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ORIGINAL ARTICLE

# The Kepler problem: the energy point, the Levi-Civita, the Burdet and the KS regularizations via the primigenial sphere

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**Abstract**

In our unitary description (Vivarelli in Meccanica 50:915–925, 2015) of the Kepler problem (obtained via the introduction of a simple structure, the primigenial sphere $S_{p^{-1}}$), we have shown that this sphere encompasses, in a sort of inbred order of its elements, several fundamental elements of the Kepler problem. In this paper, we show that also the mechanical energy of an elliptic Kepler orbit is an element embedded in the sphere through a peculiar point, the energy point $P^*$. We show that this point in its circular motion on the sphere has a velocity which is strictly linked to so-called Sundman–Levi-Civita regularizing time transformation (Levi-Civita in Opere matematiche, 1973). Moreover in this spherical scenario, we reconsider both the two regularizations of the Kepler problem, namely the Bohlin–Burdet (Burdet in Z Angew Math Phys 18:434–438, 1967) and the Kustaanheimo and Stiefel (KS) regularizations (J Reine Angew Math 218:204–219, 1965): we present a geometrical interpretation of the first one, and we show an explicit link between their regularizing fundamental equations.

**Keywords** Kepler problem · Primigenial sphere · Regularization · Mechanical energy · General mechanics

## 1 Introduction

The well-known classical Kepler problem describes the motion of a particle under an attractive force which decreases with the square of the distance from a fixed centre of attraction, i.e.

$$\mathbf{F} = -\frac{K^2}{r^3}\,\mathbf{x} \quad (r = \mid \mathbf{x} \mid) \tag{1}$$

where $K$ is a dimensional constant and the vector $\mathbf{x}$ represents the particle position vector in the Cartesian frame $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ with the origin at the attractive centre $F$ (focus) (Boccaletti and Pucacco 1996; Celletti 2002).

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The Kepler problem is not regular: when the orbiting particle falls into the centre of attraction ($r = 0$), the differential equation of the Kepler motion

$$\ddot{\mathbf{x}} + \frac{K^2}{r^3}\,\mathbf{x} = \mathbf{0} \quad \left(\dot{\mathbf{x}} = \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}t}\right) \tag{2}$$

becomes singular. This singularity leads to theoretical and practical difficulties (infinite velocities; poor accuracy in computer-aided numerical integration).

Thus, the need of a regularization procedure leads to a set of regular differential equations (Baumgarte 1972b; Boccaletti and Pucacco 1996; Breiter and Langner 2017; Deprit et al. 1994; Ferrer and Crespo 2017; Waldvogel 2006).

We focus our attention on the two celebrated regularizations: the Bohlin–Burdet and the Kustaanheimo–Stiefel (KS) regularizations that can be found, respectively, in Burdet (1967), Kustaanheimo and Stiefel (1965) and Stiefel and Scheifele (1971). [Let us observe that we call Bohlin–Burdet regularization the one commonly denoted in the literature simply as Burdet regularization: in fact Burdet employed a former idea presented by Bohlin (1911); for details, see the end of the first paragraph in Deprit et al. (1994).]

These two regularizations transform the differential singular equation of the Kepler motion (2) into that of a harmonic oscillator (respectively, in a 3-dimensional space and in a 4-dimensional space), and both use the so-called Sundman–Levi-Civita time transformation.

The aim of this paper is to show, first of all, how the particular structure we introduced in our previous work (Vivarelli 2015) (that is, the primigenial sphere $S_{p^{-1}}$ which encompasses several characteristic features of the Kepler problem) embeds also the Kepler total mechanical energy by means of the energy point $P^*$. Consequently, we show that, while a physical Kepler particle describes an elliptic orbit, the point $P^*$ describes a circle on the sphere $S_{p^{-1}}$ with a velocity which is strictly related to the Sundman–Levi-Civita regularizing time transformation.

In the same spherical arena, we recover the Bohlin–Burdet oscillator equation and we present an explicit link between the Bohlin–Burdet and the KS oscillator equations.

## 2 Levi-Civita, Burdet and KS regularizations

Let us recall the first integrals of the Kepler motion (2), namely

the constant angular momentum vector, defined by the product

$$\boldsymbol{\Gamma} = \mathbf{x} \wedge \dot{\mathbf{x}}; \tag{3}$$

the constant scalar total mechanical energy, defined by

$$E = \frac{1}{2}\,|\,\dot{\mathbf{x}}\,|^{\,2} - \frac{K^2}{r}; \tag{4}$$

the constant Laplace–Runge–Lenz vector, defined by

$$\mathbf{V} = \dot{\mathbf{x}} \wedge \boldsymbol{\Gamma} - K^2\,\frac{\mathbf{x}}{r}. \tag{5}$$

We recall that the first invariant vector $\boldsymbol{\Gamma} = \Gamma\,\mathbf{K}$ lies in the $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$-space: localized at $F$, it fixes the plane of the Kepler motion $\{\mathbf{I}, \mathbf{J}\}$ and its magnitude $\Gamma$ represents Kepler's second law.

The second scalar invariant $E$ ($E \lesseqgtr 0$) is related to the scalar eccentricity of the conic orbit $0 \leqq e \lesseqgtr 1$ by

$$2E = \Gamma^2 p^{-2}(e^2 - 1), \tag{6}$$

where the constant parameter $p > 0$ (semi-latus rectum) is given by

$$p = \Gamma^2 K^{-2}. \tag{7}$$

Both the parameter $p$ and the eccentricity $e$ define the non-degenerate $\mathbf{\Gamma} \neq 0$ vector polar equation of the plane Kepler orbit,

$$\mathbf{x} = \frac{p}{1 + e\ \cos\theta}\ \boldsymbol{\rho} \quad (\mathbf{x} = r\boldsymbol{\rho}), \tag{8}$$

which is a conic section with focus at $F$, so that for $e = 0$ the orbit is a circle, for $e < 1$ an ellipse, for $e = 1$ a parabola and for $e > 1$ an hyperbola.

The third invariant $\mathbf{V}$ lies on the fixed $\{\mathbf{I}\}$-direction: localized at $F$, it is commonly represented as the constant pericentric vector

$$\mathbf{V} = K^2\mathbf{e}, \tag{9}$$

where the vector

$$\mathbf{e} = e\,\boldsymbol{\rho}(0) = e\mathbf{I} \tag{10}$$

is the eccentricity vector along the reference line $\boldsymbol{\rho}(0) = \mathbf{I}$ of the plane polar $(r, \theta)$-coordinate system with pole at $F$: the two orthogonal unit vectors $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\theta)$ of the $\{\mathbf{I}, \mathbf{J}\}$-plane point in the direction of increasing $r$ and $\theta$.

We have already recalled that the Kepler problem (2) is a singular problem that needs regularization. A basic stone, common to the several regularizing procedures devised in the literature in order to transform the singular differential equations of the Kepler motion into regular ones, is provided by the differential Sundman–Levi-Civita regularizing time transformation (briefly Sundman–Levi-Civita time map) or

$$\mathrm{d}s = r^{-1}\mathrm{d}t, \tag{11}$$

which replaces the independent physical time variable $t$ by a new fictitious time variable $s$, related to $t$ by the scalar distance factor $r$ (see Deprit et al. 1994; Levi-Civita 1973).

The derivative with respect to the new time variable $s$ is denoted by the symbol $'$ so that

$$f' = \frac{\mathrm{d}f}{\mathrm{d}s} = r\,\frac{\mathrm{d}f}{\mathrm{d}t} = r\dot{f}. \tag{12}$$

The above time map (11) is used, in particular, by the two well-known regularizations, that is, by the Bohlin–Burdet regularization (Burdet 1967; Deprit et al. 1994) and by the Kustaanheimo–Stiefel (KS) regularization (Kustaanheimo and Stiefel 1965). These regularizations transform the differential singular equation of the Kepler motion (2) into that of a harmonic oscillator (respectively, in a 3-dimensional space and in a 4-dimensional space).

