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ORIGINAL ARTICLE

Bifurcation of frozen orbits in a gravity field with zonal harmonics

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Abstract
We propose a methodology to study the bifurcation sequences of frozen orbits when the second-order fundamental model of the satellite problem is augmented with the contribution of octupolar terms and relativistic corrections. The method is based on the analysis of twice-reduced closed normal forms expressed in terms of suitable combinations of the invariants of the Kepler problem, able to provide a clear geometric view of the problem.

Keywords Geometric reduction of the perturbed Kepler problem · Gravity field with zonal harmonics · Frozen orbits

1 Introduction

Among the manifold versions of the perturbed Kepler problem, the investigation of the gravity field expanded in multipole terms has traditionally received great attention for its relevance in applications. Therefore, several analytical tools have been developed to highlight the most important phenomena. Perturbation theory with the construction of normal forms is the standard method since the first pioneering studies (Brouwer 1959; Kozai 1962). The case in which only zonal terms are included in one of the settings in which we can obtain explicit approximations of the regular dynamics since the normal form is integrable. However, the presence of several parameters, both dynamical (or 'distinguished' in the language of the theory of integrable systems) and physical like the multipole coefficients, hinders a global description of the dynamics. More efficient geometric and group-theoretic tools have been

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exploited to study the bifurcation of invariant objects when these parameters are varied (Cushman 1983; Coffey et al. 1986, 1994; Palacián 2007).

Here, we study the bifurcation sequences of frozen orbits when the second-order fundamental model of the satellite problem is augmented with further features of a typical planetary gravity field. We consider the contribution of the octupolar term (Vinti 1963; Coffey et al. 1994) and the relativistic correction due to the quadrupolar term (Heimberger et al. 1990). We implement a twice-reduced normal form (Cushman 1988; Pucacco and Marchesiello 2014; Pucacco 2019) which allows us to obtain in an efficient way the conditions for relative equilibria corresponding to the family of periodic orbits with fixed eccentricity and inclination. The method is tested in the second-order J_2-problem in which known results are reproduced (Palacián 2007) and then applied to the above-mentioned perturbations. For the J_4-problem, interesting features around the parameter values of the 'Vinti problem' are highlighted with an additional family of stable frozen orbits. For the relativistic J_2-correction, the treatment extends and completes several results obtained by Jupp and Brumberg (1991).

The plan of the paper is as follows: In Sect. 2, we recall the model problem based on the normal from obtained after averaging with respect to the mean anomaly; in Sect. 3, we review the reduction methods adapted to the symmetries of the present model, discuss the version adopted here to cope with the structure of the Brouwer class of Hamiltonians and show how it works in locating relative equilibria; in Sect. 4, we illustrate the results in concrete cases; in Sect. 5, we conclude with some hint for possible developments and future works.

2 The model in closed normal form

We are discussing some aspects of the general problem described by a Hamiltonian of the form:

$$\mathcal{H}(L, H, G, \ell, g, h) = \sum_{j=0}^{\infty} \epsilon^j \mathcal{H}_j(L, H, G, \ell, g, h), \tag{1}$$

where $\mathcal{H}_0$ is the Kepler Hamiltonian and the canonical Delaunay variables have the following expression in terms of the standard Keplerian elements $(a, e, i, \ell, \omega, \Omega)$

$$L = \sqrt{\mu a}, \quad G = \sqrt{\mu a}\sqrt{1-e^2}, \quad H = \sqrt{\mu a}\sqrt{1-e^2}\cos i, \tag{2}$$

$$\ell = M, \qquad g = \omega, \qquad h = \Omega. \tag{3}$$

In the above equation, ϵ is a formal parameter, called *book-keeping* parameter, suitably chosen to order the hierarchy of perturbing terms (see Efthymiopoulos 2012). Therefore, we have a *perturbed Kepler problem.*

Specifically, in the even zonal artificial satellite problem, we assume to start with the 'original Hamiltonian'

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}p^2 + \mathcal{V}_{CGF} - \frac{1}{c^2}\left(\frac{p^4}{8} - \frac{\mathcal{V}_{CGF}^2}{2} - \frac{3}{2}\mathcal{V}_{CGF}\, p^2\right) \tag{4}$$

in standard Cartesian form, where $\boldsymbol{q} = \{x, y, z\}$, $\boldsymbol{p} = \{\dot{x}, \dot{y}, \dot{z}\}$, $p = |\boldsymbol{p}|$, $\mathcal{V}_{CGF}$ is the classical gravity field and c is the speed of light. We include the classical gravity field $\mathcal{V}_{CGF}$ expanded in terms of the zonal harmonics of even degree[1]

[1] In this work, we focus on the even zonal problem. Thus, only the even zonal harmonics are considered in the expansion of the gravitational potential. The complete expansion, including also tesseral terms, can be found in (Kaula 1966).

$$\mathcal{V}_{CGF} = -\frac{\mu}{r}\left[1 - \sum_{k=1}^{\infty} J_{2k}\frac{R_P^{2k}}{r^{2k}} P_{2k}(\sin\theta)\right], \tag{5}$$

where $\mu = \mathcal{G}M_P$ is the product of Newton constant and the mass of the 'planet', R_P is its radius and the P_k are the Legendre polynomials with

$$\sin\theta = \frac{z}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

We also add the first-order relativistic corrections following, for example, Weinberg (1972).

To simplify the structure of the Hamiltonian, we then perform a *closed-form normalisation* like in (Coffey et al. 1994) and (Heimberger et al. 1990). This method, inspired by works of Deprit (1981, 1982), has the advantage of avoiding expansions in the eccentricity and inclination (Palacián 2002; Cavallari and Efthymiopoulos 2022). The model in (4) is rich enough to convey several interesting dynamical features keeping the closed-form structure at the lowest level of complexity. In fact, after the Delaunay reduction and the elimination of the ascending node, we deal with a secular Hamiltonian in closed form which depends on only one degree of freedom, corresponding to the pair G and g (the argument of the perigee):

$$\mathcal{K}(L, H, G, g) = \sum_j \epsilon^j \mathcal{K}_j(L, H, G, g), \tag{6}$$

with L and H formal integrals of the motion. The zero-order term is clearly

$$\mathcal{K}_0 = \mathcal{H}_0 = -\frac{\mu^2}{2L^2}. \tag{7}$$

The first-order term is

$$\mathcal{K}_1 = \frac{\mu^4 J_2 R_P^2 (G^2 - 3H^2)}{4G^5 L^3} - \frac{\mu^4}{c^2 L^4}\left[3\frac{L}{G} - \frac{15}{8}\right]. \tag{8}$$

The second-order term $\mathcal{K}_2$ consists of two contributions:

$$\mathcal{K}_2 = \mathcal{T}_2 + \langle\mathcal{H}_2\rangle.$$

The first is related to the propagation at second order of the J_2 term in the normalising transformation (Deprit 1969; Efthymiopoulos 2012),

$$\begin{aligned}
\mathcal{T}_2 = \frac{3\mu^6 J_2^2 R_P^4}{128 L^5 G^{11}} \times \Big[& -5G^6 - 4G^5 L + 24G^3H^2L - 36GH^4L - 35H^4L^2 \\
& + G^4(18H^2 + 5L^2) - 5G^2(H^4 + 2H^2L^2) \\
& + 2(G^2 - 15H^2)(G^2 - L^2)(G^2 - H^2)\cos 2g \Big] \\
& - \frac{3\mu^6 J_2 R_P^2}{4c^2 L^5 G^7}\left[(G^2 - 3H^2)(4G^2 - 3GL - 5L^2) + (L^2 - G^2)(G^2 - H^2)\cos 2g\right].
\end{aligned} \tag{9}$$

The second is associated directly with the average of the $\mathcal{H}_2$ term:

$$\begin{aligned}
\langle\mathcal{H}_2\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_2 d\ell \\
&= \frac{\mu^6 J_2 R_P^2}{8c^2 L^5 G^7}\left[(G^2 - 3H^2)(6L^2 - 5G^2) - 3(L^2 - G^2)(G^2 - H^2)\cos 2g\right]
\end{aligned}$$

$$+\frac{3\mu^6 J_4 R_P^4}{128 L^5 G^{11}}\big[(3G^4-30G^2H^2+35H^4)(5L^2-3G^2)$$
$$-10(G^2-7H^2)(L^2-G^2)(G^2-H^2)\cos 2g\big]. \tag{10}$$

In this work, we do not consider terms of order higher than $j = 2$. Hamiltonians of this type are generally denoted as 'Brouwer's' ones (Brouwer 1959; Cushman 1983). They are characterised by the independence on the mean anomaly ℓ and the longitude of the node h (with corresponding conservation of the actions L and H), whereas the argument of perigee appears only with the harmonic $\cos 2g$. These symmetries will all be exploited in the geometric approach described in the following.

The two relativistic terms proportional to J_2/c^2 appearing in (9) and (10) have the same structure. However, in the literature (Heimberger et al. 1990; Schanner and Soffel 2018), they are usually kept separate and are, respectively, referred to as the *indirect* and *direct term* related to the non-trivial relativistic contribution of the quadrupole of the gravity field of the central body. The ordering of the perturbing terms is performed by assuming (with a certain degree of arbitrariness) the J_2 and c^{-2} terms to be of order ϵ and the J_4 term of order ϵ^2, like the J_2^2 and $J_2 \times c^{-2}$ terms.

We remark that with a slight abuse of notation, we have denoted with the same symbols the Delaunay variables appearing in (1) and (6). We have to recall that actually they are, respectively, the *original* and the *new* variables related by the normalising transformation. In the present work, we are not interested in the explicit construction of particular solutions. Therefore, we will not detail the back-transformation from the new to the original coordinates. Moreover, we are not going to investigate any issue connected with the convergence of the expansions. We rely on the asymptotic properties of these series and their ability to provide reliable approximations, especially in the cases of Earth-like gravity fields.

For the sake of completeness, the different parts of the normalised Hamiltonian $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1\epsilon + (\mathcal{T}_2 + \langle\mathcal{H}_2\rangle)\,\epsilon^2$, expressed in terms of the orbital elements (a, e, i, ω), are given by

$$\begin{aligned}
\mathcal{K}_0 &= -\frac{\mu}{2a},\\
\mathcal{K}_1 &= \frac{1}{4}\,\frac{\mu J_2 R_P^2}{a^3\eta^3}\left(1-3\cos^2 i\right) - \frac{3}{8}\,\frac{\mu^2}{c^2a^2}\left(\frac{1}{\eta}-5\right),\\
\mathcal{T}_2 &= \frac{3\mu J_2^2 R_P^4}{128\,a^5\eta^7}\Big[-\left(5\,\eta^2+36\,\eta+35\right)\sin^4 i + 8\left(-\eta^2+6\,\eta+10\right)\sin^2 i\\
&\quad +8\left(\eta^2-2\,\eta-5\right)\times\,2\sin^2 i\left(1-\eta^2\right)\left(1-15\cos^2 i\right)\cos 2\omega\Big]\\
&\quad -\frac{3\mu^2 J_2 R_P^2}{4c^2a^4\eta^5}\Big[\left(4\eta^4-3\eta-5\right)(1-3\cos^2 i) + \sin^2 i(1-\eta^2)\cos 2\omega\Big],\\
\langle\mathcal{H}_2\rangle &= \frac{\mu^2 J_2 R_P^2}{8c^2a^4\eta^5}\left[(6-5\eta^2)(1-3\cos^2 i) - 3\sin^2 i(1-\eta^2)\cos 2\omega\right]\\
&\quad +\frac{3\mu J_4 R_P^4}{128a^5\eta^7}\big[(5-3\eta^2)(35\sin^4 i-40\sin^2 i+8)\\
&\quad -10\sin^2 i(1-\eta^2)(1-7\cos^2 i)\cos 2\omega\big],
\end{aligned}$$

with $\eta = \sqrt{1-e^2}$.

3 Geometric reduction

The secular Hamiltonian in closed form in (6), while computed with an ingenious combination of tools based on the Lie transform method (Deprit 1969; Efthymiopoulos 2012) and the elimination of the parallax (Deprit 1981), is nonetheless standard in being essentially an average with respect to the mean anomaly (Deprit 1982; Palacián 2002). However, it is liable to be treated with a group theoretically approach. It can be interpreted as a suitable combination of the invariants generating the $SO(3)$ symmetry of the Kepler problem. In fact, the dynamics ensues from the *reduction* of the Hamiltonian defined on the space of the trajectories having, for the unperturbed Kepler problem with negative energy, the structure of the direct product of two spheres. The additional symmetries of the closed form of the perturbed problem are exploited to identify a regular reduced phase space with the topology of the 2-sphere. In practice, we will use a further transformation leading to a singular reduction on a surface with equivalent topology, which produces a clearer geometric view of the bifurcation sequence of frozen orbits. Here, we provide a quick reminder of the invariant theory of the Kepler problem and then apply the reduction process to perturbed Kepler problems described by Brouwer's Hamiltonians.

3.1 Invariants of the Kepler problem

Let us call $\boldsymbol{G}$ the angular momentum and $\boldsymbol{A}$ the Laplace–Runge–Lenz vector, given by

$$\boldsymbol{G} = G \begin{bmatrix} \sin i \sin h \\ -\sin i \cos h \\ \cos i \end{bmatrix}, \qquad \boldsymbol{A} = \sqrt{1 - \frac{G^2}{L^2}} \begin{bmatrix} \cos g \cos h - \sin g \sin h \cos i \\ \cos g \sin h + \sin g \cos h \cos i \\ \sin g \sin i \end{bmatrix},$$

with $i = \arccos(H/G)$ the orbital inclination. By defining

$$\boldsymbol{x} = \boldsymbol{G} + L\boldsymbol{A}, \quad \boldsymbol{y} = \boldsymbol{G} - L\boldsymbol{A}, \tag{11}$$

we get the Poisson structure of the generators of $SO(3)$

$$\begin{aligned} \{x_1, x_3\} &= x_2, \quad \{x_3, x_2\} = x_1, \quad \{x_2, x_1\} = x_3, \\ \{y_1, y_3\} &= y_2, \quad \{y_3, y_2\} = y_1, \quad \{y_2, y_1\} = y_3, \end{aligned}$$

and phase-space defined by the direct product of the two 2-spheres

$$x_1^2 + x_2^2 + x_3^2 = L^2, \quad y_1^2 + y_2^2 + y_3^2 = L^2. \tag{12}$$

It can therefore be imagined as the invariant space of the states characterised by given eccentricity, inclination, and arguments of perigee and node, but nonetheless equivalent for what pertains to the mean anomaly. In the unperturbed problem, the state is a given still point of the invariant space. The state point is kept moving on it by the action of the perturbation.

3.2 Reduction of the axial symmetry

Perturbed Kepler problems described by Hamiltonians of the form (6) are characterised by axial symmetry with H as formal third integral. In Cushman (1983) and Coffey et al. (1986), it is shown that if $0 < |H| < L$, the two-dimensional phase space of such problems is still diffeomorphic to a sphere. Two different sets of variables, both functions of the Keplerian invariants x_k, y_k $(k = 1, 2, 3)$ and suitable to analyse the dynamics, are proposed. The variables (π_1, π_2, π_3) are defined as: (Cushman 1983),

$$\begin{aligned}\pi_1 =& \frac{1}{2}(x_3 - y_3) = L(\boldsymbol{A}\cdot\boldsymbol{k}),\\ \pi_2 =& x_1 y_2 - x_2 y_1 = 2L(\boldsymbol{A}\times\boldsymbol{G})\cdot\boldsymbol{k},\\ \pi_3 =& x_1 y_1 + x_2 y_2 = |\boldsymbol{G}\times\boldsymbol{k}|^2 - L^2|\boldsymbol{A}\times\boldsymbol{k}|^2,\end{aligned}$$

where $\boldsymbol{k} = (0, 0, 1)^T$. The phase-space is then

$$\mathcal{P} = \left\{(\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3 : \pi_2^2 + \pi_3^2 = ((L + \pi_1)^2 - H^2)((L - \pi_1)^2 - H^2)\right\}. \tag{13}$$

Instead, in Coffey et al. (1986), the variables (ξ_1, ξ_2, ξ_3) are introduced, defined as:

$$\xi_1 = L(\boldsymbol{G}\times\boldsymbol{A})\cdot\boldsymbol{k}, \qquad \xi_2 = L|\boldsymbol{G}|(\boldsymbol{A}\cdot\boldsymbol{k}), \qquad \xi_3 = \frac{1}{2}\left(|\boldsymbol{G}\times\boldsymbol{k}|^2 - L^2|\boldsymbol{A}|^2\right),$$

or, in terms of Delaunay variables,

$$\begin{aligned}\xi_1 =& \sqrt{(G^2 - H^2)(L^2 - G^2)}\cos g,\\ \xi_2 =& \sqrt{(G^2 - H^2)(L^2 - G^2)}\sin g,\\ \xi_3 =& G^2 - \frac{L^2 + H^2}{2}.\end{aligned} \tag{14}$$

In this case, the phase-space is the sphere of radius $(L^2 - H^2)/2$:

$$\mathcal{S} = \left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R} : \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{(L^2 - H^2)^2}{4}\right\}. \tag{15}$$

The relation between the π_k and the ξ_k is

$$\begin{aligned}\pi_1 =& \frac{\sqrt{2}\xi_2}{\sqrt{2\xi_3 + L^2 + H^2}},\\ \pi_2 =& -2\xi_1,\\ \pi_3 =& 2\xi_3 + \frac{2\xi_2^2}{2\xi_3 + L^2 + H^2}.\end{aligned}$$

The advantage of both these sets of variables with respect to the Delaunay variables is well explained in Coffey et al. (1986) with an imaginative metaphor. In simpler words, we can say that the Kepler reduction allows us to translate the closed-form dynamics in terms of the invariants of the unperturbed problem (formal conservation of L) and the further reduction generated by the invariants ξ_k is readily apt to account for the axial symmetry associated with the formal conservation of H. Recalling the description of the states of the space defined in (12), we now have that the states of (15), given a value of H, are characterised by the eccentricity and the perigee but are nonetheless equivalent for what concerns h. The dynamical evolution of the system is then determined by the intersections of the reduced phase-space $\mathcal{S}$ with the Hamiltonian expressed in terms of the invariants, e.g. $\mathcal{K}(\xi_1, \xi_2, \xi_3)$.

Whenever one uses the (G, g) chart to analyse the dynamics of the closed form for given values of L and H, one excludes circular and equatorial orbits. Indeed, when either the orbital eccentricity or the orbital inclination is zero, the argument of the perigee g is not defined; thus, the Delaunay variables result unsuitable to evaluate the stability of such orbits, if they are periodic as typically happens in the artificial satellite problem. Following Cushman (1983), in Iñarrea et al. (2004) it is shown that when $\mathcal{K}$ possesses independent symmetries of the type

$$\begin{aligned}\mathcal{R}_1 &:(\pi_1,\pi_2,\pi_3) \to (-\pi_1,\pi_2,\pi_3),\\ \mathcal{R}_2 &:(\pi_1,\pi_2,\pi_3) \to (\pi_1,-\pi_2,\pi_3),\\ \mathcal{R}_3 &:(\pi_1,\pi_2,\pi_3) \to (-\pi_1,-\pi_2,\pi_3),\end{aligned}$$

the phase-space can be further reduced, and the variables σ_1, σ_2, defined as:

$$\sigma_1 = (L - |H|)^2 - \pi_1^2, \qquad \sigma_2 = \frac{\sqrt{L^2 + H^2 - \pi_1^2 + \pi_3}}{\sqrt{2}},$$

are introduced, where $\sigma_2 = G$. We propose here to exploit a further set of variables, which is particularly suitable when the normalised Hamiltonian possesses symmetries of the type

$$\begin{aligned}R_1 &:(\xi_1,\xi_2,\xi_3) \to (-\xi_1,\xi_2,\xi_3),\\ R_2 &:(\xi_1,\xi_2,\xi_3) \to (\xi_1,-\xi_2,\xi_3),\\ R_3 &:(\xi_1,\xi_2,\xi_3) \to (-\xi_1,-\xi_2,\xi_3).\end{aligned} \tag{16}$$

We introduce the variables (X, Y, Z) defined as:

$$\begin{aligned}X &= \xi_1^2 - \xi_2^2,\\ Y &= 2\xi_1\xi_2,\\ Z &= \xi_3,\end{aligned} \tag{17}$$

which turn the spherical phase space $\mathcal{S}$ into a *lemon* space:

$$\mathcal{L} = \left\{(X, Y, Z) \in \mathbb{R} : X^2 + Y^2 = \left(-Z^2 + \mathcal{E}^2\right)^2\right\}, \qquad \mathcal{E} = \frac{L^2 - H^2}{2}.$$

This kind of reduction was proposed for the first time by Hanßmann and Sommer (2001). It is an example of *singular* reduction (Cushman and Bates 1997) as opposed to the regular setting generated by the invariants ξ_k. This occurs here due to the appearance of *cusps* in the reduced phase-space $\mathcal{L}$ contrary to the smoothness of the 2-sphere $\mathcal{S}$. However, as it will appear clear in the following, this fact does not pose any practical issue in the induction process implemented hereafter.

