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Reduced 4D oscillators and orbital elements in Keplerian systems: Cushman–Deprit coordinates

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Abstract

We study the reduction and regularization processes of perturbed Keplerian systems from an astronomical point of view. Our approach connects axially symmetric perturbed 4-DOF oscillators with Keplerian systems, including the case of rectilinear solutions. This is done through a preliminary reduction recently studied by the authors. Then, the reduction program continues by removing the Keplerian energy. For each value of the semi-major axis, we explain the astronomical meaning of the sextuples defining the orbit space $\mathbb{S}^2 \times \mathbb{S}^2$ and its connection with the orbital elements. More precisely, we present alternative sextuple coordinates for the set of bounded Keplerian orbits that 'separate' the node of the orbital plane from the argument of perigee giving the Laplace vector in that plane. Still, the reduction of the axial symmetry defined by the third component of the angular momentum is performed. For the thrice reduced space $\Gamma_{0,L,H}$ we propose the Cushman–Deprit coordinates, a variant to the set given by Cushman. The main feature of these variables is that they are all with the same dimensions, which is convenient for the normalization procedure. As an application of the proposed scheme, we study the spatial lunar problem.

Keywords Harmonic oscillator · Kepler system · Orbital elements · Geometric reduction

1 Introduction

As it is well known, the study of perturbed Keplerian systems entered into a new paradigm during the second half of last century, built on the joint efforts coming from the numerical, analytical and geometric realms. Indeed, on the one hand the *KS* transformation made possible

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to cope with new possibilities for computing opening in those years. This transformation was key to treat numerically these nonintegrable systems as perturbed harmonic oscillators, which brought accuracy and stability when dealing with both short and long period space missions, as well as very long-term dynamics in celestial mechanics. The reader will find in the recent publication of Roa (2017) a full account of it. On the other hand, the secular perturbation theories (averaging) also received a new understanding as Lie series based on the work of Hori (1966) and Deprit (1969), now under the name of normalization. Finally, the development of the geometric program set up by Smale in the seventies dealing with dynamical systems, it was carried out by Cushman (1992) for perturbed Keplerian systems along the eighties. It was required to apply the results of regular and singular reductions obtained along those years, allowing to simplify the systems reaching the core of them, in order to classify the different types of solutions and their evolution as functions of integrals and parameters. The reader may go from the germinal papers (Meyer 1973; Marsden and Weinstein 1974) to the more recent overview of Marsden et al. (2007) for an explanation of that effort.

In the frame set up in the previous paragraph, an ingredient has to be added. Since the advent of symbolic algebraic manipulators, this tool has been an essential piece in order to carry out the steps required to produce higher order normalization theories, coping with the improvement in precision of observation data, in particular with perturbed satellite dynamics where closed form expressions were the goal, in order the dispose of a set of formulas ready for dealing with high eccentric orbits.

We really think there is still room to contribute filling the gap, which still exists, between the very active field of astrodynamics literature and the new path followed by modern celestial mechanics we have just referred to. We believe that this paper is a step in that direction. More precisely, the aim of this paper is twofold. First, we present in symplectic variables, i.e., putting it in astrodynamics context, our recent works (Ferrer and Crespo 2018; Crespo and Ferrer 2020) on a new interpretation of KS transformation dealing with perturbed Kepler systems as reduced symmetric 4D oscillators. It represents an alternative to Moser regularization which involves constrained normalization. The main contribution of this work is the proposal of Cushman–Deprit coordinates, a variant to Cushman coordinates, which helps to illustrate what is essential in Cushman variables and, at the same time, it invites the readers to search for other choices if necessary in their work. What is the interest of this proposal?

Deprit, Henrard, Coffey, and other collaborators, leading the application of symbolic manipulators to celestial mechanics, found very convenient to have at hand ways of checking intermediary heavy expressions with several variables and parameters, appearing in the computations of Lie brackets, needed to get averaged higher order theories in explicit form. One effective way to keep control of those computations was to check dimensions at different steps of the computations, simplifications, etc. That was the reason why Deprit and his team, instead of Cushman variables, they preferred to work with Coffey–Deprit–Miller (CDM) variables (Coffey et al. 1986), which are all of the same physical dimensions and why, today, they continue to be in use when no rectilinear solutions are involved. While of interest in astronomy and astrodynamics quarters, what we have just said in the above lines may be of no real concern for geometers. Moreover, some readers may see all this just as history. Here, we will like to add only two short comments. The first is that Cushman stated, without much emphasis, that his variables were just one possibility. The second comment is that Cushman–Deprit variables we propose as an alternative, which illustrate Cushman claim, have also the characteristic to be a third choice 'tuning' Cushman and CDM sets of variables; although we recognized this is a matter of opinion. In fact, the main reason for our proposal, apart from the practical issue of keeping dimensions, is that one of the variables is just G^2, the square norm of the angular momentum vector.

Section 2 summarizes the approach given in Ferrer and Crespo (2018), Crespo and Ferrer (2020) to the reduction and regularization process of Keplerian system, which is the one we follow throughout this paper. In Sect. 3, we deal with the reduced spaces obtained by removing the Keplerian energy. In Sect. 4, we deal with the singular reduction in the node. We introduce a new set of variables for the full reduced space, which are an alternative to the variables proposed by Cushman (1983, 1992). Section 5 answers the question of what families of Kepler orbits do we have associated with each point of the (*KS*-energy-reduced)-reduced space. Finally, in Sect. 6, we consider the Hill's spatial lunar problem as a benchmark to develop the alternative plan proposed. The purpose of this application is not to find new dynamical features of the model but to illustrate the setting we are proposing.

2 A Précis on the *KS*-reduction and Kepler regularization

Regarding to Keplerian systems, the reduction in the energy is only possible after regularization due to the incompleteness of the Keplerian flow. In the literature, there are mainly two different ways in which this process can be done. Namely, Moser regularization (Moser 1970) and *KS*-regularization (Kustaanheimo 1964; Kustaanheimo and Stiefel 1965), which is the chosen one in this paper. However, in Ferrer and Crespo (2018) the *KS*-regularization is revisited from a geometrical point of view. It provides a very different explanation of the whole process, which reveal the true role of the *KS*-map and the bilinear relation associated and provides a geometric insight of all these elements. Along this paper, we follow this new approach finding the Kepler system after reducing an $\mathcal{S}^1$-symmetry of the 4D isotropic oscillator.

As a result of work done along the last two decades, we have end up proposing alternatives to the program followed by Cushman, based on the use of Moser transformation and constraint normalization (van der Meer and Cushman 1986). Indeed, after several trials (van der Meer 2015; Egea et al. 2007; Díaz et al. 2010; Crespo et al. 2016) with partial progress, we have recently succeeded in presenting a reduction process by stages starting from the reparametrized 4D isotropic oscillator (Crespo and Ferrer 2020), i.e., showing that the *KS* frame, of great value for numerical purposes as we have pointed out above, it serves also to deal with reductions making more expedited the whole process, because we do not need to make use of constrained normalization required in Cushman approach. Here, we keep that part to the minimum providing an explanation of that process at each step making use of orbital elements, absent in geometric mechanics literature.

2.1 *KS*-reduction

We describe the *KS* map as the orbit map associated with a symplectic diagonal action given by

$$\chi : S^1 \times T^*\mathbb{R}_0^4 \longrightarrow T^*\mathbb{R}_0^4 \qquad \chi(\alpha, \mathbf{q}, \mathbf{p}) = (\mathcal{R}_\alpha \mathbf{q}, \mathcal{R}_\alpha \mathbf{p}), \tag{1}$$

being $\mathcal{R}_\alpha$ the rotation given by the following matrix

$$\mathcal{R}_\alpha = \begin{bmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{bmatrix}. \tag{2}$$

Moreover, the momentum map related to the named symmetry is the quadratic polynomial defining the bilinear relation

$$\varXi_0 : T^*\mathbb{R}_0^4 \longrightarrow \mathbb{R} \qquad (\mathbf{q}, \mathbf{p}) \rightarrow \varXi_0(\mathbf{q}, \mathbf{p}) = q_1 p_4 - p_1 q_4 + q_2 p_3 - p_2 q_3. \qquad (3)$$

It plays a key role in the regularization process since the Kepler system is obtained as the reduced oscillator only for the momentum value $\varXi_0 = 0$, which correspond to the classic bilinear relation. In what follows, we refer to the above symmetry of the 4D oscillator as the *KS*-symmetry. The *KS*-reduced space is a 6D manifold described with 16 invariants (see Crespo and Ferrer 2020). However, a suitable combination of these invariants allows to reduce the number of variables describing the *KS*-reduced space. Precisely, these combinations of invariants coincide with the components of the classical *KS*-map, which following Saha (2009) and using the identification of quaternions $\mathbb{H}_0 \equiv \mathbb{R}_0^4$, is expressed with a very compact expression.