This is done as follows:

Bohlin and Burdet (Burdet 1967; Deprit et al. 1994) first introduce the energy and the Laplace vector integrals into the singular equation of the Kepler motion (2) and then apply the Sundman–Levi-Civita time map (3) and finally obtain the differential equation

$$\mathbf{x}'' + \omega^2\,\mathbf{x} + \mathbf{V} = \mathbf{0}, \tag{13}$$

where $\omega^2 = -2E$ and which for $E < 0$ is the regular equation of a 3-dimensional harmonic oscillator with constant angular frequency $\omega$ and centre at $(2E^{-1})\mathbf{V}$.

Kustaanheimo and Stiefel (Kustaanheimo and Stiefel 1965; Stiefel and Scheifele 1971), after introducing the energy integral into the singular equation of the Kepler motion (2) and applying the Levi-Civita time map (11), introduce the peculiar coordinate transformation (briefly KS map) given by

$$(R^4 - \{0\}) \longrightarrow (R^3 - \{0\}) : \mathbf{u} \to \mathbf{x}, \tag{14}$$

which maps a parametric four-dimensional Euclidean space $R^4$ of real vectors $\mathbf{u} = (u_1, u_2, u_3, u_4)$ onto the ordinary three-dimensional Euclidean space or real vectors $\mathbf{x} = (x_1, x_2, x_3)$.

In a compact matrix form, the KS map is

$$\mathbf{x} = L(\mathbf{u})\mathbf{u}, \tag{15}$$

where the KS matrix $L(\mathbf{u})$ is the real matrix

$$L(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \tag{16}$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 = 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \tag{17}$$

so that, in terms of components, the KS map reads explicitly

$$\begin{cases} x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ x_2 = 2(u_1 u_2 - u_3 u_4) \\ x_3 = 2(u_1 u_3 + u_2 u_4) \end{cases}$$

where automatically $x_4 = 0$ and

$$r = \mid \mathbf{x} \mid = \sqrt{x_1^2 + x_2^2 + x_3^2} = u_1^2 + u_2^2 + u_3^2 + u_4^2 = \mid \mathbf{u} \mid^2.$$

A fundamental role is played by the subsidiary '*bilinear relation*'

$$u_4\, u_1' - u_3\, u_2' + u_2\, u_3' - u_1\, u_4' = 0. \tag{18}$$

As a consequence of the time map (11) and the coordinate KS map (15), the singular Kepler equation (2) is transformed into the regularized differential equation

$$\mathbf{u}'' + \omega^2\, \mathbf{u} = \mathbf{0}, \tag{19}$$

where $\omega^2 = -2E$ and which for $E < 0$ is the regular equation of a 4-dimensional harmonic oscillator with constant angular frequency $\omega$ in the space $R^4$.

## 3 Birth and review of the spherical $S_{p-1}$ description

In our previous work (Vivarelli 2015), we have noticed that the standard force expression (1) or, being $\mathbf{x} = r\boldsymbol{\rho}$,

$$\mathbf{F} = -\frac{K^2}{r^2}\, \boldsymbol{\rho} \tag{20}$$

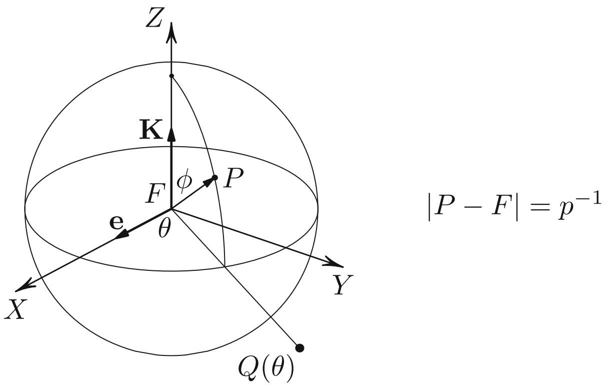



**Fig. 1** The sphere $S_{p^{-1}}$ with radius $p^{-1}$, centre $F$. A point $P(\theta, \phi)$ on the sphere and the orbiting Kepler particle $Q(\theta)$ in the $(X, Y)$-plane. The eccentricity vector $\mathbf{e}$

gives prominence to the inverse-square law character of the central force, but does not explicitly keep track of the planarity of the motion, given by the vector $\mathbf{\Gamma}$.

Therefore, we rewrote the expression (1) as

$$\mathbf{F} = -\frac{\mathbf{\Gamma} \wedge (\mathbf{\Gamma} \wedge p^{-1}\boldsymbol{\rho})}{r^2}, \tag{21}$$

whence we not only found that the new expression keeps track explicitly of the fixed oriented orbital plane by means of the constant vector $\mathbf{\Gamma}$, but also that it embodies the vector $p^{-1}\boldsymbol{\rho}$.

But this vector $p^{-1}\boldsymbol{\rho}(\theta)$ defines a circle of radius $p^{-1}$ in the orbital plane which can be interpreted as the equator of a sphere in the Euclidean space $R^3$ with Cartesian frame $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$.

This is the sphere that we have shown to encompass several characteristic features of the Kepler problem and that we denoted *'primigenial' sphere* $S_{p^{-1}}$.

Referring to Fig. 1, we recall some basic facts (Vivarelli 2015) about the sphere.

**Centre. Radius. Equatorial plane. Eccentricity Vector e**. The sphere has the centre at the attractive focus $F$ of the conic Kepler orbit. The radius $p^{-1}$ encompasses the two Kepler fundamental constant scalars $K$ and $\Gamma$ through the constant ratio

$$p = \frac{\Gamma^2}{K^2}. \tag{22}$$

The equatorial plane $(X, Y)$ of the sphere is the orbital plane of the Kepler problem. The eccentricity vector $\mathbf{e} = e\mathbf{I}$ defined in (10) lies along the $X$-axis.

**Point P of the sphere and point Q of an orbiting Kepler particle**. A point $P(\theta, \phi)$ on the sphere is described by the spherical coordinates (the longitude angle $\theta$ and the colatitude angle $\phi$), the longitude $\theta$ being the same angle of a Kepler orbiting particle $Q(\theta)$.

**Rescaling vector $\boldsymbol{\epsilon}$**. A fundamental primitive element unravelled by the sphere is the rescaling vector $\boldsymbol{\epsilon}$ (Fig. 2). Defined as the vector

$$\boldsymbol{\epsilon} = \sin\phi \;\; \boldsymbol{\rho}(\theta), \tag{23}$$

it belongs to the sphere equatorial plane $\{\mathbf{I}, \mathbf{J}\}$, it enfolds both the two spherical angles $\theta$ and $\phi$ and 'rescales' the unit vector $\boldsymbol{\rho}(\theta)$ by means of the magnitude

$$\epsilon = \mid \boldsymbol{\epsilon} \mid = \sin\phi,$$

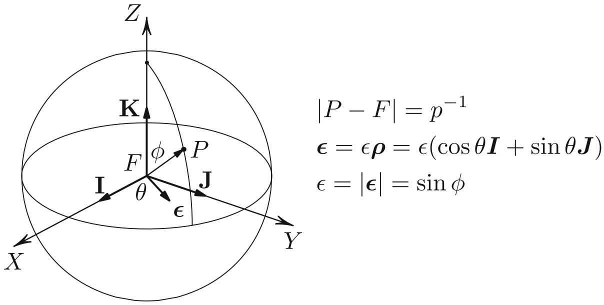



**Fig. 2** The rescaling vector $\boldsymbol{\epsilon}$

(related to the clockwise colatitude angle $\phi$), thus satisfying the range

$$0 \leq \epsilon \leq 1. \tag{24}$$

In particular, if in the definition (23) we set $\theta = 0$ (so that $\boldsymbol{\rho}(0) = \mathbf{I}$), and we choose in the plane $\{\mathbf{I}, \mathbf{K}\}$ the colatitude angle $\phi = \bar{\phi}_0$ such that

$$\sin \bar{\phi}_0 = e,$$

we recover no other than the standard eccentricity vector $\mathbf{e} = e\mathbf{I}$, that is,

$$\boldsymbol{\epsilon}_0 = \mathbf{e} = e\mathbf{I}.$$

**The sphere $S_{p^{-1}}$ as a star of vectors**.

The rescaling vector $\boldsymbol{\epsilon}$ allowed us to consider the primigenial sphere of the Kepler problem as the locus of vectors

$$P - F = p^{-1} \left(\boldsymbol{\epsilon} + \cos \phi \, \mathbf{K}\right) \tag{25}$$

in the right-handed frame $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ of $R^3$ (see Fig. 3), so that the sphere is not considered as the locus of equidistant points in space but as the *'star' of vectors* $P - F$ of the same magnitude $\mid P - F \mid = p^{-1}$ issuing from the focus $F$.

