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ORIGINAL ARTICLE

Semi-analytical model for third-body perturbations including the inclination and eccentricity of the perturbing body

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Abstract
A general third-body perturbation problem, considering the perturbing body in an elliptic and inclined orbit, is investigated by using a semi-analytical theory. Previous works have contributed to deriving the averaged third-body-perturbed dynamics, but did not provide a transformation between osculating and mean elements in the general case. In this paper, an analytical transformation between osculating and mean elements is developed explicitly using von Zeipel's method, in addition to developing the long-term dynamical equations. The resulting dynamical model is improved, because the disturbing function is averaged as a whole, instead of separating the disturbing function into many terms and averaging them independently. The simulation results indicate that the new singly averaged dynamical model behaves much better than the doubly averaged dynamics in propagating the long-term evolution of the orbital elements. Moreover, it is shown that the perturbing body's inclination and eccentricity have a vital influence on the evolution of the satellite's inclination and eccentricity.

Keywords Third-body perturbation · von Zeipel's method · Periodic corrections

1 Introduction

Third-body perturbations induce both periodic and secular variations. Secular effects change the orbital elements linearly over time or proportionally to some power of time. Secular terms are the principal contributors to the degradation of analytical theories. Kozai (1959b) derived

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the long-term dynamics of an Earth satellite including secular and long-period effects of the lunisolar perturbation and evaluated third-body effects on asteroids with large inclination and eccentricity (Kozai 1962b). Moreover, the third-body perturbation induced by Earth was analyzed when modeling lunar orbiter dynamics (Kozai 1963). Musen et al. (1961) extended the work of Kozai (1959b) by adding the parallactic term in the expansion of Legendre polynomials of the disturbing function. Blitzer (1959) and Cook (1962) averaged Lagrange's planetary equations including the lunisolar perturbations and analyzed the lunisolar perturbation effects on an Earth orbiter.

During the mid-1960s and 1970s, several authors developed satellite theories based on the method of averaging (Vallado 2001). Bertachini de Almeida Prado (2003) and Broucke (2003) introduced averaging theory to study the long-term effects of a perturbing body. They averaged the disturbing function over the period of the satellite and with respect to the period of the third body and then used Lagrange's planetary equations for developing the orbiter's dynamics. In their analysis, they assumed that the disturbing body was in a circular and equatorial orbit. Tresaco et al. (2018) used averaging to find frozen orbits around Mercury, including the third-body effects induced by the Sun. These researches used a simplified third-body perturbation model, where the perturbing body was in a circular and equatorial orbit. Domingos et al. (2008) studied the third-body perturbation including the eccentricity of the disturbing body. Liu et al. (2012) evaluated a general case, wherein the disturbing body was on an elliptic and inclined orbit. They derived a doubly averaged dynamical model. However, this model is an approximate result, because Liu et al. (2012) averaged terms of the singly averaged disturbing function separately.

All of the preceding references made fundamental contributions to the study of third-body perturbations. Most works were dedicated to evaluating the third body's long-term effects by averaging the disturbing function. This method is simple but incomplete; whereas it can obtain the orbital dynamical equations, it cannot yield the transformation between osculating elements and mean elements. The importance of periodic corrections to the mean elements has been pointed out in the initialization of formation flying missions (Schaub and Alfriend 2001; Nie et al. 2019) and in designing frozen orbits (Nie and Gurfil 2018).

In order to derive the transformation between osculating elements and mean elements, it is necessary to derive the generating function. The von Zeipel method and the Lie series theory are two main canonical transformations for developing generating functions. Brouwer (1959) first proposed using von Zeipel's method to study the first-order solution for an Earth orbiter including the J_2 perturbation. Kozai (1962a) derived the second-order solution considering the zonal spherical harmonics up to J_8. Lyddane (1963) utilized Poincaré variables instead of Delaunay variables to avoid the singularity at zero eccentricity and zero inclination in Brouwer's theory. Hori (1966) and Deprit (1969) introduced Lie series for defining the canonical transformations contemplated by von Zeipel's method. As opposed to von Zeipel's method, where the generating function was mixed, containing the original coordinates and the new momenta, the Lie transform depended explicitly on the new coordinates only, which lead to the explicit relations between the original and new variables. In addition, the Lie transform used the Poisson brackets, which were canonically invariant. Thus, the Lie transform method offers a general formula for any perturbation and is more conveniently programmable (Jupp 1982).

There are few works which contributed to deriving the transformation in the third-body problem. Roscoe et al. (2015) used the Lie theory to provide a general description of the third-body perturbation, without deriving any explicit results. Lara and Palacián (2009) derived the generating function of the third-body perturbation, but did not give the explicit transformation between osculating elements and mean elements. Giacaglia et al. (1970) included the third-body perturbation when modeling the lunar orbiter and Nie and Gurfil (2018) extended the analysis to derive complete transformations. However, these studies used a simplified model of the third-body perturbation, without considering the eccentricity and inclination of the perturbing body.

This paper evaluates the effect of third-body perturbations without significant simplifying assumptions, assuming the perturbing body in an eccentric and inclined orbit. The theory will rely on von Zeipel's method. Although von Zeipel's method yields a mixed generating function, this issue is not of major significance for the low-order problem discussed in the current paper. The main contribution includes the derivation of the analytical transformation between osculating elements and mean elements, which cannot be found in the existing literature. An improvement of the long-term dynamical model in Liu et al. (2012) is also developed, by averaging the Hamiltonian as a whole.

2 Semi-analytical dynamics

The Hamiltonian of an orbiter perturbed by a third body in an elliptic inclined orbit can be formulated as (Bertachini de Almeida Prado 2003)

$$\mathcal{H} = -\frac{\mu}{2a} - \left(\frac{Gm_3}{r_3}\right)\sum_{n=2}^{\infty}\left(\frac{r}{r_3}\right)^n P_n\left(\cos\varphi\right), \tag{1}$$

where μ is the gravitational parameter of the central body; a is the semimajor axis of the orbiter; m_3 is the mass of the perturbing body; G is the universal gravitational constant; r, r_3 are the radial distances of the orbiter and the perturbing body from the central body; and φ is the angle between the orbiter and perturbing body, determined by

$$\cos\varphi = \frac{\boldsymbol{r}}{r}\cdot\frac{\boldsymbol{r}_3}{r_3}, \tag{2}$$

where radial vectors in the inertial frame can be expressed in terms of orbital elements as (Vallado 2001)

$$\boldsymbol{r} = r\begin{bmatrix} \mathrm{c}_\Omega\mathrm{c}_\omega - \mathrm{s}_\Omega\mathrm{s}_\omega\mathrm{c}_i & -\mathrm{c}_\Omega\mathrm{s}_\omega - \mathrm{s}_\Omega\mathrm{c}_\omega\mathrm{c}_i & \mathrm{s}_\Omega\mathrm{s}_i \\ \mathrm{s}_\Omega\mathrm{c}_\omega + \mathrm{c}_\Omega\mathrm{s}_\omega\mathrm{c}_i & -\mathrm{s}_\Omega\mathrm{s}_\omega + \mathrm{c}_\Omega\mathrm{c}_\omega\mathrm{c}_i & -\mathrm{c}_\Omega\mathrm{s}_i \\ \mathrm{s}_\omega\mathrm{s}_i & \mathrm{c}_\omega\mathrm{s}_i & \mathrm{c}_i \end{bmatrix}\begin{bmatrix}\mathrm{c}_f \\ \mathrm{s}_f \\ 0\end{bmatrix}, \tag{3a}$$

$$\boldsymbol{r}_3 = r_3\begin{bmatrix} \mathrm{c}_{\Omega_3}\mathrm{c}_{\omega_3} - \mathrm{s}_{\Omega_3}\mathrm{s}_{\omega_3}\mathrm{c}_{i_3} & -\mathrm{c}_{\Omega_3}\mathrm{s}_{\omega_3} - \mathrm{s}_{\Omega_3}\mathrm{c}_{\omega_3}\mathrm{c}_{i_3} & \mathrm{s}_{\Omega_3}\mathrm{s}_{i_3} \\ \mathrm{s}_{\Omega_3}\mathrm{c}_{\omega_3} + \mathrm{c}_{\Omega_3}\mathrm{s}_{\omega_3}\mathrm{c}_{i_3} & -\mathrm{s}_{\Omega_3}\mathrm{s}_{\omega_3} + \mathrm{c}_{\Omega_3}\mathrm{c}_{\omega_3}\mathrm{c}_{i_3} & -\mathrm{c}_{\Omega_3}\mathrm{s}_{i_3} \\ \mathrm{s}_{\omega_3}\mathrm{s}_{i_3} & \mathrm{c}_{\omega_3}\mathrm{s}_{i_3} & \mathrm{c}_{i_3} \end{bmatrix}\begin{bmatrix}\mathrm{c}_{f_3} \\ \mathrm{s}_{f_3} \\ 0\end{bmatrix}, \tag{3b}$$

where $\mathrm{s}_x \triangleq \sin(x)$, $\mathrm{c}_x \triangleq \cos(x)$. Substituting Eq. (3) into Eq. (2) yields

$$\cos\varphi = \alpha\cos f + \beta\sin f, \tag{4}$$

where

$$\begin{aligned}\alpha &= (\cos\omega\cos\theta - \sin\omega\cos i\sin\theta)\cos u_3 + (\sin\omega\sin i\sin i_3 + \cos\omega\cos i_3\sin\theta \\ &\quad + \sin\omega\cos i\cos i_3\cos\theta)\sin u_3 \\ &\triangleq \alpha_1\cos u_3 + \alpha_2\sin u_3,\end{aligned} \tag{5a}$$

$$\beta = (-\sin\omega\cos\theta - \cos\omega\cos i\sin\theta)\cos u_3 + (\cos\omega\sin i\sin i_3 - \sin\omega\cos i_3\sin\theta + \cos\omega\cos i\cos i_3\cos\theta)\sin u_3 \tag{5b}$$
$$\triangleq \beta_1\cos u_3 + \beta_2\sin u_3,$$

and

$$\theta = \Omega - \Omega_3, \tag{6a}$$
$$u_3 = \omega_3 + f_3. \tag{6b}$$

Considering that the radius of the perturbing body is much larger than that of the spacecraft ($r_3 \gg r$), the Hamiltonian, after truncating the Legendre polynomials of the disturbing function up to second order, becomes

$$\mathcal{H} = -\frac{\mu}{2a} - \frac{Gm_3}{2r_3}\left(\frac{r}{r_3}\right)^2\left(3\cos^2\varphi - 1\right). \tag{7}$$

We use the relation

$$G\left(m_1 + m_3\right) = {n_3}^2{a_3}^3, \tag{8}$$

where m_1 is the mass of the central body, n_3 is the mean motion of the perturbing body, and a_3 is the semimajor axis of the perturbing body. Substituting Eq. (4) into Eq. (7) yields

$$\mathcal{H} = -\frac{\mu}{2a} - \frac{k{n_3}^2{a_3}^3}{2r_3}\left(\frac{r}{r_3}\right)^2\left(3\alpha^2\cos^2 f + 3\beta^2\sin^2 f + 3\alpha\beta\sin 2f - 1\right), \tag{9}$$

where

$$k = m_3/(m_1 + m_3). \tag{10}$$

In order to use the canonical theory, Delaunay elements $\{L, G, H, l, g, h\}$ are introduced,

$$L = \sqrt{\mu a}, \qquad l = M, \tag{11a}$$
$$G = L\sqrt{1-e^2}, \quad g = \omega, \tag{11b}$$
$$H = G\cos i, \quad h = \Omega. \tag{11c}$$

Moreover, the Hamiltonian is a function of the true anomaly, argument of perigee, right ascension of the ascending node and inclination of the perturbing body, as shown in Eqs. (5) and (9), other than the orbital elements of the orbiter. However, the periods of the argument of perigee, right ascension of ascending node and inclination of the perturbing body are much longer than the periods of the other elements. These slowly varying elements of the perturbing body can be assumed constant. Only the variation of the true anomaly of the perturbing body should be considered. This introduces one more degree of freedom. Additional two canonical variables (M_3, T) are introduced as in Giacaglia et al. (1970), satisfying

$$\frac{\mathrm{d}M_3}{\mathrm{d}t} = \frac{\partial\mathcal{H}^\times}{\partial T} = n_3, \tag{12a}$$
$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\partial\mathcal{H}^\times}{\partial M_3}, \tag{12b}$$

where M_3 is the mean anomaly of the perturbing body, so the new Hamiltonian becomes

$$\mathcal{H}^\times = \mathcal{H} + n_3 T = -\frac{\mu}{2a} + n_3 T - \frac{k{n_3}^2{a_3}^3}{2r_3}\left(\frac{r}{r_3}\right)^2\left(3\alpha^2\cos^2 f + 3\beta^2\sin^2 f + 3\alpha\beta\sin 2f - 1\right). \tag{13}$$

Comparing the new Hamiltonian with that in De Saedeleer (2006) (where the third-body perturbation, contained in the Hamiltonian, was expressed in a rotating frame) reveals that the Hamiltonians have similar forms. The term $n_3 T$ can be approximated as the Coriolis effect term $-n_3 H$ appearing in De Saedeleer (2006), where H is the same as defined in Eq. (11c). Thus, the new canonical variable represents the angular momentum, and its magnitude is approximately H. Based on the two-body relationship, the ratios between these terms are approximated by

$$k_1 = \frac{n_3 T}{-\mu/(2a)} \approx \frac{n_3 H}{\mu/(2a)} \approx \frac{n_3\sqrt{\mu a}\sqrt{1-e^2}\cos i}{\mu/(2a)} \approx \frac{n_3}{n}\left(2\sqrt{1-e^2}\cos i\right), \tag{14a}$$

$$\begin{aligned} k_2 &= \frac{\dfrac{k{n_3}^2{a_3}^3}{2r_3}\left(\dfrac{r}{r_3}\right)^2\left(3\alpha^2\cos^2 f + 3\beta^2\sin^2 f + 3\alpha\beta\sin 2f - 1\right)}{\mu/(2a)} \\ &= \frac{k n_3^2}{n^2}\frac{\left(1-e^2\right)^2(e_3\cos f_3+1)^3}{\left(1-{e_3}^2\right)^3(e\cos f+1)^2}\left(3\alpha^2\cos^2 f + 3\beta^2\sin^2 f + 3\alpha\beta\sin 2f - 1\right), \end{aligned} \tag{14b}$$

where n is the mean motion of the orbiter. Furthermore, k_1 and k_2 can be approximated as n_3/n and kn_3^2/n^2. For a lunar orbiter, perturbed by the third-body effect induced by Earth, the mass ratio shown in Eq. (10) is nearly 1, and n_3 is smaller than n. Choosing n_3/n as the small parameter, $n_3 T$ and the third-body perturbation term can be regarded as first order and second order, respectively. The Hamiltonian can be rearranged as

$$\mathcal{H}^{\times} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2, \tag{15}$$

with

$$\mathcal{H}_0 = -\frac{\mu}{2a}, \tag{16a}$$

$$\mathcal{H}_1 = n_3 T, \tag{16b}$$

$$\mathcal{H}_2 = -\frac{k{n_3}^2{a_3}^3}{2r_3}\left(\frac{r}{r_3}\right)^2\left(3\alpha^2\cos^2 f + 3\beta^2\sin^2 f + 3\alpha\beta\sin 2f - 1\right). \tag{16c}$$

As shown in Eq. (13), the Hamiltonian contains trigonometric functions of f, ω, Ω, and f_3. According to the periods of these angles, it can be divided into short-period, medium-period, and long-period variations. The terms containing f represent short-period variations, which are usually several hours. The angle f_3 has the period of the perturbing body, i.e., about one month for a lunar orbiter, which can be regarded as the medium-period variation. The periods of ω and Ω are several years for a lunar orbiter perturbed by Earth (De Saedeleer 2006), constituting the long-period variation. The orbiter dynamics including secular and long-period variations will be derived via von Zeipel's method (Kozai 1959a) in the following subsection.

