

# Relativistic satellite orbits: central body with higher zonal harmonics

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**Abstract** Satellite orbits around a central body with arbitrary zonal harmonics are considered in a relativistic framework. Our starting point is the relativistic Celestial Mechanics based upon the first post-Newtonian approximation to Einstein's theory of gravity as it has been formulated by Damour et al. (Phys Rev D 43:3273–3307, 1991; 45:1017–1044, 1992; 47:3124–3135, 1993; 49:618–635, 1994). Since effects of order  $(GM/c^2R) \times J_k$  with  $k \geq 2$  for the Earth are very small (of order  $7 \times 10^{-10} \times J_k$ ) we consider an axially symmetric body with arbitrary zonal harmonics and a static external gravitational field. In such a field the explicit  $J_k/c^2$ -terms (direct terms) in the equations of motion for the coordinate acceleration of a satellite are treated first with first-order perturbation theory. The derived perturbation theoretical results of first order have been checked by purely numerical integrations of the equations of motion. Additional terms of the same order result from the interaction of the Newtonian  $J_k$ -terms with the post-Newtonian Schwarzschild terms (relativistic terms related to the mass of the central body). These 'mixed terms' are treated by means of second-order perturbation theory based on the Lie-series method (Hori–Deprit method). Here we concentrate on the secular drifts of the ascending node  $\langle \dot{\Omega} \rangle$  and argument of the pericenter  $\langle \dot{\omega} \rangle$ . Finally orders of magnitude are given and discussed.

## List of symbols

$x^\mu = (ct, \mathbf{x})$	Space-time coordinates
$g_{\mu\nu}$	Space-time metric tensor
$\sigma$	Gravitating mass-energy density
$G$	Universal gravitational constant
$P_k$	Ordinary Legendre polynomials

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$\alpha_{k,j}$	Coefficients in the derivatives of the Legendre polynomials, see (13)
$J_k$	Dimensionless post-Newtonian zonal harmonic
$[x]$	Greatest integer that is less than or equal to $x$ (Gauss' bracket)
$X_k^{pq}(e)$	Hansen coefficient
$F_{kpq}(I)$	Kaula inclination function
$\Phi_{i,j,k,p}(f, \omega)$	Special function defined in (39a)
$\Phi_{i,j,k,p}^*(f, \omega)$	Special function defined in (39b)
$\Psi_{i,j,k,p}(f, M)$	Special function defined in (39c)
$S, T, W$	Decomposition of perturbing acceleration into radial, transversal and normal part
$(l, g, h; L, G, H) \equiv (\mathbf{y}, \mathbf{Y})$	Post-Newtonian Delaunay variables
$n$	Mean motion
$b_m$	Defined below (24)

### 1 Introduction

In this paper, we consider satellite orbits around a single isolated central body with a static external gravitational field in a relativistic framework. Our discussions are based upon the first post-Newtonian approximation to Einstein's theory of gravity. A very general relativistic Celestial Mechanics of  $N$  gravitating bodies of arbitrary time-dependent shape and composition has been formulated by Damour et al. (1991, 1992, 1993, 1994). This DSX-framework uses very efficiently a special parametrization of the metric tensor, that presents the gravitational field of the central body. In the harmonic gauge (e.g. Weinberg 1972) with space-time coordinates  $x^\mu = (ct, \mathbf{x})$ , the post-Newtonian metric tensor in the static case is written with a single gravitational potential  $w$ , generalizing the usual Newtonian potential  $U$ , i.e.  $w = U + (1/c^2)$ -terms:

$$\begin{aligned}
 g_{00} &= -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + \mathcal{O}_5 \\
 g_{0i} &= 0 \\
 g_{ij} &= \delta_{ij} \left( 1 + \frac{2w}{c^2} \right) + \mathcal{O}_4,
 \end{aligned}
 \tag{1}$$

where the symbol  $\mathcal{O}_n$  indicates that terms of order  $c^{-n}$  are neglected. The field equation for  $w$  reads:

$$\Delta w = -4\pi G \sigma,
 \tag{2}$$

where

$$\sigma = \frac{T^{00} + T^{ss}}{c^2}
 \tag{3}$$

is the gravitating mass-energy density,  $T^{\mu\nu}$  are the components of the energy-momentum tensor and  $T^{ss} = T^{11} + T^{22} + T^{33}$ .  $G$  is the universal gravitational constant. According to a Theorem by Blanchet and Damour (1989) the external potential  $w$  admits a multipole expansion of the form:

$$w(\mathbf{x}) = G \sum_{0 \leq k} \frac{(-1)^k}{k!} M_K \partial_K \left( \frac{1}{r} \right),
 \tag{4}$$

that converges outside a coordinate sphere that completely contains the central body (the energy-momentum tensor of the matter distribution). Here,  $M_K$  are the Cartesian Blanchet–Damour mass multipole moments of the central body, defined as integrals over the field generating source that explicitly contain  $1/c^2$ -terms, i.e. terms of first post-Newtonian order. The index  $K$  is a Cartesian multi-index, i.e.  $K \equiv i_1 \dots i_k$ , where each index runs over the three spatial Cartesian coordinates, and  $\partial_K \equiv \partial^k / (\partial_{i_1} \dots \partial_{i_k})$ . Assuming the central body to be axially symmetric, the gravitational potentials can be written in quasi-Newtonian form:<sup>1</sup>

$$w(\mathbf{x}) = \frac{GM}{r} - \frac{GM}{r} \sum_{k=2}^{\infty} \left(\frac{R}{r}\right)^k J_k P_k(\cos \theta), \tag{5}$$

where  $(r, \theta, \phi)$  are usual spherical coordinates,  $R$  is some suitably chosen radius of the central body (e.g. some ‘equatorial radius’ of a reference ellipsoid) and  $P_k$  are Legendre’s polynomials, given by Rodrigues’ formula

$$P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k.$$

Though expression (5) for the potential  $w$  looks perfectly Newtonian, it is a post-Newtonian result since the quantities  $J_k$ , the post-Newtonian zonal harmonics of our central body, are defined to post-Newtonian accuracy.