**The $\mathcal{C}$-cone and the vector N: a 3-dimensional characterization of Kepler orbits**.

As a consequence of the 'star view', in (Vivarelli 2015) we brought into life the $\mathcal{C}$-*cone* which generates the Kepler '*conic*' orbits.

In fact, in the star of vectors (25), we can fix constant values $\phi^*$ for the colatitude angle $\phi$: the corresponding related vectors

$$P^* - F = p^{-1} \left(\sin \phi^* \, \boldsymbol{\rho}(\theta) + \cos \phi^* \, \mathbf{K}\right) \tag{26}$$

define circular right cones with axis $\mathbf{K}$ and semi-aperture $\phi^*$ (see Fig. 3).

The particular cone with $\phi^* = \frac{\pi}{4}$ (see Fig. 3) or $P^*_{\frac{\pi}{4}} - F = \frac{\sqrt{2}}{2} p^{-1} \left(\boldsymbol{\rho} + \mathbf{K}\right)$ may be given an infinite extension (with positive parameter $\lambda \in \mathcal{R}$)

$$\mathcal{C} - F = \lambda \, p^{-1} \left(\boldsymbol{\rho} + \mathbf{K}\right). \tag{27}$$

We have called this extension circular $\mathcal{C}$-cone: it has vertex at $F(0, 0, 0)$, and its points $\mathcal{C} = (X_{\mathcal{C}}, Y_{\mathcal{C}}, Z_{\mathcal{C}})$ have coordinates $\mathcal{X}_{\mathcal{C}} = \lambda \, p^{-1} \cos \theta$, $Y_{\mathcal{C}} = \lambda \, p^{-1} \sin \theta$, $\mathcal{Z}_{\mathcal{C}} = \lambda \, p^{-1}$ which satisfy the cone scalar Cartesian equation

$$\mathcal{X}_{\mathcal{C}}^2 + \mathcal{Y}_{\mathcal{C}}^2 - \mathcal{Z}_{\mathcal{C}}^2 = 0. \tag{28}$$

The $\mathcal{C}$-*cone* (related to the physical Kepler parameter $p^{-1}$ and to the physical attractive vertex $F$) characterizes the Kepler conic orbits in $R^3$ by means of a particular vector $\mathbf{N}$

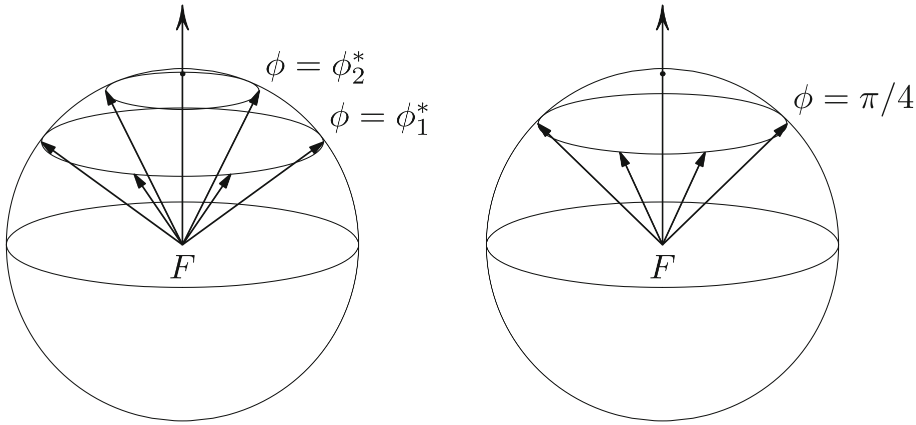



**Fig. 3** The sphere as a star of vectors. The cones for $\phi = cost$. The $\mathcal{C}$-cone for $\phi = \frac{\pi}{4}$

originated by $S_{p^{-1}}$. In fact, let us recall that the primigenial sphere relies, at the core, on the four primitive elements

$$p^{-1}, \quad \boldsymbol{\rho}, \quad \mathbf{e} = e\mathbf{I}, \quad \mathbf{K}$$

which, by means of simple linear combinations, originate immediately all the fundamental elements of the Kepler problem, as shown in Vivarelli (2015).

In particular, if (for simplicity) we denote by the symbol $\bar{\mathbf{e}}$ the eccentricity vector $\mathbf{e}$ rotated by the angle $\theta$ in the $\{X, Y\}$-plane) whence

$$\bar{\mathbf{e}} = e \cos\theta \ \mathbf{I} + e \sin\theta \, \mathbf{J} = e_X \ \mathbf{I} + e_Y \, \mathbf{J},$$

we find that the following simple linear vector combination

$$p^{-1} (\bar{\mathbf{e}} + \mathbf{K})$$

defines the peculiar 3-dimensional vector $\mathbf{N}$

$$\mathbf{N} = N - F = p^{-1} (e_X \, \mathbf{I} + e_Y \, \mathbf{J} + \mathbf{K}) \tag{29}$$

whose tip point $N$ has coordinates

$$X = p^{-1} \, e_X, \quad Y = p^{-1} e_Y, \quad Z = p^{-1} \tag{30}$$

which satisfy the scalar equation

$$X^2 + Y^2 - Z^2 = (e^2 - 1) \, p^{-2} \tag{31}$$

where $e = \mid \bar{\mathbf{e}} \mid = \sqrt{{e_X}^2 + {e_Y}^2}$ and $Z = p^{-1} > 0$ is the same third coordinate $Z = p^{-1}$ of the North pole of the (hemi)-sphere $S_{p^{-1}}$.

Thus, reminding that an ellipse, a parabola and the left branch of the hyperbola correspond to the ranges $0 \leq e \leq 1$ and $e > 1$, a comparison between (31) [with the term $(e^2 - 1)$] and the $\mathcal{C}$-cone equation (28) gives

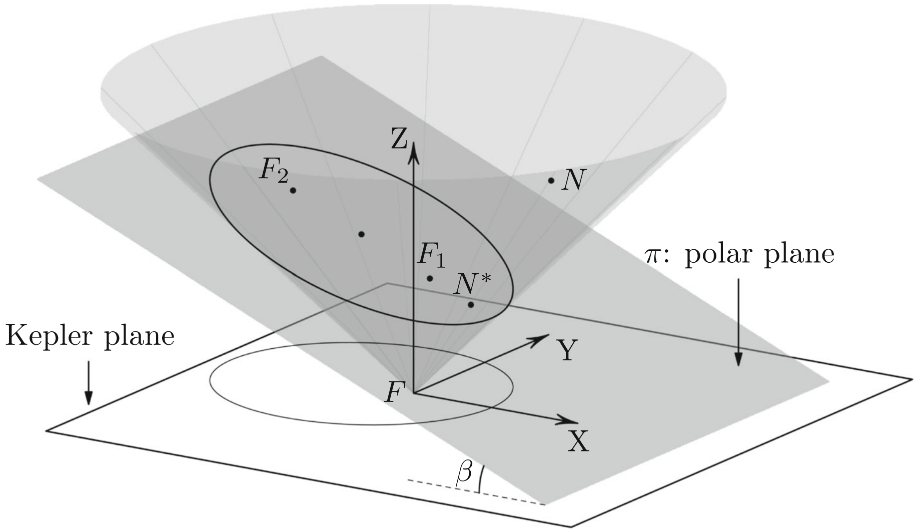



**Fig. 4** The polar plane of $N$. The projection of an elliptic section onto the Kepler plane

**the 3-dimensional characterization of the Kepler orbits:**

The tip points $N$ of the **N**-*vectors* (29) which lie inside the $\mathcal{C}$-*cone* correspond to elliptical orbits, those on the cone to parabolic ones and those outside the cone to hyperbolic orbits.