Even though the phase-space is still three-dimensional, we see that in the case in which symmetries (16) are fulfilled (such as in the problem of the geo-potential when only even zonal harmonics are retained), the transformed closed form does not depend on the variable Y: $\mathcal{K} = \mathcal{K}(X, Z)$. In particular, in the case of the Brouwer's Hamiltonian (6), $\mathcal{K}$ depends linearly on X, i.e. it is of the form:

$$\mathcal{K}(X, Z; \boldsymbol{a}) = g(Z; \boldsymbol{a}) + f(Z; \boldsymbol{a})X, \tag{18}$$

where $\boldsymbol{a}$ is the set of parameters characterising the problem, including the 'distinguished parameter' $\mathcal{E}$. For such a problem, the analysis of the intersection of the reduced phase-space $\mathcal{L}$ with the function (18) is simplified by the extra symmetry of the Brouwer's Hamiltonian since, rather than working in the full 3D-space, all significant information can be obtained by projection on the (Z, X) plane... As a matter of fact, when expressed in Delaunay variables,

(X, Y, Z) are equal to

$$\begin{aligned} X =& (G^2 - H^2)(L^2 - G^2)\cos 2g, \\ Y =& (G^2 - H^2)(L^2 - G^2)\sin 2g, \\ Z =& G^2 - \frac{L^2 + H^2}{2}, \end{aligned} \tag{19}$$

and, considering the structure of the normalised Hamiltonian presented in the previous section, the possibility of using the general form (18) appears immediately justified.

3.3 Equilibrium points

Relative equilibria of the reduced systems correspond to periodic orbits of the original closed form in (6), which in turn are approximations of the periodic orbits of the model problem in (1). Our main concern refers to frozen orbits which play a major role in shaping the phase-space structure of the system. They can be identified by locating 'contacts' between the surfaces defined by the Hamiltonian function (18) and the lemon space $\mathcal{L}$ (Pucacco and Marchesiello 2014) or in some peculiar case we will encounter in what follows if the Hamiltonian possesses a one-dimensional level set whose intersection with the phase-space produces additional (unstable) critical points. In the present subsection, we describe the general procedure to locate equilibria, postponing to the next section the details of each case.

Considering G as a function of Z, $G = \sqrt{Z + (L^2 + H^2)/2}$, the Poisson structure of the (X, Y, Z) variables is

$$\begin{aligned} \{X, Y\} =& 8GZ\sqrt{X^2 + Y^2}, \\ \{X, Z\} =& -4GY, \\ \{Y, Z\} =& 4GY. \end{aligned}$$

Henceforth, given a Hamiltonian of the form $\mathcal{K}$ in (18), the equations of motion are

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} =& \{X, \mathcal{K}\} = -4GY\frac{\partial \mathcal{K}}{\partial Z}, \\ \frac{\mathrm{d}Y}{\mathrm{d}t} =& \{Y, \mathcal{K}\} = 4G\left(-2Z\sqrt{X^2 + Y^2}\frac{\partial \mathcal{K}}{\partial X} + X\frac{\partial \mathcal{K}}{\partial Z}\right), \\ \frac{\mathrm{d}Z}{\mathrm{d}t} =& \{Z, \mathcal{K}\} = 4GY\frac{\partial \mathcal{K}}{\partial X}. \end{aligned}$$

Since we are typically interested in elliptic trajectories, which implies $G \neq 0$, there exist equilibrium points whenever

$$\begin{cases} Y = 0 \\ X\left(2Z\frac{\partial \mathcal{K}}{\partial X}\operatorname{sign}(X) + \frac{\partial \mathcal{K}}{\partial Z}\right) = 0, \end{cases} \tag{20}$$

or

$$\frac{\partial \mathcal{K}}{\partial Z} = \frac{\partial \mathcal{K}}{\partial X} = 0. \tag{21}$$

The variables X, Y, Z are particularly useful in the first case when conditions (20) are fulfilled. On the (X, Z) plane, the contour of the lemon space $\mathcal{L}$ is $\mathcal{C} = \mathcal{C}_+ \bigcup \mathcal{C}_-$, with

$$\mathcal{C}_\pm = \left\{(X, Z) \in \mathbb{R}^2 : |Z| \leq \mathcal{E},\ X = \pm\hat{X}(Z; \mathcal{E})\right\},$$

where

$$\hat{X}(Z; \mathcal{E}) = -Z^2 + \mathcal{E}^2. \tag{22}$$

For any values of the parameters $\boldsymbol{a}$, condition (20) is fulfilled if $X = 0$. Thus, the normalised Hamiltonian $\mathcal{K}$ always possesses the equilibrium points

$$E_1 = (0, 0, -\mathcal{E}), \qquad E_2 = (0, 0, \mathcal{E}).$$

From (19), $Z = -\mathcal{E}$ implies $G = H$; thus, the equilibrium point E_1 represents the family of equatorial orbits. Instead, $Z = \mathcal{E}$ implies $G = L$: the equilibrium point E_2 represents the family of circular orbits. Condition (20) is also fulfilled whenever a level curve $\tilde{X}(Z; \boldsymbol{a}, k)$ is tangent to the contour $\mathcal{C}$, with k a given level set of the Hamiltonian $\mathcal{K}(X, Z; \boldsymbol{a})$. We can therefore have an equilibrium point of coordinates $(\hat{X}(Z_+; \mathcal{E}), 0, Z_+)$ if there exists $Z = Z_+$ such that

$$\begin{cases} \frac{\mathrm{d}\tilde{X}}{\mathrm{d}Z}(Z_+; \boldsymbol{a}, k) = \frac{d\hat{X}}{dZ}(Z_+; \mathcal{E}), \\ \tilde{X}(Z; \boldsymbol{a}, k) = \hat{X}(Z_+; \mathcal{E}), \end{cases} . \tag{23}$$

where

$$\tilde{X}(Z; \boldsymbol{a}, k) = \frac{k - g(Z; \boldsymbol{a})}{f(Z; \boldsymbol{a})} \tag{24}$$

is defined by recalling (18). From (23) and (24), we obtain that Z_+ is a zero of the function $s_+(Z; \boldsymbol{a})$ equal to

$$s_+(Z; \boldsymbol{a}) = -\frac{1}{f(Z; \boldsymbol{a})} \frac{\partial \mathcal{K}}{\partial Z}\Big|_{X = \hat{X}(Z; \mathcal{E})} + 2Z. \tag{25}$$

Function $s_+(Z; \boldsymbol{p})$ can have multiple zeros corresponding to acceptable equilibrium solutions. In the following, we will refer to them as equilibrium points of type E_+. On the other hand, we can have an equilibrium point of coordinates $(-\hat{X}(Z_-; \mathcal{E}), 0, Z_-)$, if there exists $Z = Z_-$ such that

$$\begin{cases} \frac{\mathrm{d}\tilde{X}}{\mathrm{d}Z}(Z_-; \boldsymbol{a}, k) = -\frac{\mathrm{d}\hat{X}}{\mathrm{d}Z}(Z_-; \mathcal{E}), \\ \tilde{X}(Z_-; \boldsymbol{a}, k) = -\hat{X}(Z_-; \mathcal{E}). \end{cases} \tag{26}$$

In this case, Z_- results to be a zero of the function $s_-(Z; \boldsymbol{a})$ given by

$$s_-(Z; \boldsymbol{a}) = -\frac{1}{f(Z; \boldsymbol{a})} \frac{\partial \mathcal{K}}{\partial Z}\Big|_{X = -\hat{X}(Z; \boldsymbol{a})} - 2Z. \tag{27}$$

Similarly as before, equation $s_-(Z; \boldsymbol{a}) = 0$ can have multiple acceptable solutions. In this case, we are going to talk about equilibrium points of type E_-. From the first of (19), we have that equilibrium points of type E_+ correspond to the families of periodic orbits with $g = 0, \pi$, while those of type E_- correspond to the families of periodic orbits with $g = \pm\frac{\pi}{2}$.

In the second case of (21), if there exist $\bar{X} \in \mathbb{R}$ and $\bar{Z} \in \mathbb{R}$ fulfilling these conditions, one must verify whether the two resulting equilibrium points $\bar{E}_1 = (\bar{X}, \bar{Y}_1, \bar{Z})$ and $\bar{E}_2 = (\bar{X}, \bar{Y}_2, \bar{Z})$, with

$$\bar{Y}_1 = \sqrt{\left(-\bar{Z}^2 + \mathcal{E}^2\right)^2 - \bar{X}^2}, \qquad \bar{Y}_2 = -\bar{Y}_1,$$

belong to i.e. whether $\bar{Y}_1, \bar{Y}_2 \in \mathbb{R}$. It is interesting to notice that for every Y the level curves of the Hamiltonian, $\{\mathcal{K} = k\}$, given by (24), have a singularity at $Z = \bar{Z}$ as $\frac{\partial \mathcal{K}}{\partial X}(\bar{Z}) = f(\bar{Z}; \boldsymbol{a}) = 0$. The value $Z = \bar{Z}$ gives a vertical *asymptote* that is a vertical plane in the 3D space X, Y, Z.

The condition

$$\frac{\partial \mathcal{K}}{\partial Z} = g'(Z) + f'(Z)X = 0, \qquad \text{namely} \qquad X = -\frac{g'(Z)}{f'(Z)},$$

gives an oblique *asymptote*, a tilted surface in the 3D space X, Y, Z. The two surfaces cross in a straight line, orthogonal to the (Z, X) plane, which 'pierces' the lemon in the symmetric fixed points $\bar{E}_1$, $\bar{E}_2$.

Remark 1 Each equilibrium point $E_{1,2}$ in X, Y, Z, corresponds to one equilibrium in the variables (ξ_1, ξ_2, ξ_3), respectively, equal to

$$\boldsymbol{\xi}^{E_1} = (0, 0, -\mathcal{E}), \qquad \boldsymbol{\xi}^{E_2} = (0, 0, \mathcal{E}).$$

Instead, each equilibrium point of type $E_\pm$ and of type $\bar{E}_{1,2}$ corresponds to two equilibria. We have the following list of correspondences:

$$\boldsymbol{\xi}^{E_+}_{1,2} = (\pm\sqrt{\hat{X}(Z_+; \mathcal{E})}, 0, Z_+); \qquad \boldsymbol{\xi}^{E_-}_{1,2} = (0, \pm\sqrt{\hat{X}(Z_-; \mathcal{E})}, Z_-);$$

$$\boldsymbol{\xi}^{\bar{E}_1}_{1,2} = \left(\frac{\bar{Y}_1}{\bar{\xi}_2}, \pm\bar{\xi}_2, \bar{Z}\right); \qquad \boldsymbol{\xi}^{\bar{E}_2}_{1,2} = \left(\frac{\bar{Y}_2}{\bar{\xi}_2}, \pm\bar{\xi}_2, \bar{Z}\right); \qquad \bar{\xi}_2 = \sqrt{\frac{-\bar{X} + \sqrt{\bar{X}^2 + \bar{Y}_1^2}}{2}}.$$

3.4 Stability of the equilibria

To study the stability of the equilibrium points, it is more convenient to come back to the variables ξ_1, ξ_2, ξ_3 (Coffey et al. 1994). The transformed closed form is

$$\mathsf{K} = g(\xi_3; \boldsymbol{a}) + f(\xi_3; \boldsymbol{a})\left(\xi_1^2 - \xi_2^2\right).$$

Let us set $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T$. We have

$$\dot{\boldsymbol{\xi}} = \boldsymbol{F}(\boldsymbol{\xi}), \qquad \boldsymbol{F}(\boldsymbol{\xi}) = 2G\left(\frac{\partial \mathsf{K}}{\partial \boldsymbol{\xi}} \times \boldsymbol{\xi}\right).$$

We recall that $G = G(\xi_3) = \sqrt{\xi_3 + \frac{L^2+H^2}{2}}$. Let us call $\boldsymbol{\xi}_E$ an equilibrium point and $\delta\boldsymbol{\xi} = \boldsymbol{\xi} - \boldsymbol{\xi}_E$ a small displacement from it. The linearised system around the equilibrium is

$$\delta\dot{\boldsymbol{\xi}} = D\boldsymbol{F}|_{\boldsymbol{\xi}=\boldsymbol{\xi}_E}\delta\boldsymbol{\xi},$$

where

$$D\boldsymbol{F}(\boldsymbol{\xi}) = 2G\begin{bmatrix} -2\frac{\partial f}{\partial \xi_3}\xi_1\xi_2 & -\left(\frac{\partial \mathsf{K}}{\partial \xi_3} + 2f\xi_3 - 2\frac{\partial f}{\partial \xi_3}\xi_2^2\right) & -\xi_2\left(\frac{\partial^2 \mathsf{K}}{\partial \xi_3^2} + 2f + 2\frac{\partial f}{\partial \xi_3}\xi_3\right) \\ \left(\frac{\partial \mathsf{K}}{\partial \xi_3} - 2f\xi_3 + 2\frac{\partial f}{\partial \xi_3}\xi_1^2\right) & -2\frac{\partial f}{\partial \xi_3}\xi_1\xi_2 & \xi_1\left(\frac{\partial^2 \mathsf{K}}{\partial \xi_3^2} - 2f - 2\frac{\partial f}{\partial \xi_3}\xi_3\right) \\ 4f\xi_2 & 4f\xi_1 & 4\frac{\partial f}{\partial \xi_3}\xi_1\xi_2 \end{bmatrix}.$$

Since the $\boldsymbol{\xi} \in \mathcal{S}$, the solution of the previous differential system must identically satisfy the constraint

$$\xi_1\delta\xi_1 + \xi_2\delta\xi_2 + \xi_3\delta\xi_3 = 0.$$

Thus, we obtain the reduced system

$$\begin{bmatrix} \delta\dot{\xi}_1 \\ \delta\dot{\xi}_2 \end{bmatrix} = DF_R(\boldsymbol{\xi} = \boldsymbol{\xi_E}) \begin{bmatrix} \delta\xi_1 \\ \delta\xi_2 \end{bmatrix},$$

with

$$DF_R(\boldsymbol{\xi}) = 2G \begin{bmatrix} \frac{\xi_1\xi_2}{\xi_3}\left(\frac{\partial^2 \mathsf{K}}{\partial\xi_3^2} + 2f\right) & \frac{\xi_2^2}{\xi_3}\left(\frac{\partial^2 \mathsf{K}}{\partial\xi_3^2} + 2f\right) - \left(\frac{\partial \mathsf{K}}{\partial\xi_3} + 2f\xi_3 - 4\frac{\partial f}{\partial\xi_3}\xi_2^2\right) \\ \left(\frac{\partial \mathsf{K}}{\partial\xi_3} - 2f\xi_3 + 4\frac{\partial f}{\partial\xi_3}\xi_1^2\right) - \frac{\xi_1^2}{\xi_3}\left(\frac{\partial^2 \mathsf{K}}{\partial\xi_3^2} - 2f\right) & -\frac{\xi_1\xi_2}{\xi_3}\left(\frac{\partial^2 \mathsf{K}}{\partial\xi_3^2} - 2f\right) \end{bmatrix}.$$

To evaluate the stability of the equilibrium point, we have to compute the eigenvalues $\alpha_{1,2}$ of $DF_R(\boldsymbol{\xi_E})$, by solving the characteristic equation

$$\alpha^2 - \mathsf{Tr}\, DF_R(\boldsymbol{\xi_E})\alpha + \mathsf{det} DF_R(\boldsymbol{\xi_E}) = 0,$$

with $\mathsf{Tr}\, DF_R(\boldsymbol{\xi_E})$ and $\mathsf{det} DF_R(\boldsymbol{\xi_E})$ the trace and the determinant of $DF_R(\boldsymbol{\xi_E})$. By using transformation (17), we obtain that the characteristic equation for the equilibrium point E_1 is

$$\alpha^2 + 4H^2 f^2(-\mathcal{E}; \boldsymbol{a})\, s_-(-\mathcal{E}; \boldsymbol{a}) s_+(-\mathcal{E}; \boldsymbol{a}) = 0, \tag{28}$$

while the one for E_2 is

$$\alpha^2 + 4L^2 f^2(\mathcal{E}; \boldsymbol{a})\, s_-(\mathcal{E}; \boldsymbol{a}) s_+(\mathcal{E}; \boldsymbol{a}) = 0, \tag{29}$$

with $s_+(Z; \boldsymbol{a})$ and $s_-(Z; \boldsymbol{a})$ given in (25) and (27). Note that whenever the parameters $\boldsymbol{a}$ are such that an equilibrium point of either type E_+ or E_- coincides with E_1 (i.e. $Z = -\mathcal{E}$ is a zero of either $s_+(Z; \boldsymbol{a})$ or $s_-(Z; \boldsymbol{a})$) E_1 becomes degenerate. The same holds true for E_2. For an equilibrium point of type E_+ of coordinates $(\hat{X}(Z_+; \mathcal{E}), 0, Z_+)$, it can be proved that the characteristic equation is

$$\alpha^2 + 16G^2 f^2(Z_+; \boldsymbol{a})\hat{X}(Z_+; \mathcal{E})\left(\frac{\mathrm{d}^2\tilde{X}}{\mathrm{d}Z^2}(Z_+; \boldsymbol{a}, k_+) - \frac{\mathrm{d}^2\hat{X}}{\mathrm{d}Z^2}(Z_+; \mathcal{E})\right) = 0, \tag{30}$$

with k_+ the value of the Hamiltonian such that $\tilde{X}(Z_+; \boldsymbol{a}, k_+) = \hat{X}(Z_+; \mathcal{E})$. Similarly, for an equilibrium point of type E_- of coordinates $(-\hat{X}(Z_-; \mathcal{E}), 0, Z_-)$ we have

$$\alpha^2 - 16G^2 f(Z_-; \boldsymbol{a})^2\hat{X}(Z_-; \mathcal{E})\left(\frac{\mathrm{d}^2\tilde{X}}{\mathrm{d}Z^2}(Z_-; \boldsymbol{a}, k_-) + \frac{\mathrm{d}^2\hat{X}}{\mathrm{d}Z^2}(Z_-; \mathcal{E})\right) = 0, \tag{31}$$

with k_- such that $\tilde{X}(Z_-; \boldsymbol{a}, k_-) = -\hat{X}(Z_-; \mathcal{E})$. Since for $Z \neq \pm\mathcal{E}$, $\hat{X}(Z; \mathcal{E}) > 0$, the stability of the equilibrium points of type E_+ and E_- can be determined by comparing the concavities of the level curve $\tilde{X}(Z; \boldsymbol{a}, k)$ and of the contour $\mathcal{C}$ of $\mathcal{L}$ at their point of tangency. Finally, the characteristic equations for $\bar{E}_1$ and $\bar{E}_2$ are

$$\alpha^2 + 4\bar{G}^2 \frac{\bar{Y}_{1,2}^2}{\bar{Z}^2}\left(2\bar{\mathcal{K}}_{ZZ}^2 - 16\bar{f}_Z^2\bar{Z}^2\right) = 0,$$

with

$$\bar{\mathcal{K}}_{ZZ} = \frac{\partial^2\mathcal{K}}{\partial Z^2}\left(\bar{X}, \bar{Z}; \boldsymbol{p}\right), \qquad \bar{f}_Z = \frac{\partial f}{\partial Z}(\bar{Z}; \boldsymbol{p}), \qquad \bar{G} = \sqrt{\bar{Z}^2 + \frac{L^2 + H^2}{2}}.$$

As $\bar{Y}_1^2 = \bar{Y}_2^2$, the two characteristic equations coincide: $\bar{E}_1$ and $\bar{E}_2$ have the same stability. When $\bar{E}_1$ and $\bar{E}_2$ coincide since $\bar{Y}_1 = \bar{Y}_2 = 0$, the resulting equilibrium point is degenerate.

Remark 2 To evaluate the stability, we can also exploit the Poincaré–Hopf index theorem:

Let M be a compact manifold and w a smooth vector field on M with isolated zeros. The sum $\sum \iota$ of the indices of the zeros of w is equal to the Euler characteristic of M (Milnor 1965).

As the phase space is a sphere in the coordinates (ξ_1, ξ_2, ξ_3), its Euler characteristic is equal to 2. Whenever E_1 and E_2 are Lyapunov stable, in the linearised reduced system they are centres; thus, their indexes ι are both equal to $+1$. Instead, when one of them is Lyapunov unstable, it corresponds to a saddle with $\iota = -1$. Each equilibrium point of type $E_\pm$ corresponds to two equilibrium points in (ξ_1, ξ_2, ξ_3) (see Remark 1), both either Lyapunov stable or unstable. The bifurcation of a first stable pair implies a stability/instability transition of one of the cusps so that the indexes are $(+1 - 1 + 1 + 1)$. The bifurcation of a second unstable pair implies that the cusp regains stability and the indexes are $(+1+1+1+1-1-1)$. Due generalisation applies in the case of the points of type $\bar{E}_{1,2}$.