The post-Newtonian equation of motion for the coordinate position  $\mathbf{x}_s$  of a satellite in such an external gravitational field, given by (1), in harmonic coordinates takes the form (e.g. Damour et al. 1994)

$$\frac{d^2 \mathbf{x}_s}{dt^2} = \nabla w + \frac{1}{c^2} [-2\nabla(w^2) - 4(\nabla w \cdot \mathbf{v})\mathbf{v} + \mathbf{v}^2 \nabla w]. \tag{6}$$

The  $c$ -independent part of the satellite acceleration in (6) is the ‘quasi-Newtonian’ term, the rest that explicitly depends upon  $1/c^2$  describes the ‘post-Newtonian’ acceleration terms. So formally we can write the satellite acceleration,  $\mathbf{a}_s = d^2 \mathbf{x}_s / dt^2$  in the form

$$\mathbf{a}_s = \mathbf{a}_N + \mathbf{a}_{PN}, \tag{7}$$

where the quasi-Newtonian part,  $\mathbf{a}_N$ , contains the post-Newtonian zonal harmonics. We write

$$w(\mathbf{x}) = w_0(\mathbf{x}) + \sum_{k \geq 2} w_k(\mathbf{x}) \tag{8}$$

with

$$w_0(\mathbf{x}) = \frac{GM}{r}. \tag{9}$$

The potentials  $w_k(\mathbf{x})$  result from the non-spherical matter distribution of the central body. In the following, we will concentrate on one multipole moment of order  $k \geq 2$  at a time. The satellite acceleration can then be written in the form

$$\mathbf{a}_s = \mathbf{a}_N[w_0] + \mathbf{a}_N[w_k] + \mathbf{a}_{PN}[w_0] + \mathbf{a}_{PN}[w_k]. \tag{10}$$

The terms on the right hand side are: the quasi-Newtonian mass-monopole acceleration, the quasi-Newtonian acceleration of  $J_k$  with  $k \geq 2$ , the post-Newtonian Schwarzschild acceleration and the post-Newtonian acceleration of  $J_k$  with  $k \geq 2$ .

Satellite orbits in the post-Newtonian Schwarzschild field have been extensively discussed in the literature (e.g. Brumberg 1972; Soffel et al. 1987; Soffel 1989). As is well known,

<sup>1</sup> The dipole  $J_1$  vanishes because of the choice of coordinates (center of mass condition).

satellite equations of motion (for timelike-geodesics) in the exact Schwarzschild field admit exact solutions in terms of elliptic functions (e.g. Hagihara 1931; Mielnik and Plebanski 1962). Post-Newtonian effects related to the oblateness ( $J_2$ ) of the central body have been discussed by Soffel et al. (1988), Heimberger et al. (1990) and Huang and Liu (1992). In Soffel et al. (1988) the direct  $J_2/c^2$ -terms in the equations of motion (6) are discussed by using the Gauss form of the usual perturbation equations to first order. In Heimberger et al. (1990) the complete problem of direct and mixed terms is treated with the Hori–Deprit Lie-series method. Finally, Huang and Liu (1992) again discussed the complete problem of  $J_2/c^2$ -terms, using some mean elements method due to Kozai (1959). At this place also the paper by Iorio (2015) and the references cited therein should be mentioned, where the full quadrupole-problem is again discussed in great detail and for an arbitrary orientation of the symmetry axis.

The present paper focuses on relativistic (post-Newtonian) effects related to zonal harmonics  $J_k$  with  $k > 2$ , that have not been treated before. It therefore generalizes and extends the above-mentioned papers (Soffel et al. 1988; Heimberger et al. 1990; Huang and Liu 1992). The organization of this paper is as follows: Chapter 2 discusses the direct post-Newtonian  $J_k$ -terms in the harmonic equations of motion by means of first-order perturbation theory. The corresponding perturbation theoretical results will be checked by numerical integrations of the equations of motion. Chapter 3 is devoted to the mixed terms that appear from the (quasi-)Newtonian  $J_k$  terms and the post-Newtonian mass monopole (PN Schwarzschild) term as second-order terms. These mixed terms are treated by means of the Hori–Deprit Lie-series method. Finally, Chapter 4 presents a discussion based upon orders of magnitudes of these new terms.

## 2 Direct terms of order $J_k/c^2$

First, we derive direct  $J_k/c^2$ -terms from the acceleration (7) and solve the Gauss form perturbation equations via the common  $S, T, W$ -decomposition, similar to Soffel et al. (1988).

### 2.1 Derivation

The direct terms are included in  $\mathbf{a}_{\text{PN}}[w_k]$ , which is explicitly given by

$$\begin{aligned} \mathbf{a}_{\text{PN}}[w_k] &= -\frac{4}{c^2} \nabla (w_0 \cdot w_k) - \frac{4}{c^2} \mathbf{v} \cdot (\mathbf{v} \cdot \nabla) w_k + \frac{\mathbf{v}^2}{c^2} \nabla w_k \\ &\equiv \mathbf{a}_{1,k} + \mathbf{a}_{2,k} + \mathbf{a}_{3,k}, \end{aligned} \tag{11}$$

where  $w_k^2$ -terms have been neglected. Calculating the perturbing accelerations  $\mathbf{a}_{i,k}$ , we find

$$\mathbf{a}_{1,k} = -\frac{4}{c^2} \frac{\mu^2 R^k J_k}{r^{k+3}} \cdot \left[ (2+k) P_k(\cos \theta) \cdot \mathbf{e}_r + \sin \theta P'_k(\cos \theta) \cdot \mathbf{e}_\theta \right] \tag{11a}$$

$$\begin{aligned} \mathbf{a}_{2,k} &= -\frac{4}{c^2} \frac{\mu R^k J_k}{r^{k+3}} \left[ (1+k) (\mathbf{r} \cdot \mathbf{v}) P_k(\cos \theta) + (\mathbf{r} \cdot \mathbf{v}) \cdot \frac{z}{r} \cdot P'_k(\cos \theta) \right. \\ &\quad \left. - v_z \cdot r \cdot P'_k(\cos \theta) \right] \cdot \mathbf{v} \end{aligned} \tag{11b}$$

$$\mathbf{a}_{3,k} = \frac{\mu R^k J_k}{r^{k+2}} \cdot \left[ (1+k) P_k(\cos \theta) \cdot \mathbf{e}_r + \sin \theta P'_k(\cos \theta) \cdot \mathbf{e}_\theta \right] \cdot \frac{\mathbf{v}^2}{c^2}, \tag{11c}$$

from which the  $S, T, W$ -decomposition follows as

$$S = -\frac{\mu^2 R^k J_k}{c^2 r^{k+3}} \left( \left[ (14 + 10k) - 3(1 + k) \frac{r}{a} - 4(1 + k) \frac{p}{r} \right] \cdot P_k(\cos \theta) - 4e \sin f \sin I \cos(\omega + f) \cdot P'_k(\cos \theta) \right) \tag{12a}$$

$$T = \frac{\mu^2 R^k J_k}{c^2 r^{k+3}} \left[ \left( 2 + \frac{r}{a} + 4 \frac{p}{r} \right) \cdot \sin I \cos(\omega + f) \cdot P'_k(\cos \theta) - 4(1 + k) e \sin f \cdot P_k(\cos \theta) \right] \tag{12b}$$

$$W = \frac{\mu^2 R^k J_k}{c^2 r^{k+3}} \cdot \left( 2 + \frac{r}{a} \right) \cdot \cos I \cdot P'_k(\cos \theta) . \tag{12c}$$

