$. For such a h Eq. (51) predicts that $\nu_O = 0$, whereas RGF predicts that $\nu_O = \nu_e e^{-1}$. It is clear that even for a substantially smaller h, significant differences would still occur. Because of this, there is no general equivalence between an uniform field in RGF and the uniformly accelerated frame within special relativity. In fact, it is known that the equivalence principle is simply inconsistent with special relativity (Schild 1960).

Perhaps, a physically more attractive derivation of Eq. (51) goes along the following lines. By considering the accelerated metric above, one can derive the coordinate speed of light within the accelerated frame S. For a photon $ds^2 = 0$, so

$$c(z) = c\left(1 + \frac{a'_O}{c^2} z\right).$$

The law in special relativity that the speed of light c is constant is *not* a requirement relative to an *accelerated* frame. At the emitter we thus have that

$$c_e = c(-h) = c\left(1 - \frac{a'_O}{c^2} h\right)$$

relative to the observer at $z = 0$. The propagation of light becomes retarded. Viewed from an external inertial frame, however, this would instead look like the photon speed is c but the propagated distance is longer since the observer is accelerating upwards. The observer should also see a redshift due to the Doppler effect. Locally near the emitter ($z = -h$) in the accelerated frame an experimenter would say that the speed of light is c, while the observer at $z = 0$ disagrees, and instead claims that the speed is c_e. Time is slowed down and so is the antenna/emitting process according to the observer. The relation

$$\frac{c_e}{c} = \frac{\nu_O}{\nu_e}$$

then leads to exactly the same frequency shift as derived in Eq. (51), namely

$$\nu_O = \nu_e \left(1 - \frac{a'_O}{c^2} h\right).$$

It is interesting that redshifts are predicted in both an accelerated frame and in a gravitational field by RGF. However, the physical explanations in the two cases are different (energy loss versus retarded speed/time dilation). Not only are the physical explanations different, but as we showed above, the actual redshifts differ substantially unless h is small. Thus, within the RGF framework, the two situations are not equivalent.

3.6 Escape velocity from a massive object

Here, we report the escape velocity of a test particle m in a gravitational field. In GR, using the Schwarzschild metric, one can derive a radial escape velocity from a massive object (Vasiliev and Fedorov 2015)

$$v_e = \sqrt{\frac{2GM}{r}} \tag{52}$$

This is the escape velocity relative to a local observer from where the test particle was sent out. However, from another observer's perspective, far away, measuring the distance and speed from this point of view, one would find a different result. According to the Schwarzschild solution, the energy of the particle can be written as (Misner et al. 1973)

$$E = m_0 c^2 \left(1 - \frac{v^2}{c^2\left(1 - 2GM/rc^2\right)^k}\right)^{-\frac{1}{2}} \left(1 - \frac{2GM}{rc^2}\right)^{\frac{1}{2}}$$

where $k = 2$ is for a radial escape, and $k = 1$ for a tangential escape. The condition for the escape velocity can be stated as that the energy must be the same as for an object at rest at infinity, i.e., $E = m_0 c^2$. In the radial case, one then finds

$$v_e = \sqrt{\frac{2GM}{r} - \frac{2}{c^2}\left(\frac{2GM}{r}\right)^2 + \frac{1}{c^4}\left(\frac{2GM}{r}\right)^3}$$

and the tangential case leads to

$$v_e = \sqrt{\frac{2GM}{r} - \frac{1}{c^2}\left(\frac{2GM}{r}\right)^2}$$

Near the Schwarzschild radius, $r = r_s = 2GM/c^2$, the speed is given by $v_e = 0$ (in both cases) from the distant observer's point of view. This peculiar behavior is due to the enormous time dilation Δt_∞ occurring near r_s, see Eq. (68). From a local frame, however, $v_e = c$ (see Eq. (52) and Fig. 3).

In RGF, we consider a central mass M at rest and a test particle $m << M$. Then, the change in *work* dW is by definition given by

$$dW = Pdt = \frac{dE}{dt}dt = c^2\frac{dm}{dt}dt = c^2 dm,$$

where P is the *power* and $E = mc^2$. We have also from the definition of *work* that

$$dW = (\mathbf{f} - \mathbf{q}) \cdot \mathbf{dr} = \mathbf{f} \cdot \mathbf{dr} = -f dr = -\frac{GMm}{r^2}dr$$

since $-\mathbf{q} \cdot \mathbf{dr} = -\mathbf{q} \cdot \mathbf{u} dt = 0$, see Eq. (14). We thus have that

$$c^2 dm = -\frac{GMm}{r^2}dr$$

Integration leads to

$$c^2 \ln\frac{m_2}{m_1} = -GM\left(-\frac{1}{r_2} + \frac{1}{r_1}\right) \tag{53}$$

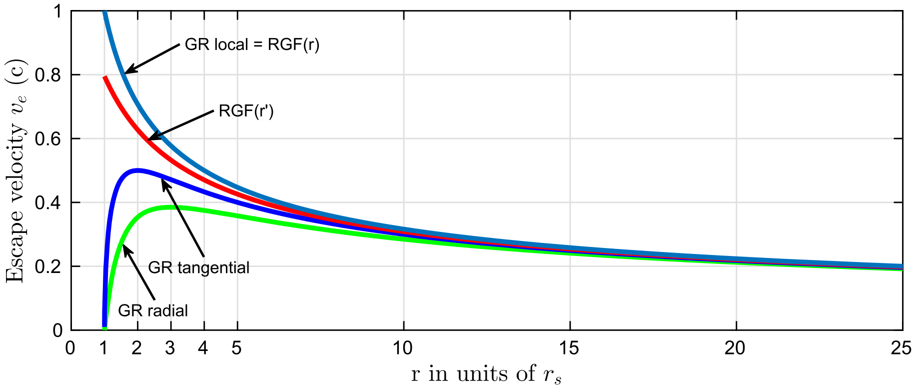

Fig. 3 Comparisons between RGF and GR. An object for which $r_s = 2GM/c^2 = 1$ is considered. The GR solutions are given in the Schwarzschild radial coordinate r. The RGF(r') solution is displayed as a function of the Euclidean distance r'. Expressed in the Schwarzschild coordinate r, the RGF(r) solution coincides with the "GR local" solution in the figure (except at $r \sim r_s$ but difficult to detect by eye)

As $r_2 \to \infty$ we have that $m_2 \to m_0$, i.e., the rest mass. By renaming $r_1 \to r'$, we get that the radial or tangential escape velocity becomes

$$v_e = c\sqrt{1 - e^{-2GM/c^2 r'}}. \tag{54}$$

Interestingly, when the RGF escape velocity is instead expressed in the Schwarzschild radial coordinate r we find that

$$v_e = c\sqrt{1 - e^{-2GM/c^2 r'}} \approx \sqrt{\frac{2GM}{r}}$$

which is essentially valid over the whole range $r \in [r_s, \infty]$, see Sect. 2.9. The result is indeed consistent with Eq. (32). The RGF escape velocity expressed in the Schwarzschild coordinate r thus to a large extent yields the same result as GR for a local observer, cf. Equation (52).

Let us now return to the simpler Euclidean coordinate r'. In the case that $r' = r_s$, a radial escape is indeed possible at $v_e = 0.8c$ according to the RGF expression in Eq. (54), see Fig. 3. Despite this, the Schwarzschild radius is still quite special in RGF. For example, a tangential escape is not possible, even for $v_e \to c$. This is because only light can briefly display a circular orbit around M at $r' = r_s$, see the next section. As will be demonstrated there, a test particle will experience an inward spiral toward the singularity (Fig. 4). The particle speed near this singularity will approach $u \to c$ so in principle it could then escape according to Eq.(54), were it not for the very presence of the singularity. As will be shown in Sect. 3.7 the test particle will aim for a collision with the singularity.

3.7 The RGF photon ring and last orbit of matter

Section 3.6 shows that a test particle can, although with considerable difficulty, still radially escape at $r' = r_s$ (Fig. 3). Apparently, $r' = r_s$ is not a special distance in RGF (an Euclidean inertial frame) so redshifted light such as radio emissions would still be expected from this region. However, we shall see below that $r' = r_s$ is still a quite special distance related to the RGF photon ring. With regard to the Schwarzschild radial coordinate at $r = r_s$, RGF still predicts basically the same escape velocity ($v_e = 0.99c$) as GR in the local frame ($v_e = c$).

We shall in the following use Euclidean coordinates (r') in the RGF derivations and then translate derived results to the Schwarzschild picture (r) to facilitate comparisons with GR Schwarzschild results. From the context, it should be clear what type of coordinates is applied.

Consider a test particle in circular orbit around a heavy object M at rest. The motion can, for example, be solved by applying Eq. (12), i.e.,

$$\mathbf{f} = \frac{2}{c^2 + u^2} (\mathbf{u} \cdot \mathbf{f}) \mathbf{u} + \frac{c^2}{c^2 + u^2} m\mathbf{a}.$$

For the test object in circular motion, we have that $\mathbf{a} = -\left(u^2/r'\right) \mathbf{e}_{r'}$; $\mathbf{f} = -\mathbf{e}_{r'} GMm/r'^2$ and $\mathbf{u} \cdot \mathbf{f} = 0$ which gives that

$$r' = GM \frac{c^2 + u^2}{c^2 u^2}. \tag{55}$$

We thus get for a photon, $u = c$, that there is one unique circular orbit with $r' = 2GM/c^2$, i.e., the Schwarzschild radius r_s. Equation 31,

$$r = r' \left(1 + \frac{r_s}{4r'}\right)^2,$$

provides a mean to translate a RGF (r') result into the GR Schwarzschild coordinate (r). We thus find that the RGF circular radius $r' = r_s$ above translates into $r = 25r_s/16 \approx 1.56r_s$ which is very close to the actual GR photon ring radius given by $r = 1.5r_s$ (Photon sphere 2021).

As in GR, the RGF photon ring is unstable. A small perturbation in the circular orbit leads to either a inward spiral toward the singularity or an outward spiral escaping the gravitational well altogether. We shall now show that the circular solution will be destroyed by any small perturbation. By rearranging Eq. (12), given above, for a photon in a gravitational field we get

$$\mathbf{a} = -\frac{2GM}{r'^2} \mathbf{e}_{r'} + \frac{2GM}{c^2 r'^2} (\mathbf{c} \cdot \mathbf{e}_{r'}) \mathbf{c}$$

Let us check the acceleration in the direction of $\mathbf{e}_{r'}$

$$a_{r'} \equiv \mathbf{a} \cdot \mathbf{e}_{r'} = -\frac{2GM}{r'^2} + \frac{2GM}{c^2 r'^2} c_{r'}^2$$

In polar coordinates $a_{r'} = \ddot{r}' - c_\varphi^2/r' = \ddot{r}' - \left(c^2 - c_{r'}^2\right)/r'$ so

$$\ddot{r}' = -\frac{2GM}{r'^2} + \frac{2GM}{c^2 r'^2} c_{r'}^2 + \left(c^2 - c_{r'}^2\right)/r'$$

The unique circular orbit of a photon can only be maintained if $\ddot{r}' = 0$, $r' = r_s$ and consequently $c_{r'} = 0$. Any small perturbation could potentially destroy the circular motion. By Taylor expansion, one can study $\ddot{r}'$ near $r' = r_s$ and $c_{r'} = 0$. Alternatively, one may study a small perturbation $r' = c_{r'} \Delta t + r_s$, where Δt is a small timestep and vary $c_{r'}$ within $[-a, a]$, where a is a small constant. A plot in MATLAB is sufficient. One then finds the following linear relationship:

$$\ddot{r}' = \lambda \left(r' - r_s\right) \tag{56}$$

where $\lambda = c^2/r_s^2$. The solution to this type of hyperbolic differential equation can be written

$$r'(t) = r_s + \frac{A}{2} \left(e^{\sqrt{\lambda} t} - e^{-\sqrt{\lambda} t}\right)$$

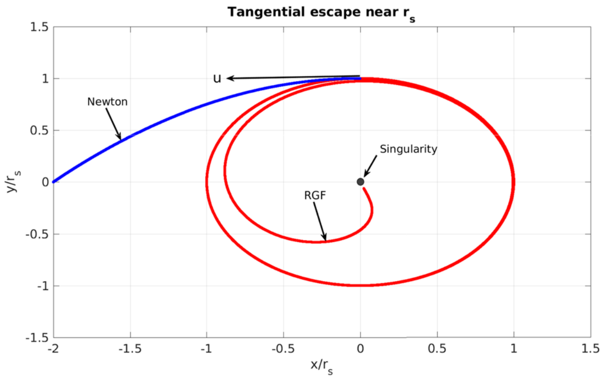

Fig. 4 N-body computation in Euclidean coordinates. Attempted tangential escape for a test particle with initial velocity $u_x = -0.9999998\,c$ at the Schwarzschild radius $y/r_s = 1$. Comparisons between RGF and classical mechanics (Newton). A central object for which $r_s = 2GM/c^2 = 1$ is considered

where $r'(0) = r_s$ and $\dot{r}'(0) = A\sqrt{\lambda}$. It is seen that depending on the sign of A (i.e., sign of $c_{r'}$) an inward or outward spiraling solution will initially occur when the circular photon orbit is perturbed. It can therefore be concluded that the photon ring in RGF is indeed unstable.

