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ORIGINAL ARTICLE

A new collision-based periodic orbit in the three-dimensional eight-body problem

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Abstract
We construct a highly-symmetric periodic orbit of eight bodies in three dimensions. In this orbit, each body collides with its three nearest neighbors in a regular periodic fashion. Regularization of the collisions in the orbit is achieved by an extension of the Levi-Civita method. Initial conditions for the orbit are found numerically. Linear stability of the orbit is then shown using a technique by Roberts. Evidence toward higher-order stability is presented as an additional result of a numerical calculation.

Keywords n-body problem · Binary collision · Regularization · Linear stability

Mathematics Subject Classification Primary 70F16 · Secondary 37N05 · 37J25 · 70F10

1 Introduction

In the *Principia Mathematica* (see Newton, updated), Newton gives mathematical equations governing the motion of point masses within their mutual gravitational field. Specifically, for n point masses in $\mathbb{R}^d$ located at $\mathbf{x}_i$ with mass m_i for $i = 1, 2, \ldots, n$, we have that

$$m_i \ddot{\mathbf{x}}_i = \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} \left(\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right). \tag{1}$$

Here, the dot represents the derivative with respect to time, and G is a constant. A suitable choice of units gives $G = 1$, which is often assumed for mathematical simplicity. (In SI units, the US National Institute of Standards and Technology[1] gives the value $G = 6.67430 \times 10^{-11} \mathrm{m}^3/\mathrm{kg} \cdot \mathrm{s}^2$, with a standard uncertainty of $0.00015 \times 10^{-11} \mathrm{m}^3/\mathrm{kg} \cdot \mathrm{s}^2$.)

Collision singularities of the n-body problem occur when $\mathbf{x}_i = \mathbf{x}_j$ for some $i \neq j$. Under suitable conditions, collisions of two bodies can be regularized. *Regularization* involves a change of temporal and spatial variables so that the collision point becomes a regular point

[1] https://physics.nist.gov/cuu/Constants/index.html.

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for the differential equations. Collision singularities have received a great deal of study. Of particular note is a result by McGehee (1973), which shows that in general, a collision of three or more bodies cannot be regularized.

Many periodic orbits featuring collisions have been produced. Existence, stability, and other properties of periodic orbits with three bodies in one spatial dimension are studied in both analytical and numerical contexts as early as 1956 in Schubart (1956) and as recently as 2019 in Kuang et al. (2019). Works between these years include (Hénon 1977; Hietarinta and Mikkola 1993; Saito and Tanikawa 2007; Moeckel 2008; Venturelli 2008; Saito and Tanikawa 2009, 2010; Shibayama 2011; Yan 2012c; Ouyang et al. 2015). Orbits with four bodies in one spatial dimension are featured in Shibayama (2011), Martínez (2012), Huang (2012), and Yan (2012a). Orbits in two spatial dimensions featuring collisions were studied as early as 1979 in Broucke (1979) and as recently as 2021 in Simmons (2021), with other notable works including (Roy and Steves 2001; Bakker et al. 2010; Sivasankaran et al. 2010; Bakker et al. 2011; Waldvogel 2012; Bakker et al. 2012; Ouyang et al. 2012; Yan 2012b; Bakker and Simmons 2015). Additionally, in Shibayama (2011) and Martínez (2012), large families of highly-symmetric orbits are given in one, two, and three dimensions, all of which can be expressed in two degrees of freedom. Additionally, three-dimensional restricted collision-based orbits are studied in Moeckel (1984), Lúcia and Claudio (2008), Brandão et al. (2017), and Guardia et al. (2021) as a case of the $e = 1$ Sitnikov problem, which can be reduced to a time-dependent two-degree-of-freedom problem.

This paper studies a three-degree of freedom, highly-symmetric, periodic orbit of eight bodies featuring collisions. The bodies form the vertices of a rectangular prism at all points in time, with edges parallel to the standard coordinate axes in $\mathbb{R}^3$. Each body collides with its three nearest neighbors in a regular periodic fashion. This appears to be the first three-degree-of-freedom collision-based periodic orbit studied.

The remainder of the paper is as follows: In Sect. 2, we set up and regularize the Hamiltonian that corresponds to the configuration being considered. Section 3 details the construction of the periodic orbit. We first describe the orbit in the regularized setting. Then, we analytically establish sufficient conditions for the orbit to exist. Finally, we complete the existence proof with a numerical calculation.

Section 4 establishes the linear stability of the orbit. We first review some preliminary details of stability, including linear stability. Next, we establish notation for the symmetries of the orbit. We next detail some results by Roberts in Roberts (2007) that allow us to establish the linear stability of the orbit in a rigorous numerical fashion in terms of these symmetries. Applications to the orbit under consideration are detailed after each result. Finally, in Sect. 5, we give results of the numerical stability calculation established in the previous section, as well as some further numerical evidence of higher-order stability of the orbit.

2 The Hamiltonian setting and regularization

2.1 Configuration

We consider the Newtonian 8-body problem with point unit masses located at $(\pm q_1, \pm q_2, \pm q_3)$, where the choices of sign are taken independently of each other, and each $q_i \geq 0$ (see Fig. 1). The positions of the bodies lie at the vertices of a rectangular prism. We will accordingly refer to this as the *rectangular prismatic configuration*, or RPC for brevity.

Note that when $q_1 = 0$, if $q_2 q_3 \neq 0$, then we have four pairs of bodies colliding in the $x = 0$ plane. Similar results hold in the $y = 0$ and $z = 0$ planes by permuting the subscripts.

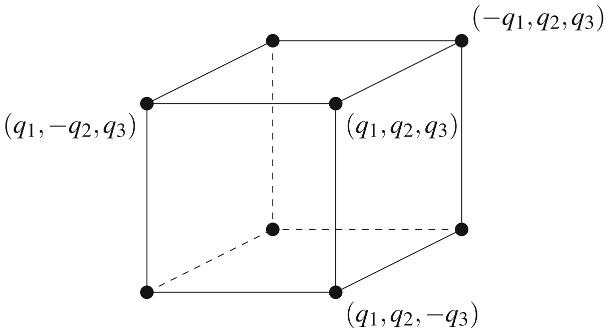

Fig. 1 The rectangular prismatic configuration (RPC)

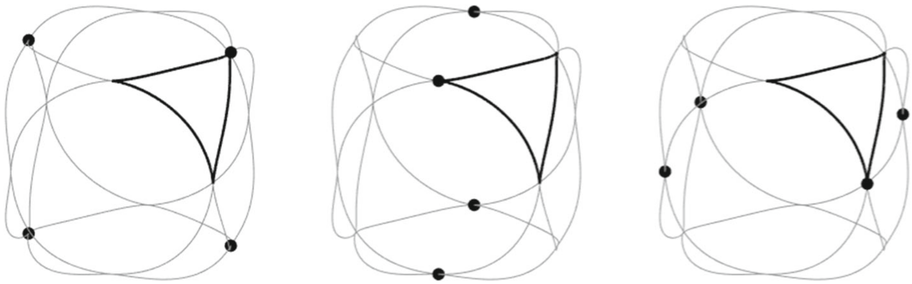

Fig. 2 The RPC orbit. Four simultaneous binary collisions occur in the $x = 0$, $y = 0$, and $z = 0$ planes in turn as pictured. For clarity, the trajectory of one of the eight bodies is highlighted

We seek an orbit possessing these four-pair collisions in the $x = 0$, $y = 0$, and $z = 0$ planes in a periodic fashion as pictured in Fig. 2.