$P'_k$  are the derivatives of Legendre’s polynomials with respect to the argument,

$$P'_k = \sum_{j=0}^{\left[ \frac{k-1}{2} \right]} \alpha_{k,j} P_{k-2j-1} , \tag{12d}$$

where the coefficients  $\alpha_{k,j}$  are given by

$$\alpha_{k,j} = 2(k - 2j) - 1 . \tag{13}$$

### 2.2 First-order perturbation theory

From the  $S, T, W$ -decomposition of the perturbing acceleration  $\mathbf{a}_{PN}[w_k]$  from (11), we obtain the right hand sides of the perturbation equations in Gauss form (e.g. Beutler 2005). These can be solved employing expansions into Kaula inclination functions (Kaula 1966) and Hansen’s coefficients (Hansen 1974). The process is straightforward, though tedious. Here we give only the most interesting parts of the solutions, the secular drifts in  $\omega$  and  $\Omega$ . Results for the other orbital elements, as well as short- and long-periodic parts in  $\omega$  and  $\Omega$ , are presented in “Appendix”. For  $k$  even, the drifts are given by ( $\eta \equiv \sqrt{1 - e^2}$ )

$$(\Delta\omega')_{\text{sec}} = \frac{\mu}{c^2 a} J_k \left( \frac{R}{a} \right)^k \frac{\eta}{e} F_{k,0,k/2}(I) \cdot \left( (14 + 10k) \cdot X_0^{-k-3,1} - 3(1 + k) \cdot X_0^{-k-2,1} - 4\eta^2(1 + k) \cdot X_0^{-k-4,1} + 2e(1 + k) \cdot \left[ X_0^{-k-3,2} - X_0^{-k-3,0} \right] + \frac{1}{\eta^2} \left( X_0^{-k-2,2} - X_0^{-k-2,0} \right) \right) \cdot nt , \tag{14}$$

$$(\Delta\Omega)_{\text{sec}} = -\frac{\mu}{c^2 a} J_k \left( \frac{R}{a} \right)^k \frac{\cot I}{\eta} \sum_{j=0}^{\left[ \frac{k-1}{2} \right]} \alpha_{k,j} \cdot F_{k-2j-1,0,k/2-j}(I) \times \left[ 2X_0^{-k-2,0} + X_0^{-k-1,0} \right] \cdot nt ,$$

where  $\omega = \omega' - \Omega \cos I$  . (15)

Here,  $a, e, I, \omega, \Omega$  are the usual osculating elements of the satellite orbit.  $X_0^{-q,m}(e)$  denotes Hansen coefficients, given by (see Hughes 1981<sup>2</sup>)

$$X_0^{-q,m}(e) = \left(\frac{e}{2}\right)^m \cdot \frac{1}{(1-e^2)^{(2q-3)/2}} \cdot \sum_{j=0}^{\lfloor \frac{q-m-2}{2} \rfloor} \binom{q-2}{2j+m} \binom{2j+m}{j} \cdot \left(\frac{e}{2}\right)^{2j}$$

where  $2 \leq q$ ,

(16)

and  $F_{k,0,m}(I)$  are Kaula inclination functions, given by (see Kaula 1966)

$$F_{k,0,m}(I) = (-1)^{\lfloor \frac{3k}{2} \rfloor} \frac{1}{2^k} \binom{k}{m} \cdot \sum_{\sigma=\sigma_1}^{\sigma_2} (-1)^\sigma \binom{2k-2m}{\sigma} \binom{2m}{k-\sigma} c^{3k-2m-2\sigma} s^{2m+2\sigma-k}$$

where  $c = \cos \frac{I}{2}$ ,  $s = \sin \frac{I}{2}$ ,  $\sigma_1 = \max \{0, k - 2m\}$  and  $\sigma_2 = \min \{k, 2k - 2m\}$ .

(17)

For  $k$  odd, there are no secular drifts in  $\omega$  and  $\Omega$ , except for the unphysical case, where one considers only odd multipoles. In this case, the usually long-periodic part of the lowest (i.e. biggest) multipole in  $\omega$  will give rise to a secular drift of the form

$$\begin{aligned} (\Delta\omega)_{\text{sec}} = & \frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{\eta}{e} \cdot nt \cdot \left[ \sum_{\substack{m=0 \\ m \neq (k/2)}}^k (-1)^k \cdot F_{k,0,m}(I) \cdot \sin(k'\omega) \right. \\ & \times \left( (14 + 10k) \cdot X_0^{-k-3,k'+1} - 3(1+k) \cdot X_0^{-k-2,k'+1} \right. \\ & - 4\eta^2(1+k) \cdot X_0^{-k-4,k'+1} + 2e(1+k) \cdot \left[ (X_0^{-k-3,k'+2} - X_0^{-k-3,k'}) \right. \\ & \left. \left. + \frac{1}{\eta^2} (X_s^{-k-2,k'+2} - X_0^{-k-2,k'}) \right] \right) \\ & + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \sin I F_{k-2j-1,0,m}(I) \cdot \sin(k''\omega) \\ & \times \left( \frac{1}{2\eta^2} \cdot [X_0^{-k-1,k''+1} - X_0^{-k-1,k''-1}] \right. \\ & + \left( \frac{1}{2} + \frac{1}{\eta^2} \right) \cdot [X_0^{-k-2,k''+1} - X_0^{-k-2,k''-1}] \\ & - \frac{2e}{\eta^2} \left[ X_0^{-k-2,k''} + \frac{1}{2} X_0^{-k-1,k''} \right] \\ & - e \cdot [X_0^{-k-3,k''+2} - X_0^{-k-3,k''-2}] \\ & + 3 \cdot [X_0^{-k-3,k''+1} - X_0^{-k-3,k''-1}] \\ & \left. \left. + 2\eta^2 \cdot [X_0^{-k-4,k''+1} - X_0^{-k-4,k''-1}] \right] \right), \end{aligned}$$

where  $k' = k - 2m$  and  $k'' = k - 2j - 2m$ ,

(18)

<sup>2</sup> It is formula (11) in Hughes (1981), which gives the Hansen coefficients for the cases relevant for us. Note however, that the formula there contains an error: it should read  $1/2^{2j}$  instead of  $1/2^j$  (Schanner 2017).

**Table 1** Initial, dressed orbital elements for the numerical integrations

$a_0$	12, 270 km	$\omega_0$	53.12°
$e_0$	0.0044333	$\Omega_0$	289.74°
$I_0$	109.84°	$M_0$	302°

The values are chosen to agree with the LAGEOS elements from Lucchesi et al. (2015)

and there will be no long-periodic perturbations in  $\omega$  from this lowest odd  $J_k$ .