2.1 Elimination of the short-period variation

According to von Zeipel's algorithm, the Hamiltonian $\mathcal{H}(L, G, H, T, l, g, h, M_3)$ is transformed into $\mathcal{H}^*(L', G', H', T', -, g', h', M_3')$ under the generating function

$$S^* = L'l + G'g + H'h + T'M_3 + S_1^* + S_2^*, \tag{17}$$

where S_1^* and S_2^* are the first-order and second-order terms in n_3, respectively. Equating like-order terms in the new Hamiltonian $\mathcal{H}^*$ and the original one $\mathcal{H}^{\times}$ yields

$$\mathcal{H}_0^* = \mathcal{H}_0(L'), \tag{18a}$$

$$\mathcal{H}_1^* = \mathcal{H}_1(T') + \frac{\partial S_1^*}{\partial l}\frac{\partial \mathcal{H}_0}{\partial L'}, \tag{18b}$$

$$\mathcal{H}_2^* = \mathcal{H}_2 + \frac{\partial S_2^*}{\partial l}\frac{\partial \mathcal{H}_0}{\partial L'} + \frac{1}{2}\left(\frac{\partial S_1^*}{\partial l}\right)^2\frac{\partial^2 \mathcal{H}_0}{\partial L'^2} + \frac{\partial S_1^*}{\partial M_3}\frac{\partial \mathcal{H}_1}{\partial T'}, \tag{18c}$$

$$\begin{aligned}\mathcal{H}_3^* = &\frac{\partial S_3^*}{\partial l}\frac{\partial \mathcal{H}_0}{\partial L'} + \frac{1}{6}\left(\frac{\partial S_1^*}{\partial l}\right)^3\frac{\partial^3 \mathcal{H}_0}{\partial L'^3} + \frac{\partial S_1^*}{\partial l}\frac{\partial S_2^*}{\partial l}\frac{\partial^2 \mathcal{H}_0}{\partial L'^2} + \frac{\partial S_2^*}{\partial M_3}\frac{\partial \mathcal{H}_1}{\partial T'}\\ &+ \frac{1}{2}\left(\frac{\partial S_1^*}{\partial M_3}\right)^2\frac{\partial^2 \mathcal{H}_1}{\partial T'^2} + \frac{\partial S_1^*}{\partial l}\frac{\partial \mathcal{H}_2}{\partial L'} + \frac{\partial S_1^*}{\partial h}\frac{\partial \mathcal{H}_2}{\partial H'} + \frac{\partial S_1^*}{\partial g}\frac{\partial \mathcal{H}_2}{\partial G'}\\ &+ \frac{\partial S_1^*}{\partial M_3}\frac{\partial \mathcal{H}_2}{\partial T'} - \frac{\partial S_1^*}{\partial G'}\frac{\partial \mathcal{H}_2^*}{\partial g'} - \frac{\partial S_1^*}{\partial H'}\frac{\partial \mathcal{H}_2^*}{\partial h'} - \frac{\partial S_1^*}{\partial T'}\frac{\partial \mathcal{H}_2^*}{\partial M_3}.\end{aligned} \tag{18d}$$

Solving Eqs. (18a) and (18b) yields

$$\mathcal{H}_0^* = -\frac{\mu}{2a'}, \quad \mathcal{H}_1^* = n_3 T', \quad S_1^* = 0. \tag{19}$$

Substituting Eq. (19) into Eq. (18c),

$$\mathcal{H}_2^* = \mathcal{H}_2 + \frac{\partial S_2^*}{\partial l}\frac{\partial \mathcal{H}_0}{\partial L'}, \tag{20}$$

the solutions are

$$\begin{aligned}\mathcal{H}_2^* &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_2 \mathrm{d}l\\ &= -\frac{k{n_3}^2 a'^2 {a_3}^3}{4{r_3}^3}\left[3\alpha'^2\left(4e'^2+1\right) - 3\beta'^2\left(e'^2-1\right) - 3e'^2 - 2\right],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathcal{S}_2^* &= \frac{L'^3}{\mu^2}\int \left(\mathcal{H}_2^* - \mathcal{H}_2\right)\mathrm{d}l\\ &= \frac{L'^3}{\mu^2}\mathcal{H}_2^* l + \frac{L'^3}{\mu^2}\frac{k{n_3}^2{a_3}^3}{2{r_3}^3}\left\{3\alpha^2\int r^2\cos^2 f\,\mathrm{d}l + 3\beta^2\int r^2\sin^2 f\,\mathrm{d}l\right.\\ &\quad\left. + 3\alpha\beta\int r^2\sin 2f\,\mathrm{d}l - \int r^2\mathrm{d}l\right\}.\end{aligned} \tag{22}$$

Using the two-body relations,

$$r = \frac{a\left(1-e^2\right)}{1+e\cos f}, \tag{23a}$$

$$\mathrm{d}l = \frac{\left(1-e^2\right)^{3/2}}{(1+e\cos f)^2}\mathrm{d}f. \tag{23b}$$

Using a symbolic algebra software, the following integrals are calculated:

$$\begin{aligned}\int r^2\cos^2 f\,\mathrm{d}l &= a^2\left(1-e^2\right)^{\frac{7}{2}}\int \frac{\cos^2 f}{(1+e\cos f)^4}\mathrm{d}f\\ &= a^2\left(4e^2+1\right)\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{f}{2}\right)\right] - \frac{a^2\sqrt{1-e^2}\sin f}{12(e\cos f+1)^3}\end{aligned}$$

$$\times \left[6\left(2e^4+9e^2-1\right)\cos f + e\left(6e^4+14e^2+\left(6e^4+10e^2-1\right)\cos 2f+25\right)\right], \tag{24a}$$

$$\begin{aligned}\int r^2 \sin^2 f \mathrm{d}l &= a^2(1-e^2)^{\frac{7}{2}} \int \frac{\sin^2 f}{(1+e\cos f)^4}\mathrm{d}f \\ &= -a^2\left(e^2-1\right)\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{f}{2}\right)\right] + \frac{a^2\sqrt{1-e^2}\left(e^2-1\right)\sin f}{12(e\cos f+1)^3} \\ &\times \left\{6\left(e^2+1\right)\cos f + e\left[\left(2e^2+1\right)\cos 2f - 2e^2+11\right]\right\},\end{aligned} \tag{24b}$$

$$\begin{aligned}\int r^2 \mathrm{d}l &= a^2(1-e^2)^{\frac{7}{2}} \int \frac{1}{(1+e\cos f)^4}\mathrm{d}f \\ &= a^2\left(3e^2+2\right)\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{f}{2}\right)\right] - \frac{a^2 e\sqrt{1-e^2}\sin f}{12(e\cos f+1)^3} \\ &\times \left[8e^4+\left(4e^2+11\right)e^2\cos 2f + 6\left(e^2+9\right)e\cos f + e^2+36\right].\end{aligned} \tag{24c}$$

Utilizing the two-body relationship between the eccentric anomaly E and true anomaly f (Vallado 2001),

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{f}{2}\right), \tag{25}$$

yields

$$\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{f}{2}\right)\right] = \frac{E}{2}. \tag{26}$$

Substituting Eq. (26) into Eq. (24) yields

$$\begin{aligned}\int r^2 \cos^2 f \mathrm{d}l = \frac{1}{2}a'^2\left(4e'^2+1\right)E - \frac{a'^2\sqrt{1-e'^2}\sin f}{12(e'\cos f+1)^3}\left[6\left(2e'^4+9e'^2-1\right)\cos f\right. \\ \left. + e'\left(6e'^4+14e'^2+\left(6e'^4+10e'^2-1\right)\cos 2f+25\right)\right],\end{aligned} \tag{27a}$$

$$\begin{aligned}\int r^2 \sin^2 f \mathrm{d}l = -\frac{1}{2}a'^2\left(e'^2-1\right)E + \frac{a'^2\sqrt{1-e'^2}\left(e'^2-1\right)\sin f}{12(e'\cos f+1)^3} \\ \times \left\{6\left(e'^2+1\right)\cos f + e'\left[\left(2e'^2+1\right)\cos 2f - 2e'^2+11\right]\right\},\end{aligned} \tag{27b}$$

$$\begin{aligned}\int r^2 \mathrm{d}l = \frac{1}{2}a'^2\left(3e'^2+2\right)E - \frac{a'^2 e'\sqrt{1-e'^2}\sin f}{12(e'\cos f+1)^3}\left[8e'^4+e'^2\right. \\ \left. + \left(4e'^2+11\right)e'^2\cos 2f + 6\left(e'^2+9\right)e'\cos f + 36\right].\end{aligned} \tag{27c}$$

It is noticed that a singularity happens when integrating $r^2 \sin 2f$ with respect to f. It can be avoided by using an eighth-order Taylor series expansion,

$$\begin{aligned}\frac{\sin 2f}{(1+e\cos f)^4} \approx \sin 2f \left\{1 - 4(e\cos f) + 10(e\cos f)^2 - 20(e\cos f)^3 + 35(e\cos f)^4\right. \\ \left. - 56(e\cos f)^5 + 84(e\cos f)^6 - 120(e\cos f)^7 + 165(e\cos f)^8\right\},\end{aligned} \tag{28}$$

which results in the approximation,

$$\int r^2 \sin 2f \mathrm{d}l \approx a'^2(1-e'^2)^{7/2}\left\{-\frac{33}{512}e'^8\left[5\left(84\cos^2 f+24\cos 4f+9\cos 6f+2\cos 8f\right)\right.\right.$$
$$\left.+\cos 10f\right]+\frac{80}{3}e'^7\cos^9 f-\frac{21}{128}e'^6\left(56\cos 2f+28\cos 4f+8\cos 6f\right.$$
$$\left.+\cos 8f\right)+16e'^5\cos^7 f-\frac{35}{96}e'^4\left(30\cos^2 f+6\cos 4f+\cos 6f\right)$$
$$\left.+8e'^3\cos^5 f-\frac{5}{8}e'^2\left(8\cos^2 f+\cos 4f\right)+\frac{8}{3}e'\cos^3 f-\frac{1}{2}\cos 2f\right\}. \tag{29}$$

Finally, substituting Eqs. (21), (27), (29) into Eq. (22) yields the short-period generating function

$$S_2^* = -\frac{kn_3{}^2a'^{7/2}\sqrt{1-e'^2}(1+e_3\cos f_3)^3}{24\mu^{1/2}(1-e_3{}^2)^3(1+e'\cos f)^3}\left\{3\alpha'^2\left(1-e'^2\right)\sin f\left[\left(6e'^2-1\right)\left(e'\cos 2f\right.\right.\right.$$
$$\left.\left.+6\cos f\right)+e'\left(6e'^2+19\right)\right]+3\beta'^2\left(e'^2-1\right)^2\sin f\left[e'(\cos 2f+5)+6\cos f\right]$$
$$-e'\left(1-e'^2\right)\sin f\left[e'^2+5e'(e'\cos 2f+6\cos f)+24\right]-36\alpha'\beta'\left(1-e'^2\right)^3$$
$$\times\left(1+e'\cos f\right)^3\left\{-\frac{33}{512}e'^8\left[5\left(84\cos^2 f+24\cos 4f+9\cos 6f+2\cos 8f\right)\right.\right.$$
$$\left.+\cos 10f\right]+\frac{80}{3}e'^7\cos^9 f-\frac{21}{128}e'^6\left(56\cos 2f+28\cos 4f+8\cos 6f+\cos 8f\right)$$
$$+16e'^5\cos^7 f-\frac{35}{96}e'^4\left(30\cos^2 f+6\cos 4f+\cos 6f\right)+8e'^3\cos^5 f$$
$$\left.\left.-\frac{5}{8}e'^2\left(8\cos^2 f+\cos 4f\right)+\frac{8}{3}e'\cos^3 f-\frac{1}{2}\cos 2f\right\}\right\}. \tag{30}$$

Then, substituting Eqs. (19), (21) and (30) into Eq. (18d) yields the third-order Hamiltonian

$$\mathcal{H}_3^* = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial S_2^*}{\partial h}\frac{\partial \mathcal{H}_1}{\partial H'}\mathrm{d}l$$
$$= -\frac{kn_3{}^3a'^{7/2}e'^2\left(1-e'^2\right)^5(e_3\cos f_3+1)^4}{65536\mu^{1/2}\left(1-e_3{}^2\right)^{9/2}}\left(5623695e'^{12}+9482544e'^{10}+10961216e'^8\right.$$
$$\left.+8736000e'^6+4227840e'^4+1971200e'^2+479232\right)\left\{\alpha_1{}'\beta_1{}'\cos(f_3+\omega_3)\right.$$
$$\times\left[-3e_3\sin f_3\cos(f_3+\omega_3)-2(e_3\cos f_3+1)\sin(f_3+\omega_3)\right]+\left(\alpha_1{}'\beta_2{}'+\alpha_2{}'\beta_1{}'\right)$$
$$\times\left(\cos(2f_3+2\omega_3)-\frac{1}{4}e_3\left[\cos(f_3+2\omega_3)-5\cos(3f_3+2\omega_3)\right]\right)$$
$$\left.+\alpha_2{}'\beta_2{}'\sin(f_3+\omega_3)\left[2(e_3\cos f_3+1)\cos(f_3+\omega_3)-3e_3\sin f_3\sin(f_3+\omega_3)\right]\right\}. \tag{31}$$

For the second-order solution, it is not necessary to obtain the third-order generating function.

2.2 Elimination of the medium-period variation

The terms in the Hamiltonian depending on M_3 induce the medium-period variation. Using the generating function

$$S^{**} = L''l' + G''g' + H''h' + S_1^{**} + S_2^{**}, \tag{32}$$

the new Hamiltonian $\mathcal{H}^{**}(L'', G'', H'', T''-, g'', h'', -)$ is determined by

$$\mathcal{H}_0^{**} = \mathcal{H}_0^{*}(L''), \tag{33a}$$

$$\mathcal{H}_1^{**} = \mathcal{H}_1^{*}(T''), \tag{33b}$$

$$\mathcal{H}_2^{**} = \mathcal{H}_2^{*} + \frac{\partial S_1^{**}}{\partial M_3'}\frac{\partial \mathcal{H}_1^{*}}{\partial T''}, \tag{33c}$$

$$\begin{aligned}\mathcal{H}_3^{**} = \mathcal{H}_3^{*} &+ \frac{\partial S_2^{**}}{\partial M_3'}\frac{\partial \mathcal{H}_1^{*}}{\partial T''} + \frac{1}{2}\left(\frac{\partial S_1^{**}}{\partial M_3'}\right)^2\frac{\partial^2 \mathcal{H}_1^{*}}{\partial T''^2} + \frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^{*}}{\partial H''}\\ &+ \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^{*}}{\partial G''} + \frac{\partial S_1^{**}}{\partial M_3'}\frac{\partial \mathcal{H}_2^{*}}{\partial T''} - \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} - \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'}.\end{aligned} \tag{33d}$$

According to Eqs. (33a) and (33b), we obtain

$$\mathcal{H}_0^{**} = -\frac{\mu}{2a'}, \tag{34a}$$

$$\mathcal{H}_1^{**} = n_3 T''. \tag{34b}$$

Based on Eq. (33c), the second-order term in the Hamiltonian is

$$\begin{aligned}\mathcal{H}_2^{**} &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_2^{*}\mathrm{d}M_3'\\ &= -\frac{kn_3{}^2a''^2}{128\left(1-e_3{}^2\right)^{3/2}}\left\{3e''^2\left\{40\sin i''\sin 2i_3\sin 2\omega'\sin\theta' + 4\sin 2i''\right.\right.\\ &\times \sin 2i_3\left(3-5\cos 2\omega'\right)\cos\theta' - 40\cos i''\sin^2 i_3\sin 2\omega'\sin 2\theta' + \sin^2 i_3\\ &\times \cos 2\theta'\left[10\left(\cos 2i''+3\right)\cos 2\omega' + 12\sin^2 i''\right] + (3\cos 2i_3+1)\\ &\times \left.\left(10\sin^2 i''\cos 2\omega' + 3\cos 2i''+1\right)\right\} + 2\left[12\sin^2 i''\sin^2 i_3\cos 2\theta'\right.\\ &\left.\left.+ 12\sin 2i''\sin 2i_3\cos\theta' + \left(3\cos 2i''+1\right)(3\cos 2i_3+1)\right]\right\},\end{aligned} \tag{35}$$

and the generating function S_1^{**} is determined by

$$\begin{aligned}S_1^{**} &= \frac{1}{n_3}\int(\mathcal{H}_2^{**} - \mathcal{H}_2^{*})\mathrm{d}M_3'\\ &= \frac{kn_3a''^2}{16\left(1-e_3{}^2\right)^{3/2}}\left\{-2\left[3\left(4e''^2+1\right)\left(\alpha_1''^2+\alpha_2''^2\right) + 3\left(1-e''^2\right)\left(\beta_1''^2+\beta_2''^2\right)\right.\right.\\ &\left.-6e''^2-4\right]\left[2\tan^{-1}\left(\sqrt{\frac{1-e_3}{1+e_3}}\tan\frac{f_3}{2}\right) - \frac{\sqrt{1-e_3{}^2}e_3\sin f_3}{(e_3\cos f_3+1)} - f_3\right.\\ &\left.-e_3\sin f_3\right] + \left[\left(\alpha_1''^2-\alpha_2''^2\right)\left(4e''^2+1\right) - \left(e''^2-1\right)\left(\beta_1''^2-\beta_2''^2\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\times \{e_3 [3\sin(f_3 + 2\omega_3) + \sin(3f_3 + 2\omega_3)] + 3\sin(2f_3 + 2\omega_3)\} \\
&- 2\left[e''^2(4\alpha_1''\alpha_2'' - \beta_1''\beta_2'') + \alpha_1''\alpha_2'' + \beta_1''\beta_2''\right]\{e_3 [3\cos(f_3 + 2\omega_3) \\
&+ \cos(3f_3 + 2\omega_3)] + 3\cos(2f_3 + 2\omega_3)\}\Big\}.
\end{aligned} \tag{36}
$$