An analogous two-degree-of-freedom orbit of four bodies in the xy plane, with bodies located at $\pm(x, y)$ and $\pm(y, x)$ with alternating collisions along the lines $y = \pm x$, was shown to exist in Ouyang et al. (2012). Linear stability of that orbit was established in Bakker et al. (2010). A four-degree-of-freedom variation of this orbit with two pairs of unequal masses was discussed in Bakker et al. (2012), in which the linear stability was shown to be dependent upon the ratio of the pairs of masses.

2.2 The Hamiltonian setting

The potential energy is the sum of 28 terms. For convenience, these are divided up into *cube diagonals*, *face diagonals*, and *edges*.

Each of the four cube diagonals (see Fig. 3) contributes a term of the form

$$\frac{1}{\sqrt{(2q_1)^2 + (2q_2)^2 + (2q_3)^2}} = \frac{1}{2\sqrt{q_1^2 + q_2^2 + q_3^2}}. \tag{2}$$

Let $\mathcal{I} = \{1, 2, 3\}$. Each face diagonal (see Fig. 4) contributes a term of the form

$$\frac{1}{\sqrt{(2q_i)^2 + (2q_j)^2}} = \frac{1}{2\sqrt{q_i^2 + q_j^2}}, \tag{3}$$

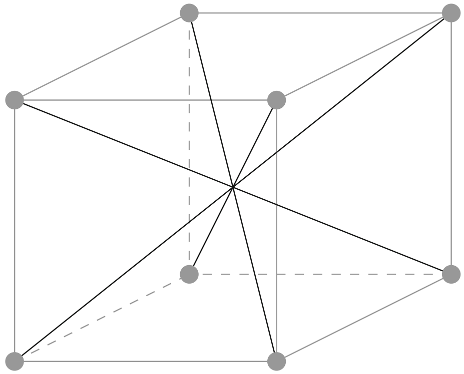

Fig. 3 Cube diagonals (4 total)

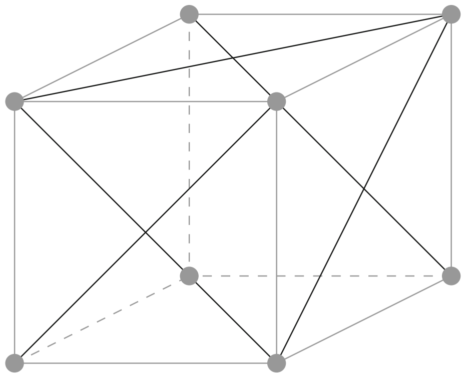

Fig. 4 Face diagonals (12 total—the remaining six are on the opposite faces of the cube)

with $i, j \in \mathcal{I}$ and $i \neq j$. Specifically are four terms for each of the three possible choices of indices. Lastly, each edge contributes a term of the form

$$\frac{1}{\sqrt{(2q_i)^2}} = \frac{1}{2q_i}, \tag{4}$$

with $i \in \mathcal{I}$. Again, for each index there are four terms. Hence, the total potential energy of the system is

$$U = \frac{2}{\sqrt{q_1^2 + q_2^2 + q_3^2}} + \frac{2}{\sqrt{q_1^2 + q_2^2}} + \frac{2}{\sqrt{q_1^2 + q_3^2}} + \frac{2}{\sqrt{q_2^2 + q_3^2}} + \frac{2}{q_1} + \frac{2}{q_2} + \frac{2}{q_3}. \tag{5}$$

Let $p_i = \dot{q}_i$ denote the components of the momentum of the bodies. The kinetic energy for the system is

$$K = \frac{8\left(\sqrt{p_1^2 + p_2^2 + p_3^2}\right)^2}{2} = 4\left(p_1^2 + p_2^2 + p_3^2\right). \tag{6}$$

The Hamiltonian for the system is then given by $H = K - U$.

It is worth noting that as long as $q_1 q_2 q_3 \neq 0$, then $\ddot{q}_i < 0$ for $i \in I$.

2.3 Regularization

We regularize the collisions that occur at $q_i = 0$ using an extension of the Levi-Civita method (see Levi-Civita 1920). Specifically, let

$$F = \sum_{i \in \mathcal{I}} \sqrt{q_i} P_i. \tag{7}$$

This generates a coordinate transformation given by

$$Q_i = \frac{\partial F}{\partial P_i} = \sqrt{q_i} \quad p_i = \frac{\partial F}{\partial q_i} = \frac{P_i}{2\sqrt{q_i}}, \tag{8}$$

or

$$q_i = Q_i^2 \quad p_i = \frac{P_i}{2Q_i}. \tag{9}$$

In these coordinates, the potential energy for the system is given by

$$\begin{aligned} \tilde{U} = & \frac{2}{\sqrt{Q_1^4 + Q_2^4 + Q_3^4}} + \frac{2}{\sqrt{Q_1^4 + Q_2^4}} + \frac{2}{\sqrt{Q_1^4 + Q_3^4}} \cdots \\ & + \frac{2}{\sqrt{Q_2^4 + Q_3^4}} + \frac{2}{Q_1^2} + \frac{2}{Q_2^2} + \frac{2}{Q_3^2}. \end{aligned} \tag{10}$$

The new kinetic energy is given by

$$\tilde{K} = \frac{P_1^2}{Q_1^2} + \frac{P_2^2}{Q_2^2} + \frac{P_3^2}{Q_3^2}. \tag{11}$$

The new Hamiltonian is given by $\tilde{H} = \tilde{K} - \tilde{U}$.

Lastly, to regularize the collisions at $Q_i = 0$, we apply a change of time satisfying

$$\frac{dt}{ds} = Q_1^2 Q_2^2 Q_3^2. \tag{12}$$

This gives the regularized Hamiltonian $\Gamma = \frac{dt}{ds}(\tilde{H} - E)$, or

$$\begin{aligned} \Gamma = & P_1^2 Q_2^2 Q_3^2 + Q_1^2 P_2^2 Q_3^2 + Q_1^2 Q_2^2 P_3^2 \\ & - \frac{2Q_1^2 Q_2^2 Q_3^2}{\sqrt{Q_1^4 + Q_2^4 + Q_3^4}} - \frac{2Q_1^2 Q_2^2 Q_3^2}{\sqrt{Q_1^4 + Q_2^4}} - \frac{2Q_1^2 Q_2^2 Q_3^2}{\sqrt{Q_1^4 + Q_3^4}} - \frac{2Q_1^2 Q_2^2 Q_3^2}{\sqrt{Q_2^4 + Q_3^4}} \\ & - 2Q_2^2 Q_3^2 - 2Q_1^2 Q_3^2 - 2Q_1^2 Q_2^2 - E Q_1^2 Q_2^2 Q_3^2, \end{aligned} \tag{13}$$

where E is the fixed energy of the system.

We now show that the system has been regularized as claimed. Let $i, j, k \in \mathcal{I}$ be distinct. Then, at the collision where $Q_i = 0$, $Q_j \neq 0$, and $Q_k \neq 0$, the condition $\Gamma = 0$ forces

$$P_i^2 Q_j^2 Q_k^2 - 2Q_j^2 Q_k^2 = (P_i^2 - 2) Q_j^2 Q_k^2 = 0. \tag{14}$$

Then $P_i = \pm\sqrt{2}$. Moreover, since

$$\dot{Q}_i = \frac{d\Gamma}{dP_i} = 2P_i Q_j^2 Q_k^2 \tag{15}$$

then $\dot{Q}_i \neq 0$ when $Q_i = 0$. Hence, the orbit can be continued past the collision. (In the regularized setting, we will use the dot notation to represent the derivative with respect to the new time variable s.)