4 Applications

4.1 The J_2-problem

We are going to apply the variables X, Y, Z to analyse a classical and well-known problem in the framework of the artificial satellite theory: the study of the secular Hamiltonian in which only the second zonal harmonic of the gravitational potential is retained, i.e. the J_2 terms. From (7), (8), (9) and (10), the resulting closed form is

$$\begin{aligned}\mathcal{K}_{J_2} = &-\frac{\mu^2}{2L^2} + \frac{\mu^4 J_2 R_P^2 (G^2 - 3H^2)}{4G^5 L^3} + \frac{3\mu^6 J_2^2 R_P^4}{128 L^5 G^{11}} \Big[- 5G^6 - 4G^5 L \\ &+ 24G^3 H^2 L - 36 G H^4 L - 35 H^4 L^2 + G^4 (18H^2 + 5L^2) \\ &- 5G^2 (H^4 + 2H^2 L^2) + 2(G^2 - 15H^2)(G^2 - L^2)(G^2 - H^2) \cos 2g \Big].\end{aligned}$$

To simplify its analysis, we make the system dimensionless by performing the following choice of units: we take the orbital semi-major axis a as unit of length and the unit of time such that $\mu = 1$. Let us call $\rho = H/L$. In the adimensional system, the Delaunay actions L and H become

$$L = 1, \qquad H = \rho,$$

where $\rho = G \cos i$, with i the orbital inclination. Moreover, the action G coincides with $\eta = \sqrt{1 - e^2}$, being e the orbital eccentricity. Since in the adimensional system the planet's radius $R_P < 1$, let us set

$$\lambda = J_2 R_P^2; \tag{32}$$

λ plays here the role of small parameter, of the same order as the book-keeping parameter ϵ. We drop the constant Keplerian term and we perform a transformation of the time variable $t \mapsto \tau$, defined as:

$$\frac{\partial \tau}{\partial t} = \lambda. \tag{33}$$

Thus, the secular Hamiltonian in closed form becomes

$$\mathcal{K}_{J_2} = \frac{(G^2-3\rho^2)}{4G^5} + \frac{3\lambda}{128G^{11}}\Big[-5G^6-4G^5+24G^3\rho^2-36G\rho^4-35\rho^4+G^4(18\rho^2+5)$$
$$-5G^2(\rho^4+2\rho^2)+2(G^2-15\rho^2)(G^2-1)(G^2-\rho^2)\cos 2g\Big].$$

In these units, the X, Y, Z variables are

$$X=(G^2-\rho^2)(1-G^2)\cos 2g,\quad Y=(G^2-\rho^2)(1-G^2)\sin 2g,\quad Z=G^2-\frac{1+\rho^2}{2}, \tag{34}$$

and we also have $\mathcal{E}=(1-\rho^2)/2$. The introduction of the variables X, Y, Z leads to a closed form with the same structure of $\mathcal{K}$ in (18), with

$$g(Z,\boldsymbol{a}) = \frac{-5\rho^2+2Z+1}{\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{5}{2}}} - \frac{3\lambda}{16\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{11}{2}}}\Big(40Z^3+\left(-84\rho^2+20\right)Z^2$$
$$+\left(-74\rho^4-44\rho^2-10\right)Z-11\rho^6+273\rho^4$$
$$-\rho^2-5\&+4\sqrt{2\rho^2+4Z+2}\left(-5\rho^2+2Z+1\right)^2\Big),$$
$$f(Z,\boldsymbol{a}) = -\frac{3}{2}\lambda\frac{\left(-29\rho^2+2Z+1\right)}{\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{11}{2}}},$$

and $\boldsymbol{a}=(\rho;\lambda)$. Note that if the terms proportional to λ are neglected, the problem has one equilibrium solution for

$$Z=\frac{9\rho^2-1}{2},\quad \forall X, Y, \tag{35}$$

which implies

$$G=G_c=\sqrt{5}\rho. \tag{36}$$

Since $\rho=G\cos i$, the orbit has then a stationary pericentre at the so-called critical inclination:

$$i_c=\arccos\frac{1}{\sqrt{5}}.$$

In the following, we study the J_2-problem for $|\rho|\in(0,1)$ and $\lambda\in(0,1)$: we discuss the existence and the stability of frozen orbits by analysing the corresponding properties of the equilibrium points of the reduced system.

First of all, we show that the equilibrium point E_1, representative of the family of equatorial orbits, is always stable. Then, we analyse the stability of the equilibrium point E_2, representative of the family of circular orbits. In particular, we determine the values ρ_+ and ρ_- of $|\rho|$ at which pitchfork bifurcations occur: for $|\rho|$ between ρ_- and ρ_+ E_2 is unstable, otherwise it is stable; moreover, there exist a stable equilibrium point of type E_+ for $|\rho|<\rho_+$ and an unstable equilibrium point of type E_- for $|\rho|<\rho_-$. At last, we show that the equilibrium points $\bar{E}_1$ and $\bar{E}_2$ do not exist for any ρ and λ.

For the J_2 problem and the other problems analysed in the following, all the equilibrium points of type E_+ are indicated with an odd integer number larger than 1 as a subscript; similarly, the subscript of the equilibrium points of type E_- is an even integer number larger than 2.

We recall that $\rho=G\cos i$ and $G=\sqrt{1-e^2}$. In the procedure we follow, we select a planet and we fix the value of the semi-major axis, on which λ depends through the dimensionless R_P. Suppose to select a value of ρ such that an equilibrium point of type E_+ exists and to compute the value of the action G of such equilibrium point; from the selected ρ and the value of G we can obtain the orbital eccentricity and inclination of the family of orbits represented by the equilibrium point itself. The same holds for all the equilibrium points. In the same way, since the eccentricity of the orbits represented by E_2 is equal to zero, from ρ_+ and ρ_- we can compute the values of the orbital inclination at which the stability of the circular orbits changes.

4.1.1 Stability of E_1

From (28), it results that the stability of the equilibrium point E_1 depends on the sign of $s_+(-\mathcal{E};\boldsymbol{a})$ and $s_-(-\mathcal{E};\boldsymbol{a})$. We have

$$s_+(-\mathcal{E};\boldsymbol{a}) = -\frac{2}{7}\frac{8\rho^4+\lambda\left(-7\rho^2+12|\rho|+31\right)}{\lambda}, \qquad s_-(-\mathcal{E},\boldsymbol{a}) = -\frac{8}{7}\frac{2\rho^4+3\lambda|\rho|+6\lambda}{\lambda}.$$

It is straightforward that $s_+(-\mathcal{E};\boldsymbol{a}) < 0$ and $s_-(-\mathcal{E},\boldsymbol{a}) < 0$ $\forall\lambda \in (0, 1)$ and $\forall|\rho| \in (0, 1)$. Thus, the equilibrium point E_1 is stable, and it does never coincide with equilibrium points of either type E_+ or E_-.

4.1.2 Stability of E_2

In analogy to E_1, from (29) we obtain that the stability of E_2 depends on the sign of $s_+(\mathcal{E};\boldsymbol{a})$ and $s_-(\mathcal{E};\boldsymbol{a})$. We have

$$s_+(\mathcal{E};\boldsymbol{a}) = -\frac{1}{2}\frac{425\lambda\rho^4-146\lambda\rho^2+80\,\rho^2+9\lambda-16}{\lambda\left(15\rho^2-1\right)},$$

and

$$s_-(\mathcal{E};\boldsymbol{a}) = -\frac{1}{2}\frac{\left(365\lambda\rho^4-82\lambda\rho^2+80\,\rho^2+5\lambda-16\right)}{\lambda\left(15\rho^2-1\right)}.$$

The function $s_+(\mathcal{E};\boldsymbol{a})$ has two real zeros $\rho = \pm\rho_+$, where

$$\rho_+ = \sqrt{\frac{1}{5}+\frac{-4(3\lambda+10)+4\sqrt{85\lambda^2+(3\lambda+10)^2}}{425\lambda}}. \tag{37}$$

Similarly, $s_-(\mathcal{E};\boldsymbol{a})$ possesses two real zeros $\rho = \pm\rho_-$, with

$$\rho_- = \sqrt{\frac{1}{5}+\frac{-4(8\lambda+10)+4\sqrt{-73\lambda^2+(8\lambda+10)^2}}{365\lambda}}. \tag{38}$$

It holds that $\rho_+ > \rho_-$, $\forall\lambda \in (0, 1)$. Thus,

- for $\rho_+ < |\rho| < 1$ and for $0 < |\rho| < \rho_-$, E_2 is stable;
- for $\rho_- < |\rho| < \rho_+$, E_2 is unstable.

Moreover, at $|\rho| = \rho_+$ the degenerate E_2 coincides with an equilibrium point of type E_+, while at $|\rho| = \rho_-$ it coincides with an equilibrium point of type E_-.

If we approximate ρ_+ and ρ_- as series in λ, we obtain

$$\rho_+ \sim \frac{1}{\sqrt{5}}\Big(1 + \frac{1}{10}\lambda - \frac{7}{200}\lambda^2 - \frac{7}{800}\lambda^3 + \mathcal{O}(\lambda^4)\Big), \tag{39}$$

$$\rho_- \sim \frac{1}{\sqrt{5}}\Big(1 - \frac{1}{10}\lambda + \frac{3}{40}\lambda^2 - \frac{299}{4000}\lambda^3 + \mathcal{O}(\lambda^4)\Big). \tag{40}$$

Thus, the bifurcations occur nearby the zero-order solution (35) when $Z = \mathcal{E}$.

4.1.3 Existence and stability of the equilibrium points of type E_+ and E_-

The equilibrium points of type E_+ correspond to the zeros of $s_+(Z; \boldsymbol{a})$. Performing the change of variable $Z \mapsto G$ using (34), we obtain

$$s_+\left(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a}\right) = \frac{S_+(G; \boldsymbol{a})}{16\lambda G(28\rho^2 + 2G)}$$

with

$$S_+(G; \boldsymbol{a}) = 32G^8 + (-160\rho^2 - 15\lambda)G^6 - 24G^5\lambda + (-98\lambda\rho^2 + 21\lambda)G^4 + 192G^3\lambda\rho^2 + (225\lambda\rho^4 + 198\lambda\rho^2)G^2 - 360G\lambda\rho^4 - 715\lambda\rho^4.$$

The zeros of $s_+(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a})$ are the zeros of $S_+(G; \boldsymbol{a})$. This is a polynomial function of degree 8 in G. Thus, finding its zeros is not straightforward. However, some pieces of information can be inferred by inverting the roles of G and ρ: we consider G as a parameter and ρ becomes the independent variable of the problem. We obtain $S_+(G; \boldsymbol{a}) = 0$ for $\rho^2 = \rho^2_{E_+1,2}$, with

$$\begin{aligned}
\rho^2_{E_+1,2} &= \frac{G^2}{5\lambda}\frac{A_+ \pm 4\sqrt{B_+}}{C_+}, \\
A_+ &= \left(49G^2 - 96\,G - 99\right)\lambda + 80\,G^4, \\
B_+ &= \frac{1}{16}\,(A_+^2 - 5\lambda C_+ D_+), \\
C_+ &= 45G^2 - 72G - 143, \\
D_+ &= 32G^4 - 15G^2\lambda - 24G\lambda + 21\lambda.
\end{aligned} \tag{41}$$

The solutions are admissible if $0 < \rho^2_{E_+1,2} < G^2$. Since $G \in (0, 1]$ and $\lambda \in (0, 1)$, it is easy to verify that $C_+ < 0$ and

$$D_+ > (32G^4 - 15G^2 - 24G + 21)\lambda > 0,$$

which implies $B_+ > A_+^2/16$. Thus, $\rho^2_{E_+1} < 0$ and is not an admissible solution. Instead, $\rho^2_{E_+2} > 0$. Since it also holds $A_+ - 5\lambda C_+ > 0$ and

$$16B_+ - (A_+ - 5\lambda C_+)^2 = 80\lambda C_+(8G^4 - 7G^2\lambda + 12G\lambda + 31\lambda) < 0$$

we have $\rho^2_{E_+2} < G^2$. It follows that $\rho^2_{E_+2}$ is admissible $\forall\lambda \in (0, 1)$ and $\forall G \in (0, 1]$. For $G = 1$ we obtain $\rho^2_{E_+2} = \rho_+$; furthermore, it can be proved that

$$\frac{\mathrm{d}\rho^2_{E_+2}}{\mathrm{d}G} > 0 \quad \forall G \in (0, 1], \quad \forall\lambda \in (0, 1), \tag{42}$$

see Appendix. It follows that for each $|\rho| \leq \rho_+$ there exists only one value of G solving $S_+(G; \boldsymbol{a}) = 0$. Thus, for each $|\rho| \leq \rho_+$ there exists only one equilibrium point of type E_+, which we call E_3 and which coincides with E_2 for $|\rho| = \rho_+$.

The value of the only meaningful solution of $S_+(G; \boldsymbol{a}) = 0$ can be approximated with a perturbation method. We observe that the solution of the 'unperturbed' problem with $\lambda = 0$ is just the critical value (36). Then, we can look for a solution of the form:

$$G_+ = \sqrt{5}\rho + \sum_{k \geq 1} a_k \lambda^k.$$

At third order in λ, we find

$$G_+ = \sqrt{5}\rho + \frac{-1 + 4\rho^2}{10\sqrt{5}\rho^3}\lambda - \frac{14 + 6\sqrt{5}\rho - 81\rho^2 - 24\sqrt{5}\rho^3 + 100\rho^4}{5000\sqrt{5}\rho^7}\lambda^2 + \frac{353 - 336\sqrt{5}\rho - 4077\rho^2 + 1944\sqrt{5}\rho^3 + 12310\rho^4 - 2400\sqrt{5}\rho^5 - 6600\rho^6}{5000000\sqrt{5}\rho^{11}}\lambda^3. \tag{43}$$

We apply the same technique to verify the existence of equilibrium points of type E_-. They correspond to the zeros of a function $S_-(G; \boldsymbol{a})$ equal to

$$S_-(G; \boldsymbol{a}) = 32G^8 + \left(-160\rho^2 - 35\,\lambda\right) G^6 - 24G^5\lambda + \left(350\,\lambda\,\rho^2 + 49\,\lambda\right) G^4 + 192\,G^3\lambda\,\rho^2 + \left(-315\,\lambda\,\rho^4 - 378\,\lambda\,\rho^2\right) G^2 - 360\,G\,\lambda\,\rho^4 - 55\,\lambda\,\rho^4.$$

We have $S_-(G; \boldsymbol{a}) = 0$ for $\rho^2 = \rho^2_{E_-1,2}$, with

$$\begin{aligned} \rho^2_{E_-1,2} &= -\frac{G^2}{5\lambda}\frac{A_- \pm 4\sqrt{B_-}}{C_-}, \\ A_- &= \left(-175\,G^2 - 96\,G + 189\right)\lambda + 80\,G^4, \\ B_- &= \frac{1}{16}\,(A_-^2 + 5\lambda C_- D_-), \\ C_- &= 63G^2 + 72G + 11, \\ D_- &= (32G^4 - 35G^2\lambda - 24G\lambda + 49\lambda. \end{aligned} \tag{44}$$

$\forall G \in (0, 1]$ and $\forall\lambda \in (0, 1)$ we have $C_- > 0$ and

$$D_- > (32G^4 - 35G^2 - 24G + 49)\lambda > 0.$$

Thus, $\rho^2_{E_-1} < 0$ and is not an admissible solution. Instead, $\rho^2_{E_-2} > 0$; as $5\lambda C_- + A_- > 0$ and

$$16B_- - (5\lambda C_- + A_-)^2 = -320\lambda C_-(2G^4 + 3G\lambda + 6\lambda) < 0,$$

we have $\rho^2_{E_-2} < G^2$: $\rho^2_{E_-2}$ is admissible $\forall G \in (0, 1]$ and $\forall\lambda \in (0, 1)$. For $G = 1$, $\rho^2_{E_-2} = \rho^2_-$; it can also be proved that

$$\frac{\mathrm{d}\rho^2_{E_-2}}{\mathrm{d}G} > 0, \quad \forall\lambda \in (0, 1), \tag{45}$$

see Appendix. As a consequence, also in this case we obtain that there exists one equilibrium point of type E_- for any $|\rho| \leq \rho_-$. We call it E_4. For $|\rho| = \rho_-$, it coincides with E_2. The solution of $S_-(G;\boldsymbol{a}) = 0$ can be approximated in analogy with what seen above. At third order in λ, we find

$$G_- = \sqrt{5}\rho + \frac{9-35\rho^2}{100\sqrt{5}\rho^3}\lambda - \frac{1305 - 108\sqrt{5}\rho - 7910\rho^2 + 420\sqrt{5}\rho^3 + 11025\rho^4}{100000\sqrt{5}\rho^7}\lambda^2 + \frac{267309 - 31320\sqrt{5}\rho - 2226905\rho^2 + 189840\sqrt{5}\rho^3 + 5775175\rho^4 - 264600\sqrt{5}\rho^5 - 4501875\rho^6}{100000000\sqrt{5}\rho^{11}}\lambda^3. \tag{46}$$

For $\rho_- < |\rho| < \rho_+$, the equilibrium E_3 is stable as a consequence of the Poincaré–Hopf theorem. By applying this last theorem, we also obtain that for $0 < |\rho| < \rho_-$, one equilibrium point between E_3 and E_4 is stable, while the other is unstable. Since E_3 does not undergo any bifurcation at $|\rho| = \rho_-$, it is stable, while E_4 is unstable.

4.1.4 About the existence of $\bar{E}_1$ and $\bar{E}_2$

The coordinates $\bar{X}$ and $\bar{Z}$ of the equilibrium points $\bar{E}_1$ and $\bar{E}_2$ are

$$\bar{X} = -\frac{1}{3}\rho^2\left(-144\sqrt{15}|\rho| - 2835\rho^2 + 307\right) - 18000\frac{\rho^6}{\lambda}, \qquad \bar{Z} = \frac{29\rho^2 - 1}{2}.$$

In order to have $\bar{Z} \in [-\mathcal{E}, \mathcal{E}]$, it is necessary that $|\rho| < 1/\sqrt{15}$. Let us call $\mathcal{Y}$ the square of Y coordinates of the equilibrium points, $\bar{Y}_{1,2}$. It holds

$$\mathcal{Y} = -\frac{\rho^4}{27\lambda^2}\Big(-10800\rho^4\sqrt{15} + 441\lambda\rho^2\sqrt{15} - 53\lambda\sqrt{15} + 432\lambda|\rho|\Big)\Big(-54000\rho^4\sqrt{15}\& + 3465\lambda\rho^2\sqrt{15} - 349\lambda\sqrt{15} + 2160\lambda|\rho|\Big).$$

We have to verify whether there exist values of $|\rho| < 1/\sqrt{15}$ such that $\mathcal{Y} \geq 0$. Since $\lambda \in (0, 1)$, we have

$$-10800\rho^4\sqrt{15} + 441\lambda\rho^2\sqrt{15} - 53\lambda\sqrt{15} + 432\lambda|\rho| < \lambda\big(-10800\rho^4\sqrt{15} + 441\rho^2\sqrt{15} - 53\sqrt{15} + 432|\rho|\big) < 0,$$

and

$$-54000\rho^4\sqrt{15} + 3465\lambda\rho^2\sqrt{15} - 349\lambda\sqrt{15} + 2160\lambda|\rho| < \lambda\big(-54000\rho^4\sqrt{15} + 3465\rho^2\sqrt{15} - 349\sqrt{15} + 2160|\rho|\big) < 0.$$

It follows that $\mathcal{Y} < 0$, $\forall\rho \in (0, 1/\sqrt{15})$ and $\forall\lambda \in (0, 1)$. Thus, the equilibrium points $\bar{E}_1$ and $\bar{E}_2$ never exist for the J_2-problem.

4.1.5 Summary and comparison with previous works

Here, we summarise the results for the J_2-problem, and we compare them with those previously obtained by Coffey et al. (1986) and Palacián (2007).

At $|\rho| = \rho_+$ and $|\rho| = \rho_-$, with ρ_+ and ρ_- defined in (37) and (38), there are two pitchfork bifurcations. In particular, we have that

- for $\rho_+ < |\rho| < 1$ there exist only the equilibrium points E_1 and E_2 and they are stable;
- at $|\rho| = \rho_+$ there is a bifurcation: E_2 is degenerate and coincides with E_3, while E_1 is still stable; E_3 is an equilibrium point of type E_+;

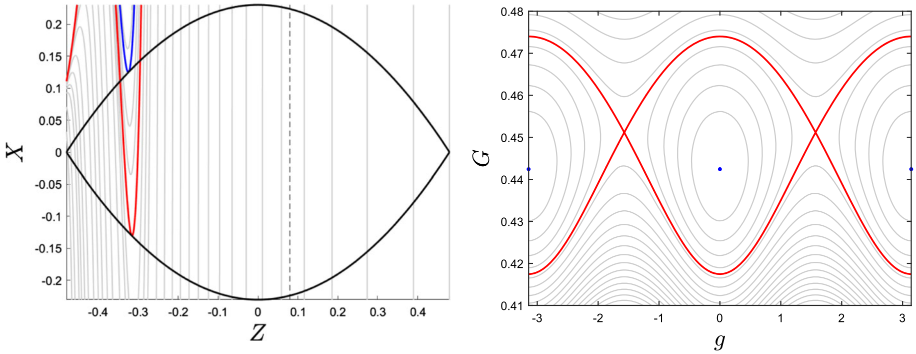


Fig. 1 Level curves for the J_2-problem. Here, $\rho = 0.2$ and $\lambda = 0.001$. On the left, they are shown in the (Z, X) plane. The black line represents the contour $\mathcal{C}$ of the *lemon* space. The blue line represents the level curve tangent to $\mathcal{C}$ at the stable equilibrium point E_3, while the red one represents the level curve tangent to $\mathcal{C}$ at the unstable E_4. The dashed black line corresponds to $Z = \bar{Z}$, for which the level curves have a singularity. On the right, the level curves are shown in an enlargement of the (g, G) plane surrounding the equilibrium points. The blue dots are the stable equilibrium points corresponding to E_3; the red curve is the separatrix of the equilibrium points at $g = \pm\frac{\pi}{2}$ corresponding to E_4

- for $\rho_- \leq |\rho| < \rho_+$ there exist the equilibrium points E_1, E_3, which are stable, and E_2 which is unstable;
- at $|\rho| = \rho_-$ there is a bifurcation: E_2 is degenerate and coincides with E_4 of type E_-, while E_1 and E_3 are still stable;
- for $|\rho| < \rho_-$ there exist the equilibrium points E_1, E_2 and E_3, which are stable, and E_4, which is unstable.