$$KS : T^*\mathbb{H}_0 \longrightarrow T^*\mathbb{H}_0 \quad (\mathbf{q}, \mathbf{p}) \rightarrow (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}),$$
$$\tilde{\mathbf{x}} = \mathbf{q}^* \, \mathbf{k} \, \mathbf{q}, \quad \tilde{\mathbf{y}} = \frac{\mathbf{q}^* \, \mathbf{k} \, \mathbf{p}}{2\mathbf{q}^* \, \mathbf{q}}, \qquad (4)$$

where $\mathbf{k}$ is the quaternion $(0, 0, 0, 1)$, $\mathbf{q}^*$ denotes de conjugate of $\mathbf{q}$ and the operation relating them is the usual quaternionic multiplication. In what follows, we consider $\mathbf{x} = Im(\tilde{\mathbf{x}}) = (x_1, x_2, x_3)$ and $\mathbf{y} = Im(\tilde{\mathbf{y}}) = (y_1, y_2, y_3)$. In Crespo and Ferrer (2020), we showed that $\mathbf{Z} = (\mathbf{x}, \mathbf{y})$ is endowed with the following non-degenerate Poisson structure

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \frac{\xi \, x_3}{2|\mathbf{x}|^3} & \frac{-\xi \, x_2}{2|\mathbf{x}|^3} \\ 0 & -1 & 0 & \frac{-\xi \, x_3}{2|\mathbf{x}|^3} & 0 & \frac{\xi \, x_1}{2|\mathbf{x}|^3} \\ 0 & 0 & -1 & \frac{\xi \, x_2}{2|\mathbf{x}|^3} & \frac{-\xi \, x_1}{2|\mathbf{x}|^3} & 0 \end{bmatrix} \qquad (5)$$

where $\varXi_0 = \xi$ is the fixed value of the momentum map. Moreover, $(\mathbf{x}, \mathbf{y})$ defines a chart for the *KS*-reduced space, which for the particular value of the momentum map $\varXi_0 = 0$ coincides with $T^*\mathbb{R}_0^3$ endowed with the standard symplectic form.

2.2 Kepler regularization

Our methodology applies for perturbed Keplerian systems. That is to say, we consider the following family of Hamiltonians made of the Kepler system plus a small perturbation

$$\tilde{\mathcal{K}}_\kappa(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{y}|^2}{2} - \frac{\kappa}{|\mathbf{x}|} + \epsilon \, \mathcal{P}(\mathbf{x}, \mathbf{y}), \qquad (6)$$

in $T^*\mathbb{R}_0^3$, where $\mathbb{R}_0^3 = \mathbb{R}^3 - \{0\}$ and κ is the (positive) mass-gravitational constant. In Ferrer and Crespo (2018); Crespo and Ferrer (2020), the authors dealt with the unperturbed case $\epsilon = 0$, which will be denoted as $\mathcal{K}_\kappa$. The connection with the 4D oscillator

$$\mathcal{H}_\omega = \frac{1}{2} \sum_i^4 (p_i^2 + \omega^2 \, q_i^2), \qquad (7)$$

defined in $T^*\mathbb{R}_0^4 = T^*(\mathbb{R}^4 - \{0\})$, being $\omega^2 \in \mathbb{R}$ a positive constant, is made by means of the *KS*-reduction. Precisely, the process starts by fixing the energy level $\mathcal{H}_\omega - h = 0$, for $h > 0$. Then, we reduce out the *KS*-symmetry of the 4D oscillator and provide the *KS*-reduced Hamiltonian expressed through the $(\mathbf{x}, \mathbf{y})$ chart as

$$H_{Aux} = \frac{1}{2}\left(4|\mathbf{y}|^2|\mathbf{x}| + \frac{\xi^2}{|\mathbf{x}|} + \omega^2|\mathbf{x}|\right) - h. \tag{8}$$

By fixing the momentum map $\varXi_0 = \xi = 0$ and subtracting the constant term $\omega^2/2$, together with the time reparametrization $d\tau = 1/(4|\mathbf{x}|)\,ds$, we finally obtain

$$\mathcal{K}_\kappa(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{y}|^2}{2} - \frac{\kappa}{|\mathbf{x}|}, \tag{9}$$

where $\kappa = h/4$. The case of a perturbed Keplerian system is tackled by considering a suitable perturbed 4D oscillator instead of (7)

$$\tilde{\mathcal{H}}_\omega = \frac{1}{2}\sum_{i=1}^n (p_i^2 + \omega^2 q_i^2) + \epsilon\, 4|\mathbf{q}|^2 \mathcal{P}^*(\mathbf{q}, \mathbf{p}), \tag{10}$$

where $\mathcal{P}^*(\mathbf{q}, \mathbf{p})$ is easily obtained by substitution of $\mathbf{x}(\mathbf{q}, \mathbf{p})$ and $\mathbf{y}(\mathbf{q}, \mathbf{p})$ in $\mathcal{P}(\mathbf{x}, \mathbf{y})$.

Remark 1 Notice that the way in which we connect the oscillator and the Kepler system is a very different operation from the originally proposed in Kustaanheimo (1964); Kustaanheimo and Stiefel (1965). Our approach establishes the connection by means of a geometric reduction in the $\varXi_0$ symmetry of the oscillator, and moreover, it reveals the true role of the bilinear relation as a particular value of the momentum map. Namely, $\varXi_0 = 0$.

Remark 2 Note that the process described above is only possible if the Hamiltonian $\tilde{\mathcal{H}}_\omega$ is also endowed with the *KS*-symmetry. However, it is always guaranteed since the perturbation is expressed as a function in the invariants $(\mathbf{x}, \mathbf{y})$.

2.3 Symplectic angular charts for the *KS*-symmetry

The previous *KS*-reduction may be easily tracked by means of classical charts based on angles coming from celestial mechanics. In this section, we recall several sets of symplectic charts incorporating the angle α, of the $\mathcal{S}^1$ action described above, among the angular variables. This feature makes the following charts particularly suitable for keeping track of the whole regularization process, since they allow to perform the *KS*-reduction by fixing one angle. These charts were previously presented in Ferrer (2010), Crespo et al. (2016), Ferrer and Crespo (2014), Crespo (2015), and here, we include a brief summary for the benefit of the reader.

2.3.1 The *KS* map in the projective Euler variables

First, we introduce the set of variables dubbed as Projective Euler variables defined by means of the following transformation

$$\mathcal{PE} : (\rho, \phi, \theta, \psi, P, \Phi, \Theta, \Psi) \rightarrow (\mathbf{q}, \mathbf{p})$$

$$q_1 = \sqrt{\rho} \, \cos \frac{\theta}{2} \sin \frac{\phi + \psi}{2},$$

$$q_3 = \sqrt{\rho} \, \sin \frac{\theta}{2} \sin \frac{\phi - \psi}{2}, \tag{11}$$

$$q_2 = \sqrt{\rho} \, \sin \frac{\theta}{2} \cos \frac{\phi - \psi}{2}, \quad q_4 = \sqrt{\rho} \, \cos \frac{\theta}{2} \cos \frac{\phi + \psi}{2},$$

with $\rho \in R^+$, $\phi + \psi \in (0, 2\pi)$, $\phi - \psi \in (-\pi, \pi)$ and $\theta \in (0, \pi)$. The associated momenta are obtained by canonical extension of (11), that is, $\sum p_i dq_i = P \, d\rho + \Phi \, d\phi + \Theta \, d\theta + \Psi \, d\psi$

$$P = \frac{1}{2\rho} (q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4),$$

$$\Theta = \frac{(q_2 p_2 + q_3 p_3)(q_1^2 + q_4^2) - (q_1 p_1 + q_4 p_1)(q_3^2 + q_2^2)}{2 \sqrt{(q_1^2 + q_4^2)(q_2^2 + q_3^2)}},$$

$$\Phi = \frac{1}{2} \big((p_1 q_4 - p_4 q_1) + (p_3 q_2 - p_2 q_3) \big), \tag{12}$$

$$\Psi = \frac{1}{2} \, \Xi_0 = \frac{1}{2} \big((p_1 q_4 - p_4 q_1) - (p_3 q_2 - p_2 q_3) \big).$$

This kind of variables is not new, see for instance (Ikeda and Miyachi 1971; Stiefel and Scheifele 1971; Barut et al. 1979; Iwai 1981; Cornish 1984), all of them being a variation of the Euler parameters. However, the set we are presenting here combines the one-dimensional square-root coordinate with the unit quaternion parametrization by Euler angles. Thus, having the advantage of connecting with the Kepler system in a straightforward manner.

In Ferrer and Crespo (2018), it is showed that the restriction of the 4D harmonic oscillator to either $\Phi = 0$ or $\Psi = 0$ in Projective Euler variables leads to the 3D Kepler system expressed in spherical coordinates.