An important feature of the regularization that can be determined from Eq. 15 is that both $\dot{Q}_i$ and P_i have the same sign at the collision time. Since P_i is continuous and non-zero at collision time, the sign of P_i is the same before and after the collision. Hence, the sign of $\dot{Q}_i$ also does not change, so Q_i must either pass from a negative to a positive value at collision, or from a positive to a negative one.

It is worth noting that the planar orbit discussed in Ouyang et al. (2012) and Bakker et al. (2010) is not "recoverable" from the RPC orbit under consideration, as forcing $z = 0$ throughout yields $Q_3 = 0$. From Eq. 14, we then have $P_3 = \pm\sqrt{2}$ or $Q_1 Q_2 = 0$. If the former, the $z = 0$ plane cannot be an invariant subspace for this Hamiltonian system. If the latter, the resulting orbit must lie entirely on the x- and y-axes, which does not describe the planar orbits discussed.

3 The periodic orbit

3.1 Description

The desired orbit passes through four simultaneous binary collisions in the $x = 0$, $y = 0$, and $z = 0$ planes in a periodic fashion, as pictured in Fig. 2. In a physical sense, we start with the bodies with (non-regularized) positions given by

$$(\pm q_1, \pm q_2, \pm q_3) = (0, \omega, \omega) \tag{16}$$

and ending at

$$(\pm q_1, \pm q_2, \pm q_3) = (\omega, 0, \omega), \tag{17}$$

for some positive number ω. The proposed orbit will then be extended by a symmetry coinciding with a rotation of 120° about the line $x = y = z$ in $\mathbb{R}^3$. In other words, the orbit continues through a sequence of collisions

$$(\pm q_1, \pm q_2, \pm q_3) : (0, \omega, \omega) \to (\omega, 0, \omega) \to (\omega, \omega, 0) \to (0, \omega, \omega) \to \dots, \tag{18}$$

with the collisions being equally-spaced in time.

In the regularized coordinates, the velocity components can also be defined. Let $\gamma(s) = (Q_1(s), Q_2(s), Q_3(s), P_1(s), P_2(s), P_3(s))^T$. At each collision time with $Q_i = 0$, the sign of Q_i changes as noted at the end of Sect. 2.3. Additionally, both $\gamma(s)$ and $-\gamma(s)$ correspond to the same setting in the original coordinates. Hence, in the regularized setting, one period of the orbit passes through six collisions rather than three.

3.2 Extension by symmetry

Lemma 1 *Suppose $\gamma(s)$ is a solution to the regularized Hamiltonian system Γ that satisfies*

$$\gamma(0) = (0, \alpha, \alpha, \sqrt{2}, -\beta, \beta)^T \tag{19}$$

and

$$\gamma(2\tau) = (\alpha, 0, \alpha, \beta, -\sqrt{2}, -\beta)^T. \tag{20}$$

for some $\tau > 0$ *and* $E < 0$. *Then* $\gamma(s)$ *extends to a* 12τ *periodic orbit for the system* Γ.

Proof We first establish the symmetries that will allow us to extend the orbit as claimed. By direct calculation, we find that the equation for $\dot{Q}_i$ is negated under the transformation $P_i \mapsto -P_i$ and remains fixed under any sign change of the remaining variables. We also find that $\dot{P}_i$ is negated under $Q_i \mapsto -Q_i$ and remains fixed under any other sign change of the remaining variables. Furthermore, for any permutation $\sigma \in S_3$, since Γ is fixed under permutation of the subscripts by σ, then the equation of motion for $\dot{Q}_i$ in terms of $Q_1, Q_2, Q_3, P_1, P_2, P_3$ is the same as that of $\dot{Q}_{\sigma(i)}$ in terms of $Q_{\sigma(1)}, Q_{\sigma(2)}, Q_{\sigma(3)}, P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}$. Similar permutation results hold for P_i.

Consider the orbit with initial conditions $\gamma(2\tau)$. By the symmetries just discussed, we have that

$$\begin{aligned}
\dot{Q}_1(2\tau) &= \dot{Q}_3(0),\\
\dot{Q}_2(2\tau) &= -\dot{Q}_1(0),\\
\dot{Q}_3(2\tau) &= \dot{Q}_2(0),\\
\dot{P}_1(2\tau) &= \dot{P}_3(0),\\
\dot{P}_2(2\tau) &= -\dot{P}_1(0),\\
\dot{P}_3(2\tau) &= \dot{P}_2(0).
\end{aligned} \tag{21}$$

Moreover, the equations of motion are the same as those on the interval $s \in [0, 2\tau]$ under permutations and sign changes as discussed above. Existence and uniqueness of solutions to differential equations gives

$$\begin{aligned}
Q_1(s+2\tau) &= Q_3(s),\\
Q_2(s+2\tau) &= -Q_1(s),\\
Q_3(s+2\tau) &= Q_2(s),\\
P_1(s+2\tau) &= P_3(s),\\
P_2(s+2\tau) &= -P_1(s),\\
P_3(s+2\tau) &= P_2(s),
\end{aligned} \tag{22}$$

is a solution to the Hamiltonian system given by Γ. Setting $s = 2\tau$, we have that

$$\gamma(4\tau) = (\alpha, -\alpha, 0, -\beta, -\beta, -\sqrt{2})^T. \tag{23}$$

Repeating the argument with initial conditions given by $\gamma(4\tau)$ gives

$$\gamma(6\tau) = (0, -\alpha, -\alpha, -\sqrt{2}, \beta, -\beta)^T. \tag{24}$$

Continuing in turn, we have that

$$\gamma(8\tau) = (-\alpha, 0, -\alpha, -\beta, \sqrt{2}, \beta)^T, \tag{25}$$

$$\gamma(10\tau) = (-\alpha, \alpha, 0, \beta, \beta, \sqrt{2})^T, \tag{26}$$

$$\gamma(12\tau) = (0, \alpha, \alpha, \sqrt{2}, -\beta, \beta)^T. \tag{27}$$

Since $\gamma(0) = \gamma(12\tau)$, the periodic orbit has been constructed as claimed. □

Physically, the orbit constructed in Lemma 1 corresponds to an orbit in which all bodies start in the $x = 0$ plane at collisions symmetrically placed along the lines $y = \pm z$. The velocity of each body projected onto the $x = 0$ plane is orthogonal to the projection of its position. The orbit then proceeds to collisions in the $y = 0$ and $z = 0$ planes with similarly symmetric positions and velocities.

Note: We do not rule out the possibility of the existence of a "less symmetric" orbit. Indeed, the arguments in Lemma 1 give the same conclusion if we assume that

$$\gamma(0) = (0, a, b, \sqrt{2}, -c, d)^T, \tag{28}$$

$$\gamma(2\tau) = (b, 0, a, d, -\sqrt{2}, -c)^T, \tag{29}$$

without the requirement that $a = b$ and $c = d$. However, for simplicity we restrict ourselves to the "reduced" case at the present time.