**The Kepler orbits as 'conic sections' of the $\mathcal{C}$-cone by the polar plane**.
Each tip point $N$ of the vector **N** defines, with respect to the unit sphere with centre at $F$, or

$$X^2 + Y^2 + Z^2 = 1, \tag{32}$$

a polar plane, that is, the plane (orthogonal to the line $FN$) which passes through the inverse point $N^*$ of $N$ so that $\mid N^* - F \mid = \frac{1}{|N-F|}$ (see Fig. 4).

This polar plane, defined by the points $(X, Y, Z)$ which satisfy the equation

$$p^{-1}\, e_X\, X + p^{-1}\, e_Y\, Y + p^{-1} Z - 1 = 0, \tag{33}$$

intersects the $\mathcal{C}$-*cone* in the locus given by the equation

$$(1 - e_X{}^2)\, X^2 + (1 - e_Y{}^2)\, Y^2 + 2pe_X X + 2pe_Y Y - 2e_X\, e_Y\, XY - p^2 = 0$$

This locus is a conic section, which projected orthogonally onto the $\{\mathbf{I}, \mathbf{J}\}$-plane gives exactly a Kepler conic section (with focus at the vertex $F(0, 0, 0)$ of the $\mathcal{C}$-*cone*, eccentricity $e$ and parameter $p$.) In Fig. 4, see an elliptic conic section.

In particular, the polar plane makes an angle $\beta$ with the **I**-axis such that

$$\tan \beta = e$$

(exactly the eccentricity of the Kepler conic orbit, see Fig. 4).

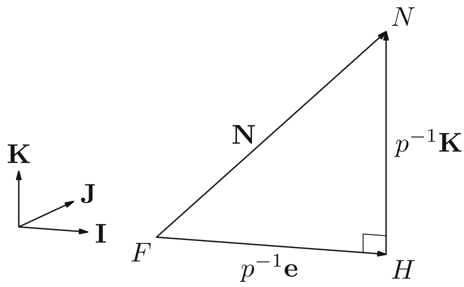



**Fig. 5** The constant vector $\mathbf{N}$ in the $\{\mathbf{I}, \mathbf{K}\}$-plane

## 4 The energy embodied by the energy point $P^*$ on the sphere

Now our basic goal is to show that also the total mechanical energy per unit of mass $E$ of a Kepler particle is deeply incorporated in the $S_{p^{-1}}$ spherical arena [via the vector $\mathbf{N}$ given by (29)].

Let us consider elliptic Kepler orbits, that is, orbits characterized by the couple of parameters $(e, p)$ with

the eccentricity $e$ in the range $0 < e < 1$;
the semi-latus rectum $p = \Gamma^2 K^{-2}$.

If, for simplicity (Fig. 5), we restrict our attention to $\theta = 0$ so that $\mathbf{e} = e\mathbf{I}$, we find that, by (29), we are considering the vector $\mathbf{N}$ in the $\{\mathbf{I}, \mathbf{K}\}$-plane or

$$\mathbf{N} = p^{-1}(\mathbf{e} + \mathbf{K}) = (H - F) + (N - H) \tag{34}$$

whose tip point $N$ has coordinates

$$N = (p^{-1}e,\ 0,\ p^{-1})$$

whence it lies inside the $\mathcal{C}$-cone (Fig. 6) (in fact, being for an ellipse $p^{-1}e < p^{-1}$, the first vector component of $\mathbf{N}$ in (34), that is, $H - F = p^{-1}\mathbf{e}$, lies inside the circle of radius $p^{-1}$, which is the intersection of the sphere with the $\{X, Z\}$-plane); the second vector component $p^{-1}\mathbf{K}$ gives exactly the North pole of the sphere (Fig. 6).

As a consequence, we can 'project' the tip point $N$ on the circle: just draw the point of intersection $P^*$ of the line $HN$ with the circle.

It is from this point $P^*$ that we find a representation of the scalar energy. In fact, its coordinate $Z_{P^*}$ (a cathetus of the triangle $FHP^*$ with $\mid P^* - F \mid = p^{-1}$) is (Figs. 6, 7)

$$Z_{P^*} = \mid P^* - H \mid = \sqrt{(p^{-1})^2 - (p^{-1}e)^2} = p^{-1}\sqrt{1 - e^2}, \tag{35}$$

but, by means of the well-known relation between $e$ and the energy $E$ given by

$$1 - e^2 = -2\, E\, \Gamma^2\, K^{-4}\,,$$

we have that (35) becomes

$$Z_{P^*} = p^{-1}\sqrt{-2E}\,\frac{\Gamma}{K^2}$$

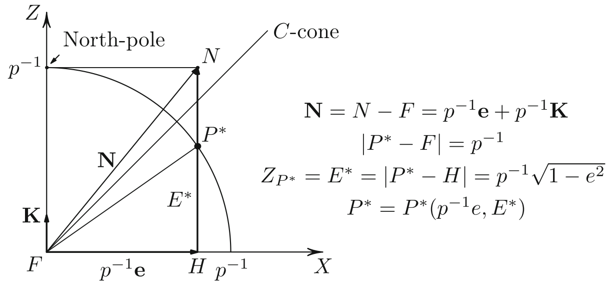



**Fig. 6** The point $N$ inside the $\mathcal{C}$-cone. The point $P^*$ and the energy segment $E^* = P^* - H$

(recall that for elliptic orbits $E < 0$). Finally, by (7), we obtain

$$Z_{P^*} = \frac{\sqrt{-2E}}{\Gamma} = E^*, \tag{36}$$

that is, by (35),

$$Z_{P^*} = \mid P^* - H \mid = E^* . \tag{37}$$

Since, by construction (Fig. 7), the point $P^*$ makes with the $Z$-axis the angle $\phi^*$ such that $\sin\phi^* = e$ we can finally state that:

**Proposition 4.1** *The constant energy ratio $E^*$ of an elliptic orbit is represented by the length of the energy segment, that is, by the coordinate $Z_{P^*}$ of the point $P^*$* [*obtained by intersecting the primigenial sphere $S_{p^{-1}}$ with the 'vertical' line passing through the tip point $N$ of* $\mathbf{N}$ (*Figs.* 6, 7)]. *The point $P^*$ has colatitude $\phi^*$ such that*

$$\sin\phi^* = e . \tag{38}$$

To further highlight the definition of $E^*$, we remark its intrinsic relation to the couple $(e, p)$ characterizing the ellipse and rewrite (36), via (33) as

$$E^* = p^{-1}\sqrt{1-e^2} = p^{-1}\sqrt{1-(\sin\phi^*)^2} . \tag{39}$$

We are now able to extend to the space $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ the 'segment' representation found for the energy of an elliptic orbit in the $\{\mathbf{I}, \mathbf{K}\}$-plane.