### 2.2.1 Checks by numerical integrations

In Soffel et al. (1988), explicit expressions for orbital perturbations resulting for the post-Newtonian quadrupole terms in the equations of motion are given, which corresponds to  $k = 2$  in our calculations. We used a symbol manipulation program (SymPy), to evaluate the general expressions for  $k = 2$  and compare the results to the formulas given there; complete agreement was found. To check the corresponding terms with  $k > 2$ , for which so far no perturbation theoretical results exist, we numerically integrate the equations of motion. As an integrator we choose an ODEX implementation in Fortran90, a description of which can be found in Hairer et al. (1993). The initial values for the orbital elements ( $a_0, e_0, \omega_0, \Omega_0, I_0, M_0$ ) are chosen in accordance with LAGEOS data and are given in Table 1.

Further parameters for numerical integration were chosen as:  $\mu = 3.986005 \times 10^{14} \text{ m}^3/\text{s}^2$ ,  $R = 6371 \text{ km}$ ,  $c = 10^5 \text{ m/s}$ ,  $J_2 = 0.0011$  so that the magnitudes of perturbing accelerations are of order  $|\mathbf{a}_N[w_0]| \sim 2.6$ ,  $|\mathbf{a}_N[w_2]|/|\mathbf{a}_N[w_0]| \sim 3.0 \times 10^{-3}$ ,  $|\mathbf{a}_{PN}[w_0]|/|\mathbf{a}_N[w_0]| \sim 3.2 \times 10^{-3}$  and  $|\mathbf{a}_{PN}[w_2]|/|\mathbf{a}_N[w_0]| \sim 9.6 \times 10^{-6}$ . From the initial elements above, the initial theoretical expressions for the perturbations are subtracted, to get the unperturbed elements for perturbation theory.

For these numerical integrations, we chose the satellite acceleration to be of the form

$$\mathbf{a}_s = -\frac{\mu}{r^2} \frac{\mathbf{x}}{r} + \mathbf{a}_{1,k} + \mathbf{a}_{2,k} + \mathbf{a}_{3,k}, \tag{19}$$

$$\text{where } \mathbf{a}_{1,k} = -\frac{4}{c^2} \frac{\mu^2 R^k J_k}{r^{k+3}} \cdot \left[ (2+k) P_k\left(\frac{z}{r}\right) \cdot \mathbf{e}_r + P'_k\left(\frac{z}{r}\right) \cdot \left(\frac{z}{r} \mathbf{e}_r - \mathbf{e}_z\right) \right] \tag{19a}$$

$$\begin{aligned} \mathbf{a}_{2,k} = & -\frac{4}{c^2} \frac{\mu R^k J_k}{r^{k+3}} \left[ (1+k) (\mathbf{r} \cdot \mathbf{v}) P_k\left(\frac{z}{r}\right) + (\mathbf{r} \cdot \mathbf{v}) \cdot \frac{z}{r} \cdot P'_k\left(\frac{z}{r}\right) \right. \\ & \left. - v_z \cdot r \cdot P'_k\left(\frac{z}{r}\right) \right] \cdot \mathbf{v} \end{aligned} \tag{19b}$$

$$\mathbf{a}_{3,k} = \frac{\mu R^k J_k}{r^{k+2}} \cdot \left[ (1+k) P_k\left(\frac{z}{r}\right) \cdot \mathbf{e}_r + P'_k\left(\frac{z}{r}\right) \cdot \left(\frac{z}{r} \mathbf{e}_r - \mathbf{e}_z\right) \right] \cdot \frac{v^2}{c^2}. \tag{19c}$$

When subtracting the perturbation theoretical results from the numerical ones, the difference was several orders of magnitudes below the first-order perturbation theoretical results which are checked in that way.

### 3 Mixed terms of order $J_k/c^2$

As already mentioned above, the second-order perturbation theory yields terms of the same order as the direct terms discussed above, due to mixing of the post-Newtonian mass monopole (the post-Newtonian Schwarzschild acceleration) and the (quasi-)Newtonian multipoles  $J_k$ . Because of this, we employ canonical perturbation theory for the full problem in this section, in analogy to Heimberger et al. (1990). The basic equation of motion (6) can be obtained from a specific Lagrangian<sup>3</sup> (e.g. Weinberg 1972)

$$\mathcal{L} = \frac{1}{2}\mathbf{v}^2 + w + \frac{1}{c^2} \left[ \frac{\mathbf{v}^4}{8} - \frac{w^2}{2} + \frac{3}{2}w\mathbf{v}^2 \right]. \tag{20}$$

The canonical momentum  $\mathbf{p}$  is then given by

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}}.$$

Note, that  $\mathbf{p} = \mathbf{v} + \delta\mathbf{p}$ , where  $\delta\mathbf{p}$  is a post-Newtonian correction to the velocity  $\mathbf{v}$ . Because of this, the orbital elements in our post-Newtonian canonical formalism, often called *contact* elements, are not osculating elements (e.g. Brumberg 1972; Kopeikin et al. 2011). The reason for this can simply be understood: The osculating elements ( $a_{\text{osc}}, e_{\text{osc}}, \omega_{\text{osc}}, \Omega_{\text{osc}}, I_{\text{osc}}, M_{\text{osc}}$ ) are equivalent to the vectorial elements

$$\begin{aligned} \mathbf{h}_{\text{osc}} &= \mathbf{x} \times \mathbf{v} \\ \mathbf{f}_{\text{osc}} &= \left( \mathbf{v}^2 - \frac{GM}{r} \right) \mathbf{x} - (\mathbf{x} \cdot \mathbf{v}) \mathbf{v}. \end{aligned}$$

$\mathbf{h}$  is the specific angular momentum vector and  $\mathbf{f}$  the Runge–Lenz vector, pointing towards pericenter. In the post-Newtonian canonical formalism, one has to replace  $\mathbf{v}$  by the canonical momentum  $\mathbf{p}$ , so that the contact elements ( $a_{\text{cont}}, e_{\text{cont}}, \omega_{\text{cont}}, \Omega_{\text{cont}}, I_{\text{cont}}, M_{\text{cont}}$ ) are determined by

$$\begin{aligned} \mathbf{h}_{\text{cont}} &= \mathbf{x} \times \mathbf{p} \\ \mathbf{f}_{\text{cont}} &= \left( \mathbf{p}^2 - \frac{GM}{r} \right) \mathbf{x} - (\mathbf{x} \cdot \mathbf{p}) \mathbf{p}. \end{aligned}$$