Lemma 2 *Suppose $\gamma(s)$ is a solution to the regularized Hamiltonian system Γ with $\gamma(0)$ defined as Eq. 19 and that satisfies*

$$\gamma(\tau) = (a, a, b, c, -c, 0)^T. \tag{30}$$

for some $\tau > 0$ and $E < 0$. Then $\gamma(s)$ extends to a 12τ periodic orbit for the system Γ.

Proof Suppose $\gamma(s)$ exists as claimed. Consider an orbit with initial conditions

$$\gamma(\tau) = (a, a, b, -c, c, 0)^T. \tag{31}$$

Note that the values of all $P_i(\tau)$ have been negated. Negating the momentum terms is equivalent to reversing time. Then for the initial conditions in Eq. 31, existence and uniqueness must give us that

$$\gamma(2\tau) = (0, \alpha, \alpha, -\sqrt{2}, \beta, -\beta)^T. \tag{32}$$

Then using a similar symmetry argument as in Lemma 1, it must be that if

$$\gamma(\tau) = (a, a, b, c, -c, 0)^T, \tag{33}$$

then

$$\gamma(2\tau) = (\alpha, 0, \alpha, \beta, -\sqrt{2}, -\beta)^T. \tag{34}$$

Applying Lemma 1 then gives the final result. □

Hence, in the regularized setting, the orbit can be constructed if we can find initial conditions that correspond to just the first twelfth of the orbit. This 12-fold symmetry is similar to that of the figure-eight orbit of Moore, Chenciner, and Montgomery (see Moore 1993; Chenciner and Montgomery 2000).

Note that at the time $\gamma(\tau)$, the vectors $\langle a, a, b\rangle$ and $\langle c, -c, 0\rangle$ (corresponding to position and velocity, respectively) are orthogonal. This type of symmetry-extension has been studied in other works as well (examples include Martínez 2012; Shibayama 2011).

3.3 Existence of the orbit

The results in this section serve to justify the numerical calculations that will ultimately be used to find the periodic orbit. For some of the proofs in this section, it will be more convenient to use the original q_i and p_i coordinates.

Lemma 3 *Suppose at some time s_0 we have that $Q_i P_i < 0$. Then if the orbit does not pass through an unregularized collision, there is some future time $s^* > s_0$ where $Q_i(s^*) = 0$.*

Proof This is most easily proven in the original q and p coordinates. The initial conditions correspond to some $q_i > 0$ and $p_i < 0$. Since $\dot{p}_i = \ddot{q}_i < 0$, then $p_i < 0$ as long as $q_i > 0$. Hence, we have a function $q_i(t)$ that is positive, decreasing, and concave down away from collision times. If the first collision after this time occurs when $q_i = 0$, then the proof is complete. On the other hand, if some other collision $q_j = 0$ occurs first, we can continue the curves for q_i and p_i through the collision by using the regularization and "patching together" the q_i and p_i curves past this time. The same inequality conditions on q_i and p_i still apply after this collision. □

Lemma 4 *Suppose at some t_0 time we have that $0 < q_i < q_j$ and $p_i < p_j$. Let $t_i > t_0$ and $t_j > t_0$ be the first times where $q_i = 0$ and $q_j = 0$, respectively. If both t_i and t_j are finite, then $t_i < t_j$.*

Proof Define $\tilde{q}(t) = q_i(t) - q_j(t)$ and $\tilde{p}(t) = p_i(t) - p_j(t)$. We will show that both $\tilde{q}$ and $\tilde{p}$ are negative and have negative derivatives with respect to time. Then the region with $\tilde{q} < 0$ and $\tilde{p} < 0$ is forward-invariant (up to collision time), which implies the result.

We first have $\frac{d\tilde{q}}{dt} = p_i(t) - p_j(t)$. By assumption, $\tilde{q}(t_0) < 0$. Similarly, since

$$\frac{dp_i}{dt} = -\frac{2q_i}{(q_1^2 + q_2^2 + q_3^2)^{3/2}} - \frac{2q_i}{(q_i^2 + q_j^2)^{3/2}} - \frac{2q_i}{(q_i^2 + q_k^2)^{3/2}} - \frac{2}{q_i^2}, \tag{35}$$

then

$$\frac{d\tilde{p}}{dt} = \frac{2q_j - 2q_i}{(q_i^2 + q_j^2 + q_k^2)^{3/2}} + \frac{2q_j - 2q_i}{(q_i^2 + q_j^2)^{3/2}} \cdots$$
$$+ \left(\frac{2q_j}{(q_j^2 + q_k^2)^{3/2}} - \frac{2q_i}{(q_i^2 + q_k^2)^{3/2}}\right) + \left(\frac{2}{q_j^2} - \frac{2}{q_i^2}\right). \tag{36}$$

Showing the sign of Eq. 36 is unfortunately not as straightforward. To simplify, we use the transformation

$$q_i = aq_j \qquad q_k = bq_j. \tag{37}$$

The assumption $0 < q_i < q_j$ gives $0 < a < 1$. The only restriction on b is $b > 0$. With this substitution, we obtain

$$\frac{d\tilde{p}}{dt} = \frac{2}{q_j^2}\left(\frac{1-a}{(a^2+b^2+1)^{3/2}} + \frac{1-a}{(a^2+1)^{3/2}} \cdots\right.$$
$$\left. + \frac{1}{(b^2+1)^{3/2}} - \frac{a}{(a^2+b^2)^{3/2}} + 1 - \frac{1}{a^2}\right). \tag{38}$$

Certainly $2q_j^{-2}$ is always positive. Define

$$f(a,b) = \frac{1-a}{(a^2+b^2+1)^{3/2}} + \frac{1-a}{(a^2+1)^{3/2}} \cdots$$
$$+ \frac{1}{(b^2+1)^{3/2}} - \frac{a}{(a^2+b^2)^{3/2}} + 1 - \frac{1}{a^2}. \tag{39}$$

It is straightforward to show that $f(1,b) = 0$ for any b and that as $a \to 0^+$, $f(a,b) \to -\infty$. Direct calculation gives

$$\frac{df}{da} = \frac{-3a(1-a)}{(a^2+b^2+1)^{5/2}} - \frac{1}{(a^2+b^2+1)^{3/2}} - \frac{3a(1-a)}{(a^2+1)^{5/2}} \cdots$$
$$- \frac{1}{(a^2+1)^{3/2}} + \frac{3a}{(a^2+b^2)^{5/2}} + \frac{2}{a^3}. \tag{40}$$

A simple exercise from calculus shows that the minimum value of the expression $-3a(1-a)$ with $a \in [0, 1]$ is $-3/4$. This gives

$$\frac{df}{da} \geq \frac{-3}{4(a^2+b^2+1)^{5/2}} - \frac{1}{(a^2+b^2+1)^{3/2}} - \frac{3}{4(a^2+1)^{5/2}} \cdots$$
$$- \frac{1}{(a^2+1)^{3/2}} + \frac{3a}{(a^2+b^2)^{5/2}} + \frac{2}{a^3}. \tag{41}$$

Since $b^2 > 0$, we can appropriately remove some b^2 terms from 41 to obtain

$$\frac{df}{da} \geq \frac{-3}{4(a^2+1)^{5/2}} - \frac{1}{(a^2+1)^{3/2}} - \frac{3}{4(a^2+1)^{5/2}} \cdots$$
$$- \frac{1}{(a^2+1)^{3/2}} + \frac{3a}{(a^2+b^2)^{5/2}} + \frac{2}{a^3}. \tag{42}$$