Substituting Eqs. (34), (35) and (36) into Eq. (33d), the third-order Hamiltonian term is

$$
\begin{aligned}
\mathcal{H}_3^{**} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_3^* \mathrm{d}\left(M_3'\right) + \frac{1}{2\pi}\int_0^{2\pi} \Bigg\{ &\frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^*}{\partial H''} + \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^*}{\partial G''} \\
&- \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} - \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'} \Bigg\} \mathrm{d}\left(M_3'\right),
\end{aligned} \tag{37}
$$

and the second-order generating function is

$$
\begin{aligned}
S_2^{**} &= \frac{1}{n_3}\int \left\{ \mathcal{H}_3^{**} - \mathcal{H}_3^* - \frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^*}{\partial H''} - \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^*}{\partial G''} + \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} + \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'} \right\} \mathrm{d}M_3' \\
&\triangleq S_{21}^{**} + S_{22}^{**},
\end{aligned} \tag{38}
$$

where

$$
S_{21}^{**} = \frac{1}{n_3}\int \left\{ \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_3^* \mathrm{d}\left(M_3'\right) - \mathcal{H}_3^* \right\} \mathrm{d}\left(M_3'\right), \tag{39a}
$$

$$
S_{22}^{**} = \frac{1}{n_3}\int \left\{ \begin{array}{l} \frac{1}{2\pi}\int_0^{2\pi} \left\{ \frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^*}{\partial H''} + \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^*}{\partial G''} - \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} - \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'} \right\} \mathrm{d}\left(M_3'\right) \\ - \frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^*}{\partial H''} - \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^*}{\partial G''} + \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} + \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'} \end{array} \right\} \mathrm{d}\left(M_3'\right). \tag{39b}
$$

According to Eq. (31), it is found that

$$
\frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_3^* \mathrm{d}M_3' = 0. \tag{40}
$$

Furthermore,

$$
\begin{aligned}
S_{21}^{**} &= -\frac{1}{n_3}\int \mathcal{H}_3^* \mathrm{d}M_3 \\
&= \frac{k{n_3}^2 a''^{7/2} e''^2 \left(1 - e''^2\right)^5}{1048576\mu^{1/2}} \left(5623695e''^{12} + 9482544e''^{10} + 10961216e''^8\right. \\
&\quad \left. + 8736000e''^6 + 4227840e''^4 + 1971200e''^2 + 479232\right) \\
&\quad \times \left\{\alpha_1''\beta_1'' P + \left(\alpha_1''\beta_2'' + \alpha_2''\beta_1''\right) Q + \alpha_2''\beta_2'' F\right\},
\end{aligned} \tag{41}
$$

where

$$
\begin{aligned}
P = &\frac{1}{\left(1 - {e_3}^2\right)^3} \left\{\cos 2\omega_3 \left\{4e_3\left({e_3}^2 + 3\right)\cos f_3 + 4\left(3{e_3}^2 + 2\right)\cos 2f_3\right.\right. \\
&\left.+ e_3\left[{e_3}^2\cos 5f_3 + 3\left({e_3}^2 + 4\right)\cos 3f_3 + 6e_3\cos 4f_3\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2e_3\left[3\left(e_3{}^2+4\right)\cos f_3 + e_3(e_3\cos 3f_3 + 6\cos 2f_3)\right]\\
&- 8\sin 2f_3 \sin 2\omega_3 (e_3\cos f_3+1)^3\Big\}, \qquad (42a)
\end{aligned}
$$

$$
\begin{aligned}
Q = &\frac{1}{\left(1-e_3{}^2\right)^3}\Big\{\sin 2\omega_3\left[e_3{}^3\cos 5f_3 + 6e_3{}^2\cos 4f_3 + 4\left(e_3{}^2+3\right)e_3\cos f_3\right.\\
&\left.+3\left(e_3{}^2+4\right)e_3\cos 3f_3 + 4\left(3e_3{}^2+2\right)\cos 2f_3\right]\\
&+8\sin 2f_3\cos 2\omega_3(e_3\cos f_3+1)^3\Big\}, \qquad (42b)
\end{aligned}
$$

$$
\begin{aligned}
F = &\frac{1}{\left(1-e_3{}^2\right)^3}\Big\{-\cos 2\omega_3\Big\{4e_3\left(e_3{}^2+3\right)\cos f_3 + 4\left(3e_3{}^2+2\right)\cos 2f_3\\
&+e_3\left[e_3{}^2\cos 5f_3 + 3\left(e_3{}^2+4\right)\cos 3f_3 + 6e_3\cos 4f_3\right]\Big\}\\
&+ 2e_3\left[3\left(e_3{}^2+4\right)\cos f_3 + e_3(e_3\cos 3f_3+6\cos 2f_3)\right]\\
&+8\sin 2f_3\sin 2\omega_3(e_3\cos f_3+1)^3\Big\}. \qquad (42c)
\end{aligned}
$$

Using the Hamiltonian $\mathcal{H}_2^{**}$ in Eq. (35) and the generating function S_1^{**} in Eq. (36), we can calculate the integral term in Eq. (39b),

$$
\begin{aligned}
&\frac{\partial S_1^{**}}{\partial h'}\frac{\partial \mathcal{H}_2^{*}}{\partial H''} + \frac{\partial S_1^{**}}{\partial g'}\frac{\partial \mathcal{H}_2^{*}}{\partial G''} - \frac{\partial S_1^{**}}{\partial G''}\frac{\partial \mathcal{H}_2^{**}}{\partial g'} - \frac{\partial S_1^{**}}{\partial H''}\frac{\partial \mathcal{H}_2^{**}}{\partial h'}\\
&= \frac{k^2 n_3{}^2 a''^{7/2}}{1024\mu^{1/2}\left(1-e_3{}^2\right)^3}\Big\{\Big\{6e''^2\left[-\sin 2i''(5\cos 2\omega'-3)\left(-2\sin^2 i_3\cos 2\theta' + 3\cos 2i_3\right.\right.\\
&\left.+1\right) - 20\sin 2\omega'\left(\sin i''\sin^2 i_3\sin 2\theta' + \cos i''\sin 2i_3\sin\theta'\right) + 4\cos 2i''\sin 2i_3\\
&\left.\times(5\cos 2\omega'-3)\cos\theta'\right] + 4\left[\sin 2i''\left(-6\sin^2 i_3\cos 2\theta' + 9\cos 2i_3 + 3\right)\right.\\
&\left.-12\cos 2i''\sin 2i_3\cos\theta'\right]\Big\} f_{gh}(A_1,B_1,C_1) + \Big\{4e''^2\left[-\sin 2i''(\cos 2i_3+3)\right.\\
&\times(5\cos 2\omega'-3)\cos 2\theta' + 4\sin 2i_3\left(\cos 2i''(5\cos 2\omega'-3)\cos\theta' - 5\cos i''\right.\\
&\left.\times\sin 2\omega'\sin\theta'\right) + 10\sin i''(\cos 2i_3+3)\sin 2\omega'\sin 2\theta' + 6\sin 2i''\sin^2 i_3\\
&\left.\times(5\cos 2\omega'-3)\right] + 8\sin 2i''(\cos 2i_3+3)\cos 2\theta' - \cos 2i''\sin 2i_3\cos\theta'\\
&-3\sin i''\cos i''\sin^2 i_3\Big\} f_{gh}\left(A_2,B_2,C_2\right) + \Big\{32\sin\theta'(\sin 2i''\cos i_3\cos\theta'\\
&-\cos 2i''\sin i_3) + 8e''^2\left[10\sin 2\omega'(\cos i''\sin i_3\cos\theta' - \sin i''\cos i_3\cos 2\theta')\right.\\
&\left.+2\cos 2i''\sin i_3(5\cos 2\omega'-3)\sin\theta' + \sin 2i''\cos i_3(3-5\cos 2\omega')\sin 2\theta'\right]\Big\}\\
&\times f_{gh}(A_3,B_3,C_3) + 3e''\left[-80\sin i''\sin 2i_3\sin 2\omega'\sin\theta' - 2\sin^2 i_3\cos 2\theta'\right.\\
&\times\left(10(\cos 2i''+3)\cos 2\omega' + 12\sin^2 i''\right) + 80\cos i''\sin^2 i_3\sin 2\omega'\sin 2\theta'\\
&+8\sin 2i''\sin 2i_3(5\cos 2\omega'-3)\cos\theta' - 2(3\cos 2i_3+1)\left(10\sin^2 i''\cos 2\omega'\right.\\
&\left.\left.+3\cos 2i''+1\right)\right] f_g(A_1,B_1,C_1) + 2e''\left[(\cos 2i_3+3)\cos 2\theta'\left(12\sin^2 i''\right.\right.\\
&\left.+10(\cos 2i''+3)\cos 2\omega'\right) + 8\sin 2i_3\left(\sin 2i''(5\cos 2\omega'-3)\cos\theta' - 10\sin i''\right.\\
&\left.\times\sin 2\omega'\sin\theta'\right) - 40\cos i''(\cos 2i_3+3)\sin 2\omega'\sin 2\theta' + 12\sin^2 i_3\left(3\cos 2i''\right.\\
&\left.\left.+10\sin^2 i''\cos 2\omega'+1\right)\right] f_g\left(A_2,B_2,C_2\right) + 8e\left[\cos i_3\sin 2\theta'\left(6\sin^2 i''\right.\right.\\
&\left.+5(\cos 2i''+3)\cos 2\omega'\right) + 20\sin 2\omega'(\cos i''\cos i_3\cos 2\theta' + \sin i''\sin i_3\cos\theta')\\
&\left.+2\sin 2i''\sin i_3(5\cos 2\omega'-3)\sin\theta'\right] f_g(A_3,B_3,C_3)\Big\}, \qquad (43)
\end{aligned}
$$

where the functions $f_{gh}(\chi,\gamma,\kappa)$, $f_g(\chi,\gamma,\kappa)$ are

$$\begin{aligned} f_{gh}(\chi,\gamma,\kappa) = \frac{1}{8\sqrt{1-e''^2}\sin i''} & \{24\chi \sin i'' \sin i_3 \sin\theta'(\sin i'' \sin i_3 \cos\theta' + \cos i'' \cos i_3) \\ & + 4\gamma\left[\sin 2i'' \sin 2i_3 \sin\theta' - \sin^2 i''(\cos 2i_3 + 3)\sin 2\theta'\right] + 8\kappa \sin i'' \\ & \times \left(\sin i'' \cos i_3 \cos 2\theta' - \cos i'' \sin i_3 \cos\theta'\right) + e''^2 \{2\cos i_3 \left[\sin i_3(3\chi + 2\gamma)\right. \\ & \times \left(5(\sin 3i'' - \sin i'')\sin 2\omega' \cos\theta' + \sin 2i''(5\cos 2\omega' + 3)\sin\theta'\right) \\ & - 5\kappa \sin 2i'' \sin i'' \sin 2\omega' \sin 2\theta' + 2\kappa \sin^2 i''(5\cos 2\omega' + 3)\cos 2\theta'\left.\right] \\ & - 5\sin^2 i'' \cos i'' \sin 2\omega'\left[\cos 2i_3(3\chi + 2\gamma)(\cos 2\theta' + 3) + 6(\chi - 2\gamma)\sin^2\theta\right] \\ & - \sin^2 i''(5\cos 2\omega' + 3)\sin 2\theta'\left[\cos 2i_3(3\chi + 2\gamma) - 3\chi + 6\gamma\right] \\ & + 2\kappa \sin i_3\left[5(\sin 3i'' - \sin i'')\sin 2\omega' \sin\theta' - \sin 2i''(5\cos 2\omega' + 3)\cos\theta'\right]\}\}, \end{aligned} \tag{44a}$$

$$\begin{aligned} f_g(\chi,\gamma,\kappa) = \frac{5}{16} e''\sqrt{1-e''^2} & \{-\sin 2\omega'\left[(\cos 2i'' + 3)\cos 2\theta'\left(\cos 2i_3(3\chi + 2\gamma) - 3\chi + 6\gamma\right)\right. \\ & + 4\sin 2i'' \sin 2i_3(3\chi + 2\gamma)\cos\theta' - 6\sin^2 i''\left(\cos 2i_3(3\chi + 2\gamma) + \chi - 2\gamma\right) \\ & + 8\kappa \sin 2i'' \sin i_3 \sin\theta' + 4\kappa(\cos 2i'' + 3)\cos i_3 \sin 2\theta\left.\right] + 8\cos i'' \cos 2\omega \\ & \times \left[2\kappa \cos i_3 \cos 2\theta' - \sin 2\theta'\left(\gamma(\cos 2i_3 + 3) - 3\chi \sin^2 i_3\right)\right] \\ & + 16\sin i'' \sin i_3 \cos 2\omega'\left[\kappa \cos\theta' - \cos i_3(3\chi + 2\gamma)\sin\theta'\right]\}, \end{aligned} \tag{44b}$$

and corresponding variables A_i, B_i, C_i $(i = 1, 2, 3)$ are

$$A_1 = 2\left(1 - \frac{2(1 + e_3\cos f_3)^3}{\left(1 - {e_3}^2\right)^{3/2}}\right)\{M_3 - e_3\sin f_3 - f_3\}, \tag{45a}$$

$$B_1 = -\frac{(1 + e_3\cos f_3)^3}{4\left(1 - {e_3}^2\right)^{3/2}}\{e_3\left[3\sin(f_3 + 2\omega_3) + \sin(3f_3 + 2\omega_3)\right] + 3\sin(2f_3 + 2\omega_3)\}, \tag{45b}$$

$$C_1 = -\frac{(1 + e_3\cos f_3)^3}{\left(1 - {e_3}^2\right)^{3/2}}\{e_3\left[3\cos(f_3 + 2\omega_3) + \cos(3f_3 + 2\omega_3)\right] + 3\cos(2f_3 + 2\omega_3)\}, \tag{45c}$$

$$\begin{aligned} A_2 = & -\{e_3\left[3\cos(f_3 + 2\omega_3) + \cos(3f_3 + 2\omega_3)\right] + 3\cos(2f_3 + 2\omega_3)\} \\ & + \frac{6\cos 2u_3(1 + e_3\cos f_3)^3}{\left(1 - {e_3}^2\right)^{3/2}}\{M_3 - e_3\sin f_3 - f_3\}, \end{aligned} \tag{45d}$$

$$B_2 = \frac{3(1 + e_3\cos f_3)^3\cos 2u_3}{8\left(1 - {e_3}^2\right)^{3/2}}\{e_3\left[3\sin(f_3 + 2\omega_3) + \sin(3f_3 + 2\omega_3)\right] + 3\sin(2f_3 + 2\omega_3)\}, \tag{45e}$$

$$C_2 = \frac{3(1 + e_3\cos f_3)^3\cos 2u_3}{2\left(1 - {e_3}^2\right)^{3/2}}\{e_3\left[3\cos(f_3 + 2\omega_3) + \cos(3f_3 + 2\omega_3)\right] + 3\cos(2f_3 + 2\omega_3)\}, \tag{45f}$$

$$A_3 = \frac{1}{2}\{e_3\left[3\sin(f_3 + 2\omega_3) + \sin(3f_3 + 2\omega_3)\right] + 3\sin(2f_3 + 2\omega_3)\}$$

$$+\frac{12(1+e_3\cos f_3)^3\sin 2u_3}{\left(1-e_3{}^2\right)^{3/2}}\left\{M_3-e_3\sin f_3-f_3\right\}, \tag{45g}$$

$$B_3=\frac{3(1+e_3\cos f_3)^3\sin 2u_3}{4\left(1-e_3{}^2\right)^{3/2}}\left\{e_3\left[3\sin(f_3+2\omega_3)+\sin(3f_3+2\omega_3)\right]+3\sin(2f_3+2\omega_3)\right\}, \tag{45h}$$

$$C_3=\frac{3(1+e_3\cos f_3)^3\sin 2u_3}{\left(1-e_3{}^2\right)^{3/2}}\left\{e_3\left[3\cos(f_3+2\omega_3)+\cos(3f_3+2\omega_3)\right]+3\cos(2f_3+2\omega_3)\right\}. \tag{45i}$$