2.3.2 Projective Andoyer variables

Although Projective Euler variables allow to see the *KS* map in a straightforward manner, they provide the Kepler system in spherical coordinates. Nevertheless, in the classical literature of Celestial Mechanics we find more convenient charts for this system as the Projective Andoyer variables will also provide a connection between both systems, but the Kepler Hamiltonian will be given in polar-nodal variables. We will use a compact quaternionic formulation, and we introduce the following notation

$$\mathbf{q}_{\mathbf{l}, \alpha} = \cos \alpha + \mathbf{l} \sin \alpha, \quad \mathbf{l} = \mathbf{i}, \mathbf{j}, \mathbf{k}$$

Then, projective Andoyer variables are given by the following transformation

$$\mathcal{PA} : (\rho, \lambda, \mu, \nu, P, \Lambda, M, N) \rightarrow (\mathbf{q}, \mathbf{p}),$$

$$\mathbf{q} = \sqrt{\rho} \, \mathbf{q}_{\mathbf{k}, \nu} \, \mathbf{q}_{\mathbf{i}, \mathcal{J}} \, \mathbf{q}_{\mathbf{k}, \mu} \, \mathbf{q}_{\mathbf{i}, \mathcal{I}} \, \mathbf{q}_{\mathbf{k}, \lambda}, \qquad\qquad \mathbf{p} = \frac{1}{4\rho^2} \, \mathbf{q} \, \mathbf{w}, \tag{13}$$

where $\rho \in R^+$, $\lambda \in (0, \pi)$, $\mu, \nu \in (0, 2\pi)$. Moreover, the angles $\mathcal{I}$ and $\mathcal{J}$ are given by $\cos \mathcal{I} = \Lambda/M$, $\cos \mathcal{J} = N/M$ and $\mathbf{w} = (\tau, \rho_1, \rho_2, \rho_3)$ denotes a quaternion whose components are

$$
\begin{aligned}
\tau &= \rho P, \\
\rho_1 &= 2\sqrt{M^2 - N^2}\, \sin \nu, \\
\rho_2 &= 2\sqrt{M^2 - N^2}\, \cos \nu, \\
\rho_3 &= 2N.
\end{aligned}
\tag{14}
$$

Note that these variables are not defined for $M = 0$ and they also incorporate the *KS*-reduction angle. Moreover, this set of variables connect with the Kepler system in polar-nodal variables. Precisely, considering the regularization $\varpi(\rho) = 1/(4\rho)$ and the Projective Andoyer variables we obtain

$$
\tilde{\mathcal{K}} = \varpi(\rho)(\mathcal{H}_\omega - h) = \frac{1}{2}\left(P^2 + \frac{M^2}{\rho^2}\right) - \frac{\kappa}{\rho},
\tag{15}
$$

where $\kappa = h/4$, in the manifold $\tilde{\mathcal{K}} = -\omega^2/8$, which is the Hamiltonian of the Kepler system in polar-nodal coordinates (see Deprit 1982). Further details are given in Ferrer and Crespo (2018).

2.3.3 Projective Delaunay variables

Up to now, we have presented two sets of symplectic coordinates that reveal the Kepler-oscillator connection in a straightforward manner. Nevertheless, the treatment of perturbed Keplerian systems calls for coordinates not only compatible with an easy way of connecting with the oscillator, but also suitable for the reduction in the Keplerian energy. These variables are implemented by means of a canonical transformation in part of the phase space of Hamilton–Jacobi type:

$$
\mathcal{D}_\kappa : (\ell, g, L, G) \rightarrow (\rho, \mu, P, M).
$$

For details of this transformation $\mathcal{D}_\kappa$ giving the relations defining Delaunay transformation, we refer to Deprit (1982). The projective Delaunay transformation is given as

$$
\mathcal{PD} : (\rho, \lambda, \mu, \nu, P, \Lambda, M, N) \rightarrow (\ell, \lambda, g, h, L, \Lambda, G, H).
$$

The relations defining Delaunay transformation read as follows

$$
\mu = g + f, \quad \nu = h, \quad N = H, \quad M = G,
\tag{16}
$$

$$
\rho = a\,(1 - e \cos E), \quad P = \frac{L\, e \sin E}{a\,(1 - e \sin E)},
\tag{17}
$$

where $a = a(L)$, $e = e(L, G)$ are given by

$$
a = L^2/\kappa, \quad \eta = G/L, \quad e = \sqrt{1 - \eta^2},
\tag{18}
$$

f and E are auxiliary angles. The symplectic variable ℓ satisfies

$$
\ell = E - e \sin E, \quad \tan f/2 = \sqrt{(1 + e)/(1 - e)}\, \tan E/2.
\tag{19}
$$

Later on, we also need $p = \rho(1 + e \cos f)$, where $p = a\eta^2$, and

$$
\rho \cos f = a(\cos E - e), \quad \rho \sin f = a\eta \sin E.
\tag{20}
$$

In Crespo et al. (2016), it is proved that the composition of $\mathcal{P}\mathcal{A}$ and $\mathcal{P}\mathcal{D}$ and the regularization $\mathrm{d}s = 1/(4\rho)\,\mathrm{d}\tau$ allows to express the Hamiltonian of the 4D isotropic oscillator as

$$\mathcal{H}_0 = -\frac{\kappa^2}{2L^2} \tag{21}$$

in the manifold $\mathcal{H}_0 = -\omega^2/8$.

For the benefit of the reader, let us mention that Moser and Zehnder (2005) introduced action-angle variables for the Kepler problem in $\mathbb{R}^n$ and called them Delaunay variables. When we restrict to $n = 4$, those variables do not coincide with the set built in this paper.

3 Astrodynamical interpretation of the *KS*-energy reduced space

In this section, we deal with the reduced spaces obtained by removing the Keplerian energy. However, the intricate way in which the reduction process behaves, usually, results in a loss of astrodynamical intuition. One of our main goals is to shed light on this process at each stage by relating invariants and orbital elements in their canonical version, i.e., Delaunay variables. These variables are the common sets used when dealing with Kepler systems; astrodynamics textbooks give a full description of them. Excluding the anomalies, when we fix a set of them, we have defined a Kepler orbit. Elementary classification of Kepler orbits usually asks for sets such as circular, equatorial, polar, rectilinear, and they are related to special values of the orbital elements, which we may see as a *label* for them. This work shows how these packages of orbits are arranged in the reduced spaces.

It is well-known that rectilinear, circular, and equatorial orbits require particular treatment in the classical approach and the new techniques we mention above have allowed to include them within the generic treatment, referring to them as singular. Nevertheless, this work is not dealing with elaborated geometric concepts. In this regard, we will make full use of ideas widely used in celestial mechanics and astronomy through centuries, which switch back to the primitive and most classical way to present symplectic reduction. Namely, we are referring to the process of choosing suitable symplectic coordinate transformations by which the number of equations and variables reduces. These simplifications are usually achieved by incorporating invariants of the geometric reduction among the new set of variables. A disadvantage is that these transformations often need several charts to analyze the whole phase space, while the geometric approach allows for a complete study.

Usually, the reduced space obtained after regularization (*KS*-reduction) and using the Keplerian energy is represented by a linear combination of the angular momentum vector $\mathbf{G} = (G_1, G_2, G_3)$ and the eccentricity vector $\mathbf{A} = (A_1, A_2, A_3)$

$$\sigma = \frac{1}{2}(\mathbf{G} + L\mathbf{A}), \quad \delta = \frac{1}{2}(\mathbf{G} - L\mathbf{A}). \tag{22}$$

Note that two of the momenta in the projective Delaunay variables are related to the angular momentum vector in the following way

$$G = |\mathbf{G}|, \qquad H = G_3. \tag{23}$$

We refer to this regularized-reduced space as the (*KS*-energy)-reduced space, which is diffeomorphic to $\mathbb{S}_L^2 \times \mathbb{S}_L^2$ and may be regarded as a way of packing orbits with the same semi-major axis. The subindex L stands for the fact that the radius of the spheres are related to L.

From the astrodynamical point of view, it would be desirable to have a simple connection between the orbital elements and the variables of the reduced space. With that idea in mind, other variables may be explored in order to obtain a better description in terms of the orbital elements. For those applications not interested in rectilinear orbits, the projective Delaunay is specially well suited. Precisely, this chart is a generalization of the classical Delaunay elements, also dubbed as dynamical orbital elements, which is completely adapted to the (*KS*-energy)-reduction process. That is to say, the starting point is the 4D oscillator expressed in these variables. Then, after time reparametrization and the *KS*-reduction we are led to the Kepler system (plus a perturbation). The reduction process in this environment involves only to constraint some of the momenta to a constant and to get rid of the corresponding cyclic angle-variable. Moreover, the remaining variables compose the corresponding reduced space.