The equations for ( $\mathbf{h}_{\text{osc}}, \mathbf{f}_{\text{osc}}$ ) and ( $\mathbf{h}_{\text{cont}}, \mathbf{f}_{\text{cont}}$ ) fix the relations between osculating and contact elements. These relations are important when comparing results from the different approaches. Evaluating such relations, that can be found in Brumberg (1972) or Kopeikin et al. (2011), we find for the here considered  $\Omega$  and  $\omega$ :

$$\Omega_{\text{osc}} = \Omega_{\text{cont}} \tag{21a}$$

$$\begin{aligned} \omega_{\text{osc}} = \omega_{\text{cont}} + \frac{\sin f}{e_{\text{osc}}} \frac{\mu}{c^2 a_{\text{osc}}} \left[ 1 - 8 \left( \frac{a}{r} \right)_{\text{osc}} \right. \\ \left. + 6J_k \left( \frac{R}{a_{\text{osc}}} \right)^k \sum_{m=0}^k F_{k,0,m}(I) \cdot \left( \frac{a}{r} \right)_{\text{osc}}^{k+1} \cos((k-2m) \cdot (\omega_{\text{osc}} + f_{\text{osc}})) \right]. \end{aligned} \tag{21b}$$

From the discussion based on  $\mathbf{h}$  and  $\mathbf{f}$ , it is clear that  $\Omega_{\text{osc}} = \Omega_{\text{cont}}$ , since in our case the orbital plane remains the same when using contact elements. Because we later concentrate

<sup>3</sup> Consider that in the Euler–Lagrange equations  $d/dt = \partial/\partial t + \mathbf{v}\nabla$ , which will lead to terms proportional to  $\dot{\mathbf{v}} = \mathbf{a}$ . To derive the direct terms of order  $J_k/c^2$  in (6) we insert  $\mathbf{a} = \nabla w + \mathcal{O}_2$  and neglect terms of order  $c^{-4}$ .



on secular drifts only, we average the second relation and find, that the differences are short- and long-periodic, i.e. for our purposes we use

$$\omega_{\text{osc}} = \omega_{\text{cont}} . \tag{21c}$$

In the following, we will often not distinguish between osculating and contact elements for the sake of convenience.

The Hamiltonian corresponding to the Lagrangian (20) reads

$$\mathcal{H} = \mathbf{v} \cdot \mathbf{p} - \mathcal{L} = \frac{1}{2} \mathbf{p}^2 - w - \frac{1}{c^2} \left[ \frac{\mathbf{p}^4}{8} - \frac{w^2}{2} + \frac{3}{2} w \mathbf{p}^2 \right] . \tag{22}$$

As canonical variables we choose post-Newtonian Delaunay variables ( $\mu \equiv \text{GM}$ )<sup>4</sup>

$$\begin{aligned} l &= M & L &= na^2 = \sqrt{\mu a} \\ g &= \omega & G &= na^2 \sqrt{1 - e^2} = \eta L \\ h &= \Omega & H &= na^2 \sqrt{1 - e^2} \cos I = \eta L \cos I , \end{aligned}$$

where we keep in mind, that  $(a, e, \omega, \Omega, I, M)$  are contact elements, that differ from osculating ones. We insert the potentials and use an expansion of Legendre’s polynomials into functions of Delaunay variables, given by Garfinkel (1965)

$$\begin{aligned} P_k(\cos \theta) &= \sum_{m=0}^k (2 - \delta_{k,2m}) \cdot b_{k-2m} \cdot \cos \left( (k - 2m) \cdot \left( f + g - \frac{\pi}{2} \right) \right) , \\ \text{where } b_m &= \frac{(l - m)!}{(l + m)!} \cdot P_l^m(0) \cdot P_l^m \left( \frac{H}{G} \right) \end{aligned} \tag{23}$$

and  $P_l^m$  are the associated Legendre polynomials

$$P_k^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_k(x) .$$

This way the Hamiltonian is written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \frac{1}{2} \mathcal{H}_2 , \tag{24}$$

where

$$\mathcal{H}_0 = -\frac{\mu^2}{2L^2} \tag{24a}$$

is the unperturbed Kepler part,

$$\begin{aligned} \mathcal{H}_1 &= \frac{J_k R^k \mu^{k+2}}{L^{2k+2}} \left( \frac{a}{r} \right)^{k+1} \sum_{m=0}^{[k/2]} (2 - \delta_{k,2m}) \cdot b_{k-2m} \cdot \cos \left( (k - 2m) \cdot \left( f + g - \frac{\pi}{2} \right) \right) \\ &\quad - \frac{\mu^4}{c^2 L^4} \left[ 3 \left( \frac{a}{r} \right)^2 - 2 \frac{a}{r} + \frac{1}{8} \right] \end{aligned} \tag{24b}$$

<sup>4</sup> Here  $M$  is the central bodies mass, not to be confused with the mean anomaly  $M$ , meant when defining the Delaunay variables.

are the quasi-Newtonian- and Schwarzschild-parts (of first order) and

$$\begin{aligned} \frac{1}{2} \mathcal{H}_2 &= \frac{J_k R^k \mu^{k+4}}{c^2 L^{2k+4}} \sum_{m=0}^{\lfloor k/2 \rfloor} (2 - \delta_{k,2m}) \cdot b_{k-2m} \cdot \cos \left( (k - 2m) \cdot \left( f + g - \frac{\pi}{2} \right) \right) \\ &\times \left[ 2 \left( \frac{a}{r} \right)^{k+2} - \frac{3}{2} \left( \frac{a}{r} \right)^{k+1} \right] \end{aligned} \tag{24c}$$

is the post-Newtonian higher-order multipole part (of second order).

### 3.1 Lie-series (Hori–Deprit) method

Hori (1966) and Deprit (1969) developed a formalism to cast the canonical equations into a simpler form by means of Lie-transformations (see e.g. Heimberger et al. 1990 for more details). Consider a Hamiltonian  $\mathcal{H}$ , given as a series in some small perturbation parameter  $\varepsilon$

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \frac{\varepsilon^2}{2} \mathcal{H}_2 + \dots, \tag{25}$$

where the solutions to the equations of motion for  $\mathcal{H}_0$  are known. Let  $(\mathbf{y}', \mathbf{Y}')$  denote the transformed canonical variables, relating to the original variables  $(\mathbf{y}, \mathbf{Y})$  via a Lie-transformation generated by a function  $W = W_1 + W_2 + \dots$ . Then, to second order the transformed Hamiltonian  $\mathcal{H}^*$  is given by Deprit (1969)

$$\mathcal{H}_0^* = \mathcal{H}_0, \tag{26a}$$

$$\mathcal{H}_1^* = \{\mathcal{H}_0; W_1\} + \mathcal{H}_1, \tag{26b}$$

$$\begin{aligned} \mathcal{H}_2^* &= \{\{\mathcal{H}_0; W_1\}; W_1\} + 2\{\mathcal{H}_1; W_1\} + \{\mathcal{H}_0; W_2\} + \mathcal{H}_2 \\ &= \{\mathcal{H}_1^* + \mathcal{H}_1; W_1\} + \{\mathcal{H}_0; W_2\} + \mathcal{H}_2. \end{aligned} \tag{26c}$$