It is noticed that the integration variable only appears in A_i, B_i, C_i. In addition, the functions $f_{gh}(\chi,\gamma,\kappa)$, $f_g(\chi,\gamma,\kappa)$ do not have coupling terms. Thus, the integration of Eq. (43) reduces to integrating A_i, B_i, C_i. The integration is conducted using true anomaly as the independent variable, utilizing the third-order approximation of the mean anomaly (Battin 1999),

$$\begin{aligned}M_3=f_3&+\frac{e_3{}^2}{3\left(\sqrt{1-e_3{}^2}+1\right)^3}\left[\left(-6e_3{}^2+9\sqrt{1-e_3{}^2}+9\right)\sin 2f_3\right.\\&\left.-2e_3\left(3\sqrt{1-e_3{}^2}+1\right)\sin 3f_3\right]-2e_3\sin f_3,\end{aligned} \tag{46}$$

and

$$\mathrm{d}M_3=\frac{\left(1-e_3{}^2\right)^{3/2}}{(1+e_3\cos f_3)^2}\mathrm{d}f_3. \tag{47}$$

In addition, it is necessary to use a Taylor series expansion to avoid singularity,

$$\mathrm{d}M_3=\left(1-e_3{}^2\right)^{3/2}\left[1-2e_3\cos f_3+3(e_3\cos f_3)^2-4(e_3\cos f_3)^3+O\left((e_3\cos f_3)^4\right)\right]\mathrm{d}f_3. \tag{48}$$

As a result,

$$\begin{aligned}S_{22}^{**}=&\frac{k^2n_3{}^2a''^{7/2}}{1024\mu^{1/2}\left(1-e_3{}^2\right)^3}\left\{\left\{6e''^2\left[-\sin 2i''(5\cos 2\omega'-3)\left(-2\sin^2 i_3\cos 2\theta'+3\cos 2i_3\right.\right.\right.\right.\\&\left.+1\right)-20\sin 2\omega'\left(\sin i''\sin^2 i_3\sin 2\theta'+\cos i''\sin 2i_3\sin\theta'\right)+4\cos 2i''\sin 2i_3\\&\times(5\cos 2\omega'-3)\cos\theta'\big]+4\left[\sin 2i''\left(-6\sin^2 i_3\cos 2\theta'+9\cos 2i_3+3\right)\right.\\&\left.-12\cos 2i''\sin 2i_3\cos\theta'\right]\big\}f_{gh}\left(A_1^{**},B_1^{**},C_1^{**}\right)+\left\{4e''^2\left[-\sin 2i''(\cos 2i_3+3)\right.\right.\\&\times(5\cos 2\omega'-3)\cos 2\theta'+4\sin 2i_3\left(\cos 2i''(5\cos 2\omega'-3)\cos\theta'-5\cos i''\right.\\&\left.\times\sin 2\omega'\sin\theta'\right)+10\sin i''(\cos 2i_3+3)\sin 2\omega'\sin 2\theta'+6\sin 2i''\sin^2 i_3\\&\times(5\cos 2\omega'-3)\big]+8\sin 2i''(\cos 2i_3+3)\cos 2\theta'-\cos 2i''\sin 2i_3\cos\theta'\\&-3\sin i''\cos i''\sin^2 i_3\big\}f_{gh}\left(A_2^{**},B_2^{**},C_2^{**}\right)+\left\{32\sin\theta'(\sin 2i''\cos i_3\cos\theta'\right.\\&-\cos 2i''\sin i_3)+8e''^2\left[10\sin 2\omega'(\cos i''\sin i_3\cos\theta'-\sin i''\cos i_3\cos 2\theta')\right.\\&\left.\left.+2\cos 2i''\sin i_3(5\cos 2\omega'-3)\sin\theta'+\sin 2i''\cos i_3(3-5\cos 2\omega')\sin 2\theta'\right]\right\}\\&\times f_{gh}\left(A_3^{**},B_3^{**},C_3^{**}\right)+3e''\left[-80\sin i''\sin 2i_3\sin 2\omega'\sin\theta'-2\sin^2 i_3\cos 2\theta'\right.\\&\times\left(10(\cos 2i''+3)\cos 2\omega'+12\sin^2 i''\right)+80\cos i''\sin^2 i_3\sin 2\omega'\sin 2\theta'\\&+8\sin 2i''\sin 2i_3(5\cos 2\omega'-3)\cos\theta'-2(3\cos 2i_3+1)\left(10\sin^2 i''\cos 2\omega'\right.\end{aligned}$$

$$+3\cos 2i''+1)\big]\, f_g\left(A_1,B_1,C_1\right)+2e''\big[(\cos 2i_3+3)\cos 2\theta'\left(12\sin^2 i''\right.$$
$$\left.+10(\cos 2i''+3)\cos 2\omega'\right)+8\sin 2i_3\left(\sin 2i''(5\cos 2\omega'-3)\cos\theta'-10\sin i''\right.$$
$$\left.\times\ \sin 2\omega'\sin\theta'\right)-40\cos i''(\cos 2i_3+3)\sin 2\omega'\sin 2\theta'+12\sin^2 i_3\left(3\cos 2i''\right.$$
$$\left.+10\sin^2 i''\cos 2\omega'+1\right)\big]\, f_g\left(A_2^{**},B_2^{**},C_2^{**}\right)+8e\left[\cos i_3\sin 2\theta'\left(6\sin^2 i''\right.\right.$$
$$\left.+5(\cos 2i''+3)\cos 2\omega'\right)+20\sin 2\omega'(\cos i''\cos i_3\cos 2\theta'+\sin i''\sin i_3\cos\theta')$$
$$\left.+2\sin 2i''\sin i_3(5\cos 2\omega'-3)\sin\theta'\right]\, f_g\left(A_3^{**},B_3^{**},C_3^{**}\right)\big\}, \tag{49}$$

where A_i^{**}, B_i^{**}, C_i^{**} $(i=1,2,3)$ are given in "Appendix A."

2.3 Transformation between osculating elements and mean elements

The elements after double averaging are commonly referred to as *mean elements*. The transformation between mean elements and osculating elements will be discussed in this section.

Under the generating functions S_2^*, S_1^{**}, and S_2^{**}, the mean elements $(L'', G'', H'', l'', g'', h'')$ are converted to the osculating elements (L, G, H, l, g, h) by

$$\begin{aligned}
L &= L''+\frac{\partial S_1^{**}}{\partial l'}+\frac{\partial S_2^{**}}{\partial l'}+\frac{\partial S_2^{*}}{\partial l}, & l &= l''-\frac{\partial S_1^{**}}{\partial L''}-\frac{\partial S_2^{**}}{\partial L''}-\frac{\partial S_2^{*}}{\partial L'},\\
G &= G''+\frac{\partial S_1^{**}}{\partial g'}+\frac{\partial S_2^{**}}{\partial g'}+\frac{\partial S_2^{*}}{\partial g}, & g &= g''-\frac{\partial S_1^{**}}{\partial G''}-\frac{\partial S_2^{**}}{\partial G''}-\frac{\partial S_2^{*}}{\partial G'},\\
H &= H''+\frac{\partial S_1^{**}}{\partial h'}+\frac{\partial S_2^{**}}{\partial h'}+\frac{\partial S_2^{*}}{\partial h}, & h &= h''-\frac{\partial S_1^{**}}{\partial H''}-\frac{\partial S_2^{**}}{\partial H''}-\frac{\partial S_2^{*}}{\partial H'}.
\end{aligned} \tag{50}$$

It is observed that the short-period generating function S_2^* in Eq. (30) is a function of osculating elements and singly averaged elements, and the medium-period generating functions S_1^{**} in Eq. (36) and S_2^{**} in Eq. (38) are functions of singly averaged elements and mean elements. To obtain an explicit conversion, the expression must only contain mean elements. The elements in the second-order generating function can be changed into mean elements directly, because the errors are of high order. However, the singly averaged elements in S_1^{**} cannot be replaced directly, because the error is of the first order. As a consequence, the replacement introduces correction terms with second-order magnitude (Nie and Gurfil 2018),

$$\begin{aligned}
L_{M2O,c} &= -\frac{\partial^2 S_1^{**}}{\partial l''^2}\frac{\partial S_1^{**}}{\partial L''}-\frac{\partial^2 S_1^{**}}{\partial l''\partial g''}\frac{\partial S_1^{**}}{\partial G''}-\frac{\partial^2 S_1^{**}}{\partial l''\partial h''}\frac{\partial S_1^{**}}{\partial H''},\\
G_{M2O,c} &= -\frac{\partial^2 S_1^{**}}{\partial g''\partial l''}\frac{\partial S_1^{**}}{\partial L''}-\frac{\partial^2 S_1^{**}}{\partial^2 g''}\frac{\partial S_1^{**}}{\partial G''}-\frac{\partial^2 S_1^{**}}{\partial g''\partial h''}\frac{\partial S_1^{**}}{\partial H''},\\
H_{M2O,c} &= -\frac{\partial^2 S_1^{**}}{\partial h''\partial l'}\frac{\partial S_1^{**}}{\partial L''}-\frac{\partial^2 S_1^{**}}{\partial h''\partial g''}\frac{\partial S_1^{**}}{\partial G''}-\frac{\partial^2 S_1^{**}}{\partial^2 h''}\frac{\partial S_1^{**}}{\partial H''},\\
l_{M2O,c} &= \frac{\partial^2 S_1^{**}}{\partial L''\partial l''}\frac{\partial S_1^{**}}{\partial L''}+\frac{\partial^2 S_1^{**}}{\partial L''\partial g''}\frac{\partial S_1^{**}}{\partial G''}+\frac{\partial^2 S_1^{**}}{\partial L''\partial h''}\frac{\partial S_1^{**}}{\partial H''},\\
g_{M2O,c} &= \frac{\partial^2 S_1^{**}}{\partial G''\partial l''}\frac{\partial S_1^{**}}{\partial L''}+\frac{\partial^2 S_1^{**}}{\partial G''\partial g''}\frac{\partial S_1^{**}}{\partial G''}+\frac{\partial^2 S_1^{**}}{\partial G''\partial h''}\frac{\partial S_1^{**}}{\partial H''},\\
h_{M2O,c} &= \frac{\partial^2 S_1^{**}}{\partial H''\partial l''}\frac{\partial S_1^{**}}{\partial L''}+\frac{\partial^2 S_1^{**}}{\partial H''\partial g''}\frac{\partial S_1^{**}}{\partial G''}+\frac{\partial^2 S_1^{**}}{\partial H''\partial h''}\frac{\partial S_1^{**}}{\partial H''}.
\end{aligned} \tag{51}$$

Finally, the mean elements can be transformed into osculating elements by

$$\begin{aligned} L &= L'' + L''_{sp2} + L''_{mp1} + L''_{mp2} + L_{M2O,c}, \quad l = l'' + l''_{sp2} + l''_{mp1} + l''_{mp2} + l_{M2O,c},\\ G &= G'' + G''_{sp2} + G''_{mp1} + G''_{mp2} + G_{M2O,c}, \quad g = g'' + g''_{sp2} + g''_{mp1} + g''_{mp2} + g_{M2O,c},\\ H &= H'' + H''_{sp2} + H''_{mp1} + H''_{mp2} + H_{M2O,c}, \quad h = h'' + h''_{sp2} + h''_{mp1} + h''_{mp2} + h_{M2O,c}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} L''_{sp2} &= \frac{\partial S_2^*}{\partial l''}, \quad G''_{sp2} = \frac{\partial S_2^*}{\partial g''}, \quad H''_{sp2} = \frac{\partial S_2^*}{\partial h''},\\ l''_{sp2} &= -\frac{\partial S_2^*}{\partial L''}, \quad g''_{sp2} = -\frac{\partial S_2^*}{\partial G''}, \quad h''_{sp2} = -\frac{\partial S_2^*}{\partial H''},\\ L''_{mp1} &= \frac{\partial S_1^{**}}{\partial l''}, \quad G''_{mp1} = \frac{\partial S_1^{**}}{\partial g''}, \quad H''_{mp1} = \frac{\partial S_1^{**}}{\partial h''},\\ l''_{mp1} &= -\frac{\partial S_1^{**}}{\partial L''}, \quad g''_{mp1} = -\frac{\partial S_1^{**}}{\partial G''}, \quad h''_{mp1} = -\frac{\partial S_1^{**}}{\partial H''},\\ L''_{mp2} &= \frac{\partial S_2^{**}}{\partial l''}, \quad G''_{mp2} = \frac{\partial S_2^{**}}{\partial g''}, \quad H''_{mp2} = \frac{\partial S_2^{**}}{\partial h''},\\ l''_{mp2} &= -\frac{\partial S_2^{**}}{\partial L''}, \quad g''_{mp2} = -\frac{\partial S_2^{**}}{\partial G''}, \quad h''_{mp2} = -\frac{\partial S_2^{**}}{\partial H''}. \end{aligned} \tag{53}$$

These expressions are given in Appendix B (provided as an electronic supplement), and the values are determined by using mean elements.