Analogous spiraling behavior can also occur for test particles. In Fig. 4, we illustrate the behavior of a test particle in a configuration very close to the RGF photon ring solution. The orbit becomes unstable and spirals inward toward the singularity. All solutions with $u < c$ ends up facing a close encounter with the singularity. A relation between the speeds and distances is easily derived from the work integral (Eq. 53) and is given by

$$\frac{c^2 - u_s^2}{c^2 - u_2^2} = e^{-(1-r_s/r_2')}.$$

Thus, as one approaches the singularity ($r_2' \to 0$), the particle speed $u_2 \to c$. According to the escape velocity given by Eq. (54), the test particle could then in principle be shot out from the singularity to infinity (if the initial speed fulfills $u_s \geq c\sqrt{1-e^{-1}}$). However, by rearranging Eq. (12) as $u_2 \to c$ we find that

$$\mathbf{a}_2 = \frac{2}{m}\mathbf{f} - \frac{2}{mc^2}(\mathbf{c}\cdot\mathbf{f})\,\mathbf{c} \tag{57}$$

By taking the dot product $\mathbf{c}\cdot\mathbf{a}_2$ on the above acceleration, we get

$$\mathbf{c}\cdot\mathbf{a}_2 = \frac{2}{m}(\mathbf{c}\cdot\mathbf{f}) - \frac{2}{m}(\mathbf{c}\cdot\mathbf{f}) = 0,$$

so either $\mathbf{a}_2 = 0$ or $\mathbf{a}_2$ and $\mathbf{c}$ are orthogonal. However, $\mathbf{a}_2$ has a component in the $\mathbf{c}$-direction (Eq. (57)), so $\mathbf{a}_2$ and $\mathbf{c}$ cannot be orthogonal unless $\mathbf{c}\cdot\mathbf{f} = 0$. Although this is the case for a photon at $r = r_s$ in the above example, it cannot be true for the test particle spiraling inwards. Also the test particle approaches the speed of light in the vicinity of the singularity and then its orbit is far from circular so the only option left is that $\mathbf{a}_2 = \mathbf{0}$ near the encounter and by inspecting Eq. (57) we then have

$$\frac{2}{m}\mathbf{f} = \frac{2}{mc^2}(\mathbf{c}\cdot\mathbf{f})\,\mathbf{c},$$

so $\mathbf{c} \parallel \mathbf{f}$. Thus the test particle will collide straight into the singularity. A photon sent out horizontally at $r' < r_s$ would also show a similar spiral inwards and with an analogous fate. On the other hand, the Newtonian solution (Fig. 4) would escape for $u_s = c\,(v_e = \sqrt{2GM/r_s})$

but as in this case of matter, $u_s < c$, the classical test particle would eventually return and forever remain in a periodic orbit.

The work integral shows that the following property α is conserved (Eq. (53))

$$\alpha = -\frac{GM}{r'} + c^2 \ln m, \; where$$
$$\ln m = \ln m_0 - \frac{1}{2} \ln \left(1 - \frac{u^2}{c^2}\right)$$

Since $\ln m_0$ is a constant, we can just as well drop it and study the constant of motion β instead

$$\beta = -\frac{GM}{r'} - \frac{c^2}{2} \ln \left(1 - \frac{u^2}{c^2}\right)$$

In the application of an effective potential of an orbit, one can separate the kinetic energy contributions in polar coordinates, i.e., $T = T_{r'} + T_\varphi$. The mechanical energy is then written as $E = T_{r'} + T_\varphi + V\left(r'\right) = T_{r'} + V_{eff}\left(r'\right)$. It thus becomes an exercise in expressing $T_\varphi = T_\varphi\left(r'\right)$. Alternatively, one can insert the condition for pure circular solutions into E, because then $V_{eff}\left(r'\right) = E_{circular}$ (since $T_{r'} = 0$). Then one can deal with the general problem $E = T_{r'} + V_{eff}\left(r'\right)$ (with this effective potential). According to Eq. (55), the speed u can be rewritten in terms of r' and r_s for circular solutions. One finds that

$$1 - \frac{u^2}{c^2} = \frac{r' - r_s}{r' - \frac{r_s}{2}}. \tag{58}$$

so then we identify the following *effective potential*

$$\beta_{eff} = -\frac{GM}{r'} - \frac{c^2}{2} \left\{\ln\left(r' - r_s\right) - \ln\left(r' - \frac{r_s}{2}\right)\right\}$$

By analyzing $d\beta_{eff}/dr' = 0$, one then finds that

$$r' = r_s + \frac{1 \pm \sqrt{5}}{2} r_s \tag{59}$$

Further analysis of the second derivative leads to the conclusion that only the positive sign in Eq. (59) corresponds to a minimum and the negative sign is an unstable orbit. Thus the innermost stable circular orbit of matter lies at the *golden ratio* away from the Schwarzschild radius, i.e., $r' = r_s + \left(1 + \sqrt{5}\right)/2\, r_s \approx 2.618 r_s$. By translating into the Schwarzschild picture using Eq. (31), one gets $r = 3.14 r_s$ which is close to the GR result $r = 3r_s$ (Misner et al. 1973). An image of the super massive black hole ($M = 6.5 \cdot 10^9 M_\odot$) in the elliptical galaxy M87 was recorded by the Event Horizon Telescope in 2019 (Akiyama et al. 2019). This image displays a circular core shadow with a radius $\sim 2.6 r_s$. Although it is nontrivial to compute what would be observed at Earth's position (relativistic ray-tracing etc. (Akiyama et al. 2019)), it is an interesting prediction by RGF that the innermost stable circular orbit of matter coincides with this observed shadow. A detailed or full image analysis is not warranted here in the context of presenting the new RGF model. According to Eq. (58) the speed of an unperturbed orbit in this region is given by $u \approx 0.486c$. Recently, the line-of-sight speed of highly ionized matter close to a black hole was observed at $v \sim 0.3c$ (Pounds 2018). These inflow velocity measurements are lower limits, due to the assumption that the inflow is aligned with the line-of-sight. The orbital configuration of this matter is also very difficult to determine. For in-falling matter from far away, this speed would correspond to a region at

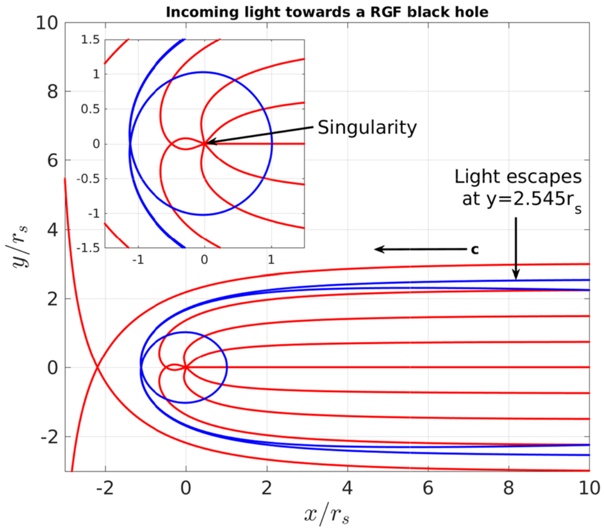

Fig. 5 Euclidean coordinates. Example of photons sent from $x = 10r_s$ with $\mathbf{c} = -c\mathbf{e}_x$ for various y-values. The photons are bent by the strong gravitational field and in many cases collide with the singularity. For $y \geq 2.545r_s$, the light is enabled to escape. The circular orbit seen is for $y = \pm 2.545r_s$. The photon orbits the singularity approximately at $r = r_s$ and then escapes. In this particular case, both photons are reflected back with nearly exchanged orbits. A central object for which $r_s = 2GM/c^2 = 1$ has been considered

$r' \approx 10r_s$, see Fig. 3. In a circular orbit, one would instead expect $r' \approx 6r_s$ according to Eq. (58). The actual orbit configuration could be more complex as indicated in Pounds (2018) where several proposals are made that could be consistent with the measurements.

In Fig. 5, we present a numerical experiment of light emitted toward the singularity. The photons are bent by the strong gravitational field and collide with the singularity unless $y' \geq 2.545r_s$. In this particular example, light is able to be reflected for $y' = \pm 2.545r_s$. More details are given in the figure text. In order to translate this $y' = 2.545r_s$ into the GR Schwarzschild coordinate, one can first note that $r' = \sqrt{10^2 + 2.545^2} \approx 10.318r_s$ and $x' = 10r_s$ (see Fig. 5). These are then translated by using Eq. (31) into: $r = 10.824r_s$ and $x = 10.506r_s$ which then yields $y \approx \sqrt{r^2 - x^2} \approx 2.6r_s$. This result agrees well with GR since the corresponding result in GR is the *critical impact parameter* which is given by $y = \sqrt{27}r_s/2 \approx 2.6r_s$ (Luminet 1979).

It is quite remarkable that RGF is able to provide quantitative agreement with GR even in the strong fields near a black hole. There are really only tiny differences between RGF and GR, and to some extent probably just related to remaining coordinate difficulties in the comparisons and/or that the derived results in GR usually are approximate due to the intrinsic nonlinearity within GR (expansions to various orders or other approximations).