The braces denote the Poisson bracket, given by

$$\{\mathcal{H}; W\} = \sum_i \left( \frac{\partial \mathcal{H}}{\partial y'_i} \frac{\partial W}{\partial Y'_i} - \frac{\partial \mathcal{H}}{\partial Y'_i} \frac{\partial W}{\partial y'_i} \right). \tag{27}$$

The time evolution of the new variables  $(\mathbf{y}', \mathbf{Y}')$  is then given by the (ideally simpler) canonical equations in  $\mathcal{H}^*$  and to second order  $(\mathbf{y}', \mathbf{Y}')$  relate to the old coordinates via

$$\mathbf{y} = \mathbf{y}' + \varepsilon \frac{\partial W_1}{\partial \mathbf{Y}'} + \frac{\varepsilon^2}{2} \left( \frac{\partial W_2}{\partial \mathbf{Y}'} + \left\{ \frac{\partial W_1}{\partial \mathbf{Y}'}; W_1 \right\} \right) \tag{28a}$$

$$\mathbf{Y} = \mathbf{Y}' - \varepsilon \frac{\partial W_1}{\partial \mathbf{y}'} - \frac{\varepsilon^2}{2} \left( \frac{\partial W_2}{\partial \mathbf{y}'} + \left\{ \frac{\partial W_1}{\partial \mathbf{y}'}; W_1 \right\} \right). \tag{28b}$$

Following Heimberger et al. (1990), we choose  $W$  in a way, that the transformed Hamiltonian  $\mathcal{H}^*$  corresponds to the  $l$ -average of the original Hamiltonian  $\mathcal{H}$ . This way, the second-order secular drifts we are interested in are contained in (26c), more precisely in  $\mathcal{H}_2$  and the Poisson bracket

$$\{\mathcal{H}_1 + \mathcal{H}_1^*; W_1\} \equiv \mathcal{T}_2. \tag{29}$$

The other Poisson bracket  $\{\mathcal{H}_0; W_2\}$  only contains short-periodic contributions and will not be considered here. The explicit calculation of  $\mathcal{T}_2$  is tedious. To simplify things, we concentrate on secular perturbations only. Formally, this is done by a second Lie-transformation,

which includes the long-periodic perturbations and eliminates them from the equations of motion. In this case, the Hori–Deprit equations read

$$\begin{aligned} \mathcal{H}_0^{**} &= \mathcal{H}_0^* , \\ \mathcal{H}_1^{**} &= \{ \mathcal{H}_0; W_1^* \} + \mathcal{H}_1^* , \\ \mathcal{H}_2^{**} &= \{ \mathcal{H}_1^{**} + \mathcal{H}_1^*; W_1^* \} + \{ \mathcal{H}_0; W_2^* \} + \mathcal{H}_2^* . \end{aligned}$$

We choose  $\mathcal{H}_1^{**}$  and  $\mathcal{H}_2^{**}$  as the averages of  $\mathcal{H}_1^*$  and  $\mathcal{H}_2^*$  with respect to  $g'$ , respectively. We find for  $\langle \langle T_2 \rangle \rangle_{g'}$ :

$$\begin{aligned} \langle \langle T_2 \rangle \rangle_{g'} &= \frac{J_k R^k \mu^{k+4}}{c^2 L'^5 G'^{2k-1}} \cdot b_0 \\ &\times \left( (k+1) \sum_{s=0}^{k+2} \sum_{q=0}^s \left( \frac{e''}{2} \right)^s \binom{k+2}{s} \binom{s}{q} \delta_{s,2q-1} \frac{3}{e''} \frac{L'^4}{G'^4} \right. \\ &- (k+1) \sum_{s=0}^{k+1} \sum_{q=0}^s \left( \frac{e''}{2} \right)^s \binom{k+1}{s} \binom{s}{q} \left[ \delta_{s,2q} \frac{9}{2} \frac{L'^4}{G'^4} \right. \\ &+ \delta_{s,2q-1} \frac{2}{e''} \frac{L'^2}{G'^2} + \delta_{s,2q-2} \frac{3}{2} \frac{L'^4}{G'^4} \left. \right] \\ &+ (k+1) \sum_{s=0}^k \sum_{q=0}^s \left( \frac{e''}{2} \right)^s \binom{k}{s} \binom{s}{q} \left[ \delta_{s,2q} \left( 3 \frac{L'^2}{G'^2} + \frac{3}{2} \frac{L'^4}{G'^4} \right) \right. \\ &+ \delta_{s,2q-1} \left( \frac{2}{e''} - \frac{3}{e''} \frac{L'}{G'} \right) - \delta_{s,2q-2} \left( \frac{3}{2} \frac{L'^4}{G'^4} - \frac{L'^2}{G'^2} \right) \left. \right] \\ &+ \sum_{s=0}^{k-1} \sum_{q=0}^s \left( \frac{e''}{2} \right)^s \binom{k-1}{s} \binom{s}{q} \\ &\times \left( \delta_{s,2q} \left[ -4k - 12 + 6k \frac{L'}{G'} + 15 \frac{L'}{G'} + (k+1)e'^2 \frac{L'^2}{G'^2} + 7 \frac{L'^2}{G'^2} \right] \right. \\ &+ \delta_{s,2q-1} \left[ 6e'' \frac{L'^2}{G'^2} + \frac{4}{e''} \frac{L'^2}{G'^2} \right] \\ &+ \delta_{s,2q-2} \left[ 3 \frac{L'^2}{G'^2} - (k+1)e'^2 \frac{L'^2}{G'^2} \right] \\ &- s \delta_{s,2q} \cdot \left[ 2 + \frac{3}{16} e'^4 \frac{L'^2}{G'^2} - 6 \frac{L'^2}{G'^2} \right] \\ &\left. + \frac{s}{s-2q} \left[ \delta_{s,2q-1} \left( \frac{2}{e''} - \frac{6}{e''} \frac{L'^2}{G'^2} \right) - \delta_{s,2q-2} 3 \frac{L'^2}{G'^2} \right] \right) \right) . \end{aligned} \tag{30}$$

$\mathcal{H}_1^{**}$  and  $\langle \langle \mathcal{H}_2 \rangle \rangle_{g'}$  follow directly:

$$\begin{aligned} \mathcal{H}_1^{**} &= \frac{J_k R^k \mu^{k+2}}{L^3 G^{2k-1}} \cdot b_0 \cdot \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \frac{e}{2} \right)^{2s} \binom{k-1}{2s} \binom{2s}{s} \\ &- \frac{\mu^4}{c^2 L^4} \left[ 3 \frac{L}{G} - \frac{15}{8} \right] \end{aligned} \tag{31}$$

$$\begin{aligned} \langle \langle \mathcal{H}_2 \rangle_l \rangle_{g'} &= \frac{J_k R^k \mu^{k+4}}{c^2 L''^5 G''^{2k-1}} \cdot b_0 \\ &\times \left[ \frac{4}{\eta^2} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \left( \frac{e''}{2} \right)^s \binom{k}{s} \binom{2s}{s} - 3 \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \left( \frac{e''}{2} \right)^s \binom{k-1}{s} \binom{2s}{s} \right]. \end{aligned} \tag{32}$$

The drifts are then given by the Hamilton equations

$$\begin{aligned} \dot{g}'' &= \frac{\partial \mathcal{H}^{**}}{\partial G''} & \dot{h}'' &= \frac{\partial \mathcal{H}^{**}}{\partial H''} \end{aligned} \tag{33a,b}$$

where  $\mathcal{H}^{**} = \mathcal{H}_0 + \mathcal{H}_1^{**} + \frac{1}{2} \left( \langle \langle \mathcal{H}_2 \rangle_l \rangle_{g'} + \langle \langle \mathcal{T}_2 \rangle_l \rangle_{g'} \right)$ .