In Fig. 1 (left panel), we show the level curves of the closed form in the (Z, X) plane when $|\rho| < \rho_-$. The blue and red lines are tangent to the contour $\mathcal{C}$ of the lemon space, respectively, at the equilibrium points E_3 and E_4. Note that the blue line has a concavity larger than that of $\mathcal{C}$ at their tangency point as E_3 is stable (see 30). Also the concavity of the red line is larger than that of $\mathcal{C}$ at their tangency point, which in this case implies that E_4 is unstable as follows from (31). In the right panel of Fig. 1, we show the level curves in an enlargement of the (G, g) plane containing the equilibrium points. Here, E_3 corresponds to the stable equilibrium points at $g = 0, \pi$ (blue dots). Instead, E_4 corresponds to the two unstable equilibrium points at $g = \pm\pi/2$: the separatrix is in red. By using (43) and (46), we are able to compute the approximated values of $G_\pm$ of the equilibrium points: $G_+ = 0.4424$ and $G_- = 0.4512$.

We remind that $\rho = H/L$. Through a transformation of variables and units, we can determine the values of $|H|$ at which the bifurcations occur in the original dimensional system. Let us call them H_+ and H_-. From (39) and (40), we obtain

$$H_+ \sim \frac{L}{\sqrt{5}}\left(1 + \frac{1}{10}\frac{J_2\mu^2 R_P^2}{L^4} - \frac{7}{200}\frac{J_2^2\mu^4 R_P^4}{L^8} - \frac{7}{800}\frac{J_2^3\mu^6 R_P^6}{L^{12}} + \mathcal{O}\left(\frac{J_2^4\mu^8 R_P^8}{L^{16}}\right)\right),$$

$$H_- \sim \frac{L}{\sqrt{5}}\left(1 - \frac{1}{10}\frac{J_2\mu^2 R_P^2}{L^4} + \frac{3}{40}\frac{J_2^2\mu^4 R_P^4}{L^8} - \frac{299}{4000}\frac{J_2^3\mu^6 R_P^6}{L^{12}} + \mathcal{O}\left(\frac{J_2^4\mu^8 R_P^8}{L^{16}}\right)\right).$$

With the same transformation, by using (43) and (46), the values $G_\pm(L, H)$ for the two bifurcated families can be expressed as series in J_2. In conclusion, by exploiting the (X, Y, Z)

variable and the geometrical approach we have recovered the results found in (Coffey et al. 1986) and in (Palacián 2007).

4.2 The J_4-problem

We study now the zonal problem containing both the J_2 and the J_4 terms. From (7), (8), (9) and (10), the closed form is

$$
\begin{aligned}
\mathcal{K}_{J_4} = & -\frac{\mu^2}{2L^2} + \frac{\mu^4 J_2 R_P^2 (G^2 - 3H^2)}{4G^5 L^3} + \frac{3\mu^6 J_2^2 R_P^4}{128 L^5 G^{11}}\big[-5G^6 - 4G^5 L \\
& + 24G^3 H^2 L - 36 G H^4 L - 35 H^4 L^2 + G^4 (18H^2 + 5L^2) \\
& - 5G^2 (H^4 + 2H^2 L^2) + 2(G^2 - 15H^2)(G^2 - L^2)(G^2 - H^2)\cos 2g\big] \\
& + \frac{3\mu^6 J_4 R_P^4}{128 L^5 G^{11}}\big[(3G^4 - 30G^2 H^2 + 35H^4)(5L^2 - 3G^2) \\
& - 10(G^2 - 7H^2)(L^2 - G^2)(G^2 - H^2)\cos 2g\big].
\end{aligned}
$$

Let us set

$$
j_4 = -\frac{J_4}{J_2^2}.
$$

After introducing it in the Hamiltonian, we adopt the same adimensional system and perform the same transformations described in Sect. 4.1. Also for this problem, we obtain a secular Hamiltonian in closed form with the structure of $\mathcal{K}$ in (18), with

$$
\begin{aligned}
g(Z, \boldsymbol{a}) = & \frac{-5\rho^2 + 2Z + 1}{\sqrt{2}\left(\rho^2 + 2Z + 1\right)^{\frac{5}{2}}} - \frac{3\lambda}{16\sqrt{2}\left(\rho^2 + 2Z + 1\right)^{\frac{11}{2}}}\Big(40Z^3 + \left(-84\rho^2 + 20\right) Z^2 \\
& + \left(-74\rho^4 - 44\rho^2 - 10\right) Z - 11\rho^6 + 273\rho^4 - \rho^2 - 5 \\
& + 4\sqrt{2\rho^2 + 4Z + 2}\left(-5\rho^2 + 2Z + 1\right)^2\Big) + \frac{3\lambda j_4}{16\sqrt{2}\left(\rho^2 + 2Z + 1\right)^{\frac{11}{2}}}\Big(-72Z^3 \\
& + 12\left(51\rho^2 + 1\right) Z^2 + 3\left(-58\rho^4 - 156\rho^2 + 22\right) Z - 249\rho^6 + 743\rho^4 - 387\rho^2 + 21\Big),
\end{aligned}
$$

$$
f(Z, \boldsymbol{a}) = -\frac{3}{2}\lambda \frac{\left(-29\rho^2 + 2Z + 1\right)}{\sqrt{2}\left(\rho^2 + 2Z + 1\right)^{\frac{11}{2}}} + \frac{15}{2}\lambda j_4 \frac{\left(-13\rho^2 + 2Z + 1\right)}{\sqrt{2}\left(\rho^2 + 2Z + 1\right)^{\frac{11}{2}}},
$$

and $\boldsymbol{a} = (\rho; \lambda, j_4)$.
In the following, we discuss the dynamical behaviour of the problem for $|\rho| \in (0, 1)$ and $j_4 \in [-6, 6]$. This range of j_4 is coherent with the book-keeping scheme used for the computation of the normalised Hamiltonian in Sect. 2 and its extent allows us to include Earth and Mars. Our results are both the outcomes of analytical considerations and numerical studies. In this case, to simplify the analysis, we fix the value of λ, taking $\lambda = 0.001$. We expect the main features of the dynamics to be qualitatively similar also for other values of λ sufficiently small.

First, we analyse the stability of the equilibrium points E_1 and E_2. Then, we discuss the existence of the equilibrium points of type E_+ and E_- and the existence of $\bar{E}_1$ and $\bar{E}_2$. Finally, we discuss their stability, and we trace a bifurcation diagram. We find out that the stability of E_1 depends on both j_4 and ρ. In particular, there are ranges of j_4 in which pitchfork bifurcations occur: they affect the stability of E_1 and can give rise to either a stable equilibrium point of type E_- or an unstable equilibrium point of type E_+. For each $j_4 \in [-6, 6]$, we also have pitchfork bifurcations affecting the stability of E_2. We determine the values ρ_+ and ρ_- of $|\rho|$ at which they occur. As in the J_2 problem, E_2 is unstable if the value of $|\rho|$ lies between ρ_- and ρ_+, otherwise it is stable. Following the pitchfork bifurcation occurring at $|\rho| = \rho_+$, an equilibrium point of type E_+ is generated; similarly, for $|\rho| < \rho_-$ there exists an equilibrium point of type E_-. The stability of these points depends on both j_4 and ρ. An interesting result is that there are ranges of j_4 where their stability changes as a consequence of further pitchfork bifurcations, which affect the existence of the equilibrium points $\bar{E}_1$ and $\bar{E}_2$. We show that, when existing, these last ones are always unstable. Finally, we find out that for some j_4 saddle-node bifurcations also occur. They can give rise to either a pair of equilibrium points of type E_+ or a pair of equilibrium points of type E_-. Independent of the type, one of the points of the pair is stable, while the other is unstable.

We recall once again that for the selected planet and the fixed value of the semi-major axis (i.e. for the given j_4 and λ), the values of ρ and G of one considered equilibrium point allow us to determine the eccentricity and the inclination of the family of orbits represented by the equilibrium point itself. In the following, we perform a general analysis not taking into account some physical limitations, for example, the fact that the orbits corresponding to a given equilibrium point may be collisional.

4.2.1 Stability of E_1

The stability of E_1 depends on the sign of the product $s_+(-\mathcal{E}, \boldsymbol{a})s_-(-\mathcal{E}, \boldsymbol{a})$. For the J_4-problem, we have

$$s_+(-\mathcal{E}, \boldsymbol{a}) = 2\,\frac{8\rho^4 + \lambda\left(31 + 12|\rho| - 7\rho^2 - 5j_4(3\rho^2 - 7)\right)}{\lambda(15j_4 - 7)},$$

and

$$s_-(-\mathcal{E}, \boldsymbol{a}) = 4\,\frac{4\rho^4 + \lambda\left(6(2 + |\rho|) - 5j_4(3\rho^2 - 5)\right)}{\lambda(15j_4 - 7)}.$$

For $j_4 \geq -31/35$, function $s_+(-\mathcal{E}, \boldsymbol{a})$ has no real zeros; instead, for $j_4 < -31/35$ there exists a real value of $|\rho|$, $|\rho| = \rho_\triangle$, solving equation $s_+(-\mathcal{E}, \boldsymbol{a}) = 0$. Similarly, if $j_4 < -12/25$ there exists one real value of $|\rho|$, $|\rho| = \rho_\triangledown$, which is a zero of $s_-(-\mathcal{E}, \boldsymbol{a})$. Thus, if $j_4 \geq -12/25$ E_1 is always stable. Instead, if $-31/35 \leq j_4 < -12/25$,

- for $\rho_\triangledown < |\rho| < 1$, E_1 is stable;
- for $|\rho| < \rho_\triangledown$, E_1 is unstable.

Finally, if $j_4 < -31/35$, it holds $\rho_\triangledown > \rho_\triangle$ so that

- for $\rho_\triangledown < |\rho| < 1$ and $0 < |\rho| < \rho_\triangle$, E_1 is stable;
- for $\rho_\triangle < |\rho| < \rho_\triangledown$, E_1 is unstable.

At $|\rho| = \rho_\triangle$, the degenerate E_1 coincides with an equilibrium point of type E_+. At $|\rho| = \rho_\triangledown$, it coincides with an equilibrium point of type E_-. In the following, we call $j_{4_{\mathrm{bif6}}} = -12/25$ and $j_{4_{\mathrm{bif9}}} = -31/35$.

4.2.2 Stability of E_2

The stability of E_2 depends on the solutions of equation (29). We have

$$s_+(\mathcal{E}, \boldsymbol{a}) = -\frac{1}{2}\,\frac{16 - 80\rho^2 - \lambda\left((420j_4 + 425)\rho^4 - (280j_4 + 146)\rho^2 + 20j_4 + 9\right)}{\lambda\left((35j_4 - 15)\rho^2 - 5j_4 + 1\right)},$$

and

$$s_-(\mathcal{E}, \boldsymbol{a}) = -\frac{1}{2}\,\frac{16 - 80\rho^2 - \lambda\left((560j_4 + 365)\rho^4 - (440j_4 + 82)\rho^2 + 40j_4 + 5\right)}{\lambda\left((35j_4 - 15)\rho^2 - 5j_4 + 1\right)}.$$

For λ sufficiently small, $\forall j_4 \in [-6, 6]$ $s_+(\mathcal{E}, \boldsymbol{a})$ possesses two real zeros at $\rho = \pm\rho_+$ with

$$\rho_+ = \sqrt{\frac{1}{5}\,\frac{\lambda(140j_4 + 73) - 40 + 4\sqrt{2}\sqrt{50 + (30 - 140j_4)\lambda + (47 + 255j_4 + 350j_4^2)\lambda^2}}{(84j_4 + 85)\lambda}}, \tag{47}$$

and $s_-(\mathcal{E}, \boldsymbol{a})$ possesses two real zeros at $\rho = \pm\rho_-$ with

$$\rho_- = \sqrt{\frac{1}{5}\,\frac{\lambda(220j_4 + 41) - 40 + 4\sqrt{100 + (160 - 540j_4)\lambda - (9 - 40j_4 - 1625j_4^2)\lambda^2}}{(112j_4 + 73)\lambda}}. \tag{48}$$

More manageable expressions are given by the series expansions

$$\rho_+ = \frac{1}{\sqrt{5}}\left(1 + \frac{1 + 6j_4}{10}\lambda - \frac{7 + 20j_4 - 132j_4^2}{200}\lambda^2\right) + \mathcal{O}\left(\lambda^3\right),$$

and

$$\rho_- = \frac{1}{\sqrt{5}}\left(1 - \frac{1 - 8j_4}{10}\lambda + \frac{15 - 166j_4 + 368j_4^2}{200}\lambda^2\right) + \mathcal{O}\left(\lambda^3\right).$$

At first order, they coincide with those found by Coffey et al. (1994). For $j_4 = j_{4_{\rm bif1}}$, with

$$j_{4_{\rm bif1}} = 1 - \frac{14}{5}\lambda + \frac{1239}{50}\lambda^2 + \mathcal{O}\left(\lambda^3\right), \tag{49}$$

it holds $\rho_+ = \rho_-$; if $j_4 > j_{4_{\rm bif1}}$, $\rho_- > \rho_+$, while for $j_4 < j_{4_{\rm bif1}}$, $\rho_- < \rho_+$. As in the J_2-problem, when the value of $|\rho|$ is between ρ_- and ρ_+ E_2 is unstable; at either $|\rho| = \rho_+$ or $|\rho| = \rho_-$, it is degenerate and coincides, respectively, with an equilibrium point of type E_+ and E_-. For all the other values of $|\rho|$, E_2 is stable.

4.2.3 Existence of the equilibrium points of type E_+ and E_-

We use here the same strategy applied for the J_2-problem. After the change of variables $Z \mapsto G$, we obtain

$$s_+\left(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a}\right) = \frac{\hat{S}_+(G; \boldsymbol{a})}{4G^2\lambda(5G^2j_4 - 35j_4\rho^2 - G^2 + 15\rho^2)},$$

with

$$\hat{S}_+ = (315G^2j_4 + 225G^2 - 360G - 1155j_4 - 715)\lambda\rho^4 - 2G^2\big(80G^4 + \lambda(35G^2j_4 + 49G^2 - 96G - 315j_4 - 99)\big)\rho^2 + G^4\big(32G^4 + \lambda(-5G^2j_4 - 15G^2 - 24G - 35j_4 + 21)\big).$$

We have that $\hat{S}_+ = 0$ for $\rho^2 = \hat{\rho}^2_{E_{+1,2}}$, where

$$\begin{aligned}
\hat{\rho}^2_{E_{+1,2}} &= \frac{G^2}{5\lambda}\,\frac{\hat{A}_+ \pm \sqrt{\hat{B}_+}}{\hat{C}_+},\\
\hat{A}_+ &= 80G^4 + \lambda\left((35G^2 - 315)\,j_4 + 49G^2 - 96G - 99\right),\\
\hat{B}_+ &= \frac{\hat{A}_+^2 - 5\lambda\hat{C}_+\hat{D}_+}{16},\\
\hat{C}_+ &= (63G^2 - 231)\,j_4 + 45G^2 - 72G - 143,\\
\hat{D}_+ &= 32G^4 + \lambda\left((-5G^2 - 35)\,j_4 - 15G^2 - 24G + 21\right).
\end{aligned}$$

For λ sufficiently small, it turns out that $0 < \hat{\rho}^2_{E_{+2}} < G^2$, $\forall G \in (0, 1]$ and $\forall j_4 \in [-6, 6]$. For $G = 1$ it holds $\hat{\rho}^2_{E_{+2}} = \rho_+$. Moreover, we numerically verified that $\hat{\rho}^2_{E_{+2}}$ is increasing with respect to G. Consequently, for each $|\rho| \in (0, \rho_+)$ there exists one equilibrium point, E_3, which coincides with E_2 for $|\rho| = \rho_+$. By applying the same perturbation method used in Sect. 4.1.3, we can determine the value of G corresponding to E_3. At third order in λ, we obtain

$$\begin{aligned}
G_+ = {} & \sqrt{5}\rho + \frac{-5 - 7j_4 + 20\rho^2 + 5j_4\rho^2}{50\sqrt{5}\rho^3}\lambda + \frac{1}{25000\sqrt{5}\rho^7}\big(70 - 384j_4 - 392j_4^2 - 30\sqrt{5}\rho\\
& - 42\sqrt{5}j_4\rho + 405\rho^2 + 1215j_4\rho^2 + 70j_4^2\rho^2 + 120\sqrt{5}\rho^3 + 30\sqrt{5}j_4\rho^3 - 500\rho^4\\
& + 475j_4\rho^4 + 150j_4^2\rho^4\big)\lambda^2 + \frac{1}{25000000\sqrt{5}\rho^{11}}\big(1765 - 16569j_4 - 70021j_4^2 - 60711j_4^3\\
& - 1680\sqrt{5}\rho - 9072\sqrt{5}j_4\rho - 9408\sqrt{5}j_4^2\rho - 20385\rho^2 + 86215j_4\rho^2 + 189665j_4^2\rho^2\\
& - 20335j_4^3\rho^2 + 9720\sqrt{5}\rho^3 + 29160\sqrt{5}j_4\rho^3 + 1680\sqrt{5}j_4^2\rho^3 + 61550\rho^4 - 19900j_4\rho^4\\
& + 265125j_4^2\rho^4 + 53375j_4^3\rho^4 - 12000\sqrt{5}\rho^5 + 11400\sqrt{5}j_4\rho^5 + 3600\sqrt{5}j_4^2\rho^5 - 33000\rho^6\\
& - 316750j_4\rho^6 - 99625j_4^2\rho^6 - 5625j_4^3\rho^6\big)\lambda^3.
\end{aligned}$$

The other solution $\hat{\rho}^2_{E_{+1}}$ is only admissible for some values of j_4. The analysis of the $\hat{\rho}^2_{E_{+1}}$ is complex, and we are forced to fix the value of λ at 0.001. Anyway, we expect similar outcomes for all values of λ sufficiently small. For $j_4 > j_{4_{\text{bif2}}}$, where $j_{4_{\text{bif2}}} \sim 0.5695$, there exists a range of values of G such that $0 < \hat{\rho}^2_{E_{+1}} < G^2$. The function $\hat{\rho}^2_{E_{+1}}$ is not monotone with respect to G. Let us set

$$\rho_\blacktriangle = \sqrt{\max_G \hat{\rho}^2_{E_{+1}}(G;\, j_4)}. \tag{50}$$

For $|\rho| = \rho_\blacktriangle$, there exists one equilibrium point E_5 of type E_+. Instead, for $|\rho| < \rho_\blacktriangle$, there are multiple equilibrium points of type E_+; they are typically two and we call them E_7 and E_9. Also for $j_4 < j_{4_{\text{bif9}}}$, it holds $\hat{\rho}^2_{E_{+1}} > 0$; through a numerical study, we observed that the function is increasing with G and that $\hat{\rho}^2_{E_{+1}} \leq G^2$ up to a certain value of G lower than 1, for which it holds $\hat{\rho}^2_{E_{+1}} = \rho^2_\triangle$. Thus, for $j_4 < j_{4_{\text{bif9}}}$ and $|\rho| \in (0, \rho_\triangle)$ there exists an equilibrium point E_{11}, which coincides with E_1 for $|\rho| = \rho_\triangle$.

Concerning the equilibrium points of type E_-, we have

$$s_-(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a}) = \frac{\hat{S}_-(G; \boldsymbol{a})}{4G^2\lambda(5G^2 j_4 - 35j_4\rho^2 - G^2 + 15\rho^2)},$$

with

$$\hat{S}_- = (1575G^2 j_4 - 315G^2 - 360G - 2695 j_4 - 55)\lambda\rho^4 - 2G^2(80G^4 + \lambda(595G^2 j_4 - 175G^2 - 96G - 1035 j_4 + 189))\rho^2 + G^4(32G^4 + \lambda(95G^2 j_4 - 35G^2 - 24G - 175 j_4 + 49)).$$

$\hat{S}_- = 0$ for $\rho^2 = \hat{\rho}^2_{E_{-1,2}} = \dfrac{G^2}{5\lambda}\dfrac{\hat{A} - \pm 4\sqrt{\hat{B}_-}}{\hat{C}_-}$, where

$$\begin{aligned}
\hat{A}_- &= 80G^4 + \lambda\left((595G^2 - 1035)j_4 - 175G^2 - 96G + 189\right),\\
\hat{B}_- &= \frac{\hat{A}_-^2 - 5\lambda\hat{C}_-\hat{D}_-}{16},\\
\hat{C}_- &= (315G^2 - 539)j_4 - 63G^2 - 72G - 11,\\
\hat{D}_- &= 32G^4 + \lambda\left((95G^2 - 175)j_4 - 35G^2 - 24G + 49\right).
\end{aligned}$$

For λ sufficiently small, solution $\hat{\rho}^2_{E_{-2}}$ is admissible $\forall G \in (0, 1]$ and $\forall j_4 \in [-6, 6]$. For $G = 1$ we have $\hat{\rho}^2_{E_{-2}} = \rho_-$. Moreover, we numerically verified that $\partial\hat{\rho}^2_{E_{-2}}/\partial G > 0$. Thus, for each $|\rho| \in (0, \rho_-)$ there exists the equilibrium point E_4 which coincides with E_2 for $|\rho| = \rho_-$ and whose value is

$$\begin{aligned}
G_- = {}& \sqrt{5}\rho + \frac{125 j_4\rho^2 - 41 j_4 - 35\rho^2 + 9}{100\sqrt{5}\rho^3}\lambda + \frac{1}{100000\sqrt{5}\rho^7}\Big(- 103125 j_4^2\rho^4\\
&+ 90450 j_4^2\rho^2 - 18573 j_4{}^2 + 68250 j_4\rho^4 + 1500\sqrt{5} j_4\rho^3 - 54320 j_4\rho^2 - 492\sqrt{5} j_4\rho\\
&+ 10022 j_4 - 11025\rho^4 - 420\sqrt{5}\rho^3 + 7910\rho^2 + 108\sqrt{5}\rho - 1305\Big)\lambda^2\\
&- \frac{1}{100000000\sqrt{5}\rho^{11}}\Big(- 106171875 j_4^3\rho^6 + 113728125 j_4^2\rho^6 + 2475000\sqrt{5} j_4^2\rho^5\\
&+ 168643125 j_4^3\rho^4 - 168503125 j_4^4 - 2170800\sqrt{5} j_4^2\rho^3\\
&- 80521425 j_4^3\rho^2 + 75097235 j_4^2\rho^2 + 445752\sqrt{5} j_4^2\rho + 12014271 j_4^3\\
&- 10434331 j_4^2 - 39598125 j_4\rho^6 - 1638000\sqrt{5} j_4\rho^5 + 54647375 j_4\rho^4\\
&+ 1303680\sqrt{5} j_4\rho^3 - 22678075 j_4\rho^2 - 240528\sqrt{5} j_4\rho + 2929289 j_4 + 4501875\rho^6\\
&+ 264600\sqrt{5}\rho^5 - 5775175\rho^4 - 189840\sqrt{5}\rho^3 + 2226905\rho^2\\
&+ 31320\sqrt{5}\rho - 267309\Big)\lambda^3.
\end{aligned}$$

Concerning the other solution $\hat{\rho}^2_{E_{-1}}$, its admissibility depends on j_4. Here too, we set $\lambda = 0.001$. Through an analysis similar to the one done for $\hat{\rho}^2_{E_{+1}}$, we reach the following conclusions:

- for $j_4 > j_{4_{\mathrm{bif5}}}$, with $j_{4_{\mathrm{bif5}}} \sim 0.2755$, at $|\rho| = \rho_\blacktriangledown$ there exists one equilibrium solution E_6 of type E_-, while for $|\rho| < \rho_\blacktriangledown$ there exist typically two equilibrium solutions of type E_- which we call E_8 and E_{10}; here,

$$\rho_\blacktriangledown = \sqrt{\max_G \hat{\rho}^2_{E_{-1}}(G; j_4)}; \tag{51}$$

- for $j_4 < j_{4_{\rm bif6}}$ and for $|\rho| < \rho_\triangledown$ there exists an equilibrium point E_{12}, which coincides with E_1 for $|\rho| = \rho_\triangledown$.