According to Eq. (50), the mean elements can be calculated by

$$\begin{aligned} L'' &= L - \frac{\partial S_1^{**}}{\partial l'} - \frac{\partial S_2^{**}}{\partial l'} - \frac{\partial S_2^*}{\partial l}, \quad l'' = l + \frac{\partial S_1^{**}}{\partial L''} + \frac{\partial S_2^{**}}{\partial L''} + \frac{\partial S_2^*}{\partial L'},\\ G'' &= G - \frac{\partial S_1^{**}}{\partial g'} - \frac{\partial S_2^{**}}{\partial g'} - \frac{\partial S_2^*}{\partial g}, \quad g'' = g + \frac{\partial S_1^{**}}{\partial G''} + \frac{\partial S_2^{**}}{\partial G''} + \frac{\partial S_2^*}{\partial G'},\\ H'' &= H - \frac{\partial S_1^{**}}{\partial h'} - \frac{\partial S_2^{**}}{\partial h'} - \frac{\partial S_2^*}{\partial h}, \quad h'' = h + \frac{\partial S_1^{**}}{\partial H''} + \frac{\partial S_2^{**}}{\partial H''} + \frac{\partial S_2^*}{\partial H'}. \end{aligned} \tag{54}$$

To obtain the explicit conversion from osculating elements to mean elements, all elements in the right-hand side of Eq. (54) should be osculating elements. For the second-order solution, the osculating elements can be substituted into all second-order terms, because the errors are of higher order; however, for the first-order terms, the errors are of second order. Taking $\frac{\partial S_1^{**}}{\partial l'}$ as an example, and using Taylor's theorem, $\frac{\partial S_1^{**}}{\partial l'}$ can be expressed as

$$\begin{aligned} \frac{\partial S_1^{**}}{\partial l'} &= \frac{\partial S_1^{**}\left(L'', G'', H'', l', g', h'\right)}{\partial l'}\\ &= \frac{\partial S_1^{**}(L, G, H, l, g, h)}{\partial l} + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial L}\left(L'' - L\right)\\ &\quad + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial G}\left(G'' - G\right) + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial H}\left(H'' - H\right)\\ &\quad + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l^2}\left(l' - l\right) + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial g}\left(g' - g\right)\\ &\quad + \frac{\partial^2 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial h}\left(h' - h\right) + \frac{1}{2}\frac{\partial^3 S_1^{**}(L, G, H, l, g, h)}{\partial l \partial L^2}\left(L'' - L\right)^2 + \cdots. \end{aligned} \tag{55}$$

According to Eq. (54), Eq. (55) can be written as

$$\begin{aligned}\frac{\partial S_1^{**}}{\partial l'} &= \frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial l} + \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial L}\left(-\frac{\partial S_1^{**}}{\partial l'} - \frac{\partial S_2^{**}}{\partial l'} - \frac{\partial S_2^{*}}{\partial l}\right) \\ &+ \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial G}\left(-\frac{\partial S_1^{**}}{\partial g'} - \frac{\partial S_2^{**}}{\partial g'} - \frac{\partial S_2^{*}}{\partial g}\right) \\ &+ \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial H}\left(-\frac{\partial S_1^{**}}{\partial h'} - \frac{\partial S_2^{**}}{\partial h'} - \frac{\partial S_2^{*}}{\partial h}\right) \\ &+ \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l^2}\left(\frac{\partial S_2^{*}}{\partial L'}\right) + \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial g}\left(\frac{\partial S_2^{*}}{\partial G'}\right) \\ &+ \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial h}\left(\frac{\partial S_2^{*}}{\partial H'}\right) \\ &+ \frac{1}{2}\frac{\partial^3 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial L^2}\left(-\frac{\partial S_1^{**}}{\partial l'} - \frac{\partial S_2^{**}}{\partial l'} - \frac{\partial S_2^{*}}{\partial l}\right)^2 + \cdots . \end{aligned} \tag{56}$$

For the second-order solution, terms are truncated up to second order, and Eq. (56) becomes

$$\begin{aligned}\frac{\partial S_1^{**}}{\partial l'} &= \frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial l} - \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial L}\left(\frac{\partial S_1^{**}}{\partial l'}\right) \\ &- \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial G}\left(\frac{\partial S_1^{**}}{\partial g'}\right) - \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial H}\left(\frac{\partial S_1^{**}}{\partial h'}\right). \end{aligned} \tag{57}$$

Furthermore, the osculating elements can be substituted into the last three terms, because these are second-order terms, so

$$\begin{aligned}\frac{\partial S_1^{**}}{\partial l'} &= \frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial l} - \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial L}\frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial l} \\ &- \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial G}\frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial g} \\ &- \frac{\partial^2 S_1^{**}(L,G,H,l,g,h)}{\partial l \partial H}\frac{\partial S_1^{**}(L,G,H,l,g,h)}{\partial h}. \end{aligned} \tag{58}$$

Finally, $\frac{\partial S_1^{**}}{\partial l'}$ can be expressed as

$$\frac{\partial S_1^{**}}{\partial l'} = \frac{\partial S_1^{**}}{\partial l} + L_{O2M,c} \tag{59}$$

with

$$L_{O2M,c} = -\frac{\partial^2 S_1^{**}}{\partial l \partial L}\frac{\partial S_1^{**}}{\partial l} - \frac{\partial^2 S_1^{**}}{\partial l \partial G}\frac{\partial S_1^{**}}{\partial g} - \frac{\partial^2 S_1^{**}}{\partial l \partial H}\frac{\partial S_1^{**}}{\partial h}. \tag{60}$$

Other correction terms can be determined in a similar way. Based on Eqs. (53), (54) and (60), the transformation from osculating elements to mean elements can be described as

$$\begin{aligned} L'' &= L - L_{sp2} - L_{mp1} - L_{mp2} - L_{O2M,c}, \quad l'' = l - l_{sp2} - l_{mp1} - l_{mp2} - l_{O2M,c}, \\ G'' &= G - G_{sp2} - G_{mp1} - G_{mp2} - G_{O2M,c}, \quad g'' = g - g_{sp2} - g_{mp1} - g_{mp2} - g_{O2M,c}, \\ H'' &= H - H_{sp2} - H_{mp1} - H_{mp2} - H_{O2M,c}, \quad h'' = h - h_{sp2} - h_{mp1} - h_{mp2} - h_{O2M,c}, \end{aligned} \tag{61}$$

where the correction terms are

$$
\begin{aligned}
L_{O2M,c} &= -\frac{\partial^2 S_1^{**}}{\partial l \partial L}\frac{\partial S_1^{**}}{\partial l} - \frac{\partial^2 S_1^{**}}{\partial l \partial G}\frac{\partial S_1^{**}}{\partial g} - \frac{\partial^2 S_1^{**}}{\partial l \partial H}\frac{\partial S_1^{**}}{\partial h}, \\
G_{O2M,c} &= -\frac{\partial^2 S_1^{**}}{\partial g \partial L}\frac{\partial S_1^{**}}{\partial l} - \frac{\partial^2 S_1^{**}}{\partial g \partial G}\frac{\partial S_1^{**}}{\partial g} - \frac{\partial^2 S_1^{**}}{\partial g \partial H}\frac{\partial S_1^{**}}{\partial h}, \\
H_{O2M,c} &= -\frac{\partial^2 S_1^{**}}{\partial h \partial L}\frac{\partial S_1^{**}}{\partial l} - \frac{\partial^2 S_1^{**}}{\partial h \partial G}\frac{\partial S_1^{**}}{\partial g} - \frac{\partial^2 S_1^{**}}{\partial h \partial H}\frac{\partial S_1^{**}}{\partial h}, \\
l_{O2M,c} &= \frac{\partial^2 S_1^{**}}{\partial L \partial L}\frac{\partial S_1^{**}}{\partial l} + \frac{\partial^2 S_1^{**}}{\partial L \partial G}\frac{\partial S_1^{**}}{\partial g} + \frac{\partial^2 S_1^{**}}{\partial L \partial H}\frac{\partial S_1^{**}}{\partial h}, \\
g_{O2M,c} &= \frac{\partial^2 S_1^{**}}{\partial G \partial L}\frac{\partial S_1^{**}}{\partial l} + \frac{\partial^2 S_1^{**}}{\partial G \partial G}\frac{\partial S_1^{**}}{\partial g} + \frac{\partial^2 S_1^{**}}{\partial G \partial H}\frac{\partial S_1^{**}}{\partial h}, \\
h_{O2M,c} &= \frac{\partial^2 S_1^{**}}{\partial H \partial L}\frac{\partial S_1^{**}}{\partial l} + \frac{\partial^2 S_1^{**}}{\partial H \partial G}\frac{\partial S_1^{**}}{\partial g} + \frac{\partial^2 S_1^{**}}{\partial H \partial H}\frac{\partial S_1^{**}}{\partial h}.
\end{aligned}
\tag{62}
$$

The explicit expressions can be found in Appendix B (provided as an electronic supplement).

2.4 Equations of motion of an orbiter perturbed by a third body

In the previous section, the short-period and medium-period variations in the Hamiltonian were nullified in succession. The singly averaged and doubly averaged Hamiltonians were developed. In this section, the singly averaged and doubly averaged dynamic equations will be derived.

The singly averaged Hamiltonian $\mathcal{H}^*$ is

$$
\mathcal{H}^* = -\frac{\mu^2}{2L'^2} + n_3 T' - \frac{k{n_3}^2 a'^2 {a_3}^3}{4{r_3}^3}\left[3\alpha'^2\left(4e'^2+1\right) - 3\beta'^2\left(e'^2-1\right) - 3e'^2 - 2\right], \tag{63}
$$

and the doubly averaged Hamiltonian $\mathcal{H}^{**}$ is

$$
\begin{aligned}
\mathcal{H}^{**} = &-\frac{\mu^2}{2L''^2} + n_3 T'' - \frac{k{n_3}^2 a''^2}{128\left(1-{e_3}^2\right)^{3/2}}\left\{3e''^2\left\{-40\cos i''\sin^2 i_3 \sin 2\omega'' \sin 2\theta''\right.\right. \\
&+ 4\sin 2i'' \sin 2i_3\left(3 - 5\cos 2\omega''\right)\cos\theta'' + 40\sin i''\sin 2i_3 \sin 2\omega''\sin\theta'' \\
&+ \sin^2 i_3 \cos 2\theta''\left[12\sin^2 i'' + 10\left(\cos 2i'' + 3\right)\cos 2\omega''\right] + \left(3\cos 2i_3 + 1\right) \\
&\times \left.\left(10\sin^2 i''\cos 2\omega'' + 3\cos 2i'' + 1\right)\right\} + 2\left[12\sin^2 i''\sin^2 i_3\cos 2\theta''\right. \\
&\left.\left. + 12\sin 2i''\sin 2i_3\cos\theta'' + \left(3\cos 2i'' + 1\right)\left(3\cos 2i_3 + 1\right)\right]\right\},
\end{aligned}
\tag{64}
$$

so the singly averaged disturbing function is

$$
\mathcal{R}^* = \frac{k{n_3}^2 a'^2 {a_3}^3}{4{r_3}^3}\left[3\alpha'^2\left(4e'^2+1\right) - 3\beta'^2\left(e'^2-1\right) - 3e'^2 - 2\right], \tag{65}
$$

and the doubly averaged disturbing function is

$$\mathcal{R}^{**}=\frac{kn_3{}^2a''^2}{128\left(1-e_3{}^2\right)^{3/2}}\left\{3e''^2\left\{-40\cos i''\sin^2 i_3\sin 2\omega''\sin 2\theta''+4\sin 2i''\sin 2i_3\right.\right.$$
$$\times\left(3-5\cos 2\omega''\right)\cos\theta''+40\sin i''\sin 2i_3\sin 2\omega''\sin\theta''+\sin^2 i_3\cos 2\theta''$$
$$\times\left[10\left(\cos 2i''+3\right)\cos 2\omega''+12\sin^2 i''\right]+\left(3\cos 2i_3+1\right)\left(10\sin^2 i''\cos 2\omega''\right.$$
$$\left.\left.+3\cos 2i''+1\right)\right\}+2\left[12\sin^2 i''\sin^2 i_3\times\cos 2\theta''+12\sin 2i''\sin 2i_3\cos\theta''+\right.$$
$$\left.\left.\times\ (3\cos 2i''+1)(3\cos 2i_3+1)\right]\right\}. \tag{66}$$

It is noticed that the singly averaged disturbing function $\mathcal{R}^*$ is identical to that in Liu et al. (2012), but the doubly averaged disturbing function is not. The difference results from the fact that Liu et al. (2012) averaged (a_3^3/r_3^3), α'^2 and β'^2 in the singly averaged disturbing function, shown in Eq. (65), separately. If we set zero inclination and eccentricity of the perturbing body, the third-body perturbation problem becomes similar to the model analyzed by Broucke (2003), and the derived doubly averaged disturbing function in Eq. (66) is changed into

$$\mathcal{R}^{**}|_{e_3=0,i_3=0}=\frac{kn_3{}^2a''^2}{32}\left[30e''^2\sin^2 i''\cos 2\omega''+\left(9e''^2+6\right)\cos 2i''+3e''^2+2\right], \tag{67}$$

which is identical to the result in Broucke (2003). However, the doubly averaged disturbing function in Liu et al. (2012) becomes

$$\bar{\bar{\mathcal{R}}}|_{e_3=0,i_3=0}=\frac{kn_3{}^2a''^2}{32}\left[15e''^2(\cos 2i''+3)\cos 2\omega''+\left(9e''^2+6\right)\cos 2i''+3e''^2+2\right]. \tag{68}$$

Compared with that in Broucke (2003), the error is

$$E_{\bar{\bar{\mathcal{R}}}|_{e_3=0,i_3=0}}=\frac{15kn_3{}^2a''^2}{8}e''^2\cos^2 i''\cos 2\omega''. \tag{69}$$

Based on the relationship between classical orbital elements and the Delaunay elements in Eqs. (11), the classical orbital elements can be determined by

$$a''=\frac{L''^2}{\mu},\quad M''=l'', \tag{70a}$$

$$e''=\sqrt{1-\frac{G''^2}{L''^2}},\quad \omega''=g'', \tag{70b}$$

$$\cos i''=\frac{H''}{G''},\quad \Omega''=h''. \tag{70c}$$

Taking the time derivative of Eqs. (70) yields

$$\dot{a}''=\frac{2L''}{\mu}\dot{L}'',\quad \dot{M}''=\dot{l}'', \tag{71a}$$

$$\dot{e}''=\frac{G''^2\dot{L}''-G''L''\dot{G}''}{L''^2\sqrt{L''^2-G''^2}},\quad \dot{\omega}''=\dot{g}'', \tag{71b}$$

$$\dot{i}''=-\frac{G''\dot{H}''-H''\dot{G}''}{G''^2\sin i''},\quad \dot{\Omega}''=\dot{h}''. \tag{71c}$$

Using Hamilton's equations

$$\begin{aligned}
\dot{L}'' &= -\frac{\partial\mathcal{H}^{**}}{\partial l''}, \quad \dot{G}'' = -\frac{\partial\mathcal{H}^{**}}{\partial g''}, \quad \dot{H}'' = -\frac{\partial\mathcal{H}^{**}}{\partial h''}, \\
\dot{l}'' &= \frac{\partial\mathcal{H}^{**}}{\partial L''}, \quad \dot{g}'' = \frac{\partial\mathcal{H}^{**}}{\partial G''}, \quad \dot{h}'' = \frac{\partial\mathcal{H}^{**}}{\partial H''},
\end{aligned} \tag{72}$$

and

$$\frac{\partial\mathcal{H}^{**}}{\partial L''} = \frac{2a''^{1/2}}{\mu^{1/2}}\frac{\partial\mathcal{H}^{**}}{\partial a''} + \frac{1}{e''}\frac{\left(1-e''^2\right)}{\sqrt{\mu a''}}\frac{\partial\mathcal{H}^{**}}{\partial e''}, \tag{73a}$$

$$\frac{\partial\mathcal{H}^{**}}{\partial G''} = -\frac{\sqrt{1-e''^2}}{\sqrt{\mu a''}e''}\frac{\partial\mathcal{H}^{**}}{\partial e''} + \frac{\cot i''}{\sqrt{\mu a''}\sqrt{1-e''^2}}\frac{\partial\mathcal{H}^{**}}{\partial i''}, \tag{73b}$$

$$\frac{\partial\mathcal{H}^{**}}{\partial H} = -\frac{1}{\sqrt{\mu a''}\sqrt{1-e''^2}\sin i''}\frac{\partial\mathcal{H}^{**}}{\partial i''}, \tag{73c}$$

equations (71) can be written in terms of classical orbital elements as

$$\begin{cases}
\dot{a}'' = -\dfrac{2\sqrt{a''}}{\sqrt{\mu}}\dfrac{\partial\mathcal{H}^{**}}{\partial M''}, \\
\dot{e}'' = \dfrac{\left(e''^2-1\right)}{\sqrt{\mu}\sqrt{a''}e}\dfrac{\partial\mathcal{H}^{**}}{\partial M''} + \dfrac{\sqrt{1-e''^2}}{\sqrt{\mu}\sqrt{a''}e''}\dfrac{\partial\mathcal{H}^{**}}{\partial\omega''}, \\
\dot{i}'' = \dfrac{1}{\sqrt{\mu}\sqrt{a''}\sqrt{1-e''^2}\sin i''}\dfrac{\partial\mathcal{H}^{**}}{\partial\Omega''} - \dfrac{\cot i''}{\sqrt{\mu}\sqrt{a''}\sqrt{1-e''^2}}\dfrac{\partial\mathcal{H}^{**}}{\partial\omega''}, \\
\dot{\Omega}'' = -\dfrac{1}{\sqrt{\mu}\sqrt{a''}\sqrt{1-e''^2}\sin i''}\dfrac{\partial\mathcal{H}^{**}}{\partial i''}, \\
\dot{\omega}'' = \dfrac{\left(e''^2-1\right)}{\sqrt{\mu}\sqrt{a''}\,e''\sqrt{1-e''^2}}\dfrac{\partial\mathcal{H}^{**}}{\partial e''} + \dfrac{\cot i''}{\sqrt{\mu}\sqrt{a''}\sqrt{1-e''^2}}\dfrac{\partial\mathcal{H}^{**}}{\partial i''}, \\
\dot{M}'' = \dfrac{\left(1-e''^2\right)}{\sqrt{\mu}\sqrt{a''}e''}\dfrac{\partial\mathcal{H}^{**}}{\partial e''} + \dfrac{2\sqrt{a''}}{\sqrt{\mu}}\dfrac{\partial\mathcal{H}^{**}}{\partial a''}.
\end{cases} \tag{74}$$