Next, we present an alternative local chart for the (*KS*-energy)-reduced space. Later on, in the following section we also introduce a global set of variables for the (*KS*-energy-axial)-reduced space. In both cases, our variables have a very direct connection with the dynamical orbital elements (Delaunay variables). The variables we are going to introduce for the (*KS*-energy)-reduced space have pros and cons. On the one hand, they provide a complete separation of orbital elements. On the other hand, these variables can only be used for non-rectilinear motions, since they are based in the Delaunay angles. Thus, this type of orbits can only be handled with the invariants σ and δ. The reader should assess the convenience of using this chart for each application. There are other alternative sets of symplectic local coordinates for the (*KS*-energy)-reduced space. For instance, see the one introduced in Iñarrea et al. (2011).

3.1 Local coordinates in the (*KS*-energy)-reduced space: separating node and perigee

A preliminary proposal of the new sorting of the orbits on $\mathbb{S}_L^2 \times \mathbb{S}_L^2$, alternative to the one associated with (σ, δ) coordinates, was already considered by one of the authors (Ferrer 2003). This new sorting allows for a complete separation of orbital elements in the (*KS*-energy)-reduced space. Nevertheless, it is not at no cost, the new variables are only locally defined and three Delaunay-type charts are needed to cover the set of non-collinear orbits. Rectilinear orbits cannot be studied with these variables.

The (*KS*-energy)-reduced space may be 'reordered' taking into account the bounded non-rectilinear Keplerian orbits, $\mathcal{O} = \Delta_c \bigcup \Delta_e$: circular orbits $\Delta_c = \{G = L\}$ and elliptic orbits $\Delta_e = \{G \mid 0 < G < L\}$. Here, we give a full description of a modified version and use it for the explanation of the reduction. The idea is to 'separate' orbital elements in the way we 'locate' the orbits in $\mathbb{S}_L^2 \times \mathbb{S}_L^2 \equiv \mathbb{S}_1^2 \times \mathbb{S}_2^2$. We use the first sphere $\mathbb{S}_1^2$ for the argument of periaxis and eccentricity. Each 'small circle of latitude' of $\mathbb{S}_1^2$ relates to orbits with the same eccentricity. We associate the north pole of $\mathbb{S}_1^2$ with the circular orbits, and the south pole with the rectilinear. The second sphere $\mathbb{S}_2^2$ gives the positions of the orbital plane. Each 'small circle of latitude' $\mathbb{S}_2^2$ corresponds to orbits with the same inclination; north and south poles of $\mathbb{S}_2^2$ relate to direct and retrograde equatorial orbits. Coordinates are denoted by $\mathbb{S}_1^2 = \{(\lambda_1, \lambda_2, \lambda_3) \mid \sum \lambda_i^2 = \frac{1}{4}L^2\}$ and $\mathbb{S}_2^2 = \{(\zeta_1, \zeta_2, \zeta_3) \mid \sum \zeta_i^2 = \frac{1}{4}L^2\}$. Following the same notation as before, by n_i and s_i we refer to the 'north' and 'south' poles of $\mathbb{S}_i^2$, respectively.

More precisely, the sorting is as follows:

(i) Δ_c = circular orbits $(G = L) : n_1 \times \mathbb{S}_2^2 - (n_2, s_2)$. The point (n_1, n_2) corresponds to the direct equatorial circular orbit, and (n_1, s_2) is the retrograde equatorial circular orbit,

which are not covered by this chart. Moreover, the set $(0, 0, \frac{1}{2}L, \zeta_1, \zeta_2, \zeta_3)$, $\zeta_3 \neq \pm\frac{1}{2}L$ are circular orbits, with the inclination $\cos I = H/L$. Thus, we obtain the argument of the node and inclination (h, I) by inverting the following expressions

$$\zeta_1 = \frac{1}{2}L \sin I \cos h, \qquad \zeta_2 = \frac{1}{2}L \sin I \sin h, \qquad \zeta_3 = \frac{1}{2}H, \tag{24}$$

where $\cos I = H/G$ and we may also write $L = \sqrt{\kappa a}$ and $\zeta_3 = \sqrt{\kappa a(1 - e^2)} \cos I$.
Finally, the set $\frac{1}{2}L(0, 0, 1, \cos h, \sin h, 0)$ are circular polar orbits, with $0 < h \leq 2\pi$.

(ii) Δ_e = elliptic orbits $(0 < G < L) : (\mathbb{S}_1^2 - \{n_1, s_1\}) \times \mathbb{S}_2^2$.
'Non-equatorial ellipses'. We make correspond the elliptic inclined (non-equatorial) orbits with the sextuples $(\mathbb{S}_1^2 - \{n_1, s_1\}) \times (\mathbb{S}_2^2 - \{n_2, s_2\})$. We establish the correspondence between the Delaunay set of variables (G, H, g, h), and our coordinates on $\mathbb{S}_1^2 \times \mathbb{S}_2^2$ as follows

$$\begin{aligned}
\lambda_1 &= \frac{G}{L}\sqrt{L^2 - G^2} \cos g, \quad \zeta_1 = \frac{1}{2}\frac{L}{G}\sqrt{G^2 - H^2} \cos h, \\
\lambda_2 &= \frac{G}{L}\sqrt{L^2 - G^2} \sin g, \quad \zeta_2 = \frac{1}{2}\frac{L}{G}\sqrt{G^2 - H^2} \sin h, \\
\lambda_3 &= \frac{1}{L}(G^2 - \frac{1}{2}L^2), \qquad \zeta_3 = \frac{1}{2}\frac{L}{G}H.
\end{aligned} \tag{25}$$

We could give to the previous variables a slight different and compact form, which could be preferred by the readers familiarized with astrodynamics

$$\begin{aligned}
\lambda_1 &= Le\sqrt{1 - e^2} \cos g, \quad \zeta_1 = \frac{1}{2}L \sin I \cos h, \\
\lambda_2 &= Le\sqrt{1 - e^2} \sin g, \quad \zeta_2 = \frac{1}{2}L \sin I \sin h, \\
\lambda_3 &= L(\frac{1}{2} - e^2), \qquad \zeta_3 = \frac{1}{2}L \cos I.
\end{aligned} \tag{26}$$

Observe that, taking into account that $L^2 = \kappa a$, the expressions of the new variables given above are explicitly given in terms of the orbital elements. Conversely, Delaunay variables (g, h, G, H) as functions of $\boldsymbol{\zeta}$ and $\boldsymbol{\lambda}$ are given by

$$\begin{aligned}
G &= \sqrt{L(\lambda_3 + \frac{L}{2})} \quad \cos g = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad \sin g = \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \\
H &= 2\zeta_3\sqrt{\frac{\lambda_3}{L} + \frac{1}{2}} \quad \cos h = \frac{\zeta_1}{\sqrt{\zeta_1^2 + \zeta_2^2}}, \quad \sin h = \frac{\zeta_2}{\sqrt{\zeta_1^2 + \zeta_2^2}}.
\end{aligned} \tag{27}$$

4 (*KS*-energy-axial)-reduced space: Cushman–Deprit coordinates

Now, we deal with the singular reduction in the node. We introduce a new set of variables for the (*KS*-Energy-axial)-reduced space, which are an alternative to the variables proposed by Cushman (1983, 1992). In our opinion, this paper contributes to complement the work done by Coffey et al. (see Sects. 2 and 3 in Coffey et al. (1986)) in their effort to bring near to the astrodynamics quarters the results on reductions done by geometers. More precisely, they left uncompleted the set of coordinates for the thrice reduced space. They proposed a set of coordinates ξ_i on $\mathbb{S}^2$, diffeomorphic to the thrice reduced space (for $H \neq 0$), as an alternative to the set π_i already chosen by Cushman (1983) based on invariant algebra.

However, the singular case $H = 0$ was not covered by the variables given in Coffey et al. (1986). Here, we consider a set of variables $(\gamma_1, \gamma_2, \gamma_3)$ based on the ones provided in Coffey et al. (1986), but being valid also for the singular case $H = 0$. Additionally, all these variables have the same dimensions, something which was not a big concern for Cushman. In our case, from the experience with normalization procedures, we prefer variables with the same dimensions. More precisely, we choose as third variable γ_3 the invariant G^2. Thus, each point in the horizontal sections of the thrice reduced space $\Gamma_{0,L,H}$, defined by $\gamma_3 = $ constant, corresponds with a family of ellipses with the same eccentricity and the same perigee, which ends in a segment when $H = 0$, where each point corresponds to a family of rectilinear solutions. This helps to see what is at the core of the third reduction process.

When Cushman carried out the reduction process he looked for a model of the (*KS*-energy-axial)-reduced space. However, there are other possibilities, from Cushman (see Cushman 1983, p. 409) we quote: "*We now carry out the reduction process on the S^1 action Ψ. To find a model for the reduced phase space…*" This section contains three sets of variables by which we can describe the (*KS*-energy-axial)-reduced space. The first set of variables given in Cushman (1983) is focussed in the mathematical treatment of the problem, while the second one (Coffey et al. 1986) concentrates on the astrodynamical description. Finally, we consider a set of coordinates that combines both approaches.