3.8 Relativistic precession rates of the planets

According to Edvardsson et al. (2002), the orbital precession rate $\dot{\phi}$ can be calculated from the longitude of the ascending node Ω, the argument of perihelion ω and the orbital inclination i

$$\dot{\phi} = \frac{d\omega}{dt} + \cos(i)\,\frac{d\Omega}{dt}$$

Rates of orbital parameters in arcseconds per century was extracted from *The Astronomical Almanac* (Explanatory (1992)). However, they list $\varpi = \omega + \Omega$ and $d\varpi/dt$ which modifies the above formula into

$$\dot{\phi} = \frac{d\overline{\omega}}{dt} + \frac{d\Omega}{dt}\left(\cos\left(i\right) - 1\right),$$

which is the expression that is applied to get the observational apsidal precession rate of a planet, see the last column of Table 3. The pure GR effect (i.e., of a two-body problem) was obtained from Will (1993)

$$\dot{\phi} = \frac{6\pi G\left(M_{\odot} + m\right)}{a\left(1 - e^2\right)c^2 P_b}$$
$$P_b = 2\pi\sqrt{\frac{a^3}{G\left(M_{\odot} + m\right)}} \tag{60}$$

where the masses $M_{\odot}$ and m are listed in Table 1, a is the semimajor axis, e is the eccentricity and P_b is the sidereal period of the orbit. The above predictions are expected to be accurate even for the case $m \sim M_{\odot}$. The orbital parameters were extracted from JPL Horizon ephemerides at MJD 51600.5 (Horizon ephemerides 2021). The GR result is listed in the third column of Table 3. The corresponding RGF results (second column) were derived by performing a quick N-body computation and compute the Runge–Lenz vector $\mathbf{A}$ as in ref. (Edvardsson et al. 2002; Goldstein 1980),

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{GM_{\odot}m\mu}{r}\mathbf{r} \tag{61}$$

where μ is the reduced mass, and $\mathbf{p} = \mu\mathbf{v}$ and $\mathbf{L} = \mu\mathbf{r} \times \mathbf{v}$ are the linear and angular momentum, i.e., $\mathbf{A}$ is computed relative to the position of $M_{\odot}$. This definition of the Runge–Lenz vector is valid in general, i.e., it also applies to the case $m \sim M_{\odot}$. In the fourth column, we also list the N-body results due to post-Newtonian expansion at the level 1PN (i.e., Equation (62)). To ensure good estimates for RGF and 1PN in columns 2 and 4, an integer number of orbital periods were studied for each planet.

Orbital elements and secular elements can easily be computed either from the barycenter point or relative to the more massive point mass M (Danby 1964). To determine the orbital parameter evolution, positions and velocities relative to any of those points can be computed in an N-body run either with a central mass $M^3/\left(m + M\right)^2$ or $m + M$, respectively. Here, we have applied the $(m + M)$-convention. For orbital precession rates $d\omega/dt$, it does not matter which orbital convention is applied.

The N-body method was then applied for the whole solar system (i.e., the Sun and all planets) and computed from MJD 51600.5 and 100 yr into the future. N-body contributions to the precession then gets automatically accounted for. Results are listed for both a pure classical and RGF computation in columns 5–6. The Runge–Lenz precession angle ϕ versus t naturally shows features from the N-body effects so in order to identify the trend a least squares fit was applied to determine $\dot{\phi}$. Equivalently, one can study the argument of perihelion ω and determine $d\omega/dt$. A comparison between columns 5–6 and observations in column 7 shows that the best agreement is seen between 6 and 7. The data in the *Astronomical Almanac* (1992) are somewhat dated and may therefore not be entirely accurate. We are therefore also listing a recent value derived from MESSENGER ranging data in column 7 (Park 2017).

General relativistic effects on the Mercury orbital elements (a, e, ω) were calculated in ref. (Balogh and Giampieri 2002). Alternatively, one can use the Post Newtonian expansion technique and run an N-body computation. Post Newtonian expansion is a well known method that provides an approximation to general relativity (PPN expansion 2021; Quinn et al. 1991; Will 1993). The expected applicability is assumed to be for systems in which motions are slow

Table 3 Orbital precession rates in arcsec/cyr

Planet	RGF	GR	1PN	Newton	RGF	Almanac
Mercury	42.983	42.9825	42.983	532.3	575.3	576.9, 575.31 Park (2017)
Venus	8.623	8.625	8.625	–	–	–
Earth-Moon	3.839	3.839	3.839	1172.6	1176.4	1198
Mars	1.351	1.351	1.351	1600.3	1601.7	1561
Jupiter	0.0633	0.0622	0.0624	–	–	–
Saturn	0.0136	0.0135	0.0135	–	–	–
Uranus	0.0024	0.0024	0.0023	–	–	–
Neptune	0.0008	0.0008	0.0008	–	–	–

The dashes (-) means no precession trend (oscillatory) in the period 1 cyr. Columns 2–4 are two-body results and 5–6 are N-body computations made in the current work. Column 7 lists observations

compared to the speed of light and where the gravitational fields are weak (characterized by the small parameter $\epsilon \sim v^2/c^2 \sim GM/c^2r$), see the refs. (Will 1993, 2014). In astrophysics such corrections to Newtonian dynamics were, for example, derived by Newhall et al. in the DE102 ephemeris paper (Newhall 1983). Perhaps the easiest presentations of the post-Newtonian expansion at the 1PN-level are provided by refs. (Hahl 2018; Blanchet 2001; Damour and Deruelle 1985).

In terms of a two-body problem one can then write the 1PN-acceleration for particle m as

$$\mathbf{a}_m = -\frac{GM}{r^2}\mathbf{e}_r\left\{1 - 5\frac{Gm}{c^2r} - 4\frac{GM}{c^2r} - \frac{3}{2c^2r}(\mathbf{r}\cdot\mathbf{u}_M)^2 + \frac{u_m^2}{c^2} - \frac{4}{c^2}\mathbf{u}_m\cdot\mathbf{u}_M + 2\frac{u_M^2}{c^2}\right\}$$
$$+\frac{GM}{c^2r^3}\{4\mathbf{r}\cdot\mathbf{u}_m - 3\mathbf{r}\cdot\mathbf{u}_M\}(\mathbf{u}_m - \mathbf{u}_M) \quad (62)$$

where $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$, $\mathbf{e}_r = \mathbf{r}/r$ and in 1PN m and M are rest masses. The corresponding equation for $\mathbf{a}_M$ is obtained by exchanging $m \longleftrightarrow M$ in Eq. (62). In the solar system where $m << M$ and $\mathbf{u}_M << \mathbf{u}_m$, one can safely neglect several terms and by changing the notation $\mathbf{u} = \mathbf{u}_m$ and $\mathbf{a} = \mathbf{a}_m$ we get

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{e}_r\left\{1 - 4\frac{GM}{c^2r} + \frac{u^2}{c^2}\right\} + \frac{GM}{c^2r^3}4(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$$

We use this acceleration (and the one for point mass M) in our N-body code to compute the GR results displayed in Fig. 6. We also present the same orbital parameters due to RGF by applying our N-body code. In the case of the semimajor axis a and eccentricity e, it is seen that the effect is more pronounced in GR (but there are no long-term trends). The change in the argument of perihelion (i.e., $d\omega/dt$) is the same as $\dot{\phi}$ (i.e., Equation (60)) in a pure two-body problem. In Fig. 6, we also note that ω has the same long-term trend in GR as in RGF which explains the similar precession rates of the planets seen in Table 3 (columns 2–3). Accurate observations could thus potentially discriminate between these model predictions. However, these fine features have still not been observed which was also noted in the GR derivations in ref. (Balogh and Giampieri 2002). This lack of data may be rectified in late 2025 with the planned arrival of the satellite BepiColombo assuming orbit around Mercury. One of its mission objects is to measure orbit details for very precise determination of the PPN parameters of the Mercury orbit.

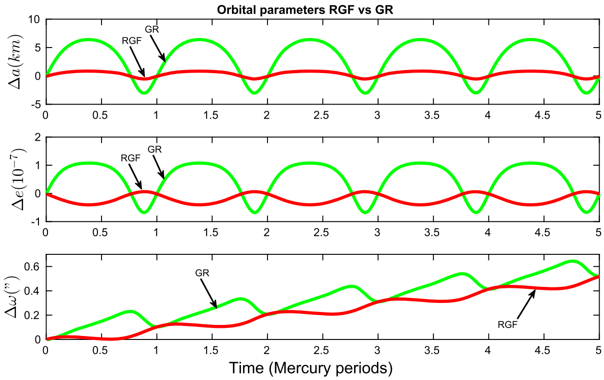

Fig. 6 Relativistic effects on Mercury orbital parameters. Comparisons between RGF (present work) and GR (1PN). Δ refers to the chosen zero (MJD 51600.5), e.g., $\Delta a = a(t) - a(0)$. The displayed differences could possibly be detected by the *Mercury Planetary Orbiter* with an arrival at Mercury in 2025 (BepiColombo)

3.9 Reduction in the general two-particle problem

It is interesting to mathematically analyze the general two-particle problem where a satellite mass $m << M$ is not assumed. We must then include the effect of propagation time (see Sect. 2.7) as this effect cannot be neglected for similar sized bodies orbiting each other. Neglecting this effect is only justified for the case $m << M$ as the point mass m then finds itself in an essentially static field. Note that the two bodies here are modeled as point masses which may be inaccurate depending on which type of binary system is considered. For example, for a star binary further effects can contribute to the orbital precession rate such as the internal properties of a star, its gravitational quadrupole moment and also tidal effects (Claret and Gimenez 2010). In Appendix 4.7–4.10 we provide some ideas about extreme effects that would only be relevant for a system such as a neutron star binary.

Relative to an inertial frame the acceleration of point mass m is given by Eq. (16)

$$\mathbf{a}_2 = \mathbf{g}_2 + \mathbf{u}_2 \times \mathbf{h}_2 - \frac{1}{c^2}\left(\mathbf{u}_2 \cdot \mathbf{g}_2\right)\mathbf{u}_2 = \mathbf{g}_2\left(1 + \frac{u_2^2}{c^2}\right) - \frac{2}{c^2}\left(\mathbf{u}_2 \cdot \mathbf{g}_2\right)\mathbf{u}_2$$

$$2 \longleftrightarrow 1$$

By investigating the retarded $\mathbf{g}_2$-field in Eq. (26), we find in normal celestial situations that the radiation term can be neglected, i.e., the expression then reduces to

$$\mathbf{g}_2 = -\frac{GM_0\gamma_{u_1}}{r^3\gamma_{u_1}^3\eta_{u_1}\left(\gamma_{u_1}^2\cos^2\theta_1 + \sin^2\theta_1\right)^{3/2}}\mathbf{r},$$

where $\mathbf{r} = \mathbf{r}_2(t) - \mathbf{r}_1(t)$ and $\cos^2\theta_1 = (\mathbf{r}\cdot\mathbf{u}_1)^2/(ru_1)^2$. After Taylor expansions and by dropping terms of order $1/c^4$ and higher, one get

$$\mathbf{g}_2 \approx -\frac{GM_0}{r^2}\mathbf{e}_r + \frac{2GM_0}{r^2}\frac{u_1^2}{c^2}\mathbf{e}_r + \frac{3}{2}\frac{GM_0}{r^4c^2}\left(\mathbf{r}\cdot\mathbf{u}_1\right)^2\mathbf{e}_r$$
$$\mathbf{g}_1 \approx \frac{Gm_0}{r^2}\mathbf{e}_r - \frac{2Gm_0}{r^2}\frac{u_2^2}{c^2}\mathbf{e}_r - \frac{3}{2}\frac{Gm_0}{r^4c^2}\left(\mathbf{r}\cdot\mathbf{u}_2\right)^2\mathbf{e}_r$$

Let us also work in a barycentric reference system so

$$\mathbf{r}_G = \left(M_0\mathbf{r}_1 + m_0\mathbf{r}_2\right)/\left(M_0 + m_0\right)$$
$$\mathbf{u}_G = \left(M_0\mathbf{u}_1 + m_0\mathbf{u}_2\right)/\left(M_0 + m_0\right).$$