4.2.4 About the existence of $\bar{E}_1$ and $\bar{E}_2$

If existing, the equilibrium points $\bar{E}_1$ and $\bar{E}_2$ have coordinates $(\bar{X}, \bar{Y}_{1,2}, \bar{Z})$, $\bar{Y}_1^2 = \bar{Y}_2^2 = \mathcal{Y}$, where

$$\bar{X} = -\frac{\rho^2}{\lambda\left(4375 j_4^5 - 5375 j_4^4 + 2550 j_4^3 - 590 j_4^2 + 67 j_4 - 3\right)}\Big(- 2000\rho^4(j_4-1)(7j_4-3)^3$$
$$+\lambda\big((5j_4-1)(55125 j_4^4\rho^2 - 28700 j_4^3\rho^2 - 14875 j_4^4 - 23130 j_4^2\rho^2 + 3350 j_4^3$$
$$+17460 j_4\rho^2 + 7000 j_4^2 - 2835\rho^2 - 2950 j_4 + 307)$$
$$+48\sqrt{\frac{7j_4-3}{5j_4-1}}\sqrt{5}|\rho|(125 j_4^4 - 250 j_4^3 + 160 j_4^2 - 38 j_4 + 3)\big)\Big),$$
$$\bar{Z} = \frac{65 j_4\rho^2 - 29\rho^2 - 5j_4 + 1}{5j_4 - 1}, \qquad \mathcal{Y} = (-\bar{Z}^2 + \mathcal{E}^2)^2 - \bar{X}^2.$$

$\bar{E}_1$ and $\bar{E}_2$ exist if

$$\bar{Z} \in [-\mathcal{E}, \mathcal{E}], \tag{52}$$
$$\mathcal{Y} \geq 0. \tag{53}$$

Let us remark that when $\bar{Y}_1 = \bar{Y}_2 = 0$ and $\bar{X} = \hat{X}(\bar{Z}; \mathcal{E})$, $\bar{E}_1$ and $\bar{E}_2$ coincide with an equilibrium point of type E_+, i.e. they correspond to zeros of $s_+(Z; \boldsymbol{a})$ defined in (25). We call $\rho_\diamond$, $\rho_{\diamond,\rm bis}$ the values of $|\rho|$ for which this occurs. Similarly, when $\bar{Y}_1 = \bar{Y}_2 = 0$ and $\bar{X} = -\hat{X}(\bar{Z}; \mathcal{E})$ $\bar{E}_1$ and $\bar{E}_2$ coincide with an equilibrium point of type E_-. In this case, we call $\rho_{\Box,}$, $\rho_{\Box,\rm bis}$ the corresponding values of $|\rho|$.

Let us set $\lambda = 0.001$. If either $j_4 \in [j_{4_{\rm bif1}}, j_{4_{\rm bif4}})$, with $j_{4_{\rm bif4}} \sim 0.546$ or $j_4 \leq j_{4_{\rm bif7}}$, with $j_{4_{\rm bif7}} \sim -0.4840$, there exists an interval of values of $|\rho| < \frac{5j_4-1}{7j_4-3}$ for which both conditions (52) and (53) are fulfilled. In particular, through a numerical study, we obtain that

- in the range $j_{4_{\rm bif3}} < j_4 \leq j_{4_{\rm bif1}}$, with $j_{4_{\rm bif3}} \sim 0.552$, $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in [\rho_\diamond, \rho_\Box]$;
- for $j_{4_{\rm bif4}} < j_4 \leq j_{4_{\rm bif2}}$ $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in (0, \rho_\Box]$;
- in the range $j_{4_{\rm bif8}} < j_4 \leq j_{4_{\rm bif7}}$, with $j_{4_{\rm bif8}} \sim -0.4886$, $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in [\rho_{\Box,\rm bis}, \rho_\Box]$;
- for $j_{4_{\rm bif10}} < j_4 \leq j_{4_{\rm bif7}}$, with $j_{4_{\rm bif10}} \sim -1.3454$, $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in (0, \rho_\Box]$;
- in the range $j_{4_{\rm bif11}} < j_4 \leq j_{4_{\rm bif10}}$, with $j_{4_{\rm bif11}} \sim -1.3533$, $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in [\rho_\diamond, \rho_\Box]$ and for $|\rho| \in (0, \rho_{\diamond,\rm bis}]$;
- for $j_4 \leq j_{4_{\rm bif11}}$ $\bar{E}_1$ and $\bar{E}_2$ exist for $|\rho| \in [\rho_\diamond, \rho_\Box]$.

We numerically verified that the equilibrium point of type E_+ coinciding with $\bar{E}_1$ and $\bar{E}_2$ at $|\rho| = \rho_\diamond$ and $|\rho| = \rho_{\diamond,\rm bis}$ is E_3. Similarly, we also verified that at $|\rho| = \rho_\Box$ and $|\rho| = \rho_{\Box,\rm bis}$ $\bar{E}_1$ and $\bar{E}_2$ coincide with E_4. Thus, $\rho_\diamond < \rho_+$ and $\rho_\Box < \rho_-$.

4.2.5 Stability analysis and bifurcation diagram

In the following, we discuss the evolution of the dynamics. We set $\lambda = 0.001$, but we expect similar outcomes for all values of λ sufficiently small.

We show in Fig. 2 the bifurcation diagram, where the colour lines represent the values of $|\rho|$ for which a bifurcation occurs. Some enlargements of interesting regions of the diagram are given in Fig. 3. In Table 1, we summarise the bifurcations sequence and list the existing points for different ranges of $j_4 \in [-6, 6]$. Through a stability analysis based on the Poincaré–Hopf theorem, we obtain that

- $|\rho| = \rho_-$ and $|\rho| = \rho_+$ are pitchfork bifurcations which cause a variation of stability of E_2 and affect the existence and the stability of the equilibrium points E_3 and E_4;
- $|\rho| = \rho_\diamond$, $|\rho| = \rho_{\diamond,\mathrm{bis}}$,$|\rho| = \rho_\Box$ and $|\rho| = \rho_{\Box,\mathrm{bis}}$ are pitchfork bifurcations, influencing the stability of E_3 and E_4 and the existence of $\bar{E}_1$ and $\bar{E}_2$: when $\bar{E}_1$ and $\bar{E}_2$ exist, they are unstable, while E_3 and E_4 are stable;
- $|\rho| = \rho_\triangle$ and $|\rho| = \rho_\triangledown$ are pitchfork bifurcations affecting the stability of E_1 and the existence and stability of E_{11} and E_{12};
- $|\rho| = \rho_\blacktriangle$ and $|\rho| = \rho_\blacktriangledown$ are saddle-node bifurcations; they have no consequence on the stability of existing equilibrium solutions, but give rise to an even number of equilibrium points of type E_+ and E_-, half of which are stable, while the other half is unstable.

To explain how to read the bifurcation diagram, let us fix a value of j_4 in the range $(j_{4_{\mathrm{bif1}}}, 6)$, which is of interest for the Earth ($j_4 \sim 1.3$) and Mars ($j_4 \sim 4$). It holds $\rho_- > \rho_+ > \rho_\blacktriangledown > \rho_\blacktriangle$. For each $|\rho|$ E_1 is stable. Moreover,

- for $|\rho| > \rho_-$, E_2 is stable;
- at $|\rho| = \rho_-$, there is a bifurcation: E_2 is degenerate and E_4 coincides with E_2;
- for $\rho_+ < |\rho| < \rho_-$, E_2 is unstable and E_4 is stable;
- at $|\rho| = \rho_+$, there is a bifurcation: E_2 is degenerate and coincides with E_3; E_4 is stable;
- for $\rho_\blacktriangledown < |\rho| < \rho_+$, E_2 and E_4 are stable, while E_3 is unstable;
- for $|\rho| = \rho_\blacktriangledown$, E_2 and E_4 are stable and E_3 is unstable; there also exists the equilibrium point E_6 which is degenerate;
- for $\rho_\blacktriangle < |\rho| < \rho_\blacktriangledown$, E_2 and E_4 are stable and E_3 is unstable; there exist the equilibrium points E_8 and E_{10}: one of them is stable, the other is unstable;
- for $|\rho| = \rho_\blacktriangle$, E_2 and E_4 are stable; E_3 is stable; one between E_8 and E_{10} is stable, while the other is unstable; there also exists the equilibrium point E_5 which is degenerate;
- $|\rho| < \rho_\blacktriangle$, E_2 and E_4 are stable; E_3 is stable; one between E_8 and E_{10} is stable, while the other is unstable; there exist the equilibrium points E_7 and E_9: one of them is stable, the other is unstable.

In Fig. 4, we show the level curves in a neighbourhood of the bifurcations $|\rho| = \rho_-$ and $|\rho| = \rho_+$. It is interesting to compare the phase portrait in Fig. 4d with the one shown in Fig. 1 for the J_2-problem: the concavities of the colour curves tangent to the contour of the lemon space are opposite. Indeed, in this case, E_3 is unstable and E_4 is stable. In Fig. 5, we show the level curves in a neighbourhood of the two bifurcations $|\rho| = \rho_\blacktriangledown$ and $|\rho| = \rho_\blacktriangle$.

Another significant range of values of j_4 is $(j_{4_{\mathrm{bif2}}}, j_{4_{\mathrm{bif1}}})$. Here, it holds $\rho_+ > \rho_- > \rho_\Box > \rho_\diamond > \rho_\blacktriangledown > \rho_\blacktriangle$. When $|\rho| > \rho_\Box$, the dynamical evolution is similar to that occurring in the J_2-problem. Instead, for $|\rho| < \rho_\diamond$, it has the same features of the one obtained for $j_4 > j_{4\mathrm{bif1}}$ when $|\rho| < \rho_+$. The link between these two situations is established by the bifurcations $|\rho| = \rho_\Box$ and $|\rho| = \rho_\diamond$, which cause a variation in the stability of the equilibrium points E_3 and E_4:

- for $\rho_\Box < |\rho| < \rho_-$, E_2 and E_3 are stable, while E_4 is unstable;
- for $|\rho| = \rho_\Box$, E_2 and E_3 are stable; the equilibrium points $\bar{E}_1$ and $\bar{E}_2$ coincide with E_4 and are degenerate;
- for $\rho_\diamond < |\rho| < \rho_\Box$ E_2, E_3 are stable; $\bar{E}_1$ and $\bar{E}_2$ are unstable, while E_4 is stable;

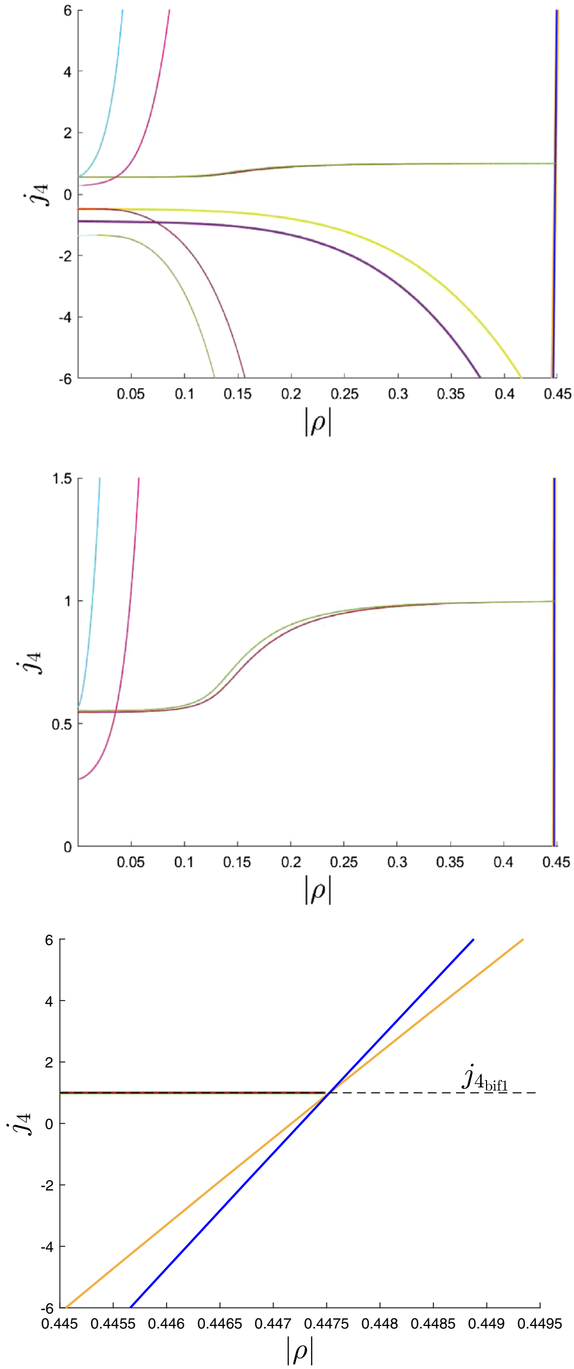


Fig. 2 In the upper panel, we show the bifurcation diagram for the J_4-problem with $\lambda = 0.001$. The lower panels show two enlargements of the diagram. Further enlargements of interesting regions are shown in Fig.3. The blue and the orange lines represent, respectively, ρ_+ defined in (47) and ρ_- defined in (48); the purple and the yellow line represent, respectively, $\rho_\triangle$ and ρ_∇, defined in Sect. 4.2.1; the light-blue line and the pink line represent, respectively, $\rho_\blacktriangle$, defined in (50), and $\rho_\blacktriangledown$, defined in (51); finally the green line, the dark-red line, the light-green line and the red line represent, respectively, $\rho_\diamond$, $\rho_\square$, $\rho_{\diamond,\mathrm{bis}}$ and $\rho_{\square,\mathrm{bis}}$, defined in Sect. 4.2.4

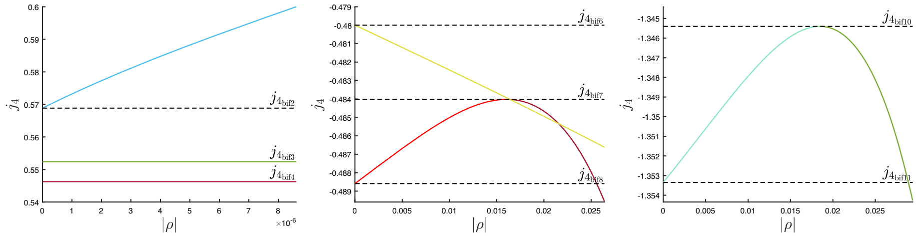


Fig. 3 We show some enlargements of interesting regions of the bifurcation diagram in Fig. 2

- for $|\rho| = \rho_\diamond$ E_2 and E_4 are stable; $\bar{E}_1$ and $\bar{E}_2$ coincide with E_3 and are degenerate;
- for $\rho_\blacktriangledown < |\rho| < \rho_\diamond$, E_2 and E_4 are stable and E_3 is unstable.

In Fig. 6, we show the levels curves in a neighbourhood of the bifurcations $|\rho| = \rho_\square$, and $|\rho| = \rho_\diamond$. At the bifurcation $|\rho| = \rho_\square$, in the (Z, X) plane there is a level curve intersecting the contour of the *lemon* space at $Z = \bar{Z}$: the intersection point is E_4, coinciding with $\bar{E}_1$ and $\bar{E}_2$. In all the range of values of $|\rho|$ such that $\bar{E}_1$ and $\bar{E}_2$ exist, there is a level curve for which $Z = \bar{Z}$ is not a singularity. When $|\rho| = \rho_\diamond$ the intersection point is E_3.

Note that for values of j_4 lower and higher than $j_{4_{\rm bif1}}$, the stability of E_3 and E_4 is different when they appear after the occurrence of the bifurcations $|\rho| = \rho_+$ and $|\rho| = \rho_-$. A similar result was also found by Coffey et al. (1994). Here, the authors argued that this change of stability occurs at $j_4 = 1$, i.e. when we deal with the so called *Vinti* problem. Instead, we observe that the variation of the stability occurs at $j_{4_{\rm bif1}}$ given by (49), which depends on λ.

To conclude, let us remark once again that the above analysis is general and it does not care about particular physical limitations. For example, one can notice that for low $|\rho|$, the value of G characterising the equilibrium points is typically small. This implies a large eccentricity. There is then the risk that the resulting distance of the pericentre is smaller than the central body's radius. In such a case, the resulting equilibrium cannot physically exist. For example, if we consider the case of Mars, the equilibrium points resulting from the bifurcations $|\rho| = \rho_\blacktriangledown$ and $|\rho| = \rho_\blacktriangle$ do not exist for $\lambda = 0.001$.

4.3 The J_2-problem with relativistic terms

We study now the zonal problem containing both the J_2 and the relativistic terms. From (7), (8), (9) and (10), the closed form is

$$
\begin{aligned}
\mathcal{K}_c = & -\frac{\mu^2}{2L^2} + \frac{\mu^4 J_2 R_P^2 (G^2 - 3H^2)}{4G^5L^3} + \frac{\mu^4}{c^2L^4G}(5G - 8L) + \frac{3\mu^6 J_2^2 R_P^4}{128 L^5 G^{11}}\big[-5G^6 - 4G^5 L \\
& + 24G^3H^2L - 36GH^4L - 35H^4L^2 + G^4(18H^2 + 5L^2) \\
& - 5G^2(H^4 + 2H^2L^2) + 2(G^2 - 15H^2)(G^2 - L^2)(G^2 - H^2)\cos 2g\big] \\
& + \frac{\mu^6 J_2 R_P^2}{8c^2L^5G^7}\big[(G^2 - 3H^2)(6L^2 - 5G^2) - 6(G^2 - 3H^2)(4G^2 - 3GL - 5L^2) \\
& - 9(L^2 - G^2)(G^2 - H^2)\cos 2g\big].
\end{aligned}
$$

We neglect here the J_4 terms to make evident the effects of the relativistic contribution.