Finally, substituting the singly averaged Hamiltonian of Eqs. (63) into Eqs. (74) results in the singly averaged orbit dynamics,

$$\dot{a}' = 0, \tag{75a}$$

$$\begin{aligned}
\dot{e}' = &-\frac{15kn_3{}^2a'^{3/2}e'\sqrt{1-e'^2}(1+e_3\cos f_3)^3}{4\mu^{1/2}\left(1-e_3^2\right)^3}\left\{2\cos 2\omega'(\cos i_3\sin\theta'\sin u_3\right. \\
&+\cos\theta'\cos u_3)\left[\cos i'(\cos i_3\cos\theta'\sin u_3-\sin\theta'\cos u_3)+\sin i'\sin i_3\sin u_3\right] \\
&-\sin 2\omega'\left[\cos u_3(\cos i'\sin\theta'+\cos\theta')-\sin u_3\left(\cos i'\cos i_3\cos\theta'+\sin i'\sin i_3\right.\right. \\
&\left.\left.-\cos i_3\sin\theta'\right)\right]\left[\sin u_3\left(\cos i_3(\cos i'\cos\theta'+\sin\theta')+\sin i'\sin i_3\right)\right. \\
&\left.\left.+\cos u_3(\cos\theta'-\cos i'\sin\theta')\right]\right\},
\end{aligned} \tag{75b}$$

$$\begin{aligned}
\dot{i}' = &\frac{3kn_3{}^2a'^{3/2}(1+e_3\cos f_3)^3}{2\mu^{1/2}\sqrt{1-e'^2}\left(1-e_3^2\right)^3}\left[\sin u_3\left(\cos i'\sin i_3-\sin i'\cos i_3\cos\theta'\right)\right. \\
&\left.+\sin i'\sin\theta'\cos u_3\right]\left\{5e'^2\sin\omega'\cos\omega'\left[\cos i'(\cos i_3\cos\theta'\sin u_3-\sin\theta'\cos u_3)\right.\right. \\
&\left.+\sin i'\sin i_3\sin u_3\right]+\left[\left(4e'^2+1\right)\cos^2\omega'-\left(e'^2-1\right)\sin^2\omega'\right] \\
&\left.\times(\cos i_3\sin\theta'\sin u_3+\cos\theta'\cos u_3)\right\},
\end{aligned} \tag{75c}$$

$$\dot{\Omega}' = \frac{3kn_3{}^2 a'^{3/2}\sin i'(1+e_3\cos f_3)^3}{2\mu^{1/2}\sqrt{1-e'^2}\left(1-e_3^2\right)^3}\left[\sin u_3(\cot i'\sin i_3-\cos i_3\cos\theta')+\sin\theta'\cos u_3\right]$$
$$\left\{\left[\left(e'^2-1\right)\cos^2\omega'-\left(4e'^2+1\right)\sin^2\omega'\right]\left[\cot i'\sin\theta'\cos u_3-\sin u_3(\cot i'\cos i_3\right.\right.$$
$$\left.\left.\times\cos\theta'+\sin i_3)\right]+5e'^2\csc i'\sin\omega'\cos\omega'\left(\cos i_3\sin\theta'\sin u_3+\cos\theta'\cos u_3\right)\right\}, \tag{75d}$$

$$\dot{\omega}' = \frac{kn_3^2 a'^{3/2}(1+e_3\cos f_3)^3}{4\mu^{1/2}\sqrt{1-e'^2}\left(1-e_3^2\right)^3}\left\{6\left(1-e'^2\right)\left\{-1-\left[\sin u_3(\cos i_3\sin\omega'\sin\theta'-\sin i'\right.\right.\right.$$
$$\times\sin i_3\cos\omega')+\cos\theta'(\sin\omega'\cos u_3-\cos i'\cos i_3\cos\omega'\sin u_3)+\cos i'$$
$$\left.\times\cos\omega'\sin\theta'\cos u_3\right]^2+4\left[\cos\omega'(\cos i_3\sin\theta'\sin u_3+\cos\theta'\cos u_3)\right.$$
$$\left.\left.+\sin\omega'\left(\cos i'(\cos i_3\cos\theta'\sin u_3-\sin\theta'\cos u_3)+\sin i'\sin i_3\sin u_3\right)\right]^2\right\}$$
$$-6\cot i'\left[\sin u_3\left(\cos i'\sin i_3-\sin i'\cos i_3\cos\theta'\right)+\sin i'\sin\theta'\cos u_3\right]$$
$$\times\left\{\left[\left(4e'^2+1\right)\sin^2\omega'-\left(e'^2-1\right)\cos^2\omega'\right]\left[\cos i'(\cos i_3\cos\theta'\sin u_3-\sin\theta'\cos u_3)\right.\right.$$
$$\left.\left.\left.+\sin i'\sin i_3\sin u_3\right]+5e'^2\sin\omega'\cos\omega'\left(\cos i_3\sin\theta'\sin u_3+\cos\theta'\cos u_3\right)\right\}\right\}, \tag{75e}$$

$$\dot{M}' = \frac{\mu^{1/2}}{a'^{3/2}}+\frac{kn_3^2 a'^{3/2}(1+e_3\cos f_3)^3}{4\mu^{1/2}\left(1-e_3^2\right)^3}\left\{6\left(e'^2-1\right)\left[\cos i'\cos\omega'\sin\theta'\cos u_3\right.\right.$$
$$+\sin u_3(\cos i_3\sin\omega'\sin\theta'-\sin i'\sin i_3\cos\omega')+\cos\theta'(\sin\omega'\cos u_3$$
$$\left.-\cos i'\cos i_3\cos\omega'\sin u_3)\right]^2-12\left(3+2e'^2\right)\left[\sin\omega'\left(\cos i'(\cos i_3\cos\theta'\right.\right.$$
$$\left.\times\sin u_3-\sin\theta'\cos u_3)+\sin i'\sin i_3\sin u_3\right)+\cos\omega'(\cos i_3\sin\theta'\sin u_3$$
$$\left.\left.+\cos\theta'\cos u_3)\right]^2+6e'^2+14\right\}. \tag{75f}$$

Similarly, the doubly averaged orbit dynamics are

$$\dot{a}'' = 0, \tag{76a}$$
$$\dot{e}'' = \frac{15kn_3{}^2 a''^{3/2}e''\sqrt{1-e''^2}}{32\mu^{1/2}\left(1-e_3{}^2\right)^{3/2}}\left\{(\cos 2i''+3)\sin^2 i_3\sin 2\omega''\cos 2\theta''+4\cos i''\sin^2 i_3\right.$$
$$\times\cos 2\omega''\sin 2\theta''-4\sin i''\sin 2i_3\cos 2\omega''\sin\theta''-2\sin 2i''\sin 2i_3\sin 2\omega''\cos\theta''$$
$$\left.+\sin^2 i''(3\cos 2i_3+1)\sin 2\omega''\right\}, \tag{76b}$$
$$\dot{i}'' = -\frac{15kn_3{}^2 a''^{3/2}e''^2\cot i''}{32\mu^{1/2}\sqrt{1-e''^2}\left(1-e_3{}^2\right)^{3/2}}\left\{\sin 2\omega''\left[(\cos 2i''+3)\sin^2 i_3\cos 2\theta''\right.\right.$$
$$\left.-2\sin 2i''\sin 2i_3\cos\theta''\right]+4\cos i''\sin^2 i_3\cos 2\omega''\sin 2\theta''$$
$$\left.-4\sin i''\sin 2i_3\cos 2\omega''\sin\theta''+\sin^2 i''(3\cos 2i_3+1)\sin 2\omega''\right\}, \tag{76c}$$
$$\dot{\Omega}'' = \frac{3kn_3{}^2 a''^{3/2}}{64\mu^{1/2}\sqrt{1-e''^2}\left(1-e_3{}^2\right)^{3/2}}\left\{\cos i''\left(5e''^2\cos 2\omega''-3e''^2-2\right)\left[\cos(2i_3-2\theta'')\right.\right.$$
$$\left.+\cos(2i_3+2\theta'')+6\cos 2i_3-2\cos 2\theta''+2\right]+4\left[5e''^2\sin 2\omega''\left(\cot i''\sin 2i_3\sin\theta''\right.\right.$$
$$\left.\left.\left.+\sin^2 i_3\sin 2\theta''\right)-\cos 2i''\csc i''\sin 2i_3\cos\theta''\left(5e''^2\cos 2\omega''-3e''^2-2\right)\right]\right\}, \tag{76d}$$

$$
\begin{aligned}
\dot{\omega}'' = & -\frac{3kn_3{}^2a''^{3/2}}{64\mu^{1/2}\sqrt{1-e''^2}\left(1-e_3{}^2\right)^{3/2}}\left\{2\sin^2 i_3\cos 2\theta''\left[5\left(2e''^2-3\right)\cos 2\omega''\right.\right.\\
& \left. + 6e''^2 + 10\cos 2i''\sin^2\omega'' - 1\right] - 20\left(e''^2-2\right)\cos i''\sin^2 i_3\sin 2\omega''\sin 2\theta''\\
& + 2\cot i''\sin 2i_3\cos\theta''\left[20\cos 2i''\sin^2\omega'' - 2\left(e''^2-1\right)\left(5\cos 2\omega''-3\right)\right]\\
& - 10\csc i''\sin 2i_3\sin 2\omega''\sin\theta''\left[\left(e''^2-2\right)\cos 2i'' - 3e''^2 + 2\right]\\
& + 5(3\cos 2i_3+1)\cos 2\omega''\left(2e''^2+\cos 2i''-1\right) - 3\cos 2i_3\\
& \left.\times\left(2e''^2+5\cos 2i''+2\right) - 2e''^2 - 5\cos 2i'' - 3\cos 2i_3 - 3\right\},
\end{aligned} \tag{76e}
$$

$$
\begin{aligned}
\dot{M}'' = & \frac{\mu^{1/2}}{a''^{3/2}} - \frac{a''^{3/2}kn_3{}^2}{64\mu^{1/2}\left(1-e_3{}^2\right)^{3/2}}\left\{120\left(e''^2+1\right)\sin i''\sin 2i_3\sin 2\omega''\sin\theta''\right.\\
& + 3\sin^2 i_3\cos 2\theta''\left[10\left(e''^2+1\right)\left(\cos 2i''+3\right)\cos 2\omega'' + 4\left(3e''^2+7\right)\sin^2 i''\right]\\
& - 120\left(e''^2+1\right)\cos i''\sin^2 i_3\sin 2\omega''\sin 2\theta'' + 12\sin 2i''\sin 2i_3\cos\theta''\\
& \times\left[-5\left(e''^2+1\right)\cos 2\omega'' + 3e''^2 + 7\right] + (3\cos 2i_3+1)\\
& \left.\times\left[30\left(e''^2+1\right)\sin^2 i''\cos 2\omega'' + \left(3e''^2+7\right)\left(3\cos 2i''+1\right)\right]\right\}.
\end{aligned} \tag{76f}
$$

When the third body is in a circular equatorial orbit ($e_3 = 0, i_3 = 0$), the dynamics become

$$
\begin{cases}
\dot{a}'' = 0,\\
\dot{e}'' = \dfrac{15kn_3{}^2a''^{3/2}}{8\mu^{1/2}}e''\sqrt{1-e''^2}\sin^2 i''\sin 2\omega'',\\
\dot{i}'' = -\dfrac{15kn_3^2a''^{3/2}}{16\mu^{1/2}}\dfrac{e''^2}{\sqrt{1-e''^2}}\sin 2i''\sin 2\omega'',\\
\dot{\Omega}'' = \dfrac{3kn_3^2a''^{3/2}}{8\mu^{1/2}\sqrt{1-e''^2}}\left(5e''^2\cos 2\omega'' - 3e''^2 - 2\right)\cos i'',\\
\dot{\omega}'' = \dfrac{3kn_3^2a''^{3/2}}{8\mu^{1/2}\sqrt{1-e''^2}}\left[\left(5\cos^2 i'' - 1 + e''^2\right) + 5\left(1-e''^2-\cos^2 i''\right)\cos 2\omega''\right],\\
\dot{M}'' = \dfrac{\mu^{1/2}}{a''^{3/2}} - \dfrac{kn_3^2a''^{3/2}}{8\mu^{1/2}}\left[\left(3e''^2+7\right)\left(3\cos^2 i''-1\right) + 15(1+e''^2)\sin^2 i''\cos 2\omega''\right],
\end{cases} \tag{77}
$$

which is in accordance with the results shown in Broucke (2003). Comparing Eq. (76) with Eq. (77), the third body's eccentricity only introduces a factor $1/(1-e_3^2)^{3/2}$ in the evolution rate of the orbital elements. However, the inclination of the third body is coupled with other orbital elements, especially the right ascension of the ascending node, which does not appear in Eq. (77). A comparison of the different dynamical models will be described in the next section.

3 Simulation

In this section, numerical examples are provided to show the influence of the eccentricity and inclination of the perturbing body, and validate the proposed semi-analytical model.

We consider a spacecraft in a three-dimensional orbit about Moon, perturbed by Earth. As shown in Nie and Gurfil (2018), the third-body perturbation is the dominant perturbation for a high-altitude lunar orbiter. Thus, the semimajor axis a is chosen to be 11738 km, and the other orbital elements are set as

$$e = 0.1, \quad i = 20^\circ, \quad \Omega = 10^\circ, \quad \omega = 50^\circ, \quad M = 20^\circ.$$

To verify the accuracy of the proposed theory, the full restricted three-body model is used as a reference, and the position of Earth in the lunar inertial frame is determined by the ephemeris date DE421 (Folkner et al. 2008). The developed dynamical models are compared with the singly averaged and doubly averaged dynamical models in previous works (Broucke 2003; Bertachini de Almeida Prado 2003), which do not consider the eccentricity and inclination of the perturbing body.

The initial singly averaged elements are obtained by subtracting short-period variations, as shown in Eq. (53),

$$\bar{a} = 11710.1928 \text{ km}, \quad \bar{e} = 0.0976, \quad \bar{i} = 20.0076^\circ, \quad \bar{\Omega} = 9.0322^\circ,$$
$$\bar{\omega} = 49.5379^\circ, \quad \bar{M} = 21.2935^\circ,$$

and the doubly averaged elements are determined by Eq. (61),

$$\bar{\bar{a}} = 11710.1928 \text{ km}, \quad \bar{\bar{e}} = 0.0944, \quad \bar{\bar{i}} = 20.4996^\circ,$$
$$\bar{\bar{\Omega}} = 9.1485^\circ, \quad \bar{\bar{\omega}} = 43.6872^\circ, \quad \bar{\bar{M}} = 27.7394^\circ.$$

The simulation results are shown in Figs. 1, 2, 3, 4, 5 and 6. The abbreviations 'SA' and 'DA' represent the singly averaged and doubly averaged dynamical models with zero eccentricity and inclination of the third body, respectively. 'SAEI' and 'DAEI' denote the singly averaged and doubly averaged dynamical models including the eccentricity and inclination of the perturbing body, respectively.

As shown in Fig. 1, only the semimajor axis has short-period variations, as indicated by Eq. (75). Its short-period variation is relatively large, about 46 km. The time history of the eccentricity consists of short-period, medium-period and long-period variations, as illustrated in Figs. 2 and 6. The short-period oscillation is noticeable, comparing the full third-body perturbation dynamical model (Full) with SAEI. However, the short-period variations for i, Ω, ω are very small, comparing with long-period variations and secular drifts. It is noticed that the inclination is with large medium-period variations, comparing the singly averaged model with the doubly averaged model. The secular drifts for Ω, ω are larger than the periodic variations.