The second term in  $\mathcal{H}^{**}$  simply gives the first-order secular drifts, namely the Schwarzschild precession in  $\omega$  and quasi-Newtonian secular drifts in both  $\omega$  and  $\Omega$ . The  $(\dots)/2$ -term contains the  $J_k/c^2$ -contributions, i.e. the direct secular and the mixed term drifts. Explicit expressions are given in ‘‘Appendix’’. For the case  $k = 2$  our results agree with those in Heimberger et al. (1990). Note that  $b_0 = 0$  for  $k$  odd, which means that our second-order discussion does not include the (unphysical) odd drifts mentioned above.

Because of several reasons, a detailed comparison of our second-order perturbation theoretical results for  $k > 2$  with those of a purely numerical integration was beyond the scope of this paper, e.g. short-periodic perturbations of second order, that play a role for the precise initial conditions in the frame of perturbation theory, have not been derived. Another point are the quasi-Newtonian secular drifts, which are here given in terms of contact elements. When converting these expressions into osculating elements, the corrections (for example in the semi-major axis) again produce terms of order  $J_k/c^2$  which have to be included carefully.

### 4 Discussion

We have discussed direct and mixed perturbations in the satellite orbit around an axisymmetric central body, that are of order  $J_k/c^2$ . Our results for the direct terms have been checked numerically. Most of the work went into the calculation of perturbations due to second-order terms, that arise from mixing of the post-Newtonian Schwarzschild- and the quasi-Newtonian multipole terms. Previous results for the quadrupole (Soffel et al. 1988; Heimberger et al. 1990) have been reproduced and correspond to the case  $k = 2$  in our calculations. We are able to give the secular drifts in the argument of pericenter  $\omega$  and the ascending node  $\Omega$  for all higher even multipole orders and for the explicit perturbations also in the (unphysical) special cases where one considers only odd multipoles. For example, the nodal drift  $\dot{\Omega}_{J_4}$  due to  $J_4$  is for the direct terms given by

$$\begin{aligned} \dot{\Omega}_{J_4, \text{direct}} &= -J_4 \cdot \frac{15}{256} \left( \frac{\mu}{c^2 a} \right) \left( \frac{R}{a} \right)^4 \cdot \frac{n}{(1-e^2)^5} \cdot (28 \cos^3(I) - 12 \cos(I)) \\ &\times (12 + 26e^2 - 3e^4). \end{aligned} \tag{34}$$

At this place, we might ask if some of the effects discussed in this paper will be measurable from the orbits of artificial Earth satellites. Since the determination of the geoid using high precession satellite data reaches an accuracy at the mm level, it is clear that one has to model satellite orbits in the framework of Einstein’s theory of gravity, at least in the first post-Newtonian approximation. It is clear from the very beginning, that relativistic effects

**Table 2** Secular drifts in the argument of pericenter  $\omega$  and the ascending node  $\Omega$  in mas/y, due to relativistic effects related to higher-order multipoles. The index ‘direct’ refers to the explicit  $J_k/c^2$ -terms in the acceleration (from Sect. 2). The index ‘mixed’ refers to second-order mixed perturbations that have been calculated in Sect. 3. We get these numbers by deriving the respective special case for each order  $k$  from the general formulas for  $\omega$  and  $\Omega$ , given in Sect. 2 and “Appendix,” and inserting the LAGEOS elements from Lucchesi et al. (2015)

Order $k$	$\dot{\omega}_{k,\text{direct}}$	$\dot{\omega}_{k,\text{mixed}}$	$\dot{\Omega}_{k,\text{direct}}$	$\dot{\Omega}_{k,\text{mixed}}$
2	0.150	- 3.675	- 0.495	2.063
4	$8.904 \times 10^{-5}$	$- 6.020 \times 10^{-4}$	$2.662 \times 10^{-4}$	$- 1.819 \times 10^{-3}$
6	$- 5.760 \times 10^{-5}$	$7.370 \times 10^{-4}$	$- 2.305 \times 10^{-5}$	$2.190 \times 10^{-4}$
8	$1.192 \times 10^{-5}$	$- 1.834 \times 10^{-4}$	$6.940 \times 10^{-7}$	$- 8.444 \times 10^{-6}$

on the orbit of artificial Earth’s satellites are small, since the Earth gravitational radius  $R_G = GM_E/c^2$  is only 0.44 cm, so that they play a role mainly for high precision orbit determinations (HPOD). If an orbit cannot be determined with cm accuracy or better, relativistic effects might be absorbed in the orbital parameters (Soffel and Frutos 2016). Such HPODs are possible for the Laser Geodynamics satellites LAGEOS, LAGEOS II and LARES. Mean Keplerian elements for these satellites for some reference epoch can be found, e.g. in Lucchesi et al. (2015). The dominant relativistic effects for these orbits are well known, concerning secular drifts of the ascending node and argument of perigee: the (secular) perigee precession  $\dot{\omega}_{\text{Schw}}$  due to the Schwarzschild field of the Earth,  $\dot{\Omega}_{\text{LT}}$  and  $\dot{\omega}_{\text{LT}}$  due to the Lense-Thirring precession (a gravito-magnetic effect caused by the Earth’s intrinsic angular momentum or spin dipole) and  $\dot{\Omega}_{\text{GP}}$ , the nodal drift due to geodetic precession. For LAGEOS the orders of magnitude for the related accelerations are in the range of  $10^{-9} \dots 10^{-12} \text{ m/s}^2$  (Schanner 2017). Drift rates can be found in Lucchesi et al. (2015).