Initial transformations of the particle coordinates are made so that $\mathbf{r}_G = \mathbf{u}_G = 0$. For slow movements the origin will remain at rest for any time t, whereas for a highly relativistic system this may only be approximately true. The level of this approximation can easily be checked by integrating the two top equations independently by an N-body computation. Now, since $\mathbf{u}_G = 0$ we have that $\mathbf{u}_1 = -(m_0/M_0)\,\mathbf{u}_2$. The acceleration of particle 2 relative to particle 1 can therefore after simplification be written

$$\begin{aligned}\mathbf{a}_{rel} &= \mathbf{a}_2 - \mathbf{a}_1 = -\frac{G}{r^2}(M_0 + m_0)\,\mathbf{e}_r - \frac{Gu_2^2}{c^2r^2}\mathbf{e}_r\frac{m_0}{M_0}\left(-2m_0 + \frac{m_0^2}{M_0} - 2M_0 + \frac{M_0^2}{m_0}\right)\\ &\quad + \frac{2G}{c^2r^3}\frac{M_0^3 + m_0^3}{M_0^2}(\mathbf{u}_2\cdot\mathbf{r})\,\mathbf{u}_2 + \frac{3G}{2r^4c^2}\frac{m_0}{M_0}(M_0 + m_0)(\mathbf{u}_2\cdot\mathbf{r})^2\,\mathbf{e}_r\end{aligned}$$

Finally by using $\mathbf{u}_2 = \mathbf{u}_{rel}M_0/(M_0 + m_0)$ one gets

$$\begin{aligned}\mathbf{a}_{rel} &\approx -\frac{G}{r^2}(M_0 + m_0)\,\mathbf{e}_r - \frac{Gu_{rel}^2}{c^2r^2}\mathbf{e}_r\frac{m_0M_0}{(M_0 + m_0)^2}\left(-2m_0 + \frac{m_0^2}{M_0} - 2M_0 + \frac{M_0^2}{m_0}\right)\\ &\quad + \frac{2G}{c^2r^3}\frac{M_0^3 + m_0^3}{(M_0 + m_0)^2}(\mathbf{u}_{rel}\cdot\mathbf{r})\,\mathbf{u}_{rel} + \frac{3G}{2r^4c^2}\frac{m_0M_0}{M_0 + m_0}(\mathbf{u}_{rel}\cdot\mathbf{r})^2\,\mathbf{e}_r \end{aligned} \tag{63}$$

This expression with the relative velocity $\mathbf{u}_{rel} = \mathbf{u}_2 - \mathbf{u}_1$ should in principle include relativistic corrections (i.e., in analogy to "relativistic addition of velocities"), i.e.,

$$\mathbf{u}_{rel} = \frac{1}{1 - \mathbf{u}_1\cdot\mathbf{u}_2/c^2}\left[\frac{1}{\gamma_{u_1}}\mathbf{u}_2 - \left(1 - \frac{\mathbf{u}_1\cdot\mathbf{u}_2}{c^2}\frac{\gamma_{u_1}}{1 + \gamma_{u_1}}\right)\mathbf{u}_1\right].$$

However, such an inclusion would only add corrections of order $1/c^4$ which is neglected here. We tested the accuracy of Eq. (63) by performing a full N-body computation and found that there is no significant deviation even for the case m=M.

3.10 Periastron precession formula

Our N-body computer code computes the Runge–Lenz vector in order to find out the exact precession rate of an orbit, see Eq. (61). However, it might be of interest to identify an approximate mathematical formula for the precession rate. In order to complete this task, it is convenient to use the Hamilton vector $\mathbf{h}$ given by Hamilton (1967) (not to be confused with the $\mathbf{h}$-field in Eq. (16))

$$\mathbf{h} = \mathbf{u}_{rel} - \frac{\alpha}{L}\mathbf{e}_\varphi,\ h = \frac{\alpha e}{L}$$

where $\mathbf{u}_{rel}$ is the velocity of m relative to M, $\alpha = GMm$, $\mathbf{L} = \mu\mathbf{r}\times\mathbf{u}_{rel}$, $\mu = Mm/(M + m)$ is the reduced mass and e is the orbital eccentricity. The relation to the Runge–Lenz vector $\mathbf{A}$ in Eq. (61) is given by

$$\frac{\mathbf{A}}{\mu} = \mathbf{h}\times\mathbf{L}$$

For a classical two-body problem, all the above vectors are conserved. In RGF, however, the Hamilton vector $\mathbf{h}$ will precess. Simple vector analysis shows that the precession rate is

$$\frac{d\phi}{dt} = \frac{|\mathbf{h}\times\dot{\mathbf{h}}|}{h^2}$$

where

$$\dot{\mathbf{h}} = \mathbf{a}_{rel} + \frac{\alpha}{L^2}|\dot{\mathbf{L}}|\mathbf{e}_\varphi + \frac{\alpha}{L}\dot{\varphi}\mathbf{e}_r$$

and $\mathbf{a}_{rel}$ is listed in Eq. (63). The angular momentum rate is determined by

$$\begin{aligned}\dot{\mathbf{L}} &= \mu\dot{\mathbf{r}} \times \mathbf{u}_{rel} + \mu\mathbf{r} \times \dot{\mathbf{u}}_{rel} = \mu\mathbf{r} \times \mathbf{a}_{rel} \\ &= \frac{\mu\sigma}{r^3c^2}\left(\mathbf{u}_{rel} \cdot \mathbf{r}\right)\mathbf{r} \times \mathbf{u}_{rel} = \frac{\mu\sigma}{c^2}\dot{r}\dot{\varphi}\mathbf{e}_z\end{aligned}$$

where σ is according to Eq. (63) given by

$$\sigma = 2G\frac{M_0^3 + m_0^3}{(M_0 + m_0)^2}$$

Since $L = \mu r^2\dot{\varphi}$, one finds that

$$\begin{aligned}\frac{\alpha}{L^2}|\dot{\mathbf{L}}| &= \alpha\frac{\frac{\mu\sigma}{c^2}\dot{r}\dot{\varphi}}{\mu^2 r^4\dot{\varphi}^2} = \frac{\alpha\sigma\dot{r}}{c^2r^2L} \\ \frac{\alpha}{L}\dot{\varphi} &= \frac{\alpha}{\mu r^2} = \frac{\beta}{r^2}\end{aligned}$$

where $\beta = G\left(M_0 + m_0\right)$. Thus, we have that

$$\dot{\mathbf{h}} = \mathbf{a}_{rel} + \frac{\alpha\sigma\dot{r}}{c^2r^2L}\mathbf{e}_\varphi + \frac{\beta}{r^2}\mathbf{e}_r$$

and $\mathbf{a}_{rel}$ can be written according to Eq. (63) as

$$\mathbf{a}_{rel} = -\frac{\beta}{r^2}\mathbf{e}_r - \gamma\frac{u_{rel}^2}{c^2r^2}\mathbf{e}_r + \frac{\sigma}{c^2r^3}\left(\mathbf{u}_{rel} \cdot \mathbf{r}\right)\mathbf{u}_{rel} + \frac{\delta}{r^4c^2}\left(\mathbf{u}_{rel} \cdot \mathbf{r}\right)^2\mathbf{e}_r$$

where

$$\begin{aligned}\gamma &= G\frac{m_0M_0}{(M_0 + m_0)^2}\left(-2m_0 + \frac{m_0^2}{M_0} - 2M_0 + \frac{M_0^2}{m_0}\right) \\ \delta &= \frac{3G}{2}\frac{m_0M_0}{M_0 + m_0}.\end{aligned}$$

The final expression becomes

$$\dot{\mathbf{h}} = -\gamma\frac{u_{rel}^2}{c^2r^2}\mathbf{e}_r + \frac{\sigma}{c^2r^3}\left(\mathbf{u}_{rel} \cdot \mathbf{r}\right)\mathbf{u}_{rel} + \frac{\delta}{r^4c^2}\left(\mathbf{u}_{rel} \cdot \mathbf{r}\right)^2\mathbf{e}_r + \frac{\alpha\sigma\dot{r}}{c^2r^2L}\mathbf{e}_\varphi.$$

It is straightforward to show that

$$\left(\mathbf{h} \times \dot{\mathbf{h}}\right)_z = \frac{\gamma u^2\dot{\varphi}}{rc^2} + \frac{2\alpha\sigma\dot{r}^2}{c^2r^2L} - \frac{\alpha\gamma u^2}{Lr^2c^2} + \frac{\delta\dot{r}^2}{r^2c^2}\left(\frac{\alpha}{L} - r\dot{\varphi}\right).$$

Now by inserting $\dot{r}^2 = u^2 - (L/r\mu)^2$ and divide by $h^2 = \left(\alpha^2e^2\right)/L^2$, one gets an expression for $\left(\mathbf{h} \times \dot{\mathbf{h}}\right)_z/h^2$. After this stage, one inserts the Keplerian instantaneous speed

$$u^2 = \frac{2\beta}{r} - \frac{\beta}{a}$$

where a is the semimajor axis. One then obtains the expression,

$$\frac{d\phi}{dt} = \frac{\left(\mathbf{h} \times \dot{\mathbf{h}}\right)_z}{h^2} = \frac{L^2}{\alpha^2e^2}\left\{\frac{k_1}{r^5} + \frac{k_2}{r^4} + \frac{k_3}{r^3} + \frac{k_4}{r^2}\right\} = \frac{L^2}{\alpha^2e^2}g\left(r\right),$$

where

$$k_1 = \frac{\delta L^3}{\mu^3 c^2}$$
$$k_2 = \frac{2(\gamma - \delta)\beta L}{c^2 \mu} - \frac{2\alpha(\sigma + \delta/2)L}{c^2\mu^2}$$
$$k_3 = -\frac{(\gamma - \delta)\beta L}{a\mu c^2} - \frac{2\alpha\beta\gamma}{Lc^2} + \frac{4\alpha(\sigma + \delta/2)\beta}{Lc^2}$$
$$k_4 = -\frac{2\alpha(\sigma + \delta/2)\beta}{c^2 La} + \frac{\alpha\beta\gamma}{c^2 La}.$$

We are now able to express the angle $\triangle\phi$ as

$$\triangle\phi = \int_0^T \frac{d\phi}{dt}dt = \frac{L^2}{\alpha^2 e^2}\int_0^T g(r)\,dt$$
$$= \frac{L^2}{\alpha^2 e^2}\int_0^{2\pi} \frac{g(r)}{\dot{\varphi}}d\varphi = \frac{L\mu}{\alpha^2 e^2}\int_0^{2\pi} \frac{k_1}{r^3} + \frac{k_2}{r^2} + \frac{k_3}{r} + k_4 d\varphi$$

where we have used $L = \mu r^2\dot{\varphi}$. One can now insert the Keplerian solution $r = p/(1 + e\cos\varphi)$ and integrate to obtain

$$\triangle\phi = \frac{L\mu 2\pi}{\alpha^2 e^2 p^2}\left(\frac{k_1}{p} + \frac{k_1}{p}\frac{3}{2}e^2 + k_2 + \frac{e^2}{2}k_2 + k_3 p + k_4 p^2\right)$$

By using the relations $L^2 = p\mu\alpha$ and $p = a\left(1 - e^2\right)$, we find several simplifications finally leading to

$$\triangle\phi = \frac{2\pi}{a\left(1 - e^2\right)c^2}(\gamma + \sigma) \tag{64}$$

The mean precession rate is given by

$$\frac{d\phi}{dt} = \frac{\triangle\phi}{P_b},\ P_b = 2\pi\sqrt{\frac{a^3}{G(M_0 + m_0)}}$$

where P_b is the classical orbital period of the two-body problem. The mass parameter can be simplified to

$$(\gamma + \sigma)_{RGF} = 3G\frac{(M_0 - m_0)^2}{M_0 + m_0} + G\frac{m_0 M_0}{M_0 + m_0}.$$

For the case that $m_0 \ll M_0$, we find that

$$(\gamma + \sigma)_{RGF} \approx 3GM_0,$$

which coincides with the general relativistic result, namely Will (1993)

$$(\gamma + \sigma)_{GR} = 3G(M_0 + m_0) \approx 3GM_0.$$

This explains why RGF and GR are in perfect agreement within the solar system.