We relax the restriction $\theta = 0$ so that now the eccentricity vector $\mathbf{e}$ makes an angle $\theta$ with the $X$-axis, or $\mathbf{e} = \mathbf{e}(\theta) = e\cos\theta\,\mathbf{I} + e\sin\theta\,\mathbf{J} = e_X\;\mathbf{I} + e_Y\,\mathbf{J}$ with $\tan\theta = \frac{e_Y}{e_X}$ and $e = \mid e \mid = \sqrt{(e_X)^2 + {e_Y}^2}$ . Of course, both the vector

$$\mathbf{N} = \mathbf{N}(\theta) = p^{-1}\,(\cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{J}) + p^{-1}\mathbf{K}$$

and its projected point $P^*$ are characterized by the same angle $\theta$ (Fig. 8), which is exactly the angle of the Kepler particle $Q(\theta)$ on the ellipse.

So, if we consider the full motion of the Kepler particle $Q(\theta)$ on the ellipse, that is, if we let the angle $\theta$ vary in the range $[0, 2\pi]$, we obtain the locus of all the vectors $P^* - F$ in the 3-space: this locus is a *cone* with vertex at $F$, height $Z_{P^*} = E^*$ and is determined by the circle of radius $p^{-1}e$ on the sphere (Fig. 8).

We call this cone the energy cone and the point $P^*(\theta, \phi^*)$ the energy point of the Kepler particle (Fig. 8). Thus, we have found a three-dimensional representation of the total mechanical energy.

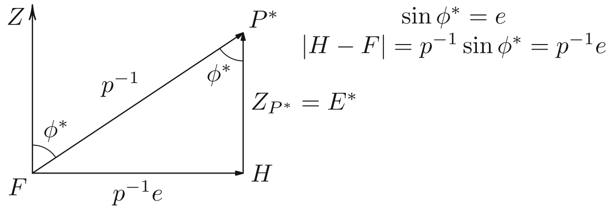



**Fig. 7** The point $P^*$, its energy coordinate $Z_{P^*}$ and its colatitude angle $\phi^*$

Motions of the points $Q(\theta)$ and $P^*(\theta, \phi^*)$. Physically, when a Kepler particle $Q(\theta)$ describes an ellipse (characterized by the constant couple $(e, p)$, the angle $\theta$ and the focus $F$ in the Kepler plane), we have that (Figs. 8, 9):

- the corresponding vector $P^* - F$ generates the energy cone with height $E^* = p^{-1}\sqrt{1-e^2}$ in the 3-dimensional space with $e = \sin\phi^*$,
- the energy point $P^*(\theta, \phi^*)$ describes a circle of radius $(p^{-1}e)$ on the sphere,
- all the three elements of the elliptic Kepler orbit (the couple $(e, p)$, the angle $\theta$ and the focus F) are simply epitomized in the energy cone and in the energy point description.

The $P^*$-description of the energy enhances the previous $\phi_0$ description. So far, we have shown that the representation of the energy $E$ of an elliptic orbit is naturally incorporated in the characteristic vector $\mathbf{N}$ related to the primigenial sphere $S_{P^{-1}}$ and to the angle $\phi^*$.

We wish to notice that in our previous paper (Vivarelli 2015) we have arrived at the energy in a different way, by choosing a particular, suitable colatitude angle $\phi_0$ which was required to satisfy $\sin\phi_0 = \sqrt{2EpK^{-1} + 1}$, the angle defining the rotation of the unit vector $\mathbf{K}$ in the $\{\mathbf{K}, \boldsymbol{\rho}\}$-plane around $F$ which brings $\mathbf{K}$ onto the $P^* - F$ direction. But, although introduced in two different ways, the angles are the same $\phi^* = \phi_0$ , so that (as a consequence) the $P^*$-description given in this paper acquires a 'rotational' flavour: the colatitude $\phi^*$ defines the rotation of the unit vector $\mathbf{K}$ in the $\{\mathbf{K}, \boldsymbol{\rho}\}$-plane around $F$ which brings $\mathbf{K}$ onto the $P^* - F$ direction.

In other words, the $P^*$-description presented in this paper (which pops up directly from the primitive element vector $\mathbf{N}$) is much more entangled to the spherical scenario and requires no 'ad hoc' introduction as the previous artificial $\phi_0$-angle of the energy,

## 5 The velocity of the energy point *P** and the Sundman–Levi-Civita time map

An interesting result stems naturally in the spherical arena from the point $P^*$.
It regards the regularizing differential time transformation (11) introduced by Sundman–Levi-Civita (Sundman–Levi-Civita map) which, as we have seen in Sect. 2, is common to both the Bohlin–Burdet and KS regularizations and which replaces the physical time variable $t$ by a new fictitious time variable $s$, related to $t$ by the scalar distance factor $r$.

**Proposition 5.1** *The Sundman–Levi-Civita time map*

$$\mathrm{d}s = r^{-1}\,\mathrm{d}t,$$

*may be rewritten as*

$$\mathrm{d}s = \sqrt{\frac{v_{P^*}}{p^{-1}e\,\Gamma}}\;\mathrm{d}t, \tag{40}$$

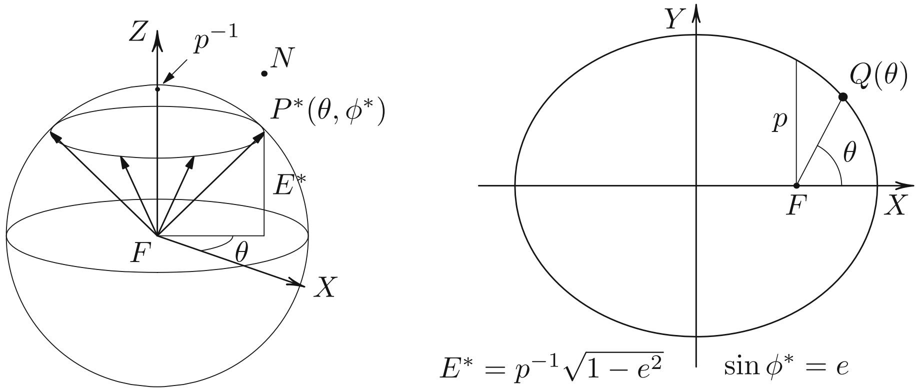



**Fig. 8** The point $P^*(\theta, \phi^*)$ for a plane elliptic Kepler orbit $(e, p, \theta)$ is the energy point on the energy cone

*where $v_{P^*}$ is the linear velocity of the energy point $P^*$ in its circular motion on the primigenial sphere $S_{P^{-1}}$.*

***Proof*** Recall that when (Figs. 8, 10) a Kepler particle $Q(r, \theta)$ describes an ellipse with focus at $F$, the corresponding energy point $P^*(\theta)$ describes a circle of radius $p^{-1}e$ on the cone, or equivalently on the sphere.