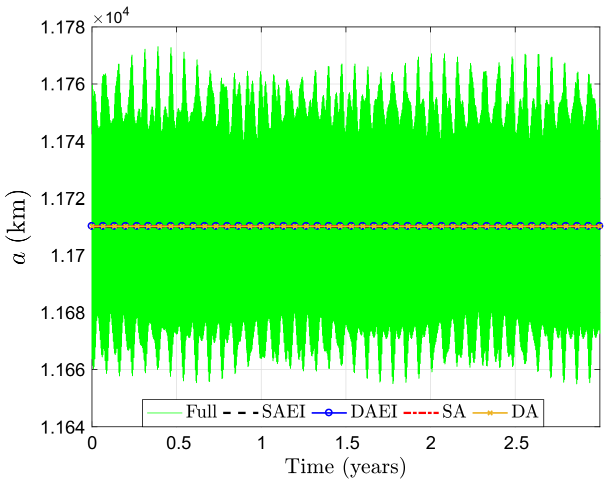

Fig. 1 Semimajor axis evolution in 2 months using different dynamical models

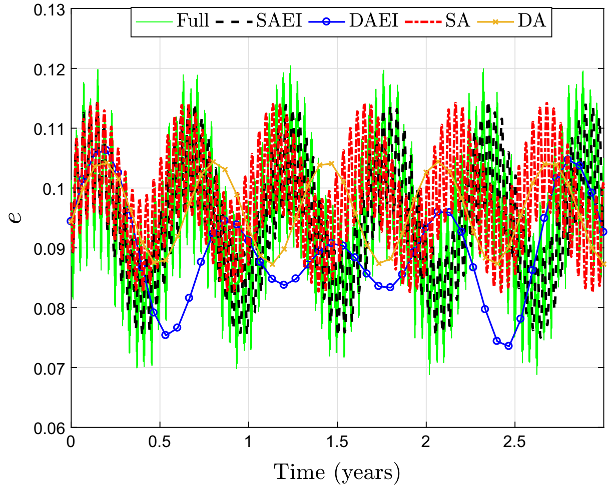

Fig. 2 Evolution of the eccentricity using different dynamical models

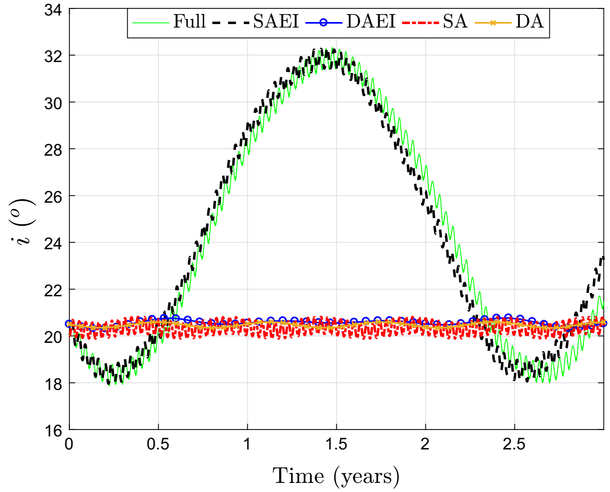

Fig. 3 Evolution of the inclination using different dynamical models

The singly averaged and doubly averaged orbital elements are determined numerically by averaging the osculating orbital elements over an orbital period of the satellite and the third body, respectively. These values are regarded as reference, and the errors are defined as the difference between the results from the semi-analytical dynamics and averaged value calculated numerically. The error variations for orbital elements are shown in Figs. 7, 8, 9, 10 and 11. The propagation results for the semimajor axis are identical for all methods. The error evolution of SAEI is the same as that of SA, and the result for DAEI is the same as that of DA in Fig. 7. It is noticed that the error oscillation for the singly averaged models is larger than that of the doubly averaged models as shown in Fig. 7. This is because the oscillation of the osculating semimajor axis is large, as shown in Fig. 1, and the reference value for determining error for the doubly averaged models is more steady, since its averaging time span is longer. The importance of including the inclination and eccentricity of the perturbing body is evident in Figs. 8, 9, 10 and 11. The singly averaged dynamical model including the eccentricity and inclination of the third body (SAEI) derived in Eq. (75) provides better

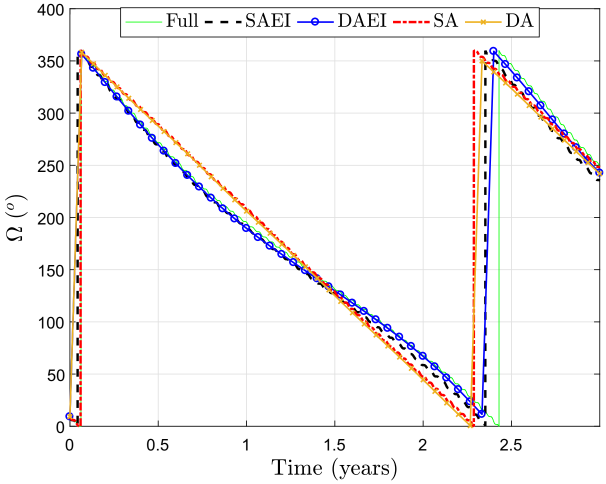

Fig. 4 Evolution of the right ascension of the ascending node using different dynamical models

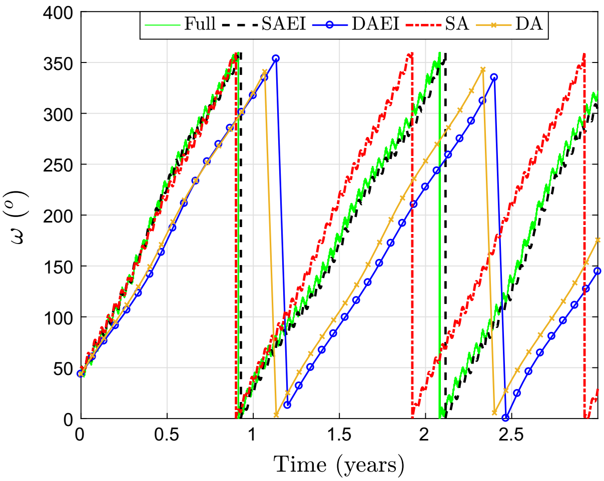

Fig. 5 Evolution of the argument of perigee using different dynamical models

results than SA, especially in the inclination. Only the SAEI model can capture the long-term evolution of the inclination well enough.

To illustrate the accuracy of the proposed transformation between osculating elements and mean elements, we compare these values with the singly averaged and doubly averaged values, determined numerically, as mentioned before. The comparison results are shown in Table. 1, where 'N-Single' and 'N-Double' represent the singly averaged and doubly averaged elements determined numerically. 'Bias' means the difference between the numerical value and the analytical value, and 'Bias-perc' means the bias percentage. It is observed that the bias of the singly averaged values is smaller than that of the doubly averaged values and that the biases for a, e and i are smaller than those of Ω and ω. This is because the averaged values, calculated numerically, include the secular drift of the perturbations. The time span of the doubly averaged values is longer, and the secular drifts of Ω and ω are larger. As a result, the biases are larger.

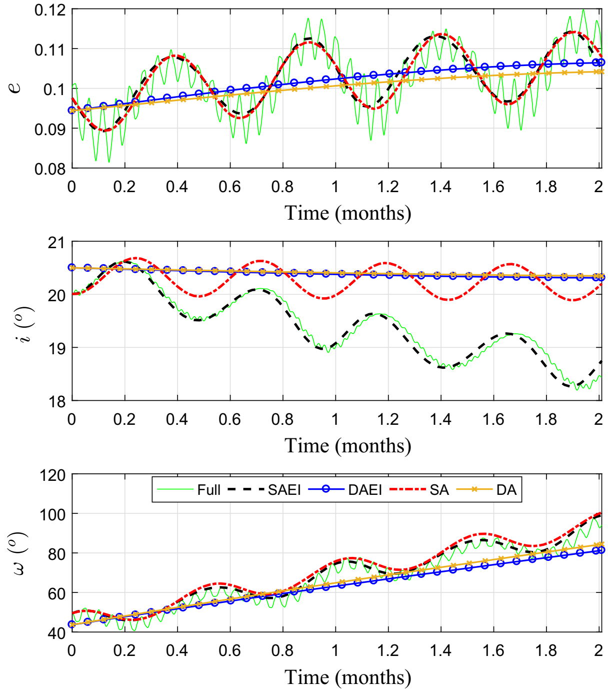

Fig. 6 Orbital element evolution during 2 months using different dynamical models

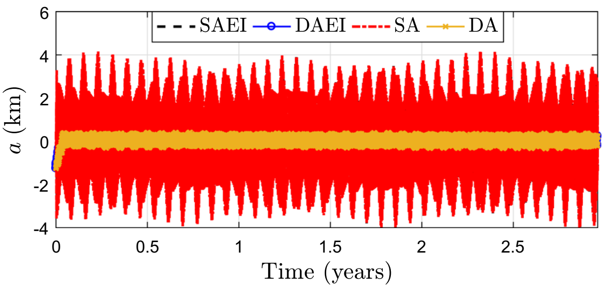

Fig. 7 Error evolution of the semimajor axis

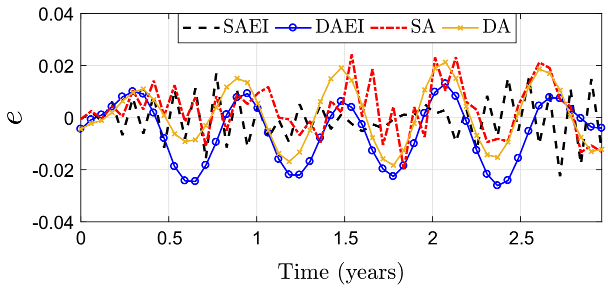

Fig. 8 Error evolution of the eccentricity

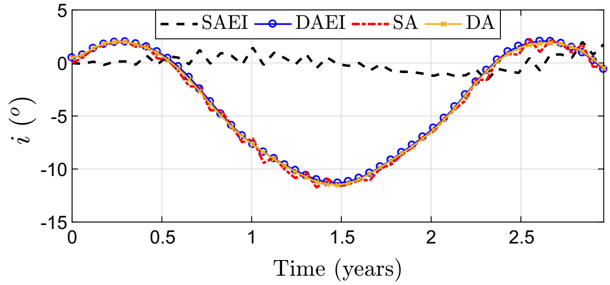

Fig. 9 Error evolution of the inclination

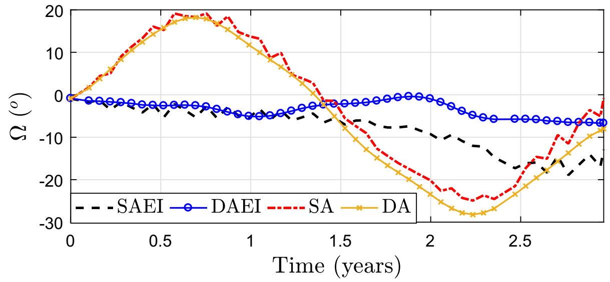

Fig. 10 Error evolution of the right ascension

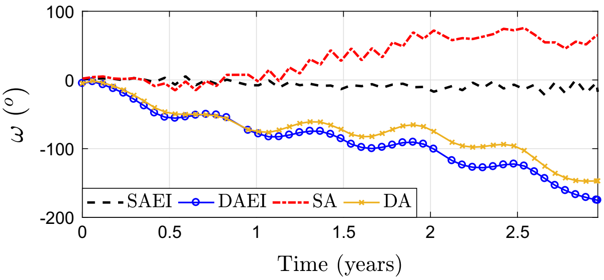

Fig. 11 Error evolution of the argument of perigee

Table 1 Comparison between the numerical value and analytical value

	a (km)	e	i	Ω	ω
N-Single	11709.7488 km	0.0981	20.0251°	9.9773°	47.5889°
Bias	0.4440 km	0.0005	0.0175°	0.9451°	1.9490°
Bias-perc	0.0038%	0.5595%	0.0874%	9.4726%	4.0955%
N-Double	11711.4483 km	0.0988	20.0145°	10.0198°	48.7104°
Bias	1.2555 km	0.0044	0.4851°	0.8713°	5.0232°
Bias-perc	0.0107%	4.4297%	2.4236%	8.6955%	10.3123%

4 Conclusions

A complete semi-analytical theory of third-body perturbations has been developed, including an explicit transformation between osculating elements, singly averaged, and doubly averaged elements, in addition to the development of the averaged dynamics. Simulation results validate the accuracy of the explicit transformation between osculating elements and mean elements.

The semimajor axis and eccentricity have large short-periodic variations; the inclination is characterized by large medium-period variation; and the secular drifts for Ω, ω are larger than the periodic variations. Ignoring the perturbing body's eccentricity and inclination would introduce propagation error to the orbital elements, especially in the inclination and argument of perigee of the orbiter. It is recommended to use the singly averaged model, including the third body's inclination and eccentricity for propagating the long-term evolution of the inclination.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest associated with the work reported in this article.