Combining terms in 42 then gives

$$\frac{df}{da} \geq \frac{-3}{2(a^2+1)^{5/2}} - \frac{2}{(a^2+1)^{3/2}} + \frac{3a}{(a^2+b^2)^{5/2}} + \frac{2}{a^3}. \tag{43}$$

The right side of 43 is certainly decreasing as b increases from zero toward infinity. Taking the limit as $b \to \infty$ then yields

$$\frac{df}{da} \geq \frac{-3}{2(a^2+1)^{5/2}} - \frac{2}{(a^2+1)^{3/2}} + \frac{2}{a^3} \tag{44}$$

for $a \in (0, 1]$. We claim that the right side of the expression in 44 is positive for all $a \in (0, 1]$. Certainly as $a \to 0^+$, this quantity is positive, as the $2a^{-3}$ term tends to infinity. Suppose there is some a at which this quantity is zero. Then we must have

$$\frac{4a^2+7}{2(a^2+1)^{5/2}} = \frac{2}{a^3}. \tag{45}$$

Clearing denominators gives

$$4a^5 + 7a^3 = 4(a^2+1)^{5/2}, \tag{46}$$

and then squaring both sides yields

$$16a^{10} + 56a^8 + 49a^6 = 16a^{10} + 80a^8 + 160a^6 + 160a^4 + 80a^2 + 16. \tag{47}$$

A solution to this must also be a solution to

$$24a^8 + 111a^6 + 160a^4 + 80a^2 + 16 = 0 \tag{48}$$

which certainly does not exist in real numbers. Hence, df/da cannot change sign, and therefore df/da (and consequently $d\tilde{p}/dt$) are positive for all a and b. Since $\tilde{p}$ increases to 0 as $a \to 1^-$ for any fixed value of b, it must then be the case that $\tilde{p}$ is negative, giving the final result. $\square$

To help facilitate future discussion, define Σ to be the set of ordered pairs (α, β) satisfying:

- $\alpha \geq 0$
- $\beta \geq 0$
- If $\gamma(0)$ is defined as in Eq. 19, then the first $s^* > 0$ for which $Q_1 Q_2 Q_3(s^*) = 0$ implies $Q_2(s^*) = 0$.

Note that for any initial condition $\gamma(0)$ defined by Eq. 19 for $\alpha > 0$ and $\beta > 0$, integrating the equations of motion on a sufficiently small interval $s \in [0, \epsilon]$ will yield unregularized conditions $0 < q_2 < q_3$ and $p_2 < p_3$. Then by Lemma 4, the first collision cannot be when $Q_3 = 0$. This definition also allows for $Q_1(s^*) = Q_2(s^*) = 0$ or $Q_2(s^*) = Q_3(s^*) = 0$, but a $Q_1 = 0$ collision cannot occur before a Q_2 collision.

Lemma 5 *For a fixed $E < 0$, there exists some $\alpha^* > 0$ so that if $\alpha > \alpha^*$, then $(\alpha, 0) \notin \Sigma$.*

Proof As $\alpha \to \infty$, with $\gamma(0)$ as defined in Eq. 19, the configuration of the four bodies approaches four decoupled binary pairs whose trajectories are line segments parallel to the x axis. Certainly for α sufficiently large it must be the case that the first collision occurs at $Q_1 = 0$, so $(\alpha, 0) \notin \Sigma$. Hence, the set of all α for which $(\alpha, 0) \in \Sigma$ is bounded above. We can then define α^* to be the supremum of this set. □

Lemma 6 *Let $\gamma(0)$ be defined as in Eq. 19 with $(\alpha, \beta) \in \Sigma$ and $E < 0$. Then there is some $\tau > 0$ with $\tau = \tau(\alpha, \beta)$ so that $Q_1(\tau) = Q_2(\tau)$.*

Proof For $s > 0$ sufficiently small, we have that the signs of the components of γ are given by

$$\gamma(s) = (+, +, +, +, -, +). \tag{49}$$

Then by Lemma 3, we have that at some future time, $Q_2(s^*) = 0$. Since $(\alpha, \beta) \in \Sigma$, it must be that $Q_1(s^*) \geq 0$. Since $Q_2(0) - Q_1(0) = \alpha > 0$ and $Q_2(s^*) - Q_1(s^*) \leq 0$ the result follows by the Intermediate Value Theorem. □

Lemma 7 *Set $\gamma(0)$ as in Eq. 19 for some $0 < \alpha < \alpha^*$ and $E < 0$, with α^* as defined in Lemma 5. Then there is some $\beta > 0$ so that $(\alpha, \beta) \in \Sigma$ and*

$$\gamma(\tau) = (a, a, b, c_1, c_2, 0)^T, \tag{50}$$

where τ is as defined in Lemma 6.

Proof Consider $\gamma(0) = (0, \alpha, \alpha, \sqrt{2}, 0, 0)$, Then by the symmetry of Γ, $Q_2(s) = Q_3(s)$ and $P_2(s) = P_3(s)$. For $s > 0$, $P_2(s)$ and $P_3(s)$ are both negative, so by Lemma 3 some future time s^* it must be that $Q_2(s^*) = Q_3(s^*) < \epsilon$ for any $\epsilon > 0$. (Note that this tends toward an unregularized collision.) Since $(\alpha, \beta) \in \Sigma$, we must have that $Q_1 > 0$ for all $0 < s < s^*$. So at the time $Q_1(\tau) = Q_2(\tau)$, it must be that $P_3(\tau) < 0$. On the other hand, as $\beta \to \infty$, then the time s^* where $Q_2(s^*) = 0$ approaches 0. Then $Q_3(\tau)$ must also be positive, as if $s^* \to 0$ then $\tau \to 0$. Certainly for β sufficiently high, $(\alpha, \beta) \in \Sigma$. The result then follows from the Intermediate Value Theorem. □

We expect that away from the potentially chaotic region near $||\gamma(0)|| = 0$, there should be a single value of β for which this holds.

Lemma 8 *Set $\gamma(0)$ as in Eq. 19 for some $E < 0$. Then there is some $\alpha > 0$ and $\beta > 0$ so that*

$$\gamma(\tau) = (a, a, b, c, -c, 0)^T, \tag{51}$$

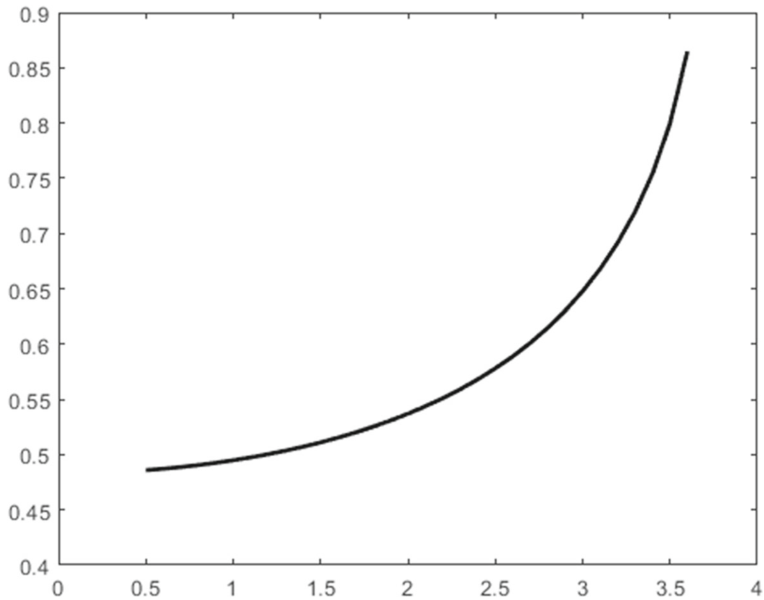

Fig. 5 The value of β (vertical axis) plotted against α (horizontal axis) for which Lemma 7 is satisfied. $E = -1$

This will be proven numerically. With $\beta = \beta(\alpha)$, we show that there is some value of α for which the quantity $P_1(\tau)+P_2(\tau)$ is negative and a second value of α for which $P_1(\tau)+P_2(\tau)$ is positive. Again, the result follows from the Intermediate Value Theorem.