4.1 Variables based on invariant theory: Cushman's choice

Cushman built his model based on the generators of the invariant algebra associated with the action defined by the axial symmetry induced by H. These generators are given as function of the components of the vectors (σ, δ) given in (22)

$$
\begin{aligned}
\tau_1 &= \sigma_1^2 + \sigma_2^2, \quad \tau_2 = \sigma_1\delta_2 - \sigma_2\delta_1, \quad \tau_3 = \sigma_3, \\
\tau_4 &= \delta_1^2 + \delta_2^2, \quad \tau_5 = \sigma_1\delta_1 + \sigma_2\delta_2, \quad \tau_6 = \delta_3
\end{aligned}
\tag{28}
$$

together with the constraints

$$
\tau_1 + \tau_3^2 = \frac{L^2}{4}, \quad \tau_4 + \tau_6^2 = \frac{L^2}{4}, \quad \tau_2^2 + \tau_5^2 = \tau_1\tau_4, \quad \tau_3 + \tau_6 = H,
\tag{29}
$$

where $H = G_3$. The above relations allow to express the (*KS*-energy-axial)-reduced space by means of the following π_i invariants

$$
\pi_1 = \frac{1}{2}(\tau_3 - \tau_6), \quad \pi_2 = \tau_2, \quad \pi_3 = \tau_5.
\tag{30}
$$

Note: The reader should be aware that Cushman along his papers used different notations for the invariants: σ_i, π_i, etc.

The surface of the (*KS*-energy-axial)-reduced space takes the form

$$
\Pi_{L,H} = \{(\pi_1, \pi_2, \pi_3) \,|\, \pi_2^2 + \pi_3^2 = (L^2 - (\pi_1 - H)^2)(L^2 - (\pi_1 + H)^2)\},
\tag{31}
$$

together with

$$
|\pi_1 - H| \leq L, \quad |\pi_1 + H| \leq L.
\tag{32}
$$

Notice that Cushman changes slightly this definition in Cushman (1992). Figure 1 shows $\Pi_{L,H}$ for several values of H.

As we said before, there is no claim that the coordinates chosen by Cushman are the only ones or even the more convenient. In fact, his choice is dictated by the algebraic process
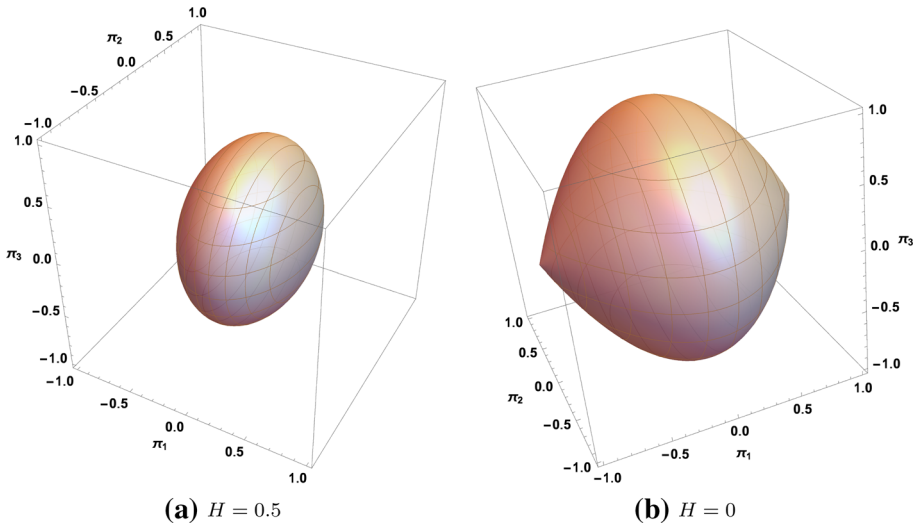
Fig. 1 Cushman's reduced space $\Pi_{L,H} \equiv V_{n\,0,\varsigma}$ for two values of H. The singular case $H = 0$ has two singular points corresponding to $\pi_1 = \pm L$ and $\pi_2 = \pi_3 = 0$

followed, taking the polynomials of lower degree, although they are of different physical dimensions. These variables may be expressed in Delaunay coordinates in the open domain where they are defined in the following way

$$
\begin{aligned}
\pi_1 &= L\,e\,s_I\,\sin g, \\
\pi_2 &= -2LG\,e\,s_I\,\cos g, \\
\pi_3 &= 2G^2 - L^2 - H^2 + L^2 e^2 s_I^2 \sin^2 g,
\end{aligned}
\tag{33}
$$

where $s_x = \sin x$ and $c_x = \cos x$.

4.2 Coffey–Deprit–Miller's variables for the non-singular case

A set of coordinates for the (*KS*-energy-axial)-reduced space, alternative to Cushman's, were proposed by Deprit and his collaborators (Coffey et al. 1986), (from now on CDM). Referring for its explanation to that paper, they are given by

$$
\begin{aligned}
\xi_1 &= LG\,e\,s_I\,\cos g, \\
\xi_2 &= LG\,e\,s_I\,\sin g, \\
\xi_3 &= G^2 - (L^2 + H^2)/2
\end{aligned}
\tag{34}
$$

which are of the same dimensions. They satisfy $\xi_1^2 + \xi_2^2 + \xi_3^2 = (L^2 - H^2)^2/4$, i.e., the reduced surface is a sphere. We may recover them from Cushman's variables by:

$$
\xi_1 = -\pi_2/2, \quad \xi_2 = \pi_1\sqrt{(\pi_3 - \pi_1^2 + L^2 + H^2)/2}, \quad \xi_3 = (\pi_3 - \pi_1^2)/2.
\tag{35}
$$

Although these coordinates are very well suited for astrodynamical applications, they have one major drawback: CDM variables cannot be used for the singular case. That is to say, for $H \neq 0$ the (*KS*-energy-axial)-reduced space is diffeomorphic to a sphere, which matches the geometry of (ξ_1, ξ_2, ξ_3). However, when $H = 0$, the reduced space is no longer

diffeomorphic, but homeomorphic to a sphere since two singular points arise. This case cannot be analyzed with CDM variables, which continue to preserve the spheric shape.

4.3 Modulus of the angular momentum as a variable: Cushman–Deprit coordinates

Here, we present an alternative model for the (KS-energy-axial)-reduced space. Namely, we introduce variables $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ defined as

$$
\begin{aligned}
\gamma_1 &= L^2 e\, s_I \sin g, \\
\gamma_2 &= LG\, e\, s_I \cos g, \\
\gamma_3 &= G^2 + c,
\end{aligned}
\tag{36}
$$

where $c \in \mathbb{R}$ is a constant that may be chosen in accordance with the problem at hand, keeping in mind that units are the same for each γ_i. For example, by choosing $c = -(L^2 + H^2)/2$ we get $\gamma_3 = \xi_3$ given in equation (34). The inverse transformation of (36) is written immediately

$$
\sin g = \frac{\gamma_1}{L^2 e s_I}, \quad \cos g = \frac{\gamma_2}{LG\, e s_I}, \quad G = \sqrt{\gamma_3 - c}
\tag{37}
$$

where

$$
e = \sqrt{1 - (\gamma_3 - c)/L^2}, \qquad s_I = \sin I = \sqrt{1 - H^2/(\gamma_3 - c)}.
\tag{38}
$$

However, this set of variables is not restricted to the Delaunay chart. Indeed, there is a diffeomorphism connecting $(\gamma_1, \gamma_2, \gamma_3)$ with Cushman's variables (Cushman 1983)

$$
\pi_1 = \frac{1}{L}\gamma_1, \quad \pi_2 = -2\gamma_2, \quad \pi_3 = 2(\gamma_3 - c) + \frac{1}{L^2}\gamma_1^2 - L^2 - H^2.
\tag{39}
$$

Note this alternative set of coordinates shares some features with Cushman's variables and with CDM variables and for that reason we dub variables (36) as Cushman–Deprit's coordinates. Namely, our alternative may be used for any value of H. Additionally, these variables are all with the same dimensions, which was not a concern for Cushman. In our case, from the experience with normalization procedures, we prefer variables with the same dimensions. Moreover, γ_3 is directly related to the eccentricity. Thus, with these invariants, a fixed value of γ_3 corresponds to a family of orbits with the same eccentricity.

In what follows, we will consider the case $c = 0$. By means of this set of invariants, the (KS-energy-axial)-reduced space is expressed by the following semialgebraic variety

$$
\Gamma_{0,L,H} = \{(\gamma_1, \gamma_2, \gamma_3) \,|\, f(\gamma) = 0, \quad 0 \le H^2 \le \gamma_3 \le L^2\},
\tag{40}
$$

where

$$
f(\gamma) = \gamma_1^2 \gamma_3 + L^2 \gamma_2^2 - L^2(L^2 - \gamma_3)(\gamma_3 - H^2).
\tag{41}
$$

This function defining the (KS-energy-axial)-reduced space is named in the context of symplectic and Poisson reduction as Casimir function. The singular case is characterized by

$$
\Gamma_{0,L,0} = \{(\gamma_1, \gamma_2, \gamma_3) \,|\, f_0(\gamma) = 0, \quad 0 = H \le \gamma_3 \le L^2\}
\tag{42}
$$

with

$$
f_0(\gamma) = \gamma_1^2 \gamma_3 + L^2 \gamma_2^2 + L^2 \gamma_3^2 - L^4 \gamma_3.
\tag{43}
$$

Since the relations (39) define a diffeomorphism in $\mathbb{R}^3$, we can assure that $\Gamma_{0,L,H}$ is diffeomorphic to a sphere for $H \neq 0$ and for the case $H = 0$ two singular points arise in $\Gamma_{0,L,0}$, which are located at $(\pm L^2, 0, 0)$, as the following proposition shows.