We adopt the same non-dimensional system described in Sect. 4.1. Let us set

$$
j_C = \frac{1}{\lambda c^2},
$$

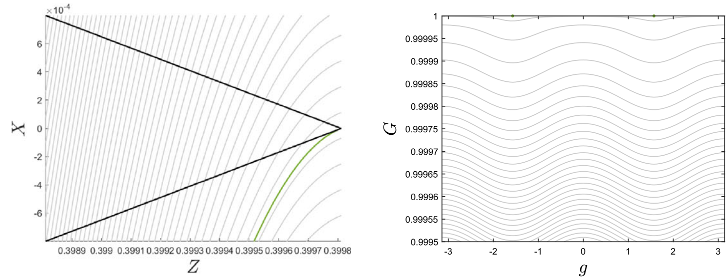
Table 1 Sequence of bifurcations and existing equilibrium points for different ranges of $j_4 \in [-6, 6]$

j_4 range	bifurcations	existing equilibrium points
$j_{4_{\rm bif1}} \le j_4 \le 6$	$\rho_- \ge \rho_+ > \rho_\blacktriangledown > \rho_\blacktriangle$	E_4 for $\vert\rho\vert \le \rho_-$; E_3 for $\vert\rho\vert \le \rho_+$; E_6 for $\vert\rho\vert = \rho_\blacktriangledown$; E_5 for $\vert\rho\vert = \rho_\blacktriangle$; E_8, E_{10} for $\vert\rho\vert < \rho_\blacktriangledown$; E_7, E_9 for $\vert\rho\vert < \rho_\blacktriangle$.
$j_{4_{\rm bif2}} < j_4 < j_{4_{\rm bif1}}$	$\rho_+ > \rho_- > \rho_\square > \rho_\diamond > \rho_\blacktriangledown > \rho_\blacktriangle$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$; $\bar{E}_{1,2}$ for $\rho_\diamond \le \vert\rho\vert \le \rho_\square$; E_6 for $\vert\rho\vert = \rho_\blacktriangledown$; E_5 for $\vert\rho\vert = \rho_\blacktriangle$; E_8, E_{10} for $\vert\rho\vert < \rho_\blacktriangledown$; E_7, E_9 for $\vert\rho\vert < \rho_\blacktriangle$.
$j_{4_{\rm bif3}} < j_4 \le j_{4_{\rm bif2}}$	$j_4 \gtrsim 0.553$: $\rho_+ > \rho_- > \rho_\square > \rho_\diamond \ge \rho_\blacktriangledown$ else: $\rho_+ > \rho_- > \rho_\square > \rho_\blacktriangledown > \rho_\diamond$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$; $\bar{E}_{1,2}$ for $\rho_\diamond \le \vert\rho\vert \le \rho_\square$; E_6 for $\vert\rho\vert = \rho_\blacktriangledown$; E_8, E_{10} for $\vert\rho\vert < \rho_\blacktriangledown$.
$j_{4_{\rm bif4}} < j_4 \le j_{4_{\rm bif3}}$	$j_4 \gtrsim 0.547$: $\rho_+ > \rho_- > \rho_\square \ge \rho_\blacktriangledown$, else $\rho_+ > \rho_- > \rho_\blacktriangledown > \rho_\square$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$; $\bar{E}_{1,2}$ for $\vert\rho\vert \le \rho_\square$; E_6 for $\vert\rho\vert = \rho_\blacktriangledown$; E_8, E_{10} for $\vert\rho\vert < \rho_\blacktriangledown$.
$j_{4_{\rm bif5}} < j_4 \le j_{4_{\rm bif4}}$	$\rho_+ > \rho_- > \rho_\blacktriangledown$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$; E_6 for $\vert\rho\vert = \rho_\blacktriangledown$; E_8, E_{10} for $\vert\rho\vert < \rho_\blacktriangledown$.
$j_{4_{\rm bif6}} \le j_4 \le j_{4_{\rm bif5}}$	$\rho_+ > \rho_-$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$.
$j_{4_{\rm bif7}} < j_4 < j_{4_{\rm bif6}}$	$\rho_+ > \rho_- > \rho_\triangledown$	E_3 for $\vert\rho\vert \le \rho_+$; E_4 for $\vert\rho\vert \le \rho_-$; E_{12} for $\vert\rho\vert \le \rho_\triangledown$.

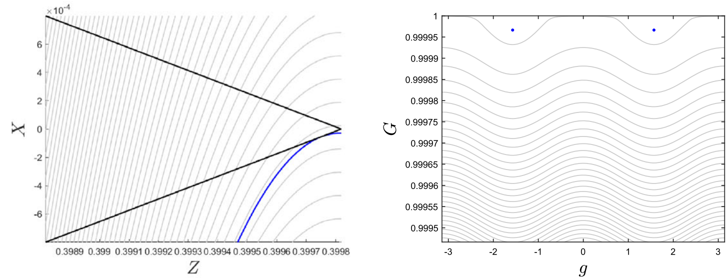
Table 1 continued

j_4 range	bifurcations	existing equilibrium points
$j_{4_{\rm bif8}} < j_4 \le j_{4_{\rm bif7}}$	$j_4 \gtrsim -0.4803$: $\rho_+ > \rho_- > \rho_\triangledown \ge \rho_\square \ge \rho_{\square,\rm bis}$ else: $\rho_+ > \rho_- > \rho_\square > \rho_\triangledown > \rho_{\square,\rm bis}$	E_3 for $\lvert\rho\rvert \le \rho_+$; E_4 for $\lvert\rho\rvert \le \rho_-$; E_{12} for $\lvert\rho\rvert \le \rho_\triangledown$; $\bar{E}_{1,2}$ for $\rho_{\square,\rm bis} \le \lvert\rho\rvert \le \rho_\square$.
$j_{4_{\rm bif9}} \le j_4 \le j_{4_{\rm bif8}}$	$j_4 \gtrsim -0.4853$: $\rho_+ > \rho_- > \rho_\square \ge \rho_\triangledown$ else: $\rho_+ > \rho_- > \rho_\triangledown > \rho_\square$	E_3 for $\lvert\rho\rvert \le \rho_+$; E_4 for $\lvert\rho\rvert \le \rho_-$; E_{12} for $\lvert\rho\rvert \le \rho_\triangledown$; $\bar{E}_{1,2}$ for $\lvert\rho\rvert \le \rho_\square$.
$j_{4_{\rm bif10}} < j_4 < j_{4_{\rm bif9}}$	$j_4 \gtrsim -0.919$: $\rho_+ > \rho_- > \rho_\triangledown > \rho_\square \ge \rho_\triangle$ else: $\rho_+ > \rho_- > \rho_\triangledown > \rho_\triangle > \rho_\square$	E_3 for $\lvert\rho\rvert \le \rho_+$; E_4 for $\lvert\rho\rvert \le \rho_-$; E_{12} for $\lvert\rho\rvert \le \rho_\triangledown$; E_{11} for $\lvert\rho\rvert \le \rho_\triangle$ $\bar{E}_{1,2}$ for $\lvert\rho\rvert \le \rho_\square$.
$j_{4_{\rm bif11}} < j_4 \le j_{4_{\rm bif10}}$	$\rho_+ > \rho_- > \rho_\triangledown > \rho_\triangle > \rho_\square > \rho_\diamond \ge \rho_{\diamond,\rm bis}$	E_3 for $\lvert\rho\rvert \le \rho_+$; E_4 for $\lvert\rho\rvert \le \rho_-$; E_{12} for $\lvert\rho\rvert \le \rho_\triangledown$; E_{11} for $\lvert\rho\rvert \le \rho_\triangle$; $\bar{E}_{1,2}$ for $\rho_\diamond \le \lvert\rho\rvert \le \rho_\square$ and for $\lvert\rho\rvert \le \rho_{\diamond,\rm bis}$
$-6 \le j_4 \le j_{4_{\rm bif11}}$	$\rho_+ > \rho_- > \rho_\triangledown > \rho_\triangle > \rho_\square > \rho_\diamond$	E_3 for $\lvert\rho\rvert \le \rho_+$; E_4 for $\lvert\rho\rvert \le \rho_-$; E_{12} for $\lvert\rho\rvert \le \rho_\triangledown$; E_{11} for $\lvert\rho\rvert \le \rho_\triangle$; $\bar{E}_{1,2}$ for $\rho_\diamond \le \lvert\rho\rvert \le \rho_\square$.

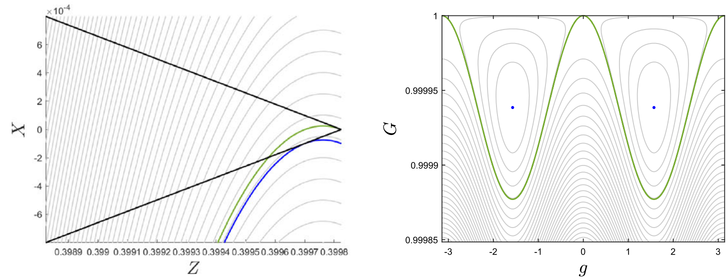
Here, $\lambda = 0.001$. Moreover, $j_{4_{\rm bif1}} \sim 0.9972$, $j_{4_{\rm bif2}} \sim 0.5695$, $j_{4_{\rm bif3}} \sim 0.552$, $j_{4_{\rm bif4}} \sim 0.546$, $j_{4_{\rm bif5}} \sim 0.2755$, $j_{4_{\rm bif6}} = -12/25$, $j_{4_{\rm bif7}} \sim -0.4840$, $j_{4_{\rm bif8}} \sim -0.4886$, $j_{4_{\rm bif9}} = -31/35$, $j_{4_{\rm bif10}} \sim -1.3454$ and $j_{4_{\rm bif11}} \sim -1.3533$



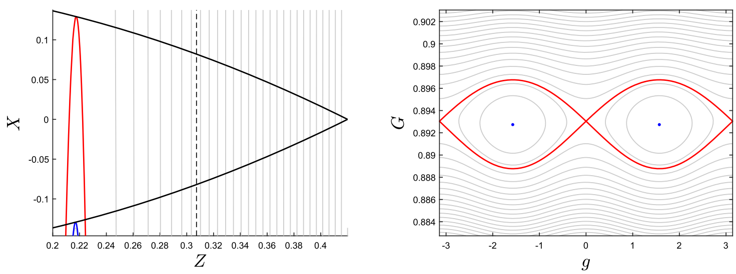
(a) $|\rho| = \rho_-$, $\rho_- \sim 0.44763$



(b) $|\rho| = 0.4472$

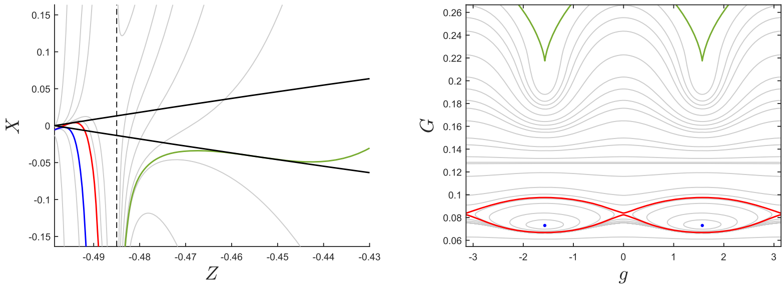


(c) $|\rho| = \rho_+$, $\rho_+ \sim 0.44761$

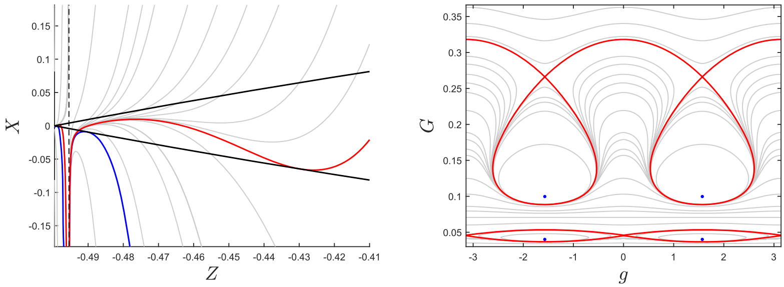


(d) $|\rho| = 0.4$

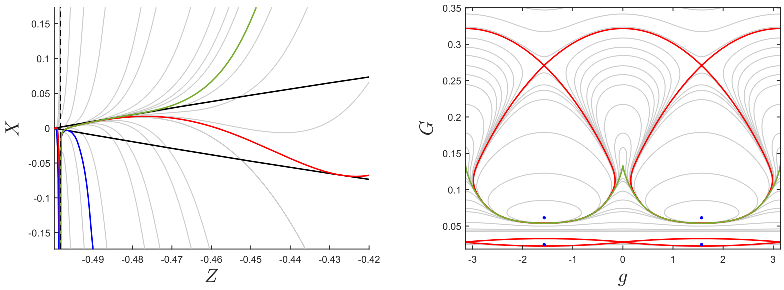
Fig. 4 Level curves for the J_4-problem with $j_4 = 1.3$ for four different values of $|\rho|$. On the left, the levels curve are represented on the (Z, X) plane. Enlargements of the regions containing the equilibrium points are performed. The black line represents the contour $\mathcal{C}$ of the *lemon* space. The dashed black line corresponds to $Z = \bar{Z}$, at which the level curves have a singularity. The coloured line is the level curves tangent to $\mathcal{C}$ at the equilibrium points: they are green if the equilibrium point is degenerate, red if it is unstable and blue if it is stable. On the right, the level curves are shown on corresponding enlargements in the (g, G) plane



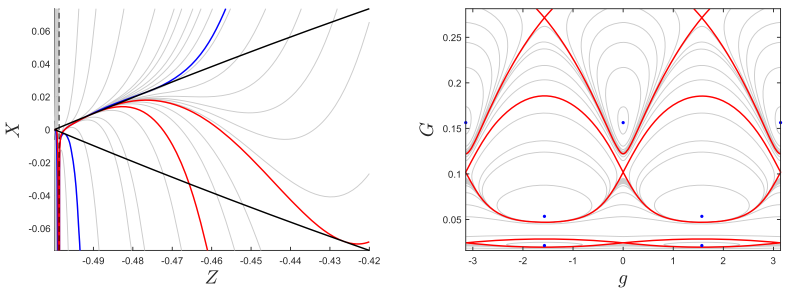
(a) $|\rho| = \rho_{\blacktriangledown}$, $\rho_{\blacktriangledown} \sim 0.054542$



(b) $|\rho| = 0.03$

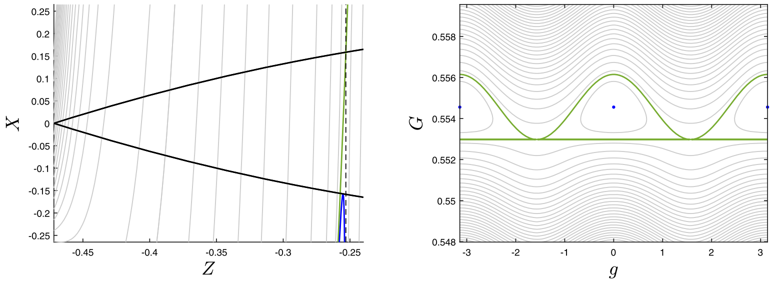


(c) $|\rho| = \rho_{\blacktriangle}$, $\rho_{\blacktriangle} \sim 0.018379$

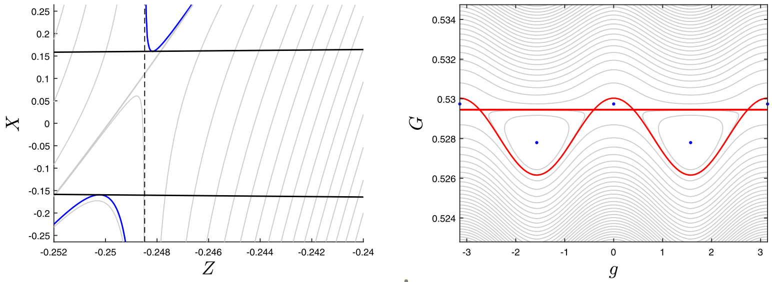


(d) $|\rho| = 0.016$

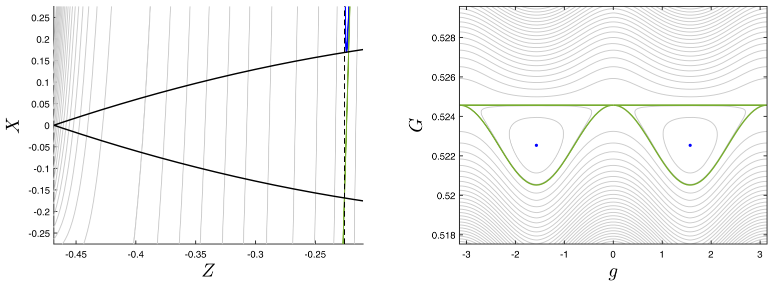
Fig. 5 Level curves for the J_4-problem with $j_4 = 1.3$ for four different values of $|\rho|$ in the (Z, X) and the (g, G) planes. Enlargements of the regions containing the equilibrium points are performed. The same colour code employed in Fig. 4 is used



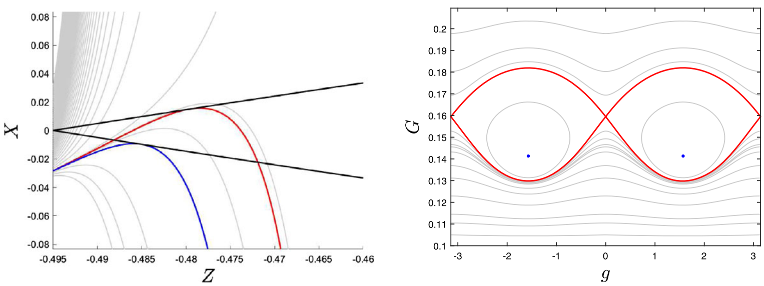
(a) $|\rho| = \rho_\square$, $\rho_\square \sim 0.25067$



(b) $|\rho| = 0.24$



(c) $|\rho| = \rho_\diamond$, $\rho_\diamond \sim 0.23779$



(d) $|\rho| = 0.1$

Fig. 6 Level curves for the J_4-problem with $j_4 = 0.95$ for four different values of $|\rho|$ in the neighbourhood of the bifurcations $|\rho| = \rho_\square$ and $|\rho| = \rho_\diamond$ on the (Z, X) and the (g, G) planes. Enlargements of the regions containing the equilibrium points are performed. The same colour code employed in Fig. 4 is used with λ defined in (32). We recall that λ was considered of the same order as the book-keeping parameter ϵ. Since the normalisation of the initial Hamiltonian was performed by assuming c^{-2} of order ϵ as well (see Sect. 2), j_C should have a value in the neighbourhood of 1 or lower. If this was not the case, the book-keeping scheme used to compute the closed form would not be suitable anymore. Let us also remark that in the adimensional system the value of c and, thus, that of j_C depend on the units of length and time, i.e. on the semi-major axis of the orbit of interest.

We introduce λ and j_C in the Hamiltonian. Then, we neglect the constant terms and we perform the time transformation (33). Also for this problem the resulting normalised Hamiltonian has the same structure of (18), with

$$\begin{aligned} g(Z,\boldsymbol{a}) =& \frac{-5\rho^2+2Z+1}{\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{5}{2}}} + j_C\,\frac{3}{8}\,\frac{5\sqrt{2\rho^2+4Z+2}-16}{\sqrt{2\rho^2+4Z+2}} - \frac{3\lambda}{16\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{11}{2}}}\Big(40Z^3 \\ &+ \left(-84\rho^2+20\right)Z^2 + \left(-74\rho^4-44\rho^2-10\right)Z - 11\rho^6+273\rho^4-\rho^2 \\ &- 5\& + 4\sqrt{2\,\rho^2+4Z+2}\left(-5\rho^2+2Z+1\right)^2\Big) + \frac{j_C\lambda(-5\rho^2+2Z+1)}{2\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{7}{2}}}(-29\rho^2 \\ &+ 18\sqrt{2\rho^2+4Z+2} - 58Z+43), \\ f(Z,\boldsymbol{a}) =& -\frac{3}{2}\,\lambda\,\frac{\left(-29\,\rho^2+2\,Z+1\right)}{\sqrt{2}\left(\rho^2+2\,Z+1\right)^{\frac{11}{2}}} - \frac{18\lambda j_C}{\sqrt{2}\left(\rho^2+2Z+1\right)^{\frac{7}{2}}}, \end{aligned}$$

and $\boldsymbol{a} = (\rho;\lambda, j_C)$. If we neglect the terms of first order in λ, we find two potential equilibrium solutions at

$$Z = \frac{1}{8}\,\frac{-4j_C\rho^2-4j_C+1\pm\sqrt{-80j_C\rho^2+1}}{j_C}, \qquad \forall X, Y. \tag{54}$$

In the following, we make some considerations about the problem considering $\lambda \in (0, 1)$. For this problem, we perform a qualitative analysis. We find out that for $j_C \ll 1$, the sequence of bifurcations is the same as in the J_2 problem. On the contrary for higher values of j_C, the dynamical evolution is more complex and depends on the values of λ and j_C. The existence of a pair of equilibrium points of type E_+, one stable and the other unstable, is triggered by a saddle-node bifurcation. The unstable point can become stable following a pitchfork bifurcation, which affects the existence of the equilibrium points $\bar{E}_1$ and $\bar{E}_2$. The stable one can disappear following a pitchfork bifurcation, which changes the stability of the equilibrium point E_2. A similar sequence of bifurcations occurs also concerning the equilibrium points of type E_-. If none of the bifurcations affecting the stability of E_2 occur, this point is always stable. The equilibrium point E_1 is always stable.

4.3.1 About the stability of E_1

We have

$$s_+(-\mathcal{E};\boldsymbol{a}) = \frac{(8\rho^6-110\lambda\rho^4+84|\rho|^3\lambda+186\lambda\rho^2)j_C+8\rho^4-7\lambda\rho^2+12\lambda|\rho|+31\lambda}{\lambda(12j_C\rho^2-7)}, \tag{55}$$

and

$$s_-(-\mathcal{E};\boldsymbol{a}) = \frac{(4\rho^6-61\lambda\rho^4+42|\rho|^3\lambda+99\lambda\rho^2)j_C+4\rho^4+6\lambda|\rho|+12\lambda}{\lambda(12j_C\rho^2-7)}. \tag{56}$$

It holds

$$\rho^2(-110\rho^2 + 84|\rho| + 186) > 0 \quad \forall \rho \in (0, 1),$$

and

$$\rho^2(-61\rho^2 + 42|\rho| + 99) > 0, \quad \forall \rho \in (0, 1).$$

Thus, if $\rho^2 > 7/12 j_C$, both $s_+(-\mathcal{E}; \boldsymbol{a})$ and $s_-(-\mathcal{E}; \boldsymbol{a})$ are positive; instead, if $\rho^2 < 7/12 j_C$ they are both negative. From (28), we can conclude that E_1 is always stable. Moreover, E_1 never coincides with an equilibrium point of either type E_+ or E_-.