As a consequence of Lemma 8, we have the following

Theorem 1 *There exists a periodic orbit corresponding to the Hamiltonian described in Sect. 2.*

3.4 Numerical results

We can find the value of α^* as defined in Lemma 5 by integrating initial conditions as defined in Eq. 19. For $E = -1$, we find $3.74 < \alpha^* < 3.75$. Specifically, for $\alpha = 3.75$ we have that $Q_1 = 0$ is the first collision for time $s > 0$, but for $\alpha = 3.74$ the conditions of Lemma 4 are met after some finite time interval (after converting to the non-regularized coordinates).

For $\alpha \in \{0.5, 0.6, \ldots, 3.5, 3.6\}$, we use a bisection method to find the value of β as defined in Lemma 7. Specifically, with initial conditions as defined in Eq. 19 we integrate the equations of motion using a fixed step-size Runge–Kutta method until the conditions of Lemma 6 are satisfied. We then vary β to find the value for which $P_3(\tau) = 0$. The results are graphed in Fig. 5.

Next, setting $\beta = \beta(\alpha)$, we compute the value $P_1(\tau) + P_2(\tau)$. The results are graphed in Fig. 6. For $\alpha = 0.5$, we have $P_1(\tau)+P_2(\tau) = 2.14$, and for $\alpha = 3.6$ we have $P_1(\tau)+P_2(\tau) = -1.63$, establishing Lemma 8. Notably, for the value $\alpha = 3.1$, we find $\beta = 0.6677$ and $P_1(\tau) + P_2(\tau) = 0.0018$.

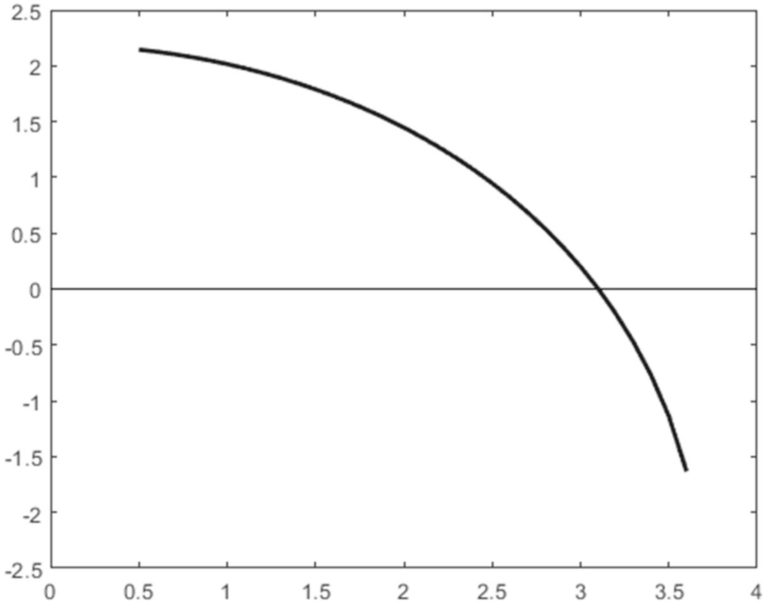

Fig. 6 The value of $P_1(\tau) + P_2(\tau)$ (vertical axis) plotted against α (horizontal axis) where $\beta = \beta(\alpha)$ from Lemma 7. $E = -1$

We then repeatedly refine the value of α (starting about $\alpha = 3.1$) and repeat the process, working to find the zero of $P_1(\tau) + P_2(\tau)$. For $E = -1$, we find the appropriate values are

$$\alpha = 3.100685, \quad \beta = 0.668162. \tag{52}$$

The full period of the regularized orbit is given by

$$12\tau = 0.124736 \tag{53}$$

A graph of numerical integration of the regularized equations of motion is given in Fig. 7.

4 Stability and symmetry

4.1 Definitions and preliminaries

Let $\mathcal{O}(\gamma_0)$ be the set of all points in $\mathbb{R}^6$ traced out in forward and backward time by the solution to the regularized Hamiltonian Γ with initial conditions γ_0. If we use the initial conditions determined by α and β in the previous section, then the time interval $0 \leq s \leq 12\tau$ captures the entire orbit and $\mathcal{O}(\gamma_0)$ is a closed loop in $\mathbb{R}^6$. This orbit is *Poincaré stable* if given any $\epsilon > 0$ there is some $\delta > 0$ so that for initial conditions $\tilde{\gamma}_0$ with $|\tilde{\gamma}_0 - \gamma_0| < \delta$, then any point on the orbit $\mathcal{O}(\tilde{\gamma}_0)$ is within ϵ of a point on the orbit $\mathcal{O}(\gamma_0)$.

Poincaré stability is generally difficult to establish in all but the simplest cases. However, there is a necessary condition that can be computed. Specifically, for a Hamiltonian system with Hamiltonian Γ and a periodic orbit $\gamma(s)$ with period T, consider the matrix differential equation

$$X' = JD^2\Gamma(\gamma(s)), \quad X(0) = I, \tag{54}$$

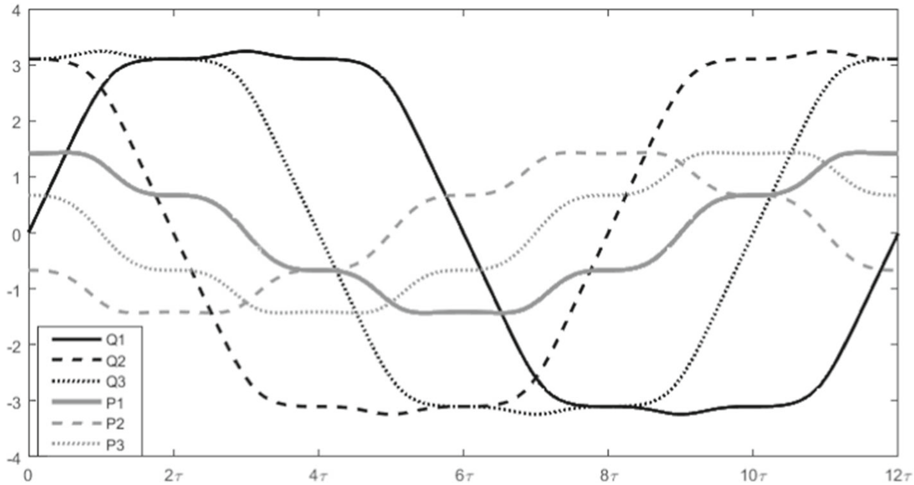

Fig. 7 Integration of the regularized equations of motion with the initial conditions given by Eq. 19, and values of α and β from Eq. 52

where D denotes the derivative, J is the symplectic matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{55}$$

with I and 0 are appropriately sized identity and zero matrices. Then the *monodromy matrix* of the orbit is the matrix $X(T)$, and the orbit is *linearly stable* if the eigenvalues of $X(T)$ all have complex modulus 1 and all have multiplicity one, apart from pairs of eigenvalues equal to 1 corresponding to first integrals of the system.