We denote by $v_{P^*}$ the linear velocity of the point $P^*$, whence

$$v_{P^*} = p^{-1}e\,\dot{\theta}.$$

By Kepler's second law ($\Gamma = r^2\,\dot{\theta}$), the angular velocity is $\dot{\theta} = \frac{\Gamma}{r^2}$, whence

$$v_{P^*} = p^{-1}e\,\frac{\Gamma}{r^2}. \tag{41}$$

Moreover, since from the Sundman–Levi-Civita map (11) we have that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{1}{r^2},$$

we finally obtain that (41) yields the relation (40) sought for. □

## 6 The Sundman–Levi-Civita map and the KS map linked by the point *P**

Let us recall that (Sect. 1) the whole Kustaanheimo–Stiefel regularization relies on the combined action of two maps (the Sundman–Levi-Civita map and the KS map). Although these maps are completely different, we find that there is a trait d'union between them given exactly by the energy point $P^*$: this particular result is obtained by considering the two maps 'on the same ground', for instance, as rotations on the sphere.
Precisely:

the time Sundman–Levi-Civita's map (11) has been related in this paper to a 'horizontal' rotation of the energy point $P^*$ on a circle of radius $p^{-1}e$ (Figs. 9, 10a);
the spatial coordinate transformation [the KS map (15)] is shown in Vivarelli (1994b, 2005, 2015) (see also Deprit et al. 1994; Ferrer and Crespo 2017; Kustaanheimo and

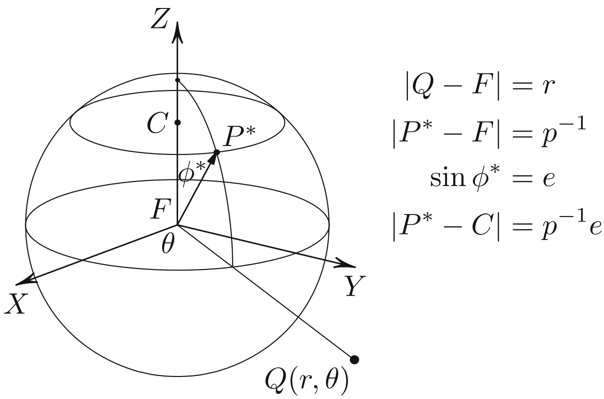



**Fig. 9** Orbiting particle $Q(r, \theta)$ in the Kepler $(X, Y)$-plane and the associated energy point $P^*$ describing a circle of radius $(p^{-1}\, e)$ on the sphere $S_{p^{-1}}$

Stiefel 1965) to represent a simple compound roto-dilation in $R^3$, that is (Fig. 10b), a rotation by the angle $\frac{\pi}{2}$ in the vertical $\{\mathbf{K}, \boldsymbol{\rho}\}$-plane through $P^*(\theta)$ that brings the North pole of the sphere $S_{p^{-1}}$ onto the point $R$ on the equator (radius $p^{-1}$), followed by a stretching of the last point position vector $R - F = p^{-1}\boldsymbol{\rho}$ by the factor $(p\, r)$ so that it finally reaches the physical particle $Q$ (position vector $\mathbf{x} = r\boldsymbol{\rho}$ on the ellipse). Concisely (Fig. 10b):

$$p^{-1}\mathbf{K} \longrightarrow R - F = p^{-1}\boldsymbol{\rho} \longrightarrow p\, r\, (p^{-1}\boldsymbol{\rho}) = r\boldsymbol{\rho}\,.$$

Summarizing: the whole KS regularization (that is Sundman–Levi-Civita map plus KS map) is naturally represented by two rotations on the primigenial sphere:

a 'horizontal' circular rotation of the energy point $P^*$ on a parallel of the sphere;

a 'vertical' rotation in the $\{\mathbf{K}, \boldsymbol{\rho}\}$-plane on the meridian of the sphere which passes through $P^*$ (followed by a 'horizontal' dilation) (Fig. 11).

In this sense, we can say that the point $P^*$ yields in a natural way the link between the two different maps which together carry out the whole KS regularization: the (horizontal) Sundman–Levi-Civita time map and the (vertical) coordinate KS map (Fig. 11).

## 7 The Bohlin–Burdet equation in the spherical scenario: the border of the S-triangle

We are now going to show that also the regular Bohlin–Burdet oscillator equation (13) pops up naturally from the sphere $S_{p^{-1}}$: as we did in Sect. 3, when we introduced the vector $\mathbf{N}$, let us consider the following simple linear vector combination

$$p^{-1}\,(\mathbf{e} + \boldsymbol{\rho}) = \mathbf{S},$$

which has brought us to define (see Vivarelli 2015) the peculiar sum vector $\mathbf{S} = \mathbf{S}(\theta)$ in the equatorial Kepler plane of the sphere. This vector was shown to give a 2-dimensional representation of the Kepler conics since it allows to epitomize the well-known standard plane polar equation of the whole family of Kepler conic orbits

$$r = \frac{p}{1 + e\ \cos\theta} \quad (\mathbf{x} = r\boldsymbol{\rho})$$

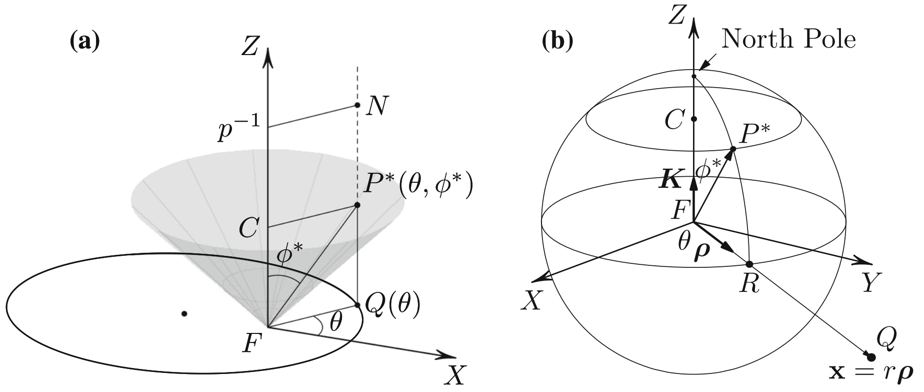



**Fig. 10** **a** The Sundman–Levi-Civita map: the elliptic motion $Q(\theta)$ and the circular motion of the energy point $P^* = P^*(\theta, \phi^*)$ with velocity $v_{P^*}$. **b** The representation of the KS map: North Pole $\to P^* \to R \to Q$

by the simple equation given by the scalar product

$$\mathbf{S} \cdot \mathbf{x} = 1. \tag{42}$$

Now, to draw $\mathbf{S}$, let us rewrite the sum vector $\mathbf{S}$ as

$$\mathbf{S} \equiv p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho}, \tag{43}$$

so that by construction (Fig. 12), if $B$ denotes the tip point of the vector $\mathbf{S}$, we have $\mathbf{S} = \mathbf{S}(\theta) = B - F$ where $p^{-1}\mathbf{e} = H - F = p^{-1}e\,\mathbf{I}$ and $B - H = p^{-1}\boldsymbol{\rho}(\theta)$.

***Remark*** Notice (Fig. 12) that $p^{-1}\boldsymbol{\rho} = B - H = R - F$, where the point $R$ (on the equator of the primigenial sphere with radius $p^{-1}$) is exactly the point which occurs in the rotation which represents the KS map (see previous Fig. 11).

We are now ready to give the following

**Definition 7.1** We define $\mathbf{S}$-triangle the triangle FHB defined by $\mathbf{S}$ (Fig. 13).

**Proposition 7.1** *The border of the* $\mathbf{S}$*-triangle epitomizes the Bohlin–Burdet equation* (13).

***Proof*** Since the Bohlin–Burdet equation (13) is expressed in terms of the vector $\mathbf{x}$ and of its derivatives, we first relate them to the two vector components (43) of $\mathbf{S}$. This is done in four steps.