4.3.2 About the stability of E_2

We have

$$s_+(\mathcal{E}; \boldsymbol{a}) = \frac{425\lambda\rho^4 + (1672 j_C\lambda - 146\lambda + 80)\rho^2 - 392 j_C\lambda + 64 j_C + 9\lambda - 16}{2(-15\rho^2 + 24 j_C + 1)\lambda}. \tag{57}$$

It holds $s_+(\mathcal{E}; \boldsymbol{p}) = 0$ for $|\rho| = \tilde{\rho}_+$, with

$$\tilde{\rho}_+^2 = \frac{-836 j_C\lambda + 73\lambda - 40 + 4\sqrt{43681 j_C^2\lambda^2 + 2784 j_C\lambda^2 + 2480 j_C\lambda + 94\lambda^2 + 60\lambda + 100}}{425\lambda}, \tag{58}$$

which is positive, thus admissible, if either $\lambda \geq \frac{8}{49}$ or $\lambda < \frac{8}{49}$ and $j_C < \frac{16-9\lambda}{64-392\lambda}$. We also have

$$s_-(\mathcal{E}; \boldsymbol{a}) = \frac{365\lambda\rho^4 + (1768 j_C\lambda - 82\lambda + 80)\rho^2 - 488 j_C\lambda + 64 j_C + 5\lambda - 16}{2(-15\rho^2 + 24 j_C + 1)\lambda}, \tag{59}$$

and $s_-(\mathcal{E}; \boldsymbol{a}) = 0$ for $|\rho| = \tilde{\rho}_-$, with

$$\tilde{\rho}_-^2 = \frac{-884 j_C\lambda + 41\lambda - 40 + 4\sqrt{48841 j_C^2\lambda^2 + 6602 j_C\lambda^2 + 2960 j_C\lambda - 9\lambda^2 + 160\lambda + 100}}{365\lambda}; \tag{60}$$

$\tilde{\rho}_-^2 > 0$ if either $\lambda \geq \frac{8}{61}$ or $\lambda < \frac{8}{61}$ and $j_C < \frac{16-5\lambda}{64-488\lambda}$. Let us remark that for $\lambda < \frac{8}{61}$ it holds $\frac{16-5\lambda}{64-488\lambda} > \frac{16-9\lambda}{64-392\lambda}$. Thus, if $\tilde{\rho}_+ > 0$ is an admissible solutions, also $\tilde{\rho}_-$ is admissible.

For each λ and j_C such that both $\tilde{\rho}_-$ and $\tilde{\rho}_+$ are admissible zeros of $s_+(\mathcal{E}, \boldsymbol{a})$ and $s_-(\mathcal{E}, \boldsymbol{a})$, it holds $\tilde{\rho}_- > \tilde{\rho}_+$ if $j_C > \tilde{j}_C$, with

$$\tilde{j}_C = \frac{397\lambda - 180 + \sqrt{142321\lambda^2 - 1800\lambda + 32400}}{\lambda}.$$

Let us now consider equation (29). If λ and j_C are such that neither $\tilde{\rho}_-$ and $\tilde{\rho}_+$ are admissible zeros, then E_2 is always stable. Also for $j_C = \tilde{j}_C$, E_2 is always stable, except when $|\rho| = \tilde{\rho}_+ = \tilde{\rho}_-$: in this case, it is degenerate. If λ and j_C are such that $\tilde{\rho}_-$ is an admissible solution, while $\tilde{\rho}_+^2 \leq 0$, E_2 is stable for $|\rho| > \tilde{\rho}_-$, it is degenerate at $|\rho| = \tilde{\rho}_-$ and is unstable for $|\rho| < \tilde{\rho}_-$. Finally, if both $\tilde{\rho}_-$ and $\tilde{\rho}_+$ are admissible solutions, E_2 is unstable when the value of $|\rho|$ lies between $\tilde{\rho}_-$ and $\tilde{\rho}_+$, it is degenerate if either $|\rho| = \tilde{\rho}_-$ or $|\rho| = \tilde{\rho}_+$ and it is stable for all the other values of $|\rho|$. When $|\rho| = \tilde{\rho}_+$, E_2 coincides with an equilibrium point of type E_+. When $|\rho| = \tilde{\rho}_-$ it coincides with an equilibrium point of type E_-.

4.3.3 About the existence of the equilibrium points of type E_+ and E_-

To discuss the existence of equilibrium points of type E_+ and E_-, we use here the same strategy adopted for the problems previously analysed.

We have

$$s_+\left(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a}\right) = \frac{\tilde{S}_+(G;\boldsymbol{a})}{4G^2\lambda(24G^4 j_C + G^2 - 15\rho^2)},$$

with

$$\begin{aligned}\tilde{S}_+(G;\boldsymbol{a}) = {} & (-225G^2\lambda + 360G\lambda + 715\lambda)\rho^4 + (-2080G^6 j_C\lambda + 1728G^5 j_C\lambda \\ & + 160G^6 + 3696G^4 j_C\lambda + 98G^4\lambda - 192G^3\lambda - 198G^2\lambda)\rho^2 \\ & + 128G^{10} j_C + 320G^8 j_C\lambda - 384G^7 j_C\lambda - 32G^8 - 720G^6 j_C\lambda \\ & + 15G^6\lambda + 24G^5\lambda - 21G^4\lambda.\end{aligned}$$

We obtain $\tilde{S}_+(G;\boldsymbol{a}) = 0$ for $\rho^2 = \tilde{\rho}^2_{E_{+1,2}}$, with

$$\begin{aligned}\tilde{\rho}^2_{E_{+1,2}} &= \frac{G^2}{5\lambda}\frac{\tilde{A}_+ \pm 4\sqrt{\tilde{B}_+}}{\tilde{C}_+}, \qquad (61)\\ \tilde{A}_+ &= -(-1040G^4 j_C + 864G^3 j_C + 1848G^2 j_C + 49G^2 - 96G - 99)\lambda - 80G^4,\\ \tilde{B}_+ &= \frac{\tilde{A}_+^2 + 5\lambda\tilde{C}_+\tilde{D}_+}{16}, \qquad \tilde{C}_+ = -45G^2 + 72G + 143,\\ \tilde{D}_+ &= (-320G^4 j_C + 384G^3 j_C + 720G^2 j_C - 15G^2 - 24G + 21)\lambda - 128G^6 j_C + 32G^4.\end{aligned}$$

Note that for $G^2 < 1/4 j_C$, $\tilde{D}_+ > 0$; instead, for $G^2 > 1/4 j_C$, $\tilde{A}_+ < 0$. Thus, $\forall\lambda, \forall j_C, \forall G$, $\tilde{\rho}^2_{E_{+2}} < 0$ and it is not admissible as solution. While $\tilde{\rho}^2_{E_{+1}} > 0$ if G, λ and j_C are such that $\tilde{D}_+ > 0$. Since $5\lambda\tilde{C}_+ - \tilde{A}_+ > 0$ and $16\tilde{B}_+ - (5\lambda\tilde{C}_+ - \tilde{A}_+)^2 < 0$, it holds $\tilde{\rho}^2_{E_{+1}} < G^2$. When admissible, $\tilde{\rho}^2_{E_{+1}}$ is generally not monotone with respect to G. However, for $j_C = 0$ it is equal to the same solution found for the J_2-problem, i.e. $\tilde{\rho}^2_{E_{+1}} = \rho^2_{E_{+2}}$ (see Sect. 4.1.3). As a consequence, we expect that for sufficiently small values of j_C, $\tilde{\rho}^2_{E_{+1}}$ is an increasing function of G in the range of interest, i.e. $G \in (0, 1]$. In this case, for $|\rho| < \tilde{\rho}_+$, there exists only one equilibrium point of type E_+. Instead, for higher values of j_C, such that $\tilde{\rho}^2_{E_{+1}}$ is not monotone, the outcome is different. Let us call $\rho_\blacklozenge$ the value of $|\rho|$ such that

$$\rho_\blacklozenge = \sqrt{\max_G \tilde{\rho}^2_{E_{+1}}}.$$

We have that

- for $|\rho| > \rho_\blacklozenge$, there is no equilibrium point of type E_+;
- for $|\rho| = \rho_\blacklozenge$, we have one equilibrium solution, which we call E_{13};
- for $|\rho| < \rho_\blacklozenge$ there exist multiple equilibrium solutions, typically two which we call E_{15} and E_{17}.

Let us suppose that the Z coordinate of E_{17} is larger than that of E_{15}. When $\lambda \geq \frac{8}{49}$ or when $\lambda < \frac{8}{49}$ and $j_C < \frac{16-9\lambda}{64-392\lambda}$, E_2 coincides with E_{17} for $|\rho| = \tilde{\rho}_+$. Thus, for $|\rho| < \tilde{\rho}_+$, the number of equilibrium solutions reduces to one: there will exist only E_{15}.

In conclusion, we can infer that reducing the value of j_C, the value $G = G_{\blacklozenge}$, corresponding to the maximum point of $|\tilde{\rho}^2_{E_{+1}}|$, increases. For a fixed λ, it exists a value of j_C such that $G_{\blacklozenge} = 1$, i.e. for which $\rho_{\blacklozenge} = \tilde{\rho}_+$. Thus, for lower values of j_C, the bifurcation $|\rho| = \rho_{\blacklozenge}$ disappears and the only existing equilibrium point of type E_+ is E_{15} for $|\rho| < \tilde{\rho}_+$.

As far as the equilibrium points of type E_-, we have

$$s_-\left(G^2 - \frac{1+\rho^2}{2}; \boldsymbol{a}\right) = \frac{\tilde{S}_-(G; \boldsymbol{a})}{4G^2\lambda(24G^4 j_C + G^2 - 15\rho^2)},$$

with

$$\begin{aligned}\tilde{S}_-(G; \boldsymbol{a}) = & (315G^2\lambda + 360G\lambda + 55\lambda)\rho^4 + (-2560G^6 j_C\lambda + 1728G^5 j_C\lambda + 160G^6 \\ & + 4368G^4 j_C\lambda - 350G^4\lambda - 192G^3\lambda + 378G^2\lambda)\rho^2 + 128G^{10} j_C \\ & + 608G^8 j_C\lambda - 384G^7 j_C\lambda - 32G^8 - 1200G^6 j_C\lambda + 35G^6\lambda \\ & + 24G^5\lambda - 49G^4\lambda.\end{aligned}$$

It holds $\tilde{S}_-(G; \boldsymbol{p}) = 0$ if $\rho^2 = \tilde{\rho}^2_{E_{-1,2}}$, with

$$\rho^2 = \tilde{\rho}^2_{E_{-1,2}} = \frac{G^2}{5\lambda}\frac{\tilde{A}_- \pm 4\sqrt{\tilde{B}_-}}{\tilde{C}_-}, \tag{62}$$

$$\begin{aligned}\tilde{A}_- &= (1280G^4 j_C - 864G^3 j_C - 2184G^2 j_C + 175G^2 + 96G - 189)\lambda - 80G^4, \\ \tilde{B}_+ &= \frac{\tilde{A}_-^2 + 5\lambda\tilde{C}_-\tilde{D}_-}{16}, \qquad \tilde{C}_- = 63G^2 + 72G + 11, \\ \tilde{D}_- &= (-608G^4 j_C + 384G^3 j_C + 1200G^2 j_C - 35G^2 - 24G + 49)\lambda - 128G^6 j_C + 32G^4.\end{aligned}$$

One can observe that for $G^2 < 1/4j_C$, $\tilde{D}_- > 0$ and that for $G^2 \geq 1/4j_C$, $\tilde{A}_- < 0$. Thus, $\forall\lambda$, $\forall j_C$ and $\forall G$, $\tilde{\rho}^2_{E_{-2}} < 0$. Instead, for G, j_C and λ such that $\tilde{D}_- > 0$, $\tilde{\rho}^2_{E_{-1}} > 0$. Since $5\lambda\tilde{C}_+ - \tilde{A}_+ > 0$ and $16\tilde{B}_+ - (5\lambda\tilde{C}_+ - \tilde{A}_+)^2 < 0$, it also holds $\tilde{\rho}^2_{E_{-1}} < G^2$. Thus, there exist values of G, j_C and λ such that $\tilde{\rho}^2_{E_{-1}}$ is an admissible solution. As $\tilde{\rho}^2_{E_{+1}}$, in general the function $\tilde{\rho}^2_{E_{-1}}$ is not monotone with respect to G. We find an outcome similar to the one obtained for the equilibrium points of type E_+. Let us consider sufficiently high values of j_C such that $\tilde{\rho}^2_{E_{-1}}$ is not monotone and let us set

$$\rho_{\blacksquare} = \sqrt{\max_G \tilde{\rho}^2_{E_{-1}}}.$$

We have that

- for $|\rho| > \rho_{\blacksquare}$, there is no equilibrium point of the type of E_-;
- for $|\rho| = \rho_{\blacksquare}$, we have one equilibrium solution, which we call E_{14};
- for $|\rho| < \rho_{\blacksquare}$ there exist multiple equilibrium solutions, typically two which we call E_{16} and E_{18}.

Suppose that E_{18} has a larger Z coordinate than E_{16}. When $\lambda \geq \frac{8}{61}$ or when $\lambda < \frac{8}{61}$ and $j_C < \frac{16-5\lambda}{64-488\lambda}$, at $|\rho| = \tilde{\rho}_-$ E_{18} coincides with E_2 and for $|\rho| < \tilde{\rho}_-$ it disappears. For a fixed λ, by considering decreasing values of j_C the value of G, $G = G_{\blacksquare}$, corresponding to the maximum point of $\tilde{\rho}^2_{E_{-1}}$, increases. Below the value of j_C for which $\rho_{\blacksquare} = \tilde{\rho}_-$, $G_{\blacksquare}$ does not belong to the admissible range of values for G. In these cases, there only exists the equilibrium point E_{16} for $|\rho| < \tilde{\rho}_-$.

4.3.4 About the existence of $\bar{E}_1$ and $\bar{E}_2$

The coordinates $\bar{X}$ and $\bar{Z}$ of the two equilibrium points of type E_+ are

$$\bar{Z} = \frac{1}{48}\frac{-24j_C\rho^2 + \sqrt{1440j_C\rho^2+1} - 24j_C - 1}{j_C},$$

and

$$\begin{aligned}\bar{X} = &\frac{1}{20736}\frac{1}{j_C^3\lambda\left((720j_C\rho^2+1)\sqrt{1440j_C\rho^2+1} - 1440j_C\rho^2 - 1\right)}\\ &\Bigg(+ 144\sqrt{3}j_C^2\lambda\big((103680j_C^2\rho^4 + 3312j_C\rho^2 + 5)\sqrt{1440j_C\rho^2+1}\\ &- 1192320j_C^2\rho^4 - 6912j_C\rho^2 - 5\big)\sqrt{\frac{\sqrt{1440j_C\rho^2+1}-1}{j_C}} + (361428480j_C^4\lambda\rho^4\\ &+ 27552960j_C^3\lambda\rho^4 + 3903552j_C^3\lambda\rho^2 - 3369600j_C^2\rho^4 + 19584j_C^2\lambda\rho^2\\ &+ 1080j_C^2\lambda - 17280j_C\rho^2 - 51j_C\lambda - 14)\sqrt{1440j_C\rho^2+1} - 5244134400j_C^4\lambda\rho^6\\ &- 2892049920j_C^4\lambda\rho^4 + 559872000j_C^3\rho^6 - 54872640j_C^3\lambda\rho^4 - 4681152j_C^3\lambda\rho^2\\ &+ 12182400j_C^2\rho^4 + 17136j_C^2\lambda\rho^2 - 1080j_C^2\lambda + 27360j_C\rho^2 + 51j_C\lambda + 14\Bigg).\end{aligned}$$

To have $\bar{Z} \in [\mathcal{E}, \mathcal{E}]$, $\rho^2 < \min\left(\frac{24j_C+1}{15}, \frac{7}{12j_C}\right)$. Let us set $\mathcal{Y} = \bar{Y}_{1,2}^2$. In general, for given j_C and λ, it can exists a subset of values of ρ such that $\mathcal{Y} > 0$, i.e. such that $\bar{E}_1$ and $\bar{E}_2$ exist. The endpoints of this range are values of ρ for which $\bar{E}_1$ and $\bar{E}_2$ coincide with either an equilibrium point of type E_+ or E_-. Let us call $\rho_\diamond$ the value of $|\rho|$ such that $\bar{E}_1$ and $\bar{E}_2$ coincide with an equilibrium point of type E_+ and $\rho_\square$ the value of $|\rho|$ such that they coincide with an equilibrium point of type E_-. We can conclude that necessarily $\rho_\diamond < \rho_\blacklozenge$ and $\rho_\square < \rho_\blacksquare$. For $j_C \to 0$ we obtain instead the same outcome found for the J_2-problem: for sufficiently small values of j_C, there does not exist any value of ρ for which $\bar{E}_1$ and $\bar{E}_2$ exist.

4.3.5 About the stability of the equilibrium points of type E_+ and E_- and of $\bar{E}_1$ and $\bar{E}_2$

Let us consider value of j_C sufficiently high, such that E_{15}, E_{16}, E_{17} and E_{18} exist. We can assume that these equilibrium points are close to the equilibrium solutions (54) of the problem at order zero in λ. With this hypothesis, we can estimate their stability. To this aim, we need to assume $j_C\rho^2 < 1/80$. At order zero in λ, we obtain the same equations for the equilibrium points E_{15} and E_{16}, i.e.

$$\begin{aligned}\frac{d^2\tilde{X}}{dZ^2} \pm \frac{d^2\hat{X}}{dZ^2} \sim 16\sqrt{-80j_C\rho^2+1}\Big(- 144000j_C^3\rho^6 + 28400j_C^2\rho^4 - 880j_C\rho^2 + 7 +\\ \sqrt{-80j_C\rho^2+1}(10000j_C^2\rho^4 - 600j_C\rho^2 + 7)\Big).\end{aligned}$$

The same holds for E_{17} and E_{18}:

$$\frac{d^2\tilde{X}}{dZ^2} \pm \frac{d^2\hat{X}}{dZ^2} \sim 16\sqrt{-80j_C\rho^2+1}\Big(144000j_C^3\rho^6 - 28400j_C^2\rho^4 + 880j_C\rho^2 - 7 + \sqrt{-80j_C\rho^2+1}(10000j_C^2\rho^4 - 600j_C\rho^2 + 7)\Big);$$

From the equations, we obtain that for $7/810 < j_C\rho^2 \leq 1/80$, E_{15} and E_{18} are unstable, while E_{16} and E_{17} are stable; instead for $j_C\rho^2 < 7/810$, E_{15} and E_{17} are both stable, while E_{16} and E_{18} are both unstable. From this zero-order analysis and by applying the Poincaré–Hopf theorem, we can infer the actual dynamical evolution:

- for $|\rho| = \rho_\blacklozenge$, there exists one equilibrium solution E_{13} which is degenerate;
- for $\rho_\diamond < |\rho| < \rho_\blacklozenge$, there exist E_{15}, which is unstable and E_{17}, which is stable;
- for $|\rho| = \rho_\diamond$, E_{15} coincides with $\bar{E}_1$ and $\bar{E}_2$ and it is degenerate; E_{17} is stable;
- for $|\rho| < \rho_\diamond$, both E_{15} and E_{17} are stable.

Something similar occurs concerning the equilibrium points E_{16} and E_{18}:

- for $|\rho| = \rho_\blacksquare$, there is one equilibrium solution E_{14} which is degenerate;
- for $\rho_\square < |\rho| < \rho_\blacksquare$, there exist the two equilibrium solutions E_{16} which is stable and E_{18} which is unstable;
- for $|\rho| = \rho_\square$, E_{16} coincides with $\bar{E}_1$ and $\bar{E}_2$ and it is degenerate; E_{18} is unstable;
- for $|\rho| < \rho_\square$, both E_{17} and E_{18} are unstable.

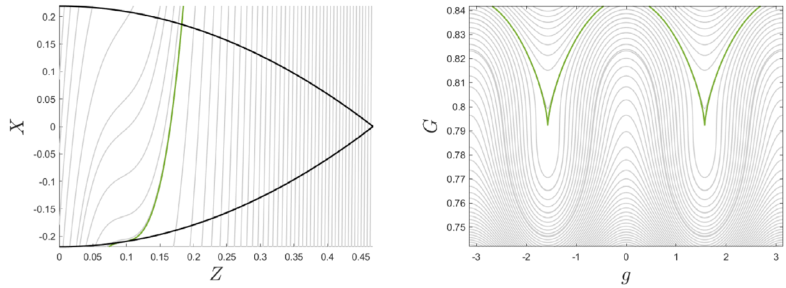
If $\rho_\square < \rho_\diamond$, $\bar{E}_1$ and $\bar{E}_2$ are unstable. On the contrary if $\rho_\square > \rho_\diamond$ $\bar{E}_1$ and $\bar{E}_2$ are stable. Finally, if $\lambda \geq \frac{8}{49}$ or if $\lambda < \frac{8}{49}$ and $j_C < \frac{16-9\lambda}{64-392\lambda}$, for $|\rho| < \tilde{\rho}_+$ E_{17} disappears, while the stability of E_{15} remains unaltered. Similarly, if $\lambda \geq \frac{8}{61}$ or if $\lambda < \frac{8}{61}$ and $j_C < \frac{16-5\lambda}{64-488\lambda}$, for $|\rho| < \tilde{\rho}_-$ E_{18} disappears, while the stability of E_{16} does not change.

In conclusion, we have that

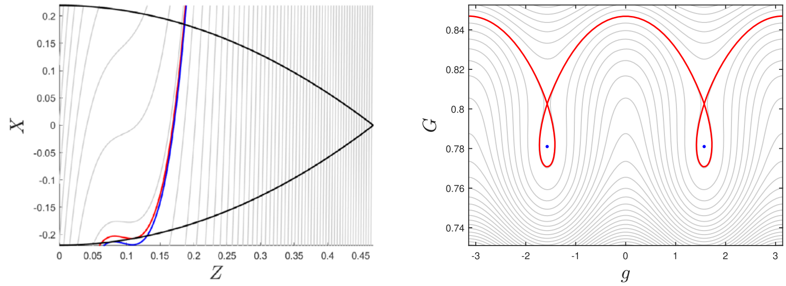
- $|\rho| = \rho_\blacksquare$ and $|\rho| = \rho_\blacklozenge$ are saddle-node bifurcations, affecting the existence of the equilibrium points E_{15}, E_{17}, E_{16} and E_{18}; for $|\rho| > \max(\rho_\blacksquare, \rho_\blacklozenge)$ no equilibrium solution exist;
- $|\rho| = \rho_\diamond$ and $|\rho| = \rho_\square$ are pitchfork bifurcation affecting the stability of the equilibrium points E_{15} and E_{16} and the existence of $\bar{E}_1$ and $\bar{E}_2$;
- if existing, $|\rho| = \tilde{\rho}_+$ and $|\rho| = \tilde{\rho}_-$ are pitchfork bifurcations affecting the stability of E_2 and the existence of E_{17} and E_{18}.