Given a binary system of similarly sized masses, RGF would normally predict a much smaller apsidal precession than GR. An interesting example of a similarly sized binary system is the star system Di Herculis where its observed precession rate is given by only 1.08”/cycle. The theoretically expected classical effect is 2.0”/cycle and the general relativistic contribution is 2.43”/cycle, thus resulting in a total of 4.43”/cycle which is much larger than the observed 1.08”/cycle. In 2009, Albrecht et al. suggested that a misalignment between the spins and the normal to the orbital plane could bring experiment and theory in to better agreement Albrecht et al. (2009). This misalignment would reduce the classical effect. However, shortly thereafter this proposal was challenged by Zimmerman et al. (2010). Tilted axes are expected to affect orbital inclination but no such effect was observed. There should also be small periodic oscillations in the eclipse timings but evidence of a light travel time anomaly has not been observed Zimmerman et al. (2010). In any case, the RGF model would predict 0.21”/cycle[3] which could be compatible with the observation of 1.08”/cycle if one would add a positive classical contribution of 0.87”/cycle to the pure RGF result. The classical contribution to the precession rate is due to a star’s internal properties, its gravitational quadrupole moment and tidal effects as described in Claret and Gimenez (2010). Although this result lends some support to the RGF model, this would need to be further studied in a separate and more detailed study.

3.11 Gravitational time dilation

Consider Eq. (48). The result was derived in a single frame of reference, i.e., a unique time coordinate t is associated with that frame. Despite this, the expression predicts that photons emitted from the emitter oscillates at a faster rate within the gravitational well than what is observed later at the far distance (see Fig. 7). Does this mean that time moves at a different rate near the emitter compared with far away? No. The fundamental reason of the different frequencies in RGF is due to the conservation law in Eq. (47), i.e.,

$$\ln \nu - \frac{GM}{c^2 r} = const. \tag{65}$$

This leads to the following photon energy detected at the distance $r = z + r_e$

$$E = E_e e^{-\frac{GM}{c^2}\left(\frac{r-r_e}{r r_e}\right)} \tag{66}$$

One can imagine events emitting photon pulses $E_e = h\nu_e$ separated by time intervals Δt by the emitter. At the receiver photons with energy $E = h\nu$ are detected, also at intervals Δt (see Fig. 7). There is no time difference in RGF and the difference in photon energy is instead due to the required work for a photon to climb the gravitational well. It is not unthinkable that the distant observer might make the *interpretation* that the experiment *behaves* like the time of the electromagnetic process (oscillator) is running at a slower pace near the emitter. However, the separation of the pulses with Δt in Fig. 7 reveals that this is only an illusion. On the other hand in the case of a continuous emission (without pulses), one could say that it behaves like time is running slow from a distant experimenter’s perspective. If one replaces ν_∞ with $1/\Delta t_\infty$ and ν_e with $1/\Delta t_e$ in Eq. (48), one gets for such an interpretation the RGF analogy to gravitational time dilation

$$\Delta t_e = \Delta t_\infty e^{-\frac{GM}{c^2 r'}} \tag{67}$$

In RGF, this is purely an observational effect which is more correctly described by the lost photon energy than gravitational time dilation. However, with regard to what actually is observed in such an experiment, both RGF and GR come precisely to the same conclusion.

[3] $M = 5.15 M_\odot$, $m = 4.52 M_\odot$, $P_b = 10.55$ days, $e = 0.489$, $a = 0.201$ AU.

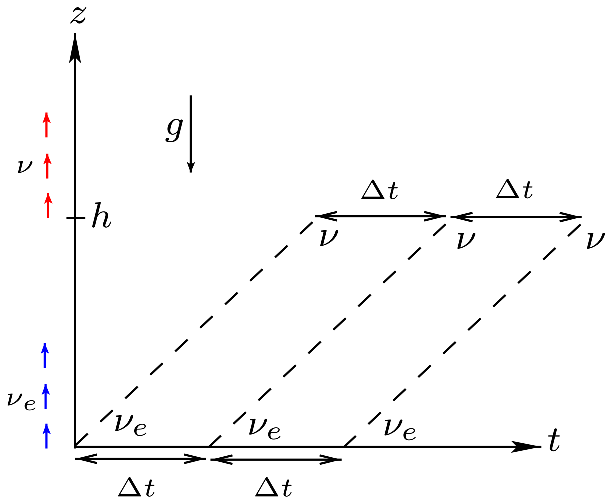

Fig. 7 Three point-like photons are sent out vertically at time intervals Δt from the emitter at $z = 0$. The photon frequency at the emitter is fixed at ν_e. They are then climbing vertically in the gravitational field $\mathbf{g} = -GM/(r_e + z)^2 \, \mathbf{e}_z$. At $z = h$ the photons are successively detected at the very same interval Δt but now with a lower frequency $\nu < \nu_e$

The GR prediction is namely given by Gravitational time dilation (2021)

$$\Delta t_e = \Delta t_\infty \sqrt{1 - \frac{2GM}{c^2 r}}. \tag{68}$$

As pointed out in Sect. 2.9, there is no detectable difference between the above expressions. Also, note that the GR result in Eq. (68) is equivalent to the GR redshift in Eq. (50) so the underlying redshift explanation in GR is entirely due to gravitational time dilation. Therefore, in contrast to the RGF physics, as soon as the photon has left the emitter, GR tells us there is no further change in its frequency as it is climbing toward the observer. However, imagine two equivalent photons emitted. Photon 1 is only measured at the far distance whereas photon 2 is detected at say half the distance. It would then be found that photon 2 is redshifted and therefore photon 1 is also redshifted since they are equivalent. This argument indicates that photon 1 is indeed behaving according to the philosophy of RGF, i.e., gradually shifting its frequency as it climbs toward the observer. We conclude that within the RGF model, true gravitational time dilation does not exist. However, we will provide a brief discussion about this lack in the last section of the Conclusions and a possible direction for a future work.

Conclusions

It has been demonstrated that special relativistic gravitation can be cast in a form which agrees well with several observations. RGF does not only reproduce some standard tests in the weak field regime, but also several tests in the strong field regime—light bending in the strong regime in Sect. 3.2, escape velocity (strong and weak regime) in Sect. 3.6 (Fig. 3: top curve is a perfect match between RGF and GR) and particularly Sect. 3.7 where even several highly relativistic orbital details close to a black hole was verified. This is unexpected, and to our knowledge is RGF unique to be able to provide almost quantitatively exact GR answers in both the weak and the strong field regime. RGF in itself is in principle valid for any field strength or speed, i.e., its starting point is not GR with some successive approximations (weak regime and moderate speeds). There are of course always small differences between RGF and GR, but sufficiently small that direct observations in celestial mechanics cannot easily separate these model predictions. Even though GR is a more complete model, RGF is by far, an improvement to Newtonian classical mechanics (and nearly as simple). An advantage of RGF is that it is very practical to deal with N-body calculations whereas it is difficult in the nonlinear tensorial GR theory.

Since RGF is a linear model expressed in flat Minkowski space, it is incredibly simple to apply in many relativistic celestial mechanics setups. For the same reason, it could be straightforward to quantize the theory. A theory of quantum gravity could reveal phenomena that may, or may not already be well known by today's experiments/observations. As is well known, a consistent theory of quantum gravity does not yet exist. It has proved extremely difficult to derive this by using GR (and its curved Riemann space) as the starting point. A successful translation of RGF into the quantum regime could therefore be very interesting. Perhaps, a revival of the ideas by Bronstein, who suggested a change of the quantization rules for the special case of gravity, would be a reasonable starting point (Bronstein 2012)?

Finally, we are well aware of that although gravitational redshift is predicted well, a gravitational time dilation is not. In a future work it would therefore be interesting to investigate if gravitational time dilation perhaps could be modeled by a consistent VSL theory (variable speed of light depending on gravitational potential) instead of curved space as in GR. In a quantum theory of gravity, it could turn out that photons can interact with gravitons and effectively display a VSL behavior, i.e., in a similar way as in quantum optics/quantum electrodynamics, where photons interact with electrons in matter (oscillator strengths) which is the true origin of the refraction index n, where the effective speed of light then varies according to $c = c_0/n$. Such an effect could therefore possibly slow down the frequency of an atomic clock ($\nu = c/\lambda$) or explain various phenomena such as the Shapiro delay.

4 Appendix

4.1 Generality of the constant τ/β

In the photon experiment performed in a homogeneous gravitational field (see Sect. 2 and Eqs. (9, 10), our findings suggested that $\tau(u)/\beta(u) = 2/c^2$. Let us check if this constant ratio can be valid for a more general setup, i.e., for an inhomogeneous field $\mathbf{g}$. According to Eq. (8), the general expression is given by

$$\mathbf{f} = \tau(u)(\mathbf{u}\cdot\mathbf{f})\mathbf{u} + \beta(u)m\mathbf{a} \tag{69}$$

Let us make the dot product with the velocity $\mathbf{u}$ on both sides of the above equation so

$$\frac{\tau(u)}{\beta(u)}u^2 = \frac{1}{\beta(u)} - m\frac{\mathbf{u}\cdot\mathbf{a}}{\mathbf{u}\cdot\mathbf{f}}.$$

Sect. 2 shows that we must require that

$$\mathbf{u}\cdot\mathbf{a} = \frac{c^2-u^2}{mc^2}\mathbf{u}\cdot\mathbf{f},$$

so we can identify the simplification

$$\frac{\tau(u)}{\beta(u)}u^2 = \frac{1}{\beta(u)} - \frac{c^2-u^2}{c^2} \tag{70}$$

The dimension of $\tau(u)/\beta(u)$ is clearly $speed^{-2}$ so the simplest appropriate ansatz is

$$\frac{\tau(u)}{\beta(u)} = \frac{a}{c^2},$$

where a is a constant. As $u \to 0$ we require that $\beta(u) \to 1$ in order for the above general force law (Eq. (69)) to approach Newton's second law. It can be seen that the constant ansatz is compatible with Eq. (70) for this case. As $u \to c$ we must invoke the second postulate in Sect. 2. The initial acceleration $\mathbf{a}_0$ refers to any situation when $\mathbf{u} \perp \mathbf{g}$, and $\mathbf{f} = m\mathbf{g}$ so according to Eq. (69) then $\mathbf{f} = \beta(c)\, m\mathbf{a}_0$. Thus, we have that

$$\beta(c)\,\mathbf{a}_0 = \mathbf{g},$$

and the second postulate tells us that $\mathbf{a}_0 = 2\mathbf{g}$ so $\beta(c) = 1/2$. Equation (70) then reads for this case

$$\frac{a}{c^2}c^2 = 2,$$

so $a = 2$. We have thus seen that a constant ansatz, $\tau/\beta = 2/c^2$, is consistent with the general formula Eq. (70) which is valid for any speed u and field $\mathbf{g}$. One may now use Eq. (70) to solve for $\beta(u)$

$$\begin{aligned}
\frac{2}{c^2}u^2 &= \frac{1}{\beta(u)} - \frac{c^2 - u^2}{c^2} \Leftrightarrow \\
\frac{1}{\beta(u)} &= \frac{c^2 - u^2 + 2u^2}{c^2} = \frac{c^2 + u^2}{c^2} \\
\frac{\tau(u)}{\beta(u)} &= \frac{2}{c^2} \Leftrightarrow \tau(u) = \frac{2}{c^2}\frac{c^2}{c^2 + u^2} = \frac{2}{c^2 + u^2}.
\end{aligned}$$

4.2 Point mass in a homogeneous gravitational field

The change in *work* dW is by definition given by

$$dW = Pdt = \frac{dE}{dt}dt = c^2\frac{dm}{dt}dt = c^2 dm,$$

where P is the *power* and $E = mc^2$. We have also from the definition of *work* that

$$dW = \mathbf{f} \cdot \mathbf{dr} = -mgdz$$

In other words

$$-gdz = c^2\frac{1}{m}dm$$

After integration, it is found that

$$c^2 \ln m + gz = const$$

which can be expressed as

$$\frac{2g}{c^2}z - \ln\left(1 - \frac{u^2}{c^2}\right) = const = A$$

and then one finds the speed

$$u^2 = u_x^2 + u_z^2 = c^2 \left(1 - e^{-\frac{2g}{c^2}|z|} e^{-A}\right)$$

For a point mass falling along the z-axis, $u_x = 0$, and given an initial condition that $u_z = 0$ at $z = 0$ one get that $e^{-A} = 1$, so

$$u_z^2 = c^2 \left(1 - e^{-\frac{2g}{c^2}|z|}\right)$$

and the acceleration is given by

$$a_z = u_z \frac{du_z}{dz} = -g e^{-\frac{2g}{c^2}|z|}$$

so for a point mass the acceleration is initially $-g$ as expected from Newtonian mechanics, but later on it will eventually decline as the speed becomes relativistic.