Proposition 1 *The algebraic variety defined by $\Gamma_{0,L,H}$ is a smooth 2D surface for $H \neq 0$. For the case $H = 0$, two singular points arise in $\Gamma_{0,L,0}$ at $(\pm L^2, 0, 0)$.*

Proof The point $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of $\Gamma_{0,L,H}$ is singular if satisfies

$$f(\gamma) = 0, \quad \mathrm{d} f(\gamma) = (0, 0, 0), \qquad 0 \leq H^2 \leq \gamma_3 \leq L^2. \tag{44}$$

Therefore, after the computation of the partial derivatives, we have the following polynomial system

$$2\gamma_1 \gamma_3 = 0, \quad 2L^2 \gamma_2 = 0, \quad \gamma_1^2 + 2L^2 \gamma_3 - L^2(L^2 - H^2) = 0. \tag{45}$$

These equations imply $\gamma_2 = 0$ and either $\gamma_1 = 0$ or $\gamma_3 = 0$. The case $\gamma_1 = 0$ and $\gamma_2 = 0$, together with the third equation, leads to $\gamma_3 = L^2 - H^2/2$, which does not satisfy condition $f(\gamma) = 0$. Then, the only remaining case is $\gamma_2 = \gamma_3 = 0$, which implies that $H = 0$ from (44). Hence, using again the third equation in (45) we have that $\gamma_1 = \pm L^2$. Therefore, the points $(\pm L^2, 0, 0)$ are singular.   □

Recently, a full study of the reduced stellar three body problem has been carried out (see Palacián et al. 2013) where the set of coordinates used (formulas (5.1), (5.2) p. 162) for the reduced space are similar to Eq. (36). In our view, the reason of that similarity, versus the ones proposed by Cushman, is due to the fact that they have chosen $\tau_1 = 2G_1^2 - \Xi_1^2$, i.e., one of the invariants is quadratic in the angular momentum, like our $\gamma_3 = G^2$.

Remark 3 For the applications, it is usually convenient to set $c = 0$ and consider dimensionless variables by dividing out L as follows

$$\gamma_1^* = \frac{\gamma_1}{L^2} = es_I \sin g, \quad \gamma_2^* = \frac{\gamma_2}{L^2} = \eta es_I \cos g, \quad \gamma_3^* = \frac{\gamma_3}{L^2} = \eta^2. \tag{46}$$

5 On the astrodynamical interpretation of $(\gamma_1, \gamma_2, \gamma_3)$

In this section, we answer the question of what families of Kepler orbits do we have associated with each point of the surface (40) defining the reduced space, see Fig. 2. In other words, we may see the (*KS*-energy-axial)-reduced space as sets of orbits and each point in the surface (40) as a label for each set. In more detail, we have that for $H \neq 0$ the surface is diffeomorphic to the sphere, but its horizontal sections have an ellipse shape, see Fig. 2a. Those parallels ellipses reduce to a point at the top $(0, 0, L^2)$ (north pole of the reduced space) when $G = L$, which corresponds to the circular orbits with inclination $\cos I = H/L$. Again, the parallels reduce also to a point at the bottom $(0, 0, H^2)$ (south pole of the reduced space), when $G = |H|$, and it corresponds to the family of equatorial elliptic orbits with eccentricity $e = \sqrt{1 - H^2/L^2}$. When the integral $H = 0$, the surface contains two singular points $(\pm L^2, 0, 0)$, and it is not diffeomorphic but homeomorphic to a sphere, see Fig. 2b. Indeed, on the surface $\Gamma_{0,L,0}$ the point $(0, 0, L^2)$ corresponds to $G = L$. Nevertheless, when $G = 0$, we are at the bottom of $\Gamma_{0,L,0}$ which no longer is a point but a segment. The extremes of the segment are the points of the surface $(-L^2, 0, 0)$ and $(L^2, 0, 0)$ which are singular. Those points represent two rectilinear orbits in the two directions of the x_3 axis. Each of the

(a) $H = 0.5$ **(b)** $H = 0$
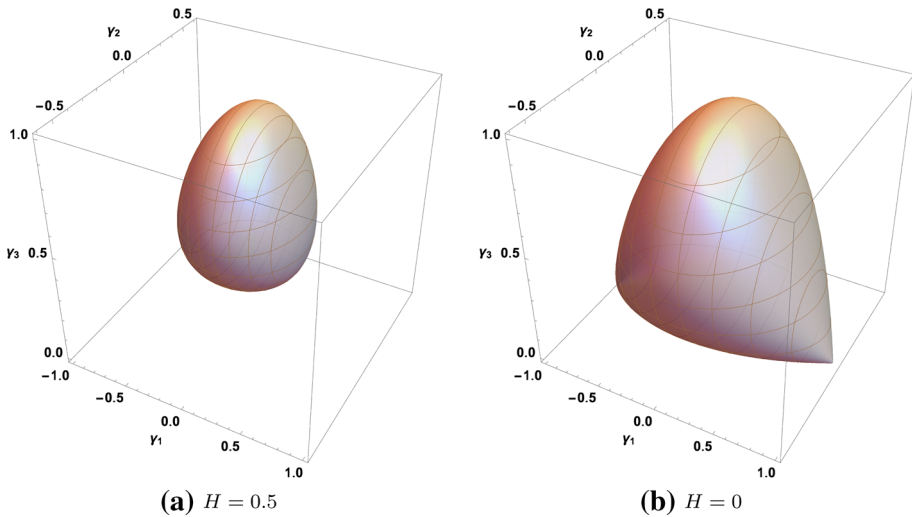
Fig. 2 (KS-energy-axial)-reduced space $\Gamma_{0,L,H}$ for two values of H. The singular case $H = 0$ has two singular points corresponding to $\gamma_1 = \pm L$ and $\gamma_2 = \gamma_3 = 0$

rest of the points of this segment, i.e., $(L^2 \sin g_0, 0, 0)$, $g_0 \in (-\pi/2, \pi/2)$ they represent families of rectilinear orbits in $s_1 \times \mathbb{S}_L^2$ (see Sect. 3.1) with the same latitude of the Laplace vector (eccentricity vector).

5.1 On the (KS-energy-axial)-reduced space and orbital elements: What about the inclination?

Referring to the (KS-energy-axial)-reduced space in astrodynamics terms, whichever set of coordinate we have chosen, we may say that we have get rid of the node angle, which may be recovered (in the reconstruction process) after we have studied the dynamics in the reduced system.

Indeed, we know the expression relating the integral H with the orbital elements

$$H = \cos I \sqrt{\kappa a (1 - e^2)}. \tag{47}$$

Thus, when we fix the value of $H = h$, the eccentricity and inclination could change according to Eq. (47). In particular, we may express the inclination as a function of the eccentricity, parametrized by L and H. Denoting $\sigma = H/L$, we have

$$I_\sigma = \arccos \frac{\sigma}{\sqrt{1 - e^2}}. \tag{48}$$

In Fig. 3, it is shown the evolution of those curves parametrized by σ. It represents a complement to Fig. 2 of the (KS-energy-axial)-reduced space $\Gamma_{0,L,H}$, where the variables (g, G) are portrayed, i.e., eccentricity and argument of perigee, but not the inclination of the orbit.