We give an example of the dynamical evolution setting $\lambda = 0.001$ and $j_C = 0.2$. This last value is not realistic, but allows us to clearly illustrate the phenomenology just described. It holds $\rho_\blacksquare > \rho_\blacklozenge > \rho_\diamond > \rho_\square > \tilde{\rho}_- > \tilde{\rho}_+$. After the saddle-node bifurcation at $|\rho| = \rho_\blacksquare$ (Fig. 7a), for $\rho_\blacklozenge < |\rho| < \rho_\blacksquare$ there exist the unstable equilibrium point E_{18} and the stable E_{16} (Fig. 7b). After the second bifurcation (Fig. 7c), for $\rho_\diamond < |\rho| < \rho_\blacklozenge$ there exist also E_{17}, which is stable, and E_{15} which is unstable (Fig. 7d). At $|\rho| = \rho_\diamond$ E_{15} coincide with $\bar{E}_1$ and $\bar{E}_2$ and it is degenerate (Fig. 8a). For $= \rho_\square < |\rho| < \rho_\diamond$, E_{15} is stable and $\bar{E}_1$ and $\bar{E}_2$ exist and are unstable (Fig. 8b). At $|\rho| = \rho_\square$, E_{17} coincides with $\bar{E}_1$ and $\bar{E}_2$ and it is degenerate (Fig. 8c). After this last bifurcation, for $\tilde{\rho}_- < |\rho| < \rho_\square$, $\bar{E}_1$ and $\bar{E}_2$ do not exist, E_{15} and E_{17} are stable, and E_{16} and E_{18} are unstable (Fig. 8d). After the last two bifurcations at $|\rho| = \tilde{\rho}_-$ and $|\rho| = \tilde{\rho}_+$, there only exist the equilibrium point E_{15}, which is stable, and E_{16} which is unstable (Fig. 9).

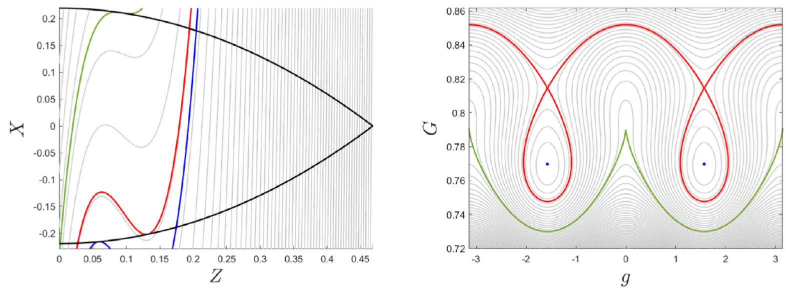
If $j_C \ll 1$, such that only the equilibrium points E_{16} and E_{15} exist, the dynamical evolution has no significant variation in comparison with the one of the J_2-problem. It is the case of the Earth problem, since the values of j_C are typically very small (of the order of 10^{-6}).



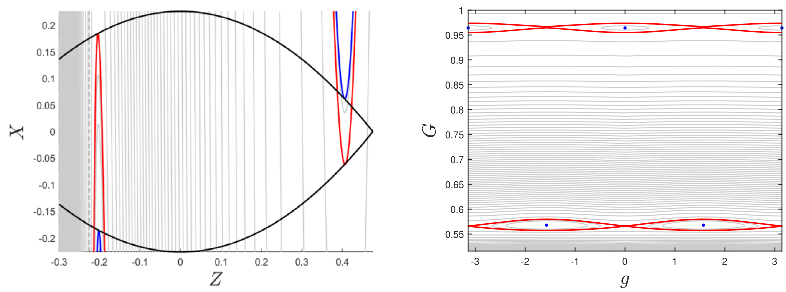
(a) $|\rho| = \rho_{\blacksquare}$, $\rho_{\blacksquare} \sim 0.2518$



(b) $|\rho| = 0.2517$

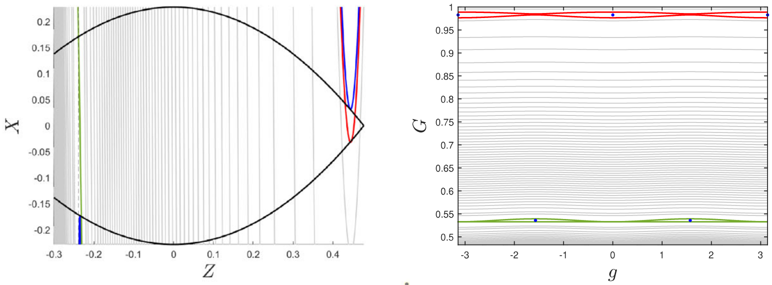


(c) $|\rho| = \rho_{\blacklozenge}$, $\rho_{\blacklozenge} \sim 0.2514$

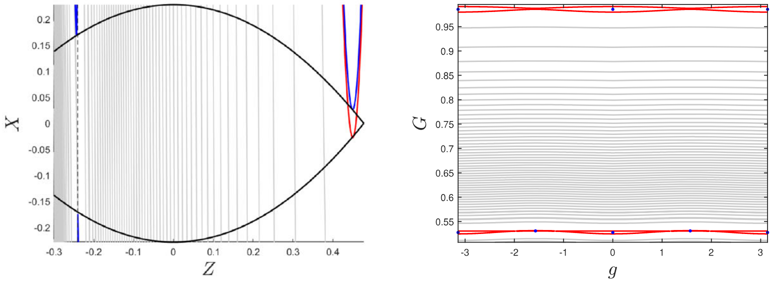


(d) $|\rho| = 0.22$

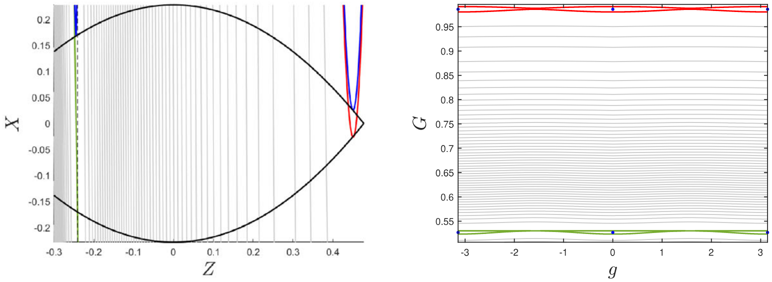
Fig. 7 Level curves for the J_2-problem with relativistic term with $j_C = 0.2$ and $\lambda = 0.001$, for four different values of $|\rho| \in (\rho_{\diamond}, \rho_{\blacksquare}]$. On the left, the levels curve are represented on the (Z, X) plane. Enlargements of the regions containing the equilibrium points are performed. On the right, the level curves are shown on corresponding enlargements in the (g, G) plane. The same colour code used in Fig. 4 is employed



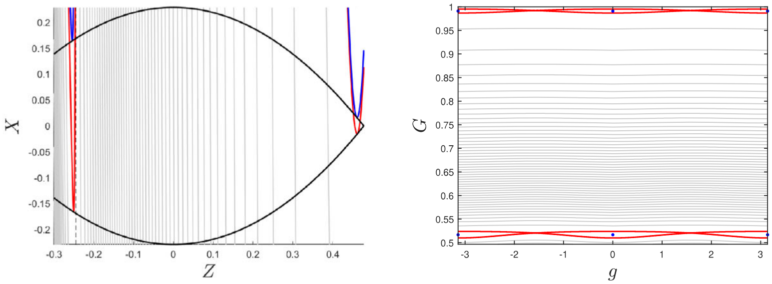
(a) $|\rho| = \rho_\diamond$, $\rho_\diamond \sim 0.2114$



(b) $|\rho| = 0.21$



(c) $|\rho| = \rho_\square$, $\rho_\square \sim 0.2098$



(d) $|\rho| = 0.207$

Fig. 8 Level curves for the J_2-problem with relativistic term with $j_C = 0.2$ and $\lambda = 0.001$, for four different values of $|\rho| \in (\tilde{\rho}_-, \rho_\diamond]$. On the left, the levels curve are represented on the (Z, X) plane. Enlargements of the regions containing the equilibrium points are performed. On the right, the level curves are shown on corresponding enlargements in the (g, G) plane. The same colour code used in Fig. 4 is employed

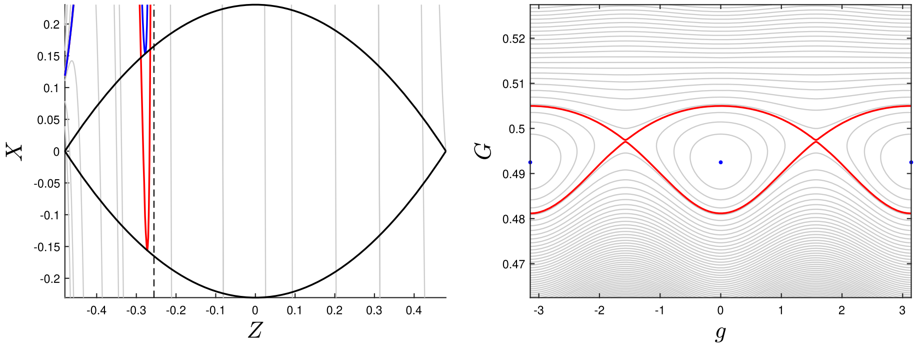


Fig. 9 Level curves for the J_2-problem with relativistic term with $j_C = 0.2$ and $\lambda = 0.001$ and for a value of $|\rho| < \tilde{\rho}_+$. Enlargements of the regions containing the equilibrium points are performed. On the right, the level curves are shown on corresponding enlargements in the (g, G) plane. The same colour code used in Fig. 4 is employed

Considering that $\rho_\blacksquare, \rho_\blacklozenge \sim \frac{1}{\sqrt{80 j_C}}$ and $\rho_\diamond, \rho_\square \sim \sqrt{\frac{7}{810 j_C}}$ in first approximation, our results are consistent with the outcomes of Jupp and Brumberg (1991).

5 Conclusions

We have described the existence and stability of frozen orbits in a gravity field expanded in even zonal terms. The main focus has been given on the power of the geometric analysis of the reduced dynamics to highlight the main features of these systems as they are determined by the presence of stable and unstable families. In this respect, the study has been limited to the J_2 and J_4 problems and to the relativistic corrections, showing the ability of the geometric invariant method to easily reproduce known results and predict new features of higher-order terms. The atlas of possible perturbations is wide, and several other terms could be added. For many of them, this approach requires very few changes and immediate results. For example, low-order tesseral terms, averaged in order to preserve Brouwer structure, can be easily analysed (Palacián 2007) without qualitative new results. Additional efforts are required for more complex perturbations. Higher-degree zonal terms (J_{2k} with $k \geq 3$) are the most promising since the symmetry of the problem is preserved. Preliminary results like those presented in Coffey et al. (1994) can be extended with a little effort. More general cases (odd zonal terms, higher-order tesserals, third-body effects, etc.) require a stronger commitment. However, in these cases, it is quite probable that difficulties arise more from the implementation of the closed-form normalisation (Palacián 2002; Cavallari and Efthymiopoulos 2022) than from the use of the reduction method.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: Proof of inequalities (42) and (45)

We start by proving relation (42). We have

$$\frac{d\rho^2_{E_+2}}{dG} = \frac{1}{5\lambda}\frac{G}{C_+^2\sqrt{B_+}}\Big(- D_+\sqrt{B_+} + \lambda E_+\sqrt{B_+} + F_+\Big),$$

where

$$\begin{aligned}
D_+ &= -14400G^6 + 28800G^5 + 68640G^4,\\
E_+ &= 4410G^4 - 14904G^3 - 14204G^2 + 48312G + 28314,\\
F_+ &= 20G^2D_+ + \lambda\big(- 21600G^8 + 154080G^7 - 285920G^6 - 237600G^5 + 1006720G^4\big)\\
&\quad + \lambda^2(-129960G^6 + 470664G^5 + 679088G^4 - 2114832G^3\\
&\quad - 2381760G^2 + 1287528G + 1774344),
\end{aligned}$$

and where B_+ is defined in (41). It holds $d\rho^2_{E_+2}/dG > 0$ if

$$-D_+\sqrt{B_+} + \lambda E_+\sqrt{B_+} + F_+ > 0.$$

It is straightforward that $D_+ > 0$ and $E_+ > 0$. Moreover,

$$\begin{aligned}
F_+ > \Big(&- 288000G^{10} + 576000G^9 + 1351200G^8 + 154080G^7 - 415880\,G^6 + 233064\,G^5\\
&+ 1685808G^4 - 2114832\,G^3 - 2381760\,G^2 + 1287528\,G + 1774344\Big)\lambda^2 > 0.
\end{aligned}$$

Thus, we need to verify whether

$$\Big(\lambda E_+\sqrt{B_+} + F_+\Big)^2 - D_+^2B_+ > 0.$$

Since $B_+ \geq (400G^8 + 240G^4\lambda + 376\lambda^2) > (20G^4 + 6\lambda)^2$, we have

$$\Big(\lambda E_+\sqrt{B_+} + F_+\Big)^2 - D_+^2B_+ > M_+,$$

with

$$M_+ = \lambda^2 E_+^2 B_+ + F_+^2 + 2\lambda E_+ F_+ (20G^4 + 6\lambda) - D_+^2 B_+$$
$$= M_+^{(4)}\lambda^4 + M_+^{(3)}\lambda^3 + M_+^{(2)}\lambda^2 + M_+^{(1)}\lambda,$$

where

$$\begin{aligned} M_+^{(4)} &= 23910365700\, G^{12} - 181225103760\, G^{11} + 120523638672\, G^{10} \\ &\quad + 1700153452368\, G^9 - 1988897297356\, G^8 - 7508533086240\, G^7 \\ &\quad + 6377312862144\, G^6 + 19130049303840\, G^5 - 4594353603924\, G^4 \\ &\quad - 23486016392784\, G^3 - 4409791599312\, G^2 \\ &\quad + 10749794943888\, G + 4994571648924, \\ M_+^{(3)} &= -160G^4(103329675G^10 - 563720310G^9 - 911841039G^8 + 7739401536G^7 \\ &\quad + 2782008506G^6 - 39130317372G^5 - 12624306282G^4 + 84930202896G^3 \\ &\quad + 49983903915G^2 - 50940897150G - 39230486295), \\ M_+^{(2)} &= 1600G^8(2772225G^8 - 11651040G^7 - 19496340G^6 + 86266224G^5 \\ &\quad + 106893822G^4 \\ &\quad - 176211648G^3 - 404205252G^2 - 24689808G + 315589417), \\ M_+^{(1)} &= 768000G^{12}(-30G^2 + 60G + 143)(45G^2 - 72G - 143)^2. \end{aligned}$$

We have $M_+^{(1)} > 0$, $M_+^{(1)} + M_+^{(2)} > 0$, $M_+^{(1)} + M_+^{(2)} + M_+^{(3)} > 0$ and $M_+^{(4)} > 0\ \forall G$; thus, using $\lambda < 1$, it holds

$$M_+ > (M_+^{(4)} + M_+^{(3)} + M_+^{(2)} + M_+^{(1)})\lambda^4 > 0.$$

Now, we prove relation (45). We have

$$\frac{d\rho_{E_-2}^2}{dG} = \frac{1}{5\lambda}\frac{G}{C_-^2\sqrt{B_-}}\Big(- D_-\sqrt{B_-} + \lambda E_-\sqrt{B_-} + F_-\Big),$$

where

$$\begin{aligned} D_- &= 20160G^6 + 28800G^5 + 5280G^4, \\ E_- &= 22050G^4 + 43848G^3 + 21524G^2 - 10440G - 4158 \\ F_- &= 20G^2 D_-(G) + \lambda\big(- 846720\, G^8 - 1441440\, G^7 + 519680\, G^6 + 1680480\, G^5 \\ &\quad + 352000\, G^4\big) + \lambda^2(617400\, G^6 + 1375920\, G^5 + 392\, G^4 \\ &\quad - 1902192\, G^3 - 968664\, G^2 + 554208\, G + 211288), \end{aligned}$$

and B_- is defined in (44). It holds $d\rho_{E_-2}^2/dG > 0$ if

$$-D_-\sqrt{B_-} + \lambda E_-\sqrt{B_-} + F_- > 0.$$

It is straightforward that $D_- > 0$ and

$$\begin{aligned} F_- > \Big(&- 1693440\, G^8 - 2882880\, G^7 + 1656760\, G^6 + 4736880\, G^5 \\ &+ 704392\, G^4 - 1902192\, G^3 - 968664\, G^2 + 554208\, G + 211288\Big)\lambda^2 > 0. \end{aligned}$$

Instead $E_- > 0\ \forall G \geq 0.5$, while its sign changes if $G < 0.5$. Let us consider $G \in (0, 0.5]$; we need to verify whether

$$F_-^2 - (-D_- + \lambda E_-)^2 B_- > 0.$$

It holds

$$F_-^2 - (-D_- + \lambda E_-)^2 B_- = P_-^{(4)}\lambda^4 + P_-^{(3)}\lambda^3 + P_-^{(2)}\lambda^2 + P_-^{(1)}\lambda,$$

with

$$\begin{aligned}
P_-^{(4)} &= 205663657500\, G^{12} + 1286808541200\, G^{11} + 2740720809456\, G^{10} \\
&\quad + 1509967832688\, G^9 - 2841942124116\, G^8 - 4558328657760\, G^7 \\
&\quad - 1319583449472\, G^6 + 1104354441696\, G^5 + 231876171316\, G^4 \\
&\quad - 430214668464\, G^3 - 18111422832\, G^2 + 89372757936\, G + 17827435780, \\
P_-^{(3)} &= -160G^4(4650179625G^10 + 21853893474G^9 + 31457092779G^8 - 4333755960G^7 \\
&\quad - 49586843494G^6 - 32290996788G^5 + 11675441446G^4 + 12857914248G^3 \\
&\quad - 2955589987G^2 - 3112286430G - 456484721), \\
P_-^{(2)} &= 1600G^8(568229823G^8 + 2010142008G^7 + 1564140924G^6 - 2245174056G^5 \\
&\quad - 3853671558G^4 - 789759288G^3 + 1195807132G^2 + 548335656G + 61908319), \\
P_-^{(1)} &= -768000G^{12}(42G^2 + 60G + 11)(10647G^4 + 11808G^3 \\
&\quad - 22398G^2 - 29664G - 5269).
\end{aligned}$$

For $G < 0.5$, we have $P_-^{(1)} > 0$, $P_-^{(2)} > 0$, $P_-^{(3)} > 0$ and $P_-^{(3)} + P_-^{(4)} > 0$; thus,

$$F_-^2 - (-D_- + \lambda E_-)^2 B_- > \lambda^4(P_-^{(3)} + P_-^{(4)}) + P_-^{(2)}\lambda^2 + P_-^{(1)}\lambda > 0.$$

Let us now consider $G \in [0.5, 1]$. In this case, we need to verify whether

$$\left(\lambda E_- \sqrt{B_-} + F_-\right)^2 - D_-^2 B_- > 0.$$

Since $B_- > 400G^8$,

$$\left(\lambda E_- \sqrt{B_-} + F_-\right)^2 - D_-^2 B_- > M_-,$$

with

$$M_- = \lambda^2 E_-^2 B_- + F_-^2 + 40\lambda E_- F_- G^4 - D_-^2 B_- = M_-^{(4)}\lambda^4 + M_-^{(3)}\lambda^3 + M_-^{(2)}\lambda^2 + M_-^{(1)}\lambda,$$

where

$$\begin{aligned}
M_-^{(4)} &= 976780822500\, G^{12} + 4476174696000\, G^{11} + 5453865257400\, G^{10} \\
&\quad - 5086142681760\, G^9 - 16601363592772\, G^8 - 7099182483456\, G^7 \\
&\quad + 11816572141456\, G^6 + 11183433731136\, G^5 - 1358260437988\, G^4 \\
&\quad - 4012690481472\, G^3 - 434866063560\, G^2 + 421070887776\, G + 86153421508, \\
M_-^{(3)} &= -320G^4(3267280800G^10 + 12799767015G^9 + 9651811113G^8 - 23608483416G^7 \\
&\quad - 41966083012G^6 - 5277158466G^5 + 28856284782G^4 + 14899437888G^3
\end{aligned}$$

$$- 4161615732G^2 - 3404287293G - 463072687),$$

$$M_-^{(2)} = 1600G^8(102880449G^8 + 321838272G^7 + 208527732G^6 - 295032528G^5 - 233557962G^4 + 563365728G^3 + 780379476G^2 + 285203952G + 31226833),$$

$$M_-^{(1)} = 768000G^12(42G^2 + 60G + 11)(63G^2 + 72G + 11)^2.$$

For $0.5 \leq G \leq 1$, we have $M_+^{(1)} > 0$, $M_+^{(2)} > 0$, $M_+^{(2)} + M_+^{(3)} > 0$ and $M_+^{(2)} + M_+^{(3)} + M_+^{(4)} > 0$; thus,

$$M_- > (M_+^{(4)} + M_+^{(3)} + M_+^{(2)})\lambda^4 + M_+^{(1)}\lambda > 0.$$

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