On the other hand, if the point mass initially has a horizontal component $u_x = u_0$ at $z = 0$ (and $u_z = 0$ as before), the initial condition becomes

$$e^{-A} = 1 - \frac{u_0^2}{c^2}$$

so in this case

$$u_x^2 + u_z^2 = c^2 - \left(c^2 - u_0^2\right) e^{-\frac{2g}{c^2}|z|} \tag{71}$$

By applying Eq. (12) in the x-direction, one finds that

$$u_x = u_0 e^{-\frac{2g}{c^2}|z|}.$$

One then inserts this result in Eq. (71)

$$u_z^2 = c^2 \left(1 - e^{-\frac{2g}{c^2}|z|}\right) + u_0^2 e^{-\frac{2g}{c^2}|z|} \left(1 - e^{-\frac{2g}{c^2}|z|}\right).$$

A Taylor expansion reveals that

$$u_z^2 \approx 2g|z| \left(1 + \frac{u_0^2}{c^2}\right)$$

which can be compared with the Newtonian result $u_z^2 = 2\,g|z|$. The acceleration is given by

$$a_z = u_z \frac{du_z}{dz} = -g e^{-\frac{2g}{c^2}|z|} - g \frac{u_0^2}{c^2} e^{-\frac{2g}{c^2}|z|} \left(2e^{-\frac{2g}{c^2}|z|} - 1\right)$$

so the initial acceleration becomes ($z = 0$)

$$a_z = -g \left(1 + \frac{u_0^2}{c^2}\right).$$

4.3 The gravitational force f at large scales

Under normal circumstances, it is assumed that the law of gravitation for a body m interacting with a central body M is given by (c.f. Equation (12))

$$\mathbf{f} = -\frac{GMm}{r^2}\mathbf{e}_r.$$

This law is experimentally known to be accurate in almost all cases (if M is static there is no retardation effect, see Sect. 2.7). A possible exception is within the regime where the distances between interacting bodies are very large. In problems dealing with galactic rotation curves or globular clusters, one will then have to include dark matter in the model or alternatively model modified dynamics through MOND (Milgrom 1983). In MOND, this is interpreted as the regime of very small accelerations ($a \ll a_0$). The critical acceleration a_0 was estimated as $1.2 \cdot 10^{-10}\, m/s^2$ by MOND (2021). Verlinde applies $a_0 = cH_0 \approx 7 \cdot 10^{-10}\, m/s^2$ in his estimate of the weak regime (Verlinde 2017). MOND was developed to provide an *ad hoc* explanation to the observed flat rotational curves of spiral galaxies and it also predicts the empirical Tully–Fisher relation (Tully and Fisher 1977). On top of that, a study of relative velocities in wide star binaries displays MOND type of behavior in the $a \ll a_0$ regime (Hernandez et al. 2012a, b). Recent studies of galaxy/star clusters lend further support to MOND (Chae 2020; Kroupa 2022). A comprehensive review of the subject is provided in Famaey and McGaugh (2012). MOND (or rather a relativistic version of it) is seen by some scientists as an alternative to the dark matter hypothesis. Some kind of MOND model may not be unreasonable because dark matter has still not *directly* been observed. However, a problem with the standard MOND formulation is that it is a classical model so well-established relativistic effects are ignored. Another problem is that there is no clear physical mechanism behind MOND. In the case of the standard MOND formulation, we have that

$$\mathbf{f} = -\frac{GMm}{\mu\left(\frac{a}{a_0}\right)r^2}\mathbf{e}_r,\ \mu\left(\frac{a}{a_0}\right) = \left(1 + \left(\frac{a_0}{a}\right)^2\right)^{-1/2} \tag{72}$$

Here, we have listed the standard interpolating function $\mu\,(a/a_0)$. Thus, in the solar system we get that

$$\mathbf{f} = -\frac{GMm}{r^2}\left(1 + \left(\frac{a_0}{a}\right)^2\right)^{1/2}\mathbf{e}_r \approx -\frac{GMm}{r^2}\left(1 + \frac{1}{2}\left(\frac{a_0}{a}\right)^2\right)\mathbf{e}_r$$

The acceleration is approximately given by $a = GM/r^2$ in the Solar system because $a \gg a_0$. We tested quickly if this force correction (besides relativity) could affect the Mercury orbital precession in any meaningful way (cf. Section 3.8). The answer is absolutely not. The ratio a_0/a is by far too tiny to be observed within the solar system. Only at distances $r \sim \sqrt{GM/a_0}$ can this term become significant. Normally this would be at the galactic scale for a galaxy, but for the Sun, it corresponds to a distance of about 7000 AU. This can be compared with the Kuiper belt which is located at 30–50 AU. As we go further out into the MOND weak regime, i.e., $a \ll a_0$, the above force transforms into

$$\mathbf{f} = -\frac{GMm}{r^2}\left(1 + \left(\frac{a_0}{a}\right)^2\right)^{1/2}\mathbf{e}_r \approx -\frac{GMm}{r^2}\frac{a_0}{a}\mathbf{e}_r$$

For example, this approximation is good already for $a_0/a = 10$ (i.e., nearly 70,000 AU from the Sun or 1 light year away). In the case of a circular orbit, one immediately finds a flat rotation curve solution (speed no longer depends on the distance r) at such a large distance

$$u^4 = GMa_0$$

The present work provides a possible relativistic version of MOND (by applying Eq.(72) and Eq. (12)). Thus, both flat rotational curves and the Tully–Fisher relation are automatically described well.

4.4 Accelerated heliocentric origin

Consider an inertial frame where the gravitational problem of two bodies 1 (M) and 2 (m) is written

$$\mathbf{f}_{21} = -\frac{GMm}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$
$$\mathbf{f}_{12} = \frac{GMm}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

According to Eq. (12), we then have

$$\mathbf{f}_{21} = \frac{2}{c^2 + u_2^2}(\mathbf{u}_2 \cdot \mathbf{f}_{21})\,\mathbf{u}_2 + \frac{c^2}{c^2 + u_2^2} m\mathbf{a}_2$$
$$\mathbf{f}_{12} = \frac{2}{c^2 + u_1^2}(\mathbf{u}_1 \cdot \mathbf{f}_{12})\,\mathbf{u}_1 + \frac{c^2}{c^2 + u_1^2} M\mathbf{a}_1$$

Let us now change the notation: $\mathbf{f} = \mathbf{f}_{21}$ and $\mathbf{f} = -\mathbf{f}_{12}$ so

$$\mathbf{f} = \frac{2}{c^2 + u_2^2}(\mathbf{u}_2 \cdot \mathbf{f})\,\mathbf{u}_2 + \frac{c^2}{c^2 + u_2^2} m\mathbf{a}_2 \tag{73}$$
$$\mathbf{f} = \frac{2}{c^2 + u_1^2}(\mathbf{u}_1 \cdot \mathbf{f})\,\mathbf{u}_1 - \frac{c^2}{c^2 + u_1^2} M\mathbf{a}_1 \approx -M\mathbf{a}_1, \tag{74}$$

where we assumed that particle M moves slowly ($u_1 << c$). The relative acceleration

$$\mathbf{a}_{rel} = \mathbf{a}_2 - \mathbf{a}_1 \approx \mathbf{a}_2 + \frac{\mathbf{f}}{M}$$

From above, we have

$$\frac{\mathbf{f}_{21}}{M} = \frac{\mathbf{f}}{M} = -\frac{Gm}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1) = -\frac{Gm}{r^2}\mathbf{e}_r$$

with the notation $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. We thus have

$$\ddot{\mathbf{r}} = \mathbf{a}_2 + \frac{\mathbf{f}}{M} = \mathbf{a}_2 - \frac{Gm}{r^2}\mathbf{e}_r$$

The last term is denoted $\mathbf{a}_{\odot} = Gm/r^2\mathbf{e}_r$ in Eq. (17).

4.5 Similarity to Electrodynamics?

There is a resemblance with the Lorentz force in electrodynamics. Consider therefore the Lorentz force acting on a point charge q_2 due to the fields from another moving point charge

q_1 (i.e., an analogous two-particle problem as in Sect. 2.3)

$$\mathbf{F}_2 = q_2 \left(\mathbf{E}_1 + \mathbf{u}_2 \times \mathbf{B}_1\right) = \dot{\mathbf{p}}_2 \tag{75}$$

The magnetic field from the moving point charge q_1 can be written (see p. 440 in Griffiths (1999))

$$\mathbf{B}_1 = -\frac{1}{c^2}\mathbf{E}_1 \times \mathbf{u}_1$$

If the situation is reversed one instead would have

$$\mathbf{F}_1 = q_1 \left(\mathbf{E}_2 + \mathbf{u}_1 \times \mathbf{B}_2\right) = \dot{\mathbf{p}}_1$$
$$\mathbf{B}_2 = -\frac{1}{c^2}\mathbf{E}_2 \times \mathbf{u}_2$$

The magnetic Lorentz force $q_i\mathbf{u}_i \times \mathbf{B}_j$ is in either case explicitly dependent on *both* $\mathbf{u}_1$ and $\mathbf{u}_2$. This is not the case for the $\mathbf{u} \times \mathbf{h}$-term in Eq. (16) where the corresponding velocities depends on the particle's eigenvelocity $\mathbf{u}$ only. Let us write a force "$\mathbf{F}_2$" in RGF to be explicit about this and directly compare with Eq. (75)

$$\mathbf{F}_2 = m_2 \left(\mathbf{g}_1 + \mathbf{u}_2 \times \mathbf{h}_{12}\right) = \dot{\mathbf{p}}_2$$
$$\mathbf{h}_{12} = \frac{1}{c^2}\mathbf{g}_1 \times \mathbf{u}_2$$

Note the difference between $\mathbf{B}_1$ and $\mathbf{h}_{12}$ where $\mathbf{B}_1$ depends only on what particle "1" is doing (inducing a magnetic field that acts on particle "2"). The $\mathbf{h}_{12}$-term tells a completely different story. There is only a $\mathbf{g}_1$-field from particle "1" and the strength depends on how fast particle "2" is passing through this field. The $\mathbf{h}_{12}$-term thus depends on both the particles. Therefore, the $\mathbf{h}_{12}$-term is *not* analogous to a "*gravitomagnetic field*" due to particle "1" (M), see (Gravitoelectromagnetism 2021). The effect is present even for an entirely static body M. The $\mathbf{u} \times \mathbf{h}$-term in Eq. (16) represents a relativistic correction for which its origin is only a consequence of the postulates stated in top of Sect. 2.