Fig. 3 Relation between e and I for families of orbits after the third reduction. Those curves are $I = I(e; L, H)$, i.e., the inclination as function of the eccentricity, parametrized by the integrals (L, H). A complement to this is Fig. 2 of the $(KS$-energy-axial)-reduced space $\Gamma_{0,L,H}$, where the variables (g, G) are portrayed, i.e., eccentricity and argument of perigee, but not the inclination of the orbit
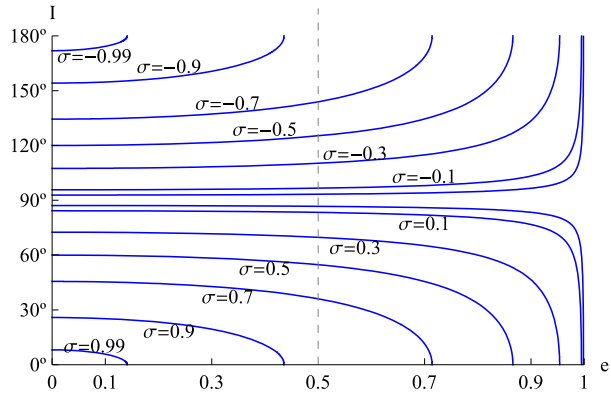
6 Application: the Hill problem

The Hill problem was originally conceived to develop a theory for the lunar motion as a first approximation of the circular restricted three body problem (CRTBP). Roughly speaking, it considers the evolution of a body with negligible mass (the Moon) attracted by two primaries (the Sun and the Earth). However, after proper scaling, this model is suitable for the description of the dynamics of a variety of systems. Hill system provided a very successful analytic theory for the lunar motion, which even today receives attention, see for example Meyer et al. (2015), where the authors determine the existence of invariant 3-tori by studying the full averaged system. Also, in Lara et al. (2010); Lara and Palacián (2009) higher order normalization through Lie transformations is performed and dynamics of the resulting system are analyzed for the case in which the scaling is adapted for an Enceladus orbiter.

In our application, we do not fix a particular scaling setting, but we consider the general model to illustrate the regularization process and usage of Cushman–Deprit variables. Thus, we study a system with two finite masses M and m, to which we refer as the primaries, moving in a Keplerian circular orbit, plus an infinitesimal body under the influence of the primaries. The origin is fixed at m, and we consider a uniformly rotating frame at the rate of the m-primary about the mass M in such a way that the positive x-axis points to the M-primary. Moreover, let R and r be the distances from the origin to the M and m primaries, respectively, the validity of the Hill model is restricted to the assumption that the third body is close to the smaller mass m. That is to say, it is assumed that the ratio $\rho = r/R$ is small and the first-order truncation of the CRTBP could be accurate enough for an analytical theory. After this process is carried out, we arrive at the Hamiltonian defining the Hill's equations, which is taken from Lara et al. (2010)

$$\mathcal{K} = \frac{1}{2}|\mathbf{y}|^2 - \frac{\kappa}{|\mathbf{x}|} - N(x_1 y_2 - x_2 y_1) + \frac{N^2}{2}(-2x_1^2 + x_2^2 + x_3^2), \qquad (49)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are the position and conjugate momenta vectors, κ is the gravitational parameter of the central body and N is the rotation rate of the system.

6.1 Regularization of the spatial lunar problem

Next, we introduce a perturbed 4D oscillator as it is instructed in Sect. 2

$$\tilde{\mathcal{H}}_\omega = \mathcal{H}_\omega - 4\,N\mathcal{P}_a + 2\,N^2\mathcal{P}_b, \tag{50}$$

defined in $T^*\mathbb{R}_0^4 = T^*(\mathbb{R}^4 - \{0\})$, where $\mathcal{H}_\omega$ is the 4D oscillator (7) and $\mathcal{P}_a$ and $\mathcal{P}_b$ are given by

$$\begin{aligned}
\mathcal{P}_a &= 2(q_2q_4 - q_1q_3)(p_2q_1 + p_1q_2 + p_4q_3 + p_3q_4) \\
&\quad - 2(q_1q_2 + q_3q_4)(p_4q_2 + p_2q_4 - p_1q_3 - p_3q_1), \\
\mathcal{P}_b &= |\mathbf{q}|^2(-12(q_2q_4 - q_1q_3)^2 + |\mathbf{q}|^4).
\end{aligned} \tag{51}$$

In what follows, we fix the energy level $\tilde{\mathcal{H}}_\omega = h > 0$. Note that the above Hamiltonian is connected with (49) through the *KS*-reduction (see Ferrer and Crespo 2018; Crespo and Ferrer 2020). That is to say, the Hamiltonian function (50) is endowed with the axial symmetry given by the polynomial

$$\Xi_0 = \frac{1}{2}\big((p_1q_4 - p_4q_1) - (p_3q_2 - p_2q_3)\big).$$

Thus, in the Ξ_0-reduced space, the regularized Lunar Hamiltonian may be expressed in terms of the invariants associated to Ξ_0. This process usually relies on invariant theory, leading to a high number of invariants. However, in Ferrer and Crespo (2018) we showed that the *KS*-map (4) provides an orbit map for this reduction and allows to express (50) in the Ξ_0-reduced space for each fixed value of $\Xi_0 = \xi$ as follows

$$\begin{aligned}
H_{Aux} = \frac{1}{2}&\left(4|\mathbf{x}|\,|\mathbf{y}|^2 + \omega^2|\mathbf{x}| + \frac{\xi^2}{|\mathbf{x}|}\right) - h \\
&- 4N|\mathbf{x}|\left((x_1y_2 - x_2y_1) - \xi\frac{x_3}{|\mathbf{x}|}\right) \\
&+ 2N^2|\mathbf{x}|(-2x_1^2 + x_2^2 + x_3^2).
\end{aligned} \tag{52}$$

Restricting to the case $\xi = 0$, it is clear that

$$H_{Aux} = 4|\mathbf{x}|\mathcal{K},$$

which implies that after time regularization given by $ds = 4|\mathbf{x}|dt$, we obtain the spatial Lunar system. That is to say, system (49) is the result of fixing the energy level $\tilde{\mathcal{H}}_\omega = h > 0$, dividing out the Ξ_0 symmetry in (50), restricting to the stratum $\xi = 0$ and a time regularization.

6.2 (*KS*-energy-axial)-reduced flow: relative equilibria and bifurcation

In this section, we carry out the elimination of the node and Keplerian energy. For this task, projective Delaunay variables are better suited. In these variables, the Hamiltonian function defining the lunar problem system Eq. (49) takes the form

$$\mathcal{K} : \mathcal{M} \longrightarrow \mathbb{R}; \quad \mathcal{K} = \mathcal{K}_\kappa + \mathcal{P}(l, g, h, L, G, H), \tag{53}$$

where $\mathcal{M} = \mathbb{T}^3 \times \Omega$, being $\Omega = \{(L, G, H)/ \mid H \mid < G < L,\ G \neq 0\}$. Note that, in the Delaunay chart, $G \neq 0$ excludes the rectilinear case. However, the set of variables we have introduced for the description of the (*KS*-energy-axial)-reduced space includes the treatment of rectilinear orbits, which correspond to $\gamma_3 = 0$.

Now, we introduce the formal LH-symmetry by means of Lie transforms in order to fully reduce the system to 1-DOF. This process gives us the following Hamiltonian

$$\mathcal{K}_m = -\frac{\kappa^2}{2L^2} + \sum_{i=1}^{m} \varepsilon^i \, \mathcal{P}_i(\_, g, \_, L, G, H). \tag{54}$$

Precisely, for the case $m = 2$ we take from San Juan and Lara (2006) the normalized Hamiltonian

$$\begin{aligned}
\mathcal{K}_2 = &-\frac{\kappa^2}{2L^2} - \varepsilon \frac{\kappa}{a} \eta \, c_I \\
&- \varepsilon^2 \frac{\kappa}{16L^2} \left(15e^2 s_I^2 \cos 2g + \left(3e^2 + 2\right)\left(2 - 3s_I^2\right)\right),
\end{aligned} \tag{55}$$

where $L^2 = \kappa a$ and using Hill units we set $\kappa = 1$, $\varepsilon = \frac{N}{n} = a^{3/2} = L^3$ and $n = \sqrt{\frac{\kappa}{a^3}}$ the mean motion of the small body. Note that the same symbols are used to refer to the new variables after the normalization to simplify notation. However, the context made clear what variables are we dealing with in each case. Thus, after some easy algebraic manipulations, eliminating constant terms and scaling the Hamiltonian, we obtain the following function in the Cushman–Deprit variables (36) that defines the (KS-energy-axial)-reduced space

$$\mathcal{K}_2 = 5\gamma_1^2 + 2\gamma_3. \tag{56}$$

Note that we are using the agreement given in Remark 3. This Hamiltonian defines a parabolic cylinder. In Fig. 4, we show the intersection of the (KS-energy-axial)-reduced space with the Hamiltonian. The associated equations of motion in the (KS-energy-axial)-reduced space read as follows

$$(\dot\gamma_1, \dot\gamma_2, \dot\gamma_3) = \nabla\mathcal{K}_2 \times \nabla f(\gamma), \tag{57}$$

which, after a convenient time reparametrization, leads to

$$\begin{aligned}
\dot\gamma_1 &= 2\gamma_2, \\
\dot\gamma_2 &= \gamma_1 \left(5\gamma_1^2 + 8\gamma_3 - 5(1 + \sigma^2)\right), \\
\dot\gamma_3 &= -10\,\gamma_1\gamma_2,
\end{aligned} \tag{58}$$

where, as it was defined before, $\sigma = H/L$. Next, we shall present the relative equilibria organized in two families of equilibria.