Appendix A: Coefficients in the generating function S_{22}^{**}

The coefficients in the generating function S_{22}^{**} are given by

$$
\begin{aligned}
A_1^{**} = &-\frac{1}{360\left(\sqrt{1-{e_3}^2}+1\right)^3}\left\{-30{e_3}^{10}\left(9\cos 4f_3+4\cos 6f_3\right)+144{e_3}^9\cos^3 f_3\right.\\
&\times\left[8\left(\sqrt{1-{e_3}^2}+3\right)\cos 2f_3+8\sqrt{1-{e_3}^2}-1\right]-5{e_3}^8\left[-8\left(\sqrt{1-{e_3}^2}+6\right)\cos 6f_3\right.\\
&\left.+9\left(20\sqrt{1-{e_3}^2}+29\right)\cos 4f_3+36\left(22\sqrt{1-{e_3}^2}+31\right)\cos 2f_3\right]-24{e_3}^7\cos f_3\\
&\times\left[-\left(41\sqrt{1-{e_3}^2}+142\right)\cos 2f_3+36\left(\sqrt{1-{e_3}^2}+2\right)\cos 4f_3-77\sqrt{1-{e_3}^2}-64\right]\\
&+5{e_3}^6\left[\sqrt{1-{e_3}^2}(501\cos 4f_3-8\cos 6f_3)+12\left(103\sqrt{1-{e_3}^2}+96\right)\cos 2f_3\right.\\
&\left.+684\cos 4f_3-24\cos 6f_3\right]+8{e_3}^5\cos f_3\left[-\left(1027\sqrt{1-{e_3}^2}+1593\right)\cos 2f_3\right.\\
&\left.+36\left(2\sqrt{1-{e_3}^2}+3\right)\cos 4f_3-229\sqrt{1-{e_3}^2}+279\right]+15{e_3}^4\left[4\left(59\sqrt{1-{e_3}^2}+113\right)\right.\\
&\left.\times\cos 2f_3-131\left(\sqrt{1-{e_3}^2}+1\right)\cos 4f_3\right]+80{e_3}^3\left[\left(35\sqrt{1-{e_3}^2}+47\right)\cos 3f_3\right.\\
&\left.\left.+54\cos f_3\right]-7560\left(\sqrt{1-{e_3}^2}+1\right){e_3}^2\cos 2f_3-8640\left(\sqrt{1-{e_3}^2}+1\right)e_3\cos f_3\right\}\\
B_1^{**} = &-\frac{3}{8}{e_3}^2M_3\sin 2\omega_3-\frac{1}{96}\left\{12\left(2{e_3}^2+3\right)\cos(2f_3+2\omega_3)+e_3\left[3e_3\cos(4f_3+2\omega_3)\right.\right.\\
&\left.\left.+4\left(-9e_3f_3\sin 2\omega_3+27\cos(f_3+2\omega_3)+5\cos(3f_3+2\omega_3)\right)\right]\right\}\\
C_1^{**} = &-\frac{3}{2}{e_3}^2M_3\cos 2\omega_3-\frac{1}{24}\left\{-e_3\left[3e_3\left(8\sin(2f_3+2\omega_3)+\sin(4f_3+2\omega_3)+12f_3\right.\right.\right.\\
&\left.\left.\left.\times\cos 2\omega_3\right)+4\left(27\sin(f_3+2\omega_3)+5\sin(3f_3+2\omega_3)\right)\right]-36\sin(2f_3+2\omega_3)\right\}\\
A_2^{**} = &\left\{\frac{{e_3}^2}{4}\left(20{e_3}^2+3\right)\left(1-{e_3}^2\right)^{3/2}\cos 2\omega_3+\frac{{e_3}^2\sin 2\omega_3}{32\left(\sqrt{1-{e_3}^2}+1\right)^3}\left[-1170\left(\sqrt{1-{e_3}^2}+1\right)\right.\right.\\
&\times{e_3}^4-168{e_3}^8-16\left(3\sqrt{1-{e_3}^2}+13\right){e_3}^2+288\left(\sqrt{1-{e_3}^2}+1\right)+3\left(57\sqrt{1-{e_3}^2}\right.\\
&\left.\left.\left.+335\right){e_3}^6\right]\right\}M_3+\frac{e_3}{887040\left(\sqrt{1-{e_3}^2}+1\right)^3}\left\{5544{e_3}^9\left[840f_3\sin 2\omega_3+420\right.\right.\\
&\times\cos(2f_3-2\omega_3)+210\cos(4f_3+2\omega_3)+140\cos(6f_3+2\omega_3)+45\cos(8f_3\\
&\left.+2\omega_3)+6\cos(10f_3+2\omega_3)+90\cos(4f_3-2\omega_3)+10\cos(6f_3-2\omega_3)\right]+48{e_3}^8
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[38115\sqrt{1-{e_3}^2}\cos(f_3-2\omega_3)+9\sqrt{1-{e_3}^2}\,(-11(-105\sin f_3-140\sin 3f_3\\
&+\ 14\sin 5f_3+65\sin 7f_3+35\sin 9f_3)\sin 2\omega_3+55(-21\cos f_3+42\cos 3f_3\\
&+28\cos 5f_3+17\cos 7f_3+7\cos 9f_3)\cos 2\omega_3+70\cos(11f_3+2\omega_3))+16\\
&\times\ \big(80(7\cos 2f_3+21\cos 4f_3+19){\cos}^7 f_3\cos 2\omega_3+7{\sin}^3 f_3(2229\cos 2f_3\\
&+750\cos 4f_3+155\cos 6f_3+15\cos 8f_3+2131)\sin 2\omega_3)]+693{e_3}^7\Big[30\left(27\sqrt{1-{e_3}^2}\right.\\
&\left.-67\right)\cos(4f_3-2\omega_3)+3\sqrt{1-{e_3}^2}\,(-2280f_3\sin 2\omega_3-240\cos(2f_3-2\omega_3)\\
&-\ 570\cos(4f_3+2\omega_3)-80\cos(6f_3+2\omega_3)+135\cos(8f_3+2\omega_3)+48\cos(10f_3+2\omega_3)\\
&+80\cos(6f_3-2\omega_3))-335\,(120f_3\sin 2\omega_3+48\cos(2f_3-2\omega_3)+30\cos(4f_3+2\omega_3)\\
&+16\cos(6f_3+2\omega_3)+3\cos(8f_3+2\omega_3))]+88{e_3}^6\Big[21\sqrt{1-{e_3}^2}\,(160(-40\\
&\times\ \cos 2f_3+17\cos 4f_3+33){\cos}^5 f_3\cos 2\omega_3+8{\sin}^3 f_3(1059\cos 2f_3+480\\
&\times\cos 4f_3+85\cos 6f_3+536)\sin 2\omega_3)+160\,(-2500\cos 2f_3+77\cos 4f_3\\
&+1077)\cos^5 f_3\cos 2\omega_3-8{\sin}^3 f_3\,(78681\cos 2f_3+9420\cos 4f_3-385\cos 6f_3\\
&+118924)\sin 2\omega_3]+270270\left(\sqrt{1-{e_3}^2}+1\right){e_3}^5\,[48\cos(2f_3-2\omega_3)+120f_3\\
&\times\ \sin 2\omega_3++30\cos(4f_3+2\omega_3)+16\cos(6f_3+2\omega_3)+3\cos(8f_3+2\omega_3)\\
&+6\cos(4f_3-2\omega_3)]+6336{e_3}^4\,\{-2730\cos(f_3+2\omega_3)+910\cos(3f_3+2\omega_3)\\
&+\ 861\cos(5f_3+2\omega_3)+225\cos(7f_3+2\omega_3)+525\cos(3f_3-2\omega_3)+15\\
&\times\ \Big[\sqrt{1-{e_3}^2}\,(-210\cos(f_3+2\omega_3)+70\cos(3f_3+2\omega_3)+63\cos(5f_3+2\omega_3)\\
&+15\cos(7f_3+2\omega_3)+35\cos(3f_3-2\omega_3))+7\left(45\sqrt{1-{e_3}^2}+41\right)\cos(f_3-2\omega_3)\Big]\Big\}\\
&-\ 36960{e_3}^3\Big[3\left(4\left(3\sqrt{1-{e_3}^2}+1\right)\cos(2f_3-2\omega_3)+\sqrt{1-{e_3}^2}\,[4\,(\cos(6f_3+2\omega_3)\right.\\
&\left.-3f_3\sin 2\omega_3)-3\cos(4f_3+2\omega_3)]\right)-39\cos(4f_3+2\omega_3)+4\,(-39f_3\sin 2\omega_3\\
&+\ \cos(6f_3+2\omega_3))]-88704{e_3}^2\Big[3\sqrt{1-{e_3}^2}\,(45\cos(f_3+2\omega_3)+\cos(5f_3+2\omega_3)\\
&-15\cos(3f_3+2\omega_3))+5\left(3\sqrt{1-{e_3}^2}-5\right)\cos(f_3-2\omega_3)-5\,(-63\cos(f_3+2\omega_3)\\
&+21\cos(3f_3+2\omega_3)+\cos(5f_3+2\omega_3))]-1995840\left(\sqrt{1-{e_3}^2}+1\right)e_3\,[4f_3\\
&\times\ \sin 2\omega_3+\cos(4f_3+2\omega_3)]-10644480\left(\sqrt{1-{e_3}^2}+1\right)[\cos(3f_3+2\omega_3)\\
&-3\cos(f_3+2\omega_3)]\}-\frac{1}{240}\left(1-{e_3}^2\right)^{3/2}\{20{e_3}^4\,[9\,(\sin(2f_3-2\omega_3)+60f_3\cos 2\omega_3\\
&+4\sin(2f_3+2\omega_3)+\sin(4f_3+2\omega_3))+\sin(6f_3+2\omega_3)]+{e_3}^3\left(96{\cos}^3 f_3\right.\\
&\left.\times(3\cos 2f_3-2)\sin 2\omega_3-96{\sin}^3 f_3(3\cos 2f_3+7)\cos 2\omega_3\right)+15{e_3}^2\,[12f_3\\
&\times\ \cos 2\omega_3-4\sin(2f_3+2\omega_3)-5\sin(4f_3+2\omega_3)]+160e_3\sin(3f_3+2\omega_3)\\
&\left.-360\sin(2f_3+2\omega_3)\right\}
\end{aligned}
$$

$$B_2^{**} = -\frac{1}{64}\left[-6{e_3}^2\cos(4f_3+4\omega_3) - 9{e_3}^2\cos(2f_3+4\omega_3) - {e_3}^2\cos(6f_3+4\omega_3)\right.$$
$$+\ 6{e_3}^2\cos 2f_3 - 18e_3\cos(3f_3+4\omega_3) - 6e_3\cos(5f_3+4\omega_3) + 24e_3\cos f_3$$
$$\left. -\,9\cos(4f_3+4\omega_3)\right]$$

$$C_2^{**} = \frac{3}{4}\left(2{e_3}^2+3\right)M_3 - \frac{1}{16}\left[{e_3}^2\left(6\sin(4f_3+4\omega_3) + 9\sin(2f_3+4\omega_3)\right.\right.$$
$$+\ \sin(6f_3+4\omega_3) + 24f_3 + 12\sin 2f_3) + 6e_3\left(3\sin(3f_3+4\omega_3)\right.$$
$$\left.+\ \sin(5f_3+4\omega_3) + 14\sin f_3) + 9\sin(4f_3+4\omega_3) + 36f_3\right]$$

$$A_3^{**} = \left\{\frac{{e_3}^2\cos 2\omega_3}{16\left(\sqrt{1-{e_3}^2}+1\right)^3}\left[168{e_3}^8 + 16\left(3\sqrt{1-{e_3}^2}+13\right){e_3}^2 - 288\left(\sqrt{1-{e_3}^2}+1\right) - 3\right.\right.$$
$$\times\left(57\sqrt{1-{e_3}^2}+335\right){e_3}^6 + 1170\left(\sqrt{1-{e_3}^2}+1\right){e_3}^4\Big] - \frac{{e_3}^2}{8}\left(1-{e_3}^2\right)^{3/2}\left(20{e_3}^2+3\right.$$
$$\times\ \sin 2\omega_3\}\,M_3 - \frac{1}{2}\left(1-{e_3}^2\right)^{3/2}\left\{\frac{1}{12}{e_3}^4\left[-60f_3\sin 2\omega_3 - 9\cos(2f_3-2\omega_3) + 36\right.\right.$$
$$\times\ \cos(2f_3+2\omega_3) + 9\cos(4f_3+2\omega_3) + \cos(6f_3+2\omega_3)] + \frac{2}{5}{e_3}^3\left[(3\cos 2f_3 - 2)\right.$$
$$\times\cos^3 f_3\cos 2\omega_3 + \sin^3 f_3(3\cos 2f_3+7)\sin 2\omega_3\Big] - \frac{1}{16}{e_3}^2\left[12f_3\sin 2\omega_3 + 4\right.$$
$$\left.\left.\times\ \cos(2f_3+2\omega_3) + 5\cos(4f_3+2\omega_3)\right] + \frac{2}{3}e_3\cos(3f_3+2\omega_3) - \frac{3}{2}\cos(2f_3+2\omega_3)\right\}$$
$$-\ \frac{e_3}{443520\left(\sqrt{1-{e_3}^2}+1\right)^3}\left\{5544{e_3}^9\left[-6\sin(10f_3+2\omega_3) - 5(-84\sin 2f_3 + 24\right.\right.$$
$$\times\ \sin 4f_3 + 26\sin 6f_3 + 9\sin 8f_3 - 168f_3)\cos 2\omega_3 - 15(28\cos 2f_3 + 20\cos 4f_3$$
$$+\,10\cos 6f_3 + 3\cos 8f_3)\sin 2\omega_3] + 48{e_3}^8\left(38115\sqrt{1-{e_3}^2}\sin(f_3-2\omega_3) - 9\right.$$
$$\times\ \sqrt{1-{e_3}^2}\left[(-1155\sin f_3 - 1540\sin 3f_3 + 154\sin 5f_3 + 715\sin 7f_3 + 385\right.$$
$$\times\ \sin 9f_3 + 70\sin 11f_3)\cos 2\omega_3 + 5\left(-231\cos f_3 + 462\cos 3f_3 + 308\cos 5f_3\right.$$
$$+\,187\cos 7f_3 + 77\cos 9f_3 + 14\cos 11f_3)\sin 2\omega_3] - 1280\left(19 + 7\cos 2f_3\right.$$
$$+\,21\cos 4f_3)\cos^7 f_3\sin 2\omega_3 + 112\sin^3 f_3\left(2229\cos 2f_3 + 750\cos 4f_3\right.$$
$$+\,155\cos 6f_3 + 15\cos 8f_3 + 2131)\cos 2\omega_3) - 693{e_3}^7\left[3\left(40f_3\left(57\sqrt{1-{e_3}^2}\right.\right.\right.$$
$$+\,335)\cos 2\omega_3 + \sqrt{1-{e_3}^2}(48\sin(10f_3+2\omega_3) + 5(48\sin 2f_3 - 168\sin 4f_3$$
$$-\ 32\sin 6f_3 + 27\sin 8f_3)\cos 2\omega_3 + 15(-16\cos 2f_3 - 20\cos 4f_3 + 9\cos 8f_3)$$
$$\times\ \sin 2\omega_3)) - 335\left(-48\sin(2f_3-2\omega_3) + 30\sin(4f_3+2\omega_3) + 16\sin(6f_3+2\omega_3)\right.$$
$$+\,3\sin(8f_3+2\omega_3) - 6\sin(4f_3-2\omega_3))] + 88{e_3}^6\left[21\sqrt{1-{e_3}^2}\left(8\sin^3 f_3\right.\right.$$
$$\times\ (1059\cos 2f_3 + 480\cos 4f_3 + 85\cos 6f_3 + 536)\cos 2\omega_3 - 160\cos^5 f_3$$
$$\times\,(-\,40\cos 2f_3 + 17\cos 4f_3 + 33)\sin 2\omega_3) - 160(-2500\cos 2f_3 + 77\cos 4f_3$$
$$+\ 1077)\cos^5 f_3\sin 2\omega_3 - 8\sin^3 f_3\left(78681\cos 2f_3 + 9420\cos 4f_3 - 385\right.$$

$$
\begin{aligned}
&\times \cos 6 f_3 + 118924) \cos 2\omega_3] + 270270 \left(\sqrt{1-{e_3}^2}+1\right) {e_3}^5 \left[48 \sin(2 f_3 - 2\omega_3)\right. \\
&- 30 \sin(4 f_3 + 2\omega_3) - 16 \sin(6 f_3 + 2\omega_3) - 3 \sin(8 f_3 + 2\omega_3) + 6 \sin(4 f_3 - 2\omega_3) \\
&+ 120 f_3 \cos 2\omega_3] + 6336 {e_3}^4 \{2730 \sin(f_3 + 2\omega_3) - 910 \sin(3 f_3 + 2\omega_3) - 861 \\
&\times \sin(5 f_3 + 2\omega_3) - 225 \sin(7 f_3 + 2\omega_3) + 525 \sin(3 f_3 - 2\omega_3) + 15 \left[\sqrt{1-{e_3}^2}\right. \\
&\times (210 \sin(f_3 + 2\omega_3) - 70 \sin(3 f_3 + 2\omega_3) - 63 \sin(5 f_3 + 2\omega_3) - 15 \\
&\left.\left.\times \sin(7 f_3 + 2\omega_3) + 35 \sin(3 f_3 - 2\omega_3)) + 7 \left(45\sqrt{1-{e_3}^2} + 41\right) \sin(f_3 - 2\omega_3)\right]\right\} \\
&+ 36960 {e_3}^3 \{-12 \sin(2 f_3 - 2\omega_3) - 39 \sin(4 f_3 + 2\omega_3) + 4 \sin(6 f_3 + 2\omega_3) \\
&+ 3 \left(\sqrt{1-{e_3}^2}\, [4 \sin(6 f_3 + 2\omega_3) - 3\,(4 \sin(2 f_3 - 2\omega_3) + \sin(4 f_3 + 2\omega_3))]\right. \\
&\left.\left. + 4 \left(3\sqrt{1-{e_3}^2} + 13\right) f_3 \cos 2\omega_3\right)\right\} + 88704 {e_3}^2 \left[3\sqrt{1-{e_3}^2}\, (45 \sin(f_3 + 2\omega_3)\right. \\
&- 15 \sin(3 f_3 + 2\omega_3) + \sin(5 f_3 + 2\omega_3)) + 5 \left(5 - 3\sqrt{1-{e_3}^2}\right) \sin(f_3 - 2\omega_3) \\
&- 5\,(-63 \sin(f_3 + 2\omega_3) + 21 \sin(3 f_3 + 2\omega_3) + \sin(5 f_3 + 2\omega_3))] \\
&+ 1995840 \left(\sqrt{1-{e_3}^2}+1\right) e_3\, [\sin(4 f_3 + 2\omega_3) - 4 f_3 \cos 2\omega_3] \\
&\left. + 10644480 \left(\sqrt{1-{e_3}^2}+1\right) [\sin(3 f_3 + 2\omega_3) - 3 \sin(f_3 + 2\omega_3)]\right\}
\end{aligned}
$$

$$
\begin{aligned}
B_3^{**} = {} & \frac{3}{16} \left(2{e_3}^2 + 3\right) M_3 - \frac{1}{64} \left\{{e_3}^2 \left[-6 \sin(4 f_3 + 4\omega_3) - 9 \sin(2 f_3 + 4\omega_3) + 12 \sin 2 f\right.\right. \\
&\left. - \sin(6 f_3 + 4\omega_3) + 24 f_{33}\right] - 6 e_3\, [3 \sin(3 f_3 + 4\omega_3) + \sin(5 f_3 + 4\omega_3) \\
&- 14 \sin f_3] - 9 \sin(4 f_3 + 4\omega_3) + 36 f_3\}
\end{aligned}
$$

$$
\begin{aligned}
C_3^{**} = {} & -\frac{1}{8} \left\{{e_3}^2\, [-6 \cos(4 f_3 + 4\omega_3) - 9 \cos(2 f_3 + 4\omega_3) - \cos(6 f_3 + 4\omega_3) - 6 \cos 2 f_3]\right. \\
&- 6 e_3\, [3 \cos(3 f_3 + 4\omega_3) + \cos(5 f_3 + 4\omega_3) + 4 \cos f_3] - 9 \cos(4 f_3 + 4\omega_3)\}
\end{aligned}
$$

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