- First, the vector $p^{-1}\boldsymbol{\rho}$ in (43) can be reformulated as

$$p^{-1}\boldsymbol{\rho} = -\frac{r^2\ddot{\mathbf{x}}}{\Gamma^2}, \tag{44}$$

  since the classical equation of motion (2), by means of $\frac{\mathbf{x}}{r} = \boldsymbol{\rho}$ and of (9), can be expressed in the simple form $r^2\ddot{\mathbf{x}} + \Gamma^2(p^{-1}\boldsymbol{\rho}) = \mathbf{0}$.
- On the other hand, the same vector $p^{-1}\boldsymbol{\rho}$ is related to the energy: simply multiply (4) on the right by $\mathbf{x}$ so that $E\mathbf{x} = \frac{1}{2} \mid \dot{\mathbf{x}} \mid^2 \mathbf{x} - \Gamma^2 p^{-1}\boldsymbol{\rho}$ whence

$$p^{-1}\boldsymbol{\rho} = \frac{\mid \dot{\mathbf{x}} \mid^2 \mathbf{x} - 2E\,\mathbf{x}}{2\,\Gamma^2}. \tag{45}$$

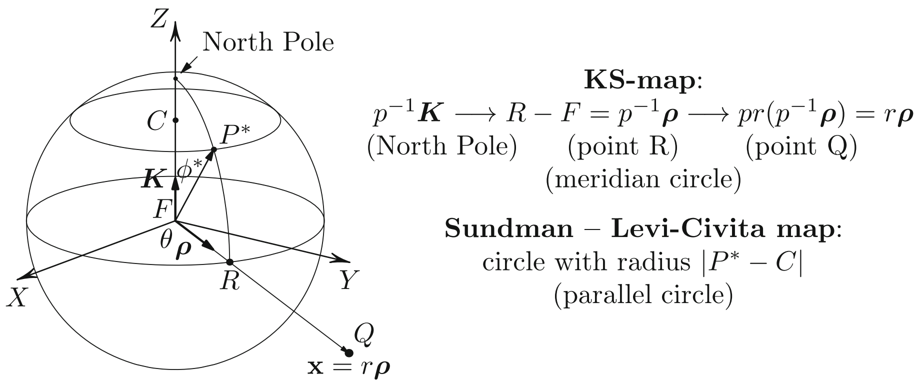



**Fig. 11** The KS map (a rotation through $P^*$ and a dilation). The Sundman–Levi-Civita map (a rotation of $P^*$ on the circle of centre $C$). The two maps linked by the energy point $P^*$

– The other vector $p^{-1}\mathbf{e}$ in (43) can be reformulated as

$$p^{-1}\mathbf{e} = \frac{|\,\dot{\mathbf{x}}\,|^2\,\mathbf{x} - r\dot{r}\dot{\mathbf{x}}}{\Gamma^2} - p^{-1}\boldsymbol{\rho} \tag{46}$$

obtained by expression (5) of the vector $\mathbf{V}$, where the vector $\mathbf{\Gamma}$ may be written as (3), whence, by a well-known vector property of the wedge product $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, expression (9) can be recast by (5) as $\mathbf{V} = K^2\mathbf{e} =|\,\dot{\mathbf{x}}\,|^2\,\mathbf{x} - r\dot{r}\dot{\mathbf{x}} - K^2\frac{\mathbf{x}}{r}$ which by (7) yields exactly (46).

– Finally, the vector $\mathbf{S}$, by substituting (46) in the definition (43), is related to the velocity by the expression

$$\mathbf{S} = p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho} = \frac{|\,\dot{\mathbf{x}}\,|^2\,\mathbf{x} - r\dot{r}\dot{\mathbf{x}}}{\Gamma^2}. \tag{47}$$

We are now ready to consider the border of the $\mathbf{S}$-*triangle*, given by the vector expression (Fig. 13)

$$p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho} - \mathbf{S} = \mathbf{0}. \tag{48}$$

This border, after substituting the definition (43) of $\mathbf{S}$ and after collecting, becomes

$$p^{-1}\mathbf{e} + (p^{-1}\boldsymbol{\rho} - p^{-1}\mathbf{e}) - p^{-1}\boldsymbol{\rho} = \mathbf{0}. \tag{49}$$

But the vector between brackets may be written as

$$(p^{-1}\boldsymbol{\rho} - p^{-1}\mathbf{e}) = 2p^{-1}\boldsymbol{\rho} - (p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho}),$$

whence the border relation (49) becomes

$$p^{-1}\mathbf{e} + 2p^{-1}\boldsymbol{\rho} - (p^{-1}e - p^{-1}\boldsymbol{\rho}) - p^{-1}\boldsymbol{\rho} = \mathbf{0},$$

so that (after substituting (47) in the brackets, after expressing $2p^{-1}\boldsymbol{\rho}$ by (45) and $p^{-1}\boldsymbol{\rho}$ by (44) and after simplifying) the border relation now reads

$$p^{-1}\mathbf{e} - \frac{2E\mathbf{x} + r\dot{r}\dot{\mathbf{x}}}{\Gamma^2} + \frac{r^2\ddot{\mathbf{x}}}{\Gamma^2} = \mathbf{0}$$

or, after rearranging the order of the derivates,

$$r^2\ddot{\mathbf{x}} + r\dot{r}\dot{\mathbf{x}} - 2E\mathbf{x} + \Gamma^2 p^{-1}\mathbf{e} = \mathbf{0}. \tag{50}$$

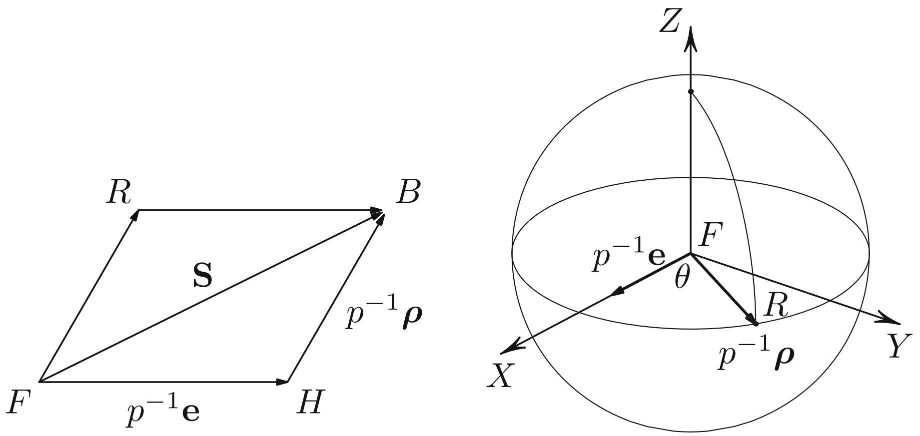
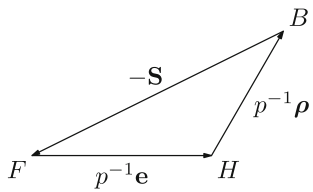



**Fig. 12** The sum vector $\mathbf{S} \equiv p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho}$. The two vector components $p^{-1}\mathbf{e}$ and $p^{-1}\boldsymbol{\rho}$ as elements of $S_{p^{-1}}$




$(p^{-1}\mathbf{e} + p^{-1}\boldsymbol{\rho} - \mathbf{S} \equiv \mathbf{0})$

**Fig. 13** The **S**-triangle and its border

Finally, in order to introduce the time $s$, we recall that, by (12), the second derivatives with respect to $t$ and $s$ are related by

$$r^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) + r\dot{r}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}, \tag{51}$$

so that expression (50) now reads

$$\mathbf{x}'' - 2E\mathbf{x} + \Gamma^2 p^{-1}\mathbf{e} = \mathbf{0} \tag{52}$$

or, by recalling that $\omega^2 = -2E$ and by (9), it finally becomes

$$\mathbf{x}'' + \omega^2\mathbf{x} + \mathbf{V} = \mathbf{0}, \tag{53}$$

which is exactly the regular oscillator Bohlin–Burdet equation (13). □

Thus, we may state that:

> The 3-dimensional Bohlin–Burdet oscillator equation acquires a simple geometrical interpretation given by the border of the **S**-*triangle*.

## 8 The harmonic oscillator equations: Bohlin–Burdet meets KS

In the previous section, we have presented a geometrical interpretation of the 3-dimensional Bohlin–Burdet oscillator equation (13).