– *Rectilinear orbits:* We search first for possible rectilinear solutions. They satisfy $\gamma_3 = 0$. Moreover, from the third equation in (56) we identify two possibilities: (i) $\gamma_1 = \gamma_2 = 0$ which are rectilinear equatorial orbits; (ii) $\gamma_3 = \gamma_2 = 0$ and $\gamma_1 = \pm 1$, i.e., singular rectilinear polar orbits.

– *Polar equilibria:* These are equilibria located at the γ_3 axis also dubbed as North-South equilibria. They always exist for any value of the ratio $\sigma = H/L$. Assuming $\gamma_1 = \gamma_2 = 0$, we obtain the equilibria $E_1 = (0, 0, L^2)$ and $E_2 = (0, 0, H^2)$ in the γ_3 axis. These points are located at the top and bottom of the reduced space and for this reason are dubbed as north and south pole of the reduced space. Since $G = L$ for E_1, this equilibrium corresponds to a family of circular orbits with inclination given by $\cos I = \sigma$. For E_2, we have $G = H$, which leads to the family of elliptic equatorial orbits with eccentricity $e = \sqrt{1 - \sigma^2}$. Note that for $G = H = 0$, E_2 correspond to the family of equatorial rectilinear orbits, which we have previously identified.

(a) $H = 0.6$ **(b)** $H = 0.35$

(c) $H = 0.1$ **(d)** $H = 0$
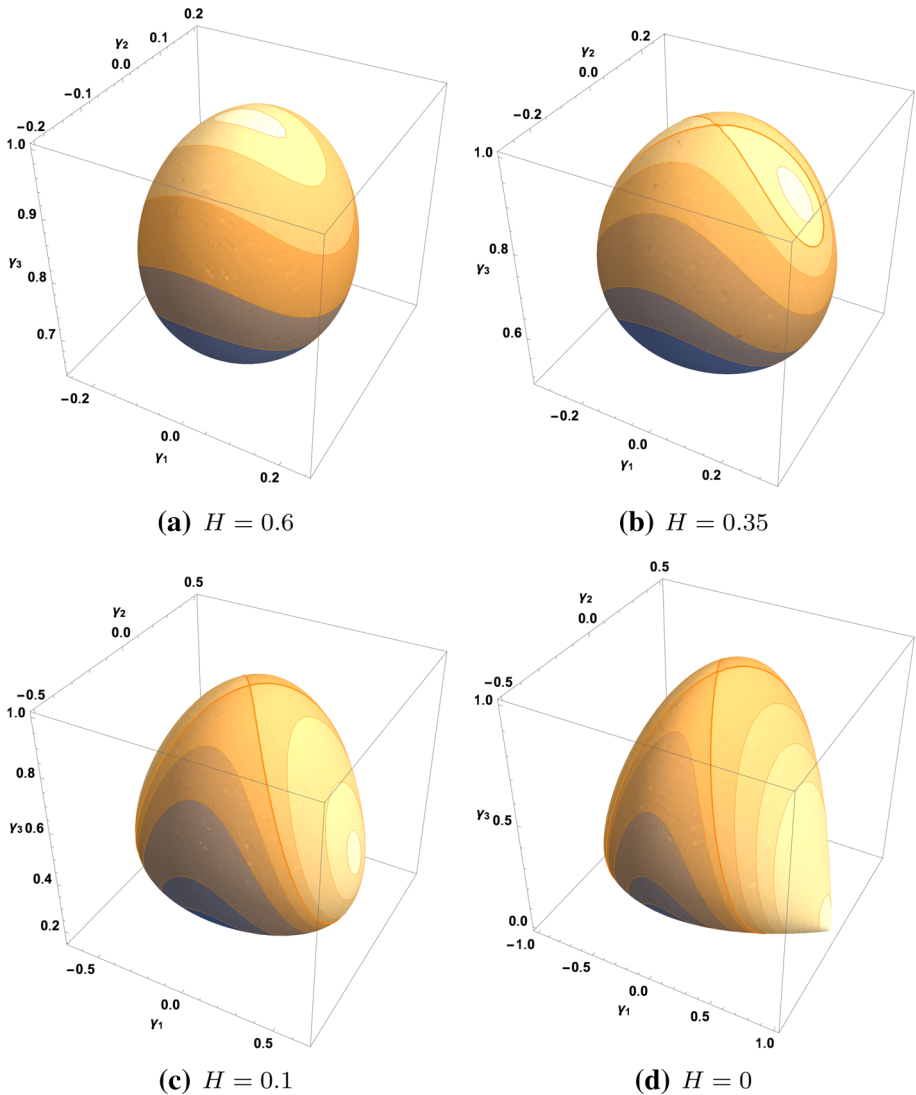
Fig. 4 The reduced flow is given by the level curves result of the intersection of the reduced space $\Gamma_{0,L,H}$ with the Hamiltonian surface given by constant values of $\mathcal{K}_2$. Negative values of H are analogous. Notice that the reduced space for different values of H has been rescaled to the same size

– *Equilibria in the $\gamma_1\gamma_3$-plane and bifurcation at the north pole:* A pitchfork bifurcation occurs at the north pole for $\sigma_0 = \pm\sqrt{3/5}$, which in astronomical terms corresponds to the "critical inclination"

$$\cos I = \pm\sqrt{\frac{3}{5}}. \tag{59}$$

Then, for $\sigma \in (-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}})$ we have the equilibria $E_3 = (\gamma_1, \gamma_2, \gamma_3)$ and $E_4 = (-\gamma_1, \gamma_2, \gamma_3)$, where

$$(\gamma_1, \gamma_2, \gamma_3) = L^2 \left(\sqrt{1 + |\sigma|\left(|\sigma| - \frac{8}{\sqrt{15}}\right)}, \ 0, \ \sqrt{\frac{5}{3}}|\sigma| \right). \tag{60}$$

That is to say, the equilibria $(0, 0, L^2)$, corresponding with circular orbits with inclination $\cos I = \sigma$, bifurcates into elliptic orbits. Moreover, as H decreases the eccentricity increases and for $H = 0$ these equilibria are located in the singular points $(\pm L^2, 0, 0)$, which correspond to polar rectilinear orbits, see Fig. 4, where we plot the intersection of the reduced space with the level surfaces defined by the Hamiltonian Eq. (56) for several values of the integral H.

One may readily get the astrodynamical meaning to the relative equilibria taking into account Eq. (38). Indeed, making use of them, we have that when $\sigma_0 < |\sigma| < L$ for any initial conditions the families of normalized ellipses undergo circulation of the perigee meanwhile the eccentricity oscillates. This regime changes when $|\sigma| < \sigma_0$ because then, the separatrix defines two regions. The one related to the lower part of the reduced space continues to have a circulation pattern for the perigee. Nevertheless, associated with the two relative equilibria bifurcating from the 'north pole' of the reduced space, we have two regions where perigee and eccentricity are librating. These features may be seen in Fig. 4a–d. We also like to mention that the equatorial rectilinear solutions found in this analysis require further investigation.

7 Conclusions and future work

This work presents an alternative to the usual itinerary in the regularization process of perturbed Keplerian systems. Our approach relies on the reduction of symmetries on perturbed 4D oscillators. Precisely, the *KS* map is just the orbit map associated with an axial symmetry of the oscillator and allows to identify perturbed Keplerian systems as subsystems in 4D perturbed oscillators. Moreover, this methodology avoids the use of additional techniques as constraint dynamics (Cushman 1992), which is needed when the regularization is made through Moser procedure (Moser 1970). However, another way to avoid the use of constraint dynamics in the setting of Moser regularization is given in Meyer et al. (2018), where the authors work directly with the invariants of the Kepler reduction plus an extra angle, which when n = 2, 3, is essentially the eccentric anomaly. Alternatively, we provide several symplectic charts allowing to track the regularization process in a straightforward way. Additionally, once the regularization is carried out, the remaining system is normalized and the Keplerian energy and the node are eliminated in consecutive stages. This work presents an astrodynamical interpretation of each stage, and alternative variables are presented for the full reduced space. These variables have the advantage of having the same physical dimensions, which from our experience, is of great practical interest.

The reconstruction of the points in the (*KS*-energy-axial)-reduced space is left for a forthcoming paper. This process can be made in several stages, reverting each of the reductions performed. Thus, a point in the (*KS*-energy-axial)-reduced space, generically corresponds with a S^1 fiber in the (*KS*-energy)-reduced space. However, for the singular case, there are two points (associated with the rectilinear polar orbits), which, each one, correspond with a single point in the (*KS*-energy)-reduced space. Further progress in the reconstruction process

toward the (*KS*)-reduced and full spaces increases the dimension of the fiber to T^2 and T^3, respectively (except for the singular points). Moreover, along with this process, it is necessary to establish the main features (in terms of the existence of periodic solutions, invariant tori of different dimensions, bifurcations, etc.) of the departure system obtained through the reduction. This process is delicate as a truncation of the remainder in the normalization is done; hence, one has to prove the persistence of the solutions for the departure system, see (Reeb 1952; Yanguas et al. 2008).

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