Linear stability can be established by considering conditions other that $X(0) = I$ as well. Specifically, if we let $X(0) = Y_0$ be the initial condition to Eq. 54, then $Y(s) = X(s)Y_0$, so $X(T) = Y(T)Y_0^{-1}$. Hence, $Y_0^{-1}Y(T)$ is similar to the monodromy matrix $X(T)$, and linear stability can be determined from either matrix as similarity preserves eigenvalues.

In the RPC setting, our choice of coordinates has already forced the integrals corresponding to center of mass, net momentum, and angular momentum to be zero. Hence, the monodromy matrix corresponding to the periodic RPC orbit should contain one pair of eigenvalues 1 corresponding to the fixed value of the Hamiltonian. Further, it will be shown that a particular choice of Y_0 simplifies the calculation.

4.2 Symmetries of the orbit

A technique by Roberts allows us to further simplify this calculation by "factoring" the monodromy matrix in terms of the symmetries of the orbit. We establish the symmetries of the orbit in this section.

By construction of the orbit as given in Lemma 1, we have that

$$\gamma(s + 2\tau) = S_f \gamma(s), \tag{56}$$

where

$$S_f = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{57}$$

So S_f is a time-preserving symmetry of the orbit. This symmetry corresponds to a 120° rotation about the line $x = y = z$, coupled with an appropriate sign change which arises in the regularized setting.

A time-reversing symmetry of the orbit is given by

$$\gamma(-s + 2\tau) = S_r \gamma(s), \tag{58}$$

where

$$S_r = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]. \tag{59}$$

This can be proven using a similar technique as shown in Lemma 1. Setting $s = \tau$ gives $\gamma(\tau) = S_r \gamma(\tau)$, implying that $\gamma(\tau)$ is an eigenvector of S_r with eigenvalue 1. Directly computing this, we find that $\gamma(\tau)$ must be of the form

$$\gamma(\tau) = (a, a, b, c, -c, 0)^T \tag{60}$$

for suitable values of a, b, and c. Note that this is exactly what was found in Lemma 2.

4.3 Roberts's symmetry-reduction technique

The general results in this section are presented, with proof, in Section 2 of Roberts (2007). Statements of the results are included here for convenience. The application of each result to the RPC orbit is given after each statement. Results similar to Lemmas 11–13 also appear in Roberts (2007), but the form presented in this section is specifically applied to the RPC orbit.

Lemma 9 (Lemma 2.1 from Roberts (2007)) *Suppose that $\gamma(s)$ is a T-periodic solution of a Hamiltonian system with Hamiltonian Γ and time-preserving symmetry S such that*

(1) There exists some $N \in \mathbb{N}$ such that $\gamma(s + T/N) = S\gamma(s)$ for all s,
(2) $\Gamma(Sx) = \Gamma(x)$,
(3) $SJ = JS$, and
(4) S is orthogonal.

Then the fundamental matrix solution $X(s)$ to the linearization problem $\dot{X} = JD^2\Gamma(\gamma(s))X$ with $X(0) = I$ satisfies

$$X(s + T/N) = SX(s)S^T X(T/N). \tag{61}$$

We note that the matrix $S = S_f$ from Eq. 57 satisfies all of these hypotheses with $T = 12\tau$ and $N = 6$.

Corollary 1 (Corollary 2.2 from Roberts (2007)) *Given the hypotheses of Lemma 9, the fundamental matrix solution $X(s)$ satisfies*

$$X(kT/N) = S^k(S^T X(T/N))^k \tag{62}$$

for any $k \in \mathbb{N}$.

In the case of the RPC orbit, this gives us that $X(12\tau) = (S_f^T X(2\tau))^6$, as $S_f^6 = I$.

Lemma 10 (Lemma 2.4 from Roberts (2007)) *Suppose that $\gamma(s)$ is a T-periodic solution of a Hamiltonian system with Hamiltonian Γ and time-reversing symmetry S such that*

(1) There exists some $N \in \mathbb{N}$ such that $\gamma(-s + T/N) = S\gamma(s)$ for all s,
(2) $\Gamma(Sx) = \Gamma(x)$,
(3) $SJ = -JS$, and
(4) S is orthogonal.

Then the fundamental matrix solution $X(s)$ to the linearization problem $\dot{X} = JD^2\Gamma(\gamma(s))X$ with $X(0) = I$ satisfies

$$X(-s + T/N) = SX(s)S^T X(T/N). \tag{63}$$

The matrix $S = S_r$ from Eq. 59 satisfies all of these hypotheses with $T = 12\tau$ and $N = 6$.

Corollary 2 (Corollary 2.5 from Roberts (2007)) *Given the hypotheses of Lemma 10,*

$$X(T/N) = SA^{-1}S^T A, \quad A = X(T/2N). \tag{64}$$

In the case of the RPC orbit, noting that $S_r^T = S_r$ gives $X(2\tau) = S_r A^{-1} S_r A$ with $A = X(\tau)$. Combining this with the earlier result, this gives us that the monodromy matrix of the RPC orbit is $X(12\tau) = (S_f^T S_r A^{-1} S_r A)^6$. Hence, we can evaluate the stability of the orbit by evaluating the relevant differential equations along only a twelfth of the orbit.

Roberts also gives similar results for the case where the initial conditions given in Eq. 54 are not the identity matrix. These are listed below.

Corollary 3 (Remark following Corollary 2.2 in Roberts (2007)) *If $Y(s)$ is the fundamental matrix solution with $X(0) = Y_0$, then*

$$Y(s + T/N) = SY(s)Y_0^{-1}S^T Y(T/N), \tag{65}$$

and so

$$Y(kT/N) = S^k Y_0 (Y_0^{-1} S^T Y(T/N))^k \tag{66}$$

Corollary 4 (Remark following Corollary 2.5 in Roberts (2007)) *If $Y(s)$ is the fundamental matrix solution with $X(0) = Y_0$, then*

$$Y(-s + T/n) = SY(s)Y_0^{-1}S^T Y(T/N), \tag{67}$$

and so

$$Y(T/N) = SY_0 B^{-1} S^T B, \quad B = Y(T/2N). \tag{68}$$

Combining these with previous results gives that for an arbitrary $X(0) = Y_0$, the resulting matrix solution $Y(s)$ to Eq. 54 satisfies

$$Y(12\tau) = Y_0(Y_0^{-1} S_f^T S_r Y_0 B^{-1} S_r B)^6, \tag{69}$$

so

$$X(12\tau) = Y_0(Y_0^{-1} S_f^T S_r Y_0 B^{-1} S_r B)^6 Y_0^{-1}, \tag{70}$$

where $B = Y(\tau)$.

Define $W = Y_0^{-1} S_f^T S_r Y_0 B^{-1} S_r B$. Then $X(12\tau) = Y_0 W^6 Y_0^{-1}$, and stability of the RPC orbit is thus reduced to determining the eigenvalues of W.

For a properly chosen initial condition matrix Y_0, some additional simplification of the calculation can be done.

Lemma 11 (Lemma 4.1 from Roberts (2007)) *Suppose that W is a symplectic matrix satisfying*

$$\frac{1}{2}(W + W^{-1}) = \begin{bmatrix} K & 0 \\ 0 & K^T \end{bmatrix}. \tag{71}$$

Then W has all eigenvalues on the unit circle if and only if the eigenvalues of K lie in the real interval $[-1, 1]$.