The aim of this section is to display an explicit link between this Bohlin–Burdet oscillator equation (13) and the 4-dimensional KS oscillator equation (19), which, for perusal simplicity, we collect together:

$$\mathbf{x}'' + \omega^2 \mathbf{x} + \mathbf{V} = \mathbf{0} \tag{54}$$

$$\mathbf{u}'' + \omega^2 \mathbf{u} = \mathbf{0}\,. \tag{55}$$

To obtain the link, we find it useful to compare these two different equations (which live in different spaces) by rewriting them in the same mathematical language. Explicitly, we adopt the quaternion language, suggested by the fact that, as we have shown, the KS map is a rotation and that it is well known that a rotation acquires a simple, concise description through a quaternion formulation.

In fact, as we have demonstrated in our previous works (Vivarelli 1983, 1994a, b, 2005, 2015), we recall that:

- the vector KS equation (55) can be rewritten in a simple quaternion form by considering the map $\mathbf{u} \to q$ given by

$$q = (u_1 + u_2\mathbf{I} + u_3\mathbf{J}) + u_4\mathbf{K}, \tag{56}$$

  where the real quaternion $q$ (called in Breiter and Langner 2017 the Stiefel–Scheifele–Vivarelli quaternion) has the term between brackets denoting its 3-vector part. In other words, in the 3-dimensional space $(R^3 - \{0\}) = \mathrm{im}\,\{1, \mathbf{I}, \mathbf{J}\}$, a 3-vector $\mathbf{x}$ is a quaternion with a null fourth component, or $\mathbf{x} = x_1 + x_2\mathbf{I} + x_3\mathbf{J} + 0$.
- It follows that the KS map $\mathbf{u} \to \mathbf{x}$, given by the matrix expression (15), is expressed explicitly by the simple quaternion product

$$\mathbf{x} = qq_*, \tag{57}$$

  where $q_*$ is the anti-involute of $q$ or

$$q_* = u_1 + u_2\mathbf{I} + u_3\mathbf{J} - u_4\mathbf{K} \tag{58}$$

  whence

$$\mathbf{x} = qq_* = (u_1 + u_2\mathbf{I} + u_3\mathbf{J} + u_4\mathbf{K})(u_1 + u_2\mathbf{I} + u_3\mathbf{J} - u_4\mathbf{K}).$$

- With the new map $\mathbf{u} \to q$ given by (56), the vector four-dimensional regular KS equation of motion (55) may be rewritten in quaternion form as:

$$q'' + \omega^2 q = 0. \tag{59}$$

Now we are able to rewrite in quaternion language also the Bohlin–Burdet equation. Just recall (Vivarelli 1983, 1994a, b) the following important quaternion relation:

$$q' q_* = q q'_* \tag{60}$$

(obtained by simple but tedious calculations and by taking into account the bilinear relation (18)), whence, from (57), we have $\mathbf{x}' = q'q_* + qq'_* = qq'_* + qq'_* = 2qq'_*$ and finally the formula

$$\mathbf{x}'' = 2q'q'_* + 2qq''_*.$$

By the above formula and by (57), we can express the Bohlin–Burdet vector equation (54) as:

$$(2q'q'_* + 2qq''_*) + (\omega^2\, qq_*) + V = 0\,. \tag{61}$$

By adding and subtracting the term $\omega^2 qq_*$, we obtain

$$2q'q'_* + 2qq''_* + 2\omega^2 qq_* - \omega^2 qq_* + V = 0,$$

whence, by collecting the factor $2q$ and by suitably ordering, we finally find

$$[\,(2q\,(q''_* + \omega^2 q_*)\,] + [\,2q'q'_* - \omega^2\, qq_* + V\,] = 0 \tag{62}$$

which is the quaternion form of the Bohlin–Burdet equation (54) sought for.

Now, notice that Eq. (62) expresses nothing, but the sum of two quaternion functions given by the square brackets say $f(q) + g(q) = 0$, which means that $g(q) = -f(q)$.

Of course, if we make the particular choice $f(q) = 0$, the whole Eq. (62) is automatically satisfied.

But this particular choice means that

$$q''_* + \omega^2 q_* = 0 \quad (q \neq 0)$$

or, by recalling the quaternion anti-involute property $(q_*)_* = q$,

$$q'' + \omega^2 q = 0,$$

which is exactly the quaternion regular KS equation (59) sought for.

## 9 Concluding remarks and outlook

Consider a particle which, under the action of a Newtonian gravitational force (Kepler problem), describes an ellipse characterized by the triplet $(p, e, \theta)$, where $p$ and $e$ are, respectively, the semi-latus rectum and the eccentricity of the ellipse, while $\theta$ is the polar angle of the particle on the ellipse.

In this paper, we have shown that the elliptic motion of this particle can be related to the circular motion of a particular point $P^*$ which lies on a sphere with radius $p^{-1}$, that is, on the primigenial sphere $S_{p^{-1}}$, we introduced in our previous work (Vivarelli 2015) and that was shown to encompass several geometrical and physical features of the Kepler motion.

The point $P^*$ is characterized by the spherical coordinates $(\theta, \phi^*)$, where the longitude $\theta$ is exactly the polar angle of the physical particle in its elliptic motion and the latitude $\phi^*$ is related to the eccentricity $e$ of the ellipse by the relation $e = \sin\phi^*$. The point $P^*$ is called the energy point since it embeds, by means of its third Cartesian coordinate, the total mechanical energy of the physical particle.

But there is more than meets the eye, for this energy point $P^*$ is peculiar in that:

- it moves (in its circular motion on the sphere) with a velocity that relates the real and the fictitious times which appear in the Sundman–Levi-Civita regularizing time map;
- it gives a ‘link’ between the two geometrical representations which we have introduced, respectively, for the regularizing Kustaanheimo–Stiefel coordinate map and the regularizing Sundman–Levi-Civita time map: the first map is related to a meridian circle of the sphere and the second map to a parallel circle (the two circles intersecting exactly in the energy point $P^*$).

On the other hand, the primigenial sphere $S_{p^{-1}}$ is shown to concern also the two following particular items of the Kepler problem, that is,

- the regularizing Bohlin–Burdet oscillator equation, which acquires a geometrical flavour in the $S_{p^{-1}}$ spherical scenario, since it is strictly related to the border of a triangle which lies in the equatorial plane of the sphere;
- the Bohlin–Burdet and Kustaanheimo–Stiefel regularizing oscillator equations, which are shown to be explicitly linked in a natural quaternion-rotation perspective.

These results confirm the effectiveness of the primigenial sphere structure which we think is more than a fortuitous invention since it succeeds in gathering together all the characteristic elements of the Kepler motion: suitably developed, the concept of a primigenial structure may help in suggesting and finding the main aspects and the evolution of other well-known dynamical systems.

We wish to end this paper by pointing out that we have restricted to elliptic Kepler motion (which is notoriously non-degenerate, that is, $\mathbf{\Gamma} \neq \mathbf{0}$) and that we have worked in the context of pure Kepler motion (no perturbations). So let us anticipate that we have in mind to extend these investigations not only to degenerate orbits ($\mathbf{\Gamma} = \mathbf{0}$), but also to consider, instead of the Newtonian force $\mathbf{F}$ in (21), a perturbing force of the type $\mathbf{F} = -\frac{\partial U}{\partial \mathbf{x}} + \mathbf{P}$ (where $U(\mathbf{x}, t)$ is a perturbing potential and $\mathbf{P}$ an additional force), following the line suggested by Baumgarte (1972a, b) where numerical stabilization devices are shown to improve the accuracy of computation of perturbed Kepler orbits. Of course, if perturbations occur, we must rewrite anew expression (21) of the Newtonian force $\mathbf{F}$ (from which the primigenial structure arises): in so doing, we find that the geometrical elements of the sphere (say the eccentricity) are subject to variations, and so the sphere undergoes a sort of 'deformation'.

### Compliance with ethical standards

**Conflict of interest** The author declares that she has no conflict of interest.

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