4.6 The principle of relativity

Consider the expression given by Eq. (16)

$$\mathbf{F} = \mathbf{g} + \mathbf{u} \times \mathbf{h}$$

We are interested to see if the 3-vector $\mathbf{F}$ can be used to form a proper 4-vector. Consider the following candidate 4-vector

$$\mathbb{F} = \gamma_u \left(\frac{\mathbf{F}\cdot\mathbf{u}}{c}, \mathbf{F}\right)$$

and also the standard 4-velocity (Steane 2012)

$$\mathbb{U} = \gamma_u \left(c, \mathbf{u}\right).$$

It is obvious that $\mathbf{F}\cdot\mathbf{u} = \mathbf{g}\cdot\mathbf{u}$ so

$$\mathbb{F} = \gamma_u \left(\frac{\mathbf{g}\cdot\mathbf{u}}{c}, \mathbf{g} + \mathbf{u} \times \mathbf{h}\right)$$

If the vector $\mathbb{F}$ can be expressed in terms of a covariant tensor and the 4-velocity $\mathbb{U}$, then it is clear that $\mathbb{F}$ itself is a proper 4-vector. The following antisymmetric tensor is given in the metric convention $\eta = (-1,\ 1,\ 1,\ 1)$

$$\mathcal{F} = \begin{bmatrix} 0 & g_x/c & g_y/c & g_z/c \\ -g_x/c & 0 & h_z & -h_y \\ -g_y/c & -h_z & 0 & h_x \\ -g_z/c & h_y & -h_x & 0 \end{bmatrix}$$

and it turns out that $\mathbb{F}$ indeed can be expressed as

$$\mathbb{F} = \mathcal{F}\mathbb{U}.$$

The definition of a covariant tensor in special relativity is a object that transforms according to

$$\left(\mathcal{F}_{\mu\nu}\right)' = \Lambda_\mu^\alpha \Lambda_\nu^\beta \mathcal{F}_{\alpha\beta}$$

where Λ is a general Lorentz matrix. This shows that the vector $\mathbb{F}$ is a legitimate 4-vector which has the same form in all Lorentz transformed frames.

4.7 Field mass of a compact object

The dense structure of a compact object such as a neutron star or a white dwarf is so extreme that the energy of the gravitational field itself needs to be taken into account. This effect results in an additional mass component—the *field mass*. Here, we shall outline a simple calculation inspired by the analogy of interaction delays in electrodynamics, see, e.g., Sections 8, 28 in Feynman (1963) and Eq. 17.30 in Jackson (1962). In Sect. 2.5 we showed that the classical potential energy of type $U = -GmM/r$ is incorrect in a relativistic world. However, it is still a good approximation for slowly moving particles. A classical derivation of the field mass effect can thus be made as follows. The total potential energy U of a mass distribution is given by

$$U = -\frac{1}{2}G \int\int \frac{\rho\left(\mathbf{x}_1\right)\rho\left(\mathbf{x}_2\right)}{r_{12}} dV_1 dV_2$$

A discretized particle version reads

$$U = -\sum_{i<j} Gm_i m_j / r_{ij}.$$

If $m_i m_j > 0$ it is clear that U is a negative energy and clearly if Einstein's relation, $E = mc^2$, is taken to be correct for any form of energy, one must conclude that the system mass component due to the gravitational potential energy U must be negative as well. For example, the system mass M then becomes

$$M = \sum_i m_i - \frac{1}{c^2}\sum_{i<j} Gm_i m_j / r_{ij}.$$

Further, for the continuous problem one can write

$$U = \frac{1}{2}\int \rho\left(\mathbf{x}_1\right)\phi\left(\mathbf{x}_1\right) dV_1$$

where the gravitational potential is given by

$$\phi(\mathbf{x}_1) = -G \int \frac{\rho(\mathbf{x}_2)}{r_{12}} dV_2.$$

The gravitational field $\mathbf{g}$ is given by

$$\mathbf{g} = -\nabla\phi$$

and the mass density ρ can due to Gauss's law and the superposition principle be expressed as

$$\rho = -\frac{1}{4\pi G}\nabla \cdot \mathbf{g} = \frac{1}{4\pi G}\nabla^2\phi.$$

The potential energy can therefore be written as

$$\begin{aligned} U &= \frac{1}{8\pi G}\int \phi\nabla^2\phi dV \\ &= \frac{1}{8\pi G}\int \nabla \cdot (\phi\nabla\phi)\, dV - \frac{1}{8\pi G}\int \nabla\phi \cdot \nabla\phi dV \\ &= -\frac{1}{8\pi G}\int \mathbf{g}\cdot\mathbf{g}\, dV. \end{aligned}$$

The first integral integrated over all space is zero due to Gauss's theorem:

$$\begin{aligned} &\frac{1}{8\pi G}\int \nabla \cdot (\phi\nabla\phi)\, dV = \frac{1}{8\pi G}\int (\phi\nabla\phi)\cdot d\mathbf{A} \\ &\sim \frac{1}{8\pi G}(\phi\nabla\phi)\, 4\pi R^2 \sim \frac{1}{R}\frac{1}{R^2}R^2 \to 0. \end{aligned}$$

Thus, classically the potential energy of the system can be calculated according to

$$U = -\frac{1}{8\pi G}\int \mathbf{g}\cdot\mathbf{g}\, dV$$

In RGF, the gravitational energy density is written

$$\rho_U = -\frac{1}{8\pi G}\left(\mathbf{g}\cdot\mathbf{g} + c^2\mathbf{h}\cdot\mathbf{h}\right)$$

where $\mathbf{h}$ is a relativistic component from Eq. (16). We have that

$$c^2\mathbf{h}\cdot\mathbf{h} = \frac{1}{c^2}(\mathbf{g}\times\mathbf{u})\cdot(\mathbf{g}\times\mathbf{u}) = \frac{1}{c^2}\left(g^2u^2 - (\mathbf{g}\cdot\mathbf{u})^2\right).$$

This relativistic energy density can safely be neglected in most cases. The total energy integrated over all space then becomes the same as in the classical picture

$$U = -\frac{1}{8\pi G}\int \mathbf{g}\cdot\mathbf{g}\, dV.$$

Now consider a spherical body whose radius is R. The application of Gauss's law

$$M_0(r) = -\frac{1}{4\pi G}\int \mathbf{g}\cdot d\mathbf{A} = \frac{1}{4\pi G}g(r)\, 4\pi r^2$$

gives that the field strength $g(r) = GM_0(r)/r^2$. If the mass density is homogeneous then $M_0(r) = M_0 r^3/R^3$. Then, the field strength at an arbitrary radius r is given by

$$g(r) = \frac{GM_0}{R^3} r, \ if \ r \leq R$$
$$g(r) = \frac{GM_0}{r^2}, \ if \ r > R$$

and the potential energy due to the internal field becomes

$$U_{int} = -\frac{1}{2G}\int_0^R \mathbf{g}\cdot\mathbf{g} r^2 dr = -\frac{1}{10}\frac{GM_0^2}{R}.$$

The external field energy is given by

$$U_{ext} = -\frac{1}{2G}\int_R^\infty \mathbf{g}\cdot\mathbf{g} r^2 dr = -\frac{1}{2}\frac{GM_0^2}{R}$$

The total field energy thus corresponds to the field mass

$$M_{field} = \frac{U_{int} + U_{ext}}{c^2} = -\frac{3}{5}\frac{GM_0^2}{Rc^2} = -\alpha\frac{GM_0^2}{Rc^2}, \tag{76}$$

where M_0 is the rest mass and $3/5$ is replaced by α for a general mass distribution. In the case of a neutron star, $\alpha > 3/5$. The fact that the field mass is negative is simply a consequence from Einstein's energy mass equivalence and that the gravitational interaction energy is negative. This is also in agreement with (Sebens 2022). The effective mass is given by $M_{eff} = M_0 + M_{field}$ and we argue in Appendix 4.8 that M_{eff} only affects the inertial mass ($M_0 \rightarrow M_{eff}$), while the gravitational mass remains intact (fixed at M_0). In RGF, one could say that the field mass is inertial in its character. Thus, M_{field} in RGF is not a source of any additional gravitational fields (as it would be in general relativity).

Further, the above $M_{eff} = M_0 + M_{field}$ is of course appropriate for a nonrelativistic speed only. The general expression should be given by: $M_{eff} = \left(M_0 + M_{field}\right)/\sqrt{1 - u^2/c^2}$. Also since $M_{field} < 0$, one can but wonder if there is some exotic physics involved here. However, the mass $M_0 + M_{field} > 0$ for any reasonable object. The radius would need to become smaller than the Schwarzschild radius for a negative mass object ($R < \alpha GM_0/c^2$). Although the field mass effect is usually expected to be negligible it cannot be neglected when the object size R is small and simultaneously the mass M_0 is large. One such object is the case of a neutron star (or quark star) where $R \sim 10$ km and its mass is exceeding that of the Sun. The structural parameter α depends on the assumed inner mass/energy density structure of the spherically symmetric object. The constant α could therefore become larger than for the above homogeneous case ($\alpha = 3/5$) if the core density is much higher than the surrounding material. Indeed, a recent study suggests that heavy neutron stars ($\sim 2\ M_\odot$) may have a quark matter core which is over 40 times more dense than the surrounding hadronic matter (Annala 2020).

4.8 Interactions with the field itself?

Here, we ask the question how the field mass is relevant for the inertial mass versus the gravitational mass. We will show that in RGF the field mass effect is connected only to the inertial mass. We start by assuming that there is an interaction with the gravitational field itself and show that such an assumption in RGF would contradict experiment.

Thus, consider the gravitational interaction acting on a test particle at a distance r due to a central object with rest mass M_0. One now need to deal with the situation that r may not be at a infinitely large distance. This has an interesting effect on M_{field}. For a nearly static external field $\mathbf{g}$ from the central object, the energy density of its field is given by

$$\rho_U = -\frac{1}{8\pi G}\mathbf{g}\cdot\mathbf{g}$$

and the corresponding *mass* density of the external field according to Einstein's relation, $E = mc^2$, then becomes

$$\rho_{ext} = -\frac{1}{8\pi Gc^2}\mathbf{g}\cdot\mathbf{g}.$$

Gauss law for a spherical symmetrical mass density tells us that these external mass densities from the central body build up a field mass $M_{ext}(r)$ that depends on the distance r to the test particle. At this point, the external part of the $\mathbf{g}$-field itself makes an additional contribution forming the total field

$$\mathbf{g}_{tot} = \mathbf{g} + \mathbf{g}_{ext} = -\frac{G}{r^2}\left(M + M_{ext}(r)\right)\mathbf{e}_r$$

where $M = M_0 + M_{int}$, i.e., the rest mass plus the internal field mass which was described in the previous section. The external field mass is given by

$$M_{ext}(r) = -\frac{1}{8\pi Gc^2}\int_R^r g^2 4\pi r^2 dr = -\frac{1}{2}\frac{GM^2}{Rc^2} + \frac{1}{2}\frac{GM^2}{rc^2}$$

where R is the radius of the central object. An iterative procedure for computing $M_{ext}(r)$ should now in principle be performed since the field has been updated into $\mathbf{g}_{tot}$ but it is quite clear that this effect is sufficiently small to neglect. The mass for use in a calculation then becomes

$$\begin{aligned} M + M_{ext}(r) &= M_0 + M_{int} + M_{ext}(r) \\ &= M_0 + M_{field} + \frac{1}{2}\frac{GM^2}{rc^2} = M_{eff} + \frac{1}{2}\frac{GM^2}{rc^2} \end{aligned}$$