Proper choice of the matrix Y_0 will give W of the required form.

Lemma 12 *Setting $\delta = \sqrt{2}/2$ and*

$$Y_0 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta & 0 & \delta & 0 \\ 0 & 0 & \delta & 0 & \delta & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\delta & 0 & 0 & 0 & -\delta \\ 0 & -\delta & 0 & 0 & 0 & \delta \end{array}\right] \tag{72}$$

gives a matrix W of the form in Lemma 11.

Proof Let

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{73}$$

where I and 0 represent 3×3 identity and zero matrices, respectively. Then direct calculation yields $-Y_0^{-1} S_f^T S_r Y_0 = \Lambda$.

Set $D = -B^{-1} S_r B$. Then by definition of W we have that $W = \Lambda D$. Since $D^2 = \Lambda^2 = I$, then we know that $W^{-1} = D\Lambda$. Since B is symplectic, writing

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \quad \text{and} \quad S_r = \begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}, \tag{74}$$

then the formula for the inverse of a symplectic matrix gives

$$B^{-1} = \begin{bmatrix} B_4^T & -B_2^T \\ -B_3^T & B_1^T \end{bmatrix}. \tag{75}$$

Directly computing D gives

$$D = \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix} \tag{76}$$

with K, L_1, and L_2 defined up to sign by matrix multiplication. Then

$$W = \Lambda D = \begin{bmatrix} K^T & L_1 \\ L_2 & K \end{bmatrix} \quad \text{and} \quad W^{-1} = D\Lambda = \begin{bmatrix} K^T & -L_1 \\ -L_2 & K \end{bmatrix} \tag{77}$$

and

$$\frac{1}{2}(W + W^{-1}) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix} \tag{78}$$

as required. □

As noted earlier, our coordinate system has already made use of the first integrals corresponding to center of mass, net momentum, and angular momentum in this setting. There is an additional pair of eigenvalues 1 in the monodromy matrix corresponding to the remaining first integral, the Hamiltonian itself. These can be found, with eigenvector, as shown below.

Lemma 13 *The matrix K^T has a right eigenvector $[1\ 0\ 0]^T$ with corresponding eigenvalue* 1.

Proof Let $v = Y_0^{-1}\gamma'(0)/||\gamma'(0)|| = Y_0^T\gamma'(0)/||\gamma'(0)||$. Since Y_0 is orthogonal and S_r is symmetric, we have

$$W = Y_0^{-1} S_f^T S_r Y_0 B^{-1} S_r B = Y_0^T S_f^T S_r Y_0 B^{-1} S_r^T B = Y_0^T S_f^T Y(2\tau) \tag{79}$$

by Corollary 4 with $s = 0$.

Define $\gamma(s)$ to be the periodic orbit with initial conditions defined in Sect. 3. Since $\gamma'(s)$ is a solution to $\dot{\xi} = JD^2\Gamma(\gamma(s))\xi$ and

$$\gamma'(0) = Y(0)Y_0^{-1}\gamma'(0) = Y(0)v,$$

then

$$\gamma'(s) = Y(s)Y_0^{-1}\gamma'(0) = Y(s)v. \tag{80}$$

This implies

$$Y_0^{-1} S_f^T \gamma'(2\tau) = Y_0^T S_f^T Y(2\tau)v = Wv. \tag{81}$$

Since

$$\gamma'(0) = (2\sqrt{2}\alpha^4, 0, 0, 0, 0, 0) \tag{82}$$

and

$$\gamma'(2\tau) = (0, -2\sqrt{2}\alpha^4, 0, 0, 0, 0) \tag{83}$$

with α as defined in Eq. 52, we have

$$S_f^T \gamma'(2\tau) = \gamma'(0). \tag{84}$$

Then

$$Wv = Y_0^{-1} S_f^T \gamma'(2\tau) = Y_0^T S_f^T S_f \gamma'(0) = Y_0^T Y_0 v = v.$$

So v is an eigenvector of W with eigenvalue 1.

Since $\gamma'(0)$ is known, we have that $v = Y_0^{-1} e_1$, where

$$e_1 = [1\ 0\ 0\ 0\ 0\ 0]^T. \tag{85}$$

Direct calculation gives that $v = e_1$. Then, since W satisfies the relation given in Lemma 11, K^T must have eigenvector $[1\ 0\ 0]^T$ with eigenvalue 1 as claimed. □

As a consequence, we know that the matrix K must be of the form

$$K = \begin{bmatrix} 1 & 0 & 0 \\ * & k_{22} & k_{23} \\ * & k_{32} & k_{33} \end{bmatrix} \tag{86}$$

and so the eigenvalues of the lower-right 2×2 block will determine stability.

5 Stability results

Using the matrix Y_0 from Eq. 72, we find the matrix $B = Y(\tau)$ numerically with the initial conditions from Eq. 52. Then the matrix K is given numerically by

$$K = \begin{bmatrix} 1.0007 & 0.0004 & -0.0001 \\ -0.9038 & 0.3487 & 0.1926 \\ 1.7654 & -1.1211 & -1.1241 \end{bmatrix} \tag{87}$$

The values given for the k_{12} and k_{13} entries are the result of propagation of numerical error in the calculation. Assuming they are zero as proven earlier, the eigenvalues from the lower-right 2×2 block of K are given by a simple application of the quadratic formula. We find

$$\lambda_1 = 0.1836, \quad \lambda_2 = -0.95899 \tag{88}$$

As a consequence of Lemma 11, we have the following

Theorem 2 *The RPC orbit described throughout this paper is linearly stable.*

We seek to give evidence of higher-order stability of the RPC orbit. Using $E = -1$ and the values of α and β from Eq. 52, we set

$$\begin{aligned} \gamma_0 =&(0, \alpha + r\cos(a)\cos(b), \alpha + r\cos(a)\sin(b), \ldots \\ &\sqrt{2}, \beta + r\sin(a)\cos(c), -\beta + r\sin(a)\sin(c)) \end{aligned} \tag{89}$$

where

$$a, b, c \in \{0, \pi/6, \pi/3, \pi/2, \ldots, 11\pi/6\}, \tag{90}$$

$$r \in \{0.005, 0.010, 0.015, \ldots, 0.100\}. \tag{91}$$

The equations of motion are run up to 200 collisions at $Q_1 = 0$ for each possible combination of a, b, c, and r. Integration is preemptively terminated after a time length of $s = 1$ has occurred since the last $Q_1 = 0$ collision. This time cutoff value seems reasonable given the length of the period $12\tau = 0.124736$. We track the distance from $\pm\gamma(0)$ at those collision times. For all values of r for which all 200 collisions were achieved on all values of a, b, and c tested, the maximum distance from $\pm\gamma(0)$ at collision is given in the table below.

r	$\text{dist}_{\max}$
0.005	0.0391
0.010	0.0824
0.015	0.1289
0.020	0.1833
0.025	0.2574

(92)

For all values of $r \geq 0.030$, there is at least one value of a, b, and c for which fewer than 200 $Q_1 = 0$ instances occur. For example, when $r = 0.030$ and $a = b = \pi/6$, $c = 11\pi/6$, only 34 instances of $Q_1 = 0$ are recorded before the integration is terminated, giving evidence of instability at this distance from the periodic orbit.

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Data Availability All data generated or analyzed during this study are included in this published article.

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