The effective mass M_{eff} is just a constant (independent of r) so it would just be absorbed in what we normally would call rest mass of the central object (fitting orbits, etc.). What has been changed here is a small relativistic additional r-dependent term that can effect the fine features of an orbit (e.g., orbital precession). A good approximation in the solar system is that $M_{eff} \approx M \approx M_0$ so

$$\begin{aligned} \mathbf{g}_{tot} &= -\frac{G}{r^2}\left(M + M_{ext}(r)\right)\mathbf{e}_r \\ &\approx -\frac{G}{r^2}\left(M_0 + \frac{1}{2}\frac{GM_0^2}{rc^2}\right)\mathbf{e}_r = -\frac{GM_0}{r^2}\mathbf{e}_r - \frac{1}{2}\frac{G^2M_0^2}{r^3c^2}\mathbf{e}_r \end{aligned}$$

and the force acting on the test particle is $\mathbf{f} = m\mathbf{g}_{tot}$, where m is the relativistic mass. To order $1/c^2$, this leads to an additional acceleration term

$$\mathbf{a}_{field} = -\frac{1}{2}\frac{G^2M_0^2}{r^3c^2}\mathbf{e}_r$$

which in the case of Mercury would contribute an additional 3.58 arcsec/cyr to its orbital precession. However, RGF without this field effect already perfectly predicts 42.983 arcsec/cyr as can be seen in Table 3 of Sect. 3.8. One has to conclude that in RGF the test particle m only interacts directly with the central object and not its gravitational field mass components. A consistent interpretation is that the inertial mass of the central object is indeed affected ($M_0 \rightarrow M_0 + M_{field}$) while its gravitational mass is not (i.e., it remains fixed to M_0). There are no gravitational interactions between the field mass components of the central object M_0 and the point mass m. Of course for the Sun, the M_{field} contribution to its inertial mass is negligible anyway but for a object such as a neutron star this effect could potentially play a significant role for its inertial mass which would affect the dynamical evolution of a binary system.

4.9 Self-force on a compact object

Due to gravitational propagation delays within the structure of a compact object (e.g., a neutron star, white dwarf, etc.), one can speculate that self-forces not only could occur, but could even be important. In analogy with relativistic electrodynamics, as a body M_0 with radius R is accelerated, a self-force due to propagation delays arise, see p. 588 in Jackson (1962):

$$\mathbf{f}_{self} = -M_{field}\mathbf{a} = \alpha \frac{GM_0^2}{Rc^2}\mathbf{a} \tag{77}$$

4.10 Radiation damping

Here, we consider a point mass m_0 moving at nonrelativistic speeds in order to simplify calculations. Even for a neutron star pair in close orbit it is expected that both their speeds fulfill $u << c$. As the object $m = m_0$ accelerates, positive gravitational energy waves are emitted in various directions. The flux S, i.e., the total amount of gravitational radiation energy that passes per unit area and unit time is easy to derive and is given by

$$S = \frac{c}{4\pi G} < g_{rad}^2 > .$$

The radiation field $\mathbf{g}_{rad}$ is listed in Sect. 2.7. The symbol $<>$ denotes a time average in the case of oscillatory motion. We here assume that Feynman's proposed general expression for the total radiated power is correct (see p. 124 (Feynman 1995)). An appropriate expression for the total gravitational power emitted in circular orbit then becomes

$$P_{rad} = \frac{2}{3}\frac{u^2}{c^2}\frac{Gm^2}{c^3}\mathbf{a}\cdot\mathbf{a} - \frac{2}{3}\frac{u^2}{c^2}\frac{Gm^2}{c^3}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{a}) = -\frac{2}{3}\frac{u^2}{c^2}\frac{Gm^2}{c^3}\mathbf{u}\cdot\dot{\mathbf{a}}$$

where from now on m is the rest mass. For an oscillatory motion, the second term will be negligible since the average $\left\langle \frac{d}{dt}(\mathbf{u}\cdot\mathbf{a})\right\rangle = 0$. For a simple circular motion, the term vanishes completely because $\mathbf{u}\cdot\mathbf{a} = 0$. Thus, only the $\mathbf{a}\cdot\mathbf{a}$-term is important for oscillatory motion which agrees with the classical result in electrodynamics (Larmor formula). However, by using Feynman's general expression (the term with the *jerk* $\dot{\mathbf{a}}$), the below derivation becomes unusually straightforward. One can then also see that radiation is *not* predicted for a body under constant acceleration $\mathbf{a}$.

As positive radiative power P_{rad} is emitted, a corresponding negative work W will be performed on the object itself. This work of the object can therefore be written

$$W = \frac{2}{3}\frac{Gm^2}{c^3}\int \frac{u^2}{c^2}\mathbf{u}\cdot\dot{\mathbf{a}}dt \equiv \int \mathbf{f}_{rad}\cdot\mathbf{u}dt$$

or

$$\int \left(\mathbf{f}_{rad} - \frac{2}{3}\frac{u^2}{c^2}\frac{Gm^2}{c^3}\dot{\mathbf{a}} \right) \cdot \mathbf{u} dt = 0.$$

In order for this integral to be zero for general motion, we must require that

$$\mathbf{f}_{rad} = \frac{2}{3}\frac{u^2}{c^2}\frac{Gm^2}{c^3}\dot{\mathbf{a}}.$$

We thus arrive at the classical result that the component acceleration due to the emission of gravitational radiation is approximately given by

$$\mathbf{a}_{rad} = \frac{2}{3}\frac{u^2}{c^2}\frac{Gm}{c^3}\dot{\mathbf{a}}.$$

Computationally (i.e., numerical solution), this may be solved by keeping track of $\mathbf{a}(t)$ and $\mathbf{a}(t - \Delta t)$ due to the gravitational forces only. Then estimate $\dot{\mathbf{a}}(t) \approx (\mathbf{a}(t) - \mathbf{a}(t - \Delta t))/\Delta t$, compute $\mathbf{a}_{rad}(t)$ and finally adding it to $\mathbf{a}(t)$. For the simple case of a circular orbit, it can easily be derived that

$$\dot{\mathbf{a}} = -\frac{u^2}{r^2}\mathbf{u}.$$

Thus, a test particle m initially in circular orbit about a central object M, will lose energy which becomes manifested as an inward spiral toward the object M. This energy loss is of course balanced by a simultaneous emission of positive gravitational radiation energy per unit time

$$P_{rad} = \int S dA.$$

4.11 The general two-particle problem

It is of interest to see what happens if both particles m_1 and m_2 are allowed to move. In Sect. 2.5, we found that if m_1 (there denoted M) is held fixed, then the other mass m_2 (there denoted m) is given by

$$m_2 = m_2(0)\, e^{Gm_1(-1/r(0)+1/r)/c^2},$$

where at $t = 0$ the distance $r = r(0)$ and the mass $m_2 = m_2(0)$. As m_1 was kept fixed in Sect. 2.5, m_1 is a rest mass and m_2 is a relativistic mass in the above expression. However, if both point masses are allowed to move we will show below that then

$$m_1 = m_1(0)\, e^{aGm_2(-1/r(0)+1/r)/c^2}$$
$$m_2 = m_2(0)\, e^{bGm_1(-1/r(0)+1/r)/c^2} \tag{78}$$

where the constants are given by $a = m_2(0)/(m_1(0) + m_2(0))$ and $b = m_1(0)/(m_1(0) + m_2(0))$. In this case, both the masses m_1 and m_2 are relativistic masses.

According to the superposition principle, the change in the work can be written as

$$dW = P_1 dt + P_2 dt = c^2 dm_1 + c^2 dm_2,$$

where P is the power (cf. Section 3.6). We also can write

$$dW = \mathbf{f}_1 \cdot d\mathbf{r}_1 + \mathbf{f}_2 \cdot d\mathbf{r}_2.$$

As we here are neglecting propagation effects (Sect. 2.7 and 2.8), we have that $\mathbf{f}_1 = -\mathbf{f}_2$. Thus, we approximately have that

$$dW = \mathbf{f}_1 \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) .$$

Now since

$$\begin{aligned}\mathbf{r}(t) = \mathbf{r}_2(t) - \mathbf{r}_1(t) &= \mathbf{r}_2(t-dt) + d\mathbf{r}_2 - \mathbf{r}_1(t-dt) - d\mathbf{r}_1 \\ &= \mathbf{r}(t-dt) + d\mathbf{r}_2 - d\mathbf{r}_1.\end{aligned}$$

Thus,

$$d\mathbf{r} = \mathbf{r}(t) - \mathbf{r}(t-dt) = d\mathbf{r}_2 - d\mathbf{r}_1$$

so the work becomes

$$dW = -\mathbf{f}_1 \cdot d\mathbf{r} = -\frac{Gm_1m_2}{r^2}dr.$$

We thus have the following equation from the infinitesimal work

$$\frac{c^2}{m_1m_2}dm_1 + \frac{c^2}{m_1m_2}dm_2 = -\frac{G}{r^2}dr$$

and integration leads to

$$\frac{c^2}{m_2}\int_{m_1(0)}^{m_1}\frac{1}{m_1}dm_1 + \frac{c^2}{m_1}\int_{m_2(0)}^{m_2}\frac{1}{m_2}dm_2 = -G\int_{r(0)}^{r}\frac{1}{r^2}dr$$

which results in

$$\frac{c^2}{m_2}\ln\frac{m_1}{m_1(0)} + \frac{c^2}{m_1}\ln\frac{m_2}{m_2(0)} = -G\left(-\frac{1}{r} + \frac{1}{r(0)}\right) \tag{79}$$

By inserting the masses from Eq. (78), we indeed see that this is a solution to Eq. (79). Also, Eq. (79) shows that if m_1 is kept fixed we get

$$\frac{c^2}{m_1}\ln\frac{m_2}{m_2(0)} = G\left(-\frac{1}{r(0)} + \frac{1}{r}\right)$$

which is equivalent to

$$m_2 = m_2(0)\,e^{Gm_1(-1/r(0)+1/r)/c^2}.$$

This, of course, is just the first equation given in this appendix which was derived differently in Sect. 2.5.

Let us now multiply Eq. (79) with m_1m_2 so we get

$$m_1c^2\ln\frac{m_1}{m_1(0)} + m_2c^2\ln\frac{m_2}{m_2(0)} = G\frac{m_1m_2}{r} - G\frac{m_1m_2}{r(0)} \tag{80}$$

If one considers low velocities, a Taylor expansion reveals that for example

$$m_1c^2\ln\frac{m_1}{m_1(0)} \approx \frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1u_1(0)^2 = T_1 - T_1(0),$$

where here T_1 has the classical form of kinetic energy for the point mass m_1. If we now also let the relativistic masses approach their rest masses, Eq. (80) becomes

$$T_1 - T_1(0) + T_2 - T_2(0) = -U + U(0)$$

This thus leads to the energy conservation law of classical mechanics

$$T_1 + T_2 + U = T_1(0) + T_2(0) + U(0).$$

Although the relativistic Eq. (80) cannot be separated in the same way and form an analogous relativistic energy, one can from Eq. (79) get

$$\frac{c^2}{m_2}\ln\frac{m_1}{m_1(0)} + \frac{c^2}{m_1}\ln\frac{m_2}{m_2(0)} - \frac{G}{r} = -\frac{G}{r(0)} = const.$$

so given the initial conditions this equation provides information of the future dynamics in a similar way as the conservation of energy.

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