These numbers imply that for a measurement of relativistic effects by means of SLR data from a single satellite, the even zonal harmonics of the Earth would have to be known with extreme precision. A comprehensive list for the secular nodal drifts of the two LAGEOS satellites has been published in Iorio (2006). To get an idea of the orders of magnitude of effects due to the Newtonian  $J_k$ -zonal harmonics of the LAGEOS II orbit:  $\dot{\Omega}_k \approx - 8.284 \times 10^8, + 5.933 \times 10^2, + 2.794$  for  $k = 2, 10$  and  $16$ , respectively (values are in mas/y = 1 milli-arc-second per year). Corresponding orders of magnitude for  $\dot{\omega}_k$  are:  $+ 5.733 \times 10^8, + 3.723 \times 10^3, + 2.721$  (Soffel and Frutos 2016). At such level of precision tidal perturbations and a variety of non-gravitational forces acting on a satellite have to be modelled with sufficient accuracy: atmospheric drag, solar radiation pressure, albedo radiation pressure, thermal emission, dynamic solid Earth tide, dynamic ocean tide (e.g. Milani et al. 1987), etc. These accelerations are for LAGEOS of orders between  $10^{-6} \dots 10^{-19} \text{ m/s}^2$ . Explicit numbers can be found in Lucchesi et al. (2015). These numbers suggest that for the modelling of the LAGEOS orbits there exists some fundamental noise limit of the order  $10^{-15} \text{ m/s}^2$ , depending not only upon the orbit itself, but also upon the physical properties of the satellite, where certain ‘Newtonian’ forces can no longer be controlled and modelled sufficiently well (Soffel and Frutos 2016). Modelling with an accuracy beyond this critical noise floor becomes practically impossible.

Typical satellite accelerations related to  $J_k/c^2$  that are considered in this paper are roughly of order

$$\left(\frac{GM_E}{a^2}\right) \left(\frac{GM_E}{c^2 R_E}\right) \left(\frac{R_E}{a}\right)^{k+1} J_k.$$

The first term is the Earth’s monopole contribution of  $2.7 \text{ m/s}^2$ . The second term has a value of  $6.96 \times 10^{-10}$  and  $R_E/a$  for the LAGEOS orbits is given by 0.53. So for the  $k = 4$  terms with  $J_4 = -1.5 \times 10^{-6}$ , we obtain an acceleration of order  $10^{-16} \text{ m/s}^2$ , that is below the noise floor and likely can never be detected in the data for the LAGEOS orbits. Effects from higher values of  $k$  are even smaller. Nonetheless, we conclude by giving the drifts in  $\omega$  and  $\Omega$  for higher  $k$  values, calculated in this paper (see Table 2).

**Compliance with ethical standards**

**Conflict of interest** The authors confirm that they have no conflict of interest to declare.

**Appendix**

**A. Orbital averages**

In this paper, certain averages of functions  $F$  over one complete revolution in the unperturbed Keplerian orbit are employed:

$$\langle F \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} F \, dM = \frac{1}{2\pi\eta} \int_0^{2\pi} F \left(\frac{a}{r}\right)^{-2} \, df, \tag{35}$$

where  $M$  and  $f$  are the mean and true anomaly, respectively,  $\eta \equiv \sqrt{1 - e^2}$  and

$$\frac{a}{r} = \frac{1 + e \cos f}{\eta^2}. \tag{36}$$

The following averages are used:

$$\left\langle \left(\frac{a}{r}\right)^n \right\rangle = \frac{1}{\eta^{2n-3}} \sum_{s=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(\frac{e}{2}\right)^{2s} \binom{n-2}{2s} \binom{2s}{s} \tag{37a}$$

$$\left\langle \left(\frac{a}{r}\right)^n \cos f \right\rangle = \frac{1}{\eta^{2n-3}} \sum_{s=0}^{n-2} \sum_{q=0}^s \left(\frac{e}{2}\right)^s \binom{n-2}{s} \binom{s}{q} \delta_{s,2q+1}. \tag{37b}$$

From these relations, the following special cases can be derived:

$$\left\langle \left(\frac{a}{r}\right)^4 \right\rangle = \frac{e^2 + 2}{2} \frac{L^5}{G^5} \tag{38a}$$

$$\left\langle \left(\frac{a}{r}\right)^3 \right\rangle = \frac{L^3}{G^3} \tag{38b}$$

$$\left\langle \left(\frac{a}{r}\right)^2 \right\rangle = \frac{L}{G} \tag{38c}$$

$$\left\langle \left(\frac{a}{r}\right) \right\rangle = 1 \tag{38d}$$

$$\left\langle \left( \frac{a}{r} \right)^5 \cos f \right\rangle = \frac{3e^3 + 12e}{8} \frac{L^7}{G^7} \tag{38e}$$

$$\left\langle \left( \frac{a}{r} \right)^4 \cos f \right\rangle = e \frac{L^5}{G^5} \tag{38f}$$

$$\left\langle \left( \frac{a}{r} \right)^3 \cos f \right\rangle = \frac{e}{2} \frac{L^3}{G^3} \tag{38g}$$

$$\left\langle \left( \frac{a}{r} \right)^2 \cos f \right\rangle = 0 \tag{38h}$$

$$\left\langle \left( \frac{a}{r} \right)^4 \sin^2 f \right\rangle = \frac{e^2 + 4}{8} \frac{L^5}{G^5} \tag{38i}$$

$$\left\langle \left( \frac{a}{r} \right)^3 \sin^2 f \right\rangle = \frac{1}{2} \frac{L^3}{G^3} \tag{38j}$$

$$\left\langle \left( \frac{a}{r} \right)^2 \sin^2 f \right\rangle = \frac{1}{2} \frac{L}{G}, \tag{38k}$$

with

$$\eta = \sqrt{1 - e^2} = \frac{G}{L},$$

where  $G$  and  $L$  are Delaunay elements.

### B. Solutions for the direct terms

Here, we list the results for all orbital elements. The subscript SP stands for short-periodic, LP for long-periodic and  $S$  for secular perturbations. To condense the short-periodic results, we use the functions

$$\begin{aligned} \Phi_{i,j,k,l}(f, \omega) &= \sum_{s=0}^{k+i} \sum_{q=0}^s \binom{k+i}{s} \binom{s}{q} \left( \frac{e}{2} \right)^s \frac{1}{l+j+s-2q} \\ &\times \begin{cases} \sin \\ \cos \end{cases}_{k \text{ odd}}^{k \text{ even}} ((l+j+s-2q)f + l\omega) \end{aligned} \tag{39a}$$

$$\begin{aligned} \Phi_{i,j,k,l}^*(f, \omega) &= \sum_{s=0}^{k+i} \sum_{q=0}^s \binom{k+i}{s} \binom{s}{q} \left( \frac{e}{2} \right)^s \frac{1}{l+j+s-2q} \\ &\times \begin{cases} \cos \\ \sin \end{cases}_{k \text{ odd}}^{k \text{ even}} ((l+j+s-2q)f + l\omega) \end{aligned} \tag{39b}$$

$$\Psi_{i,j,k,l}(f, M) = \sum_{s=0}^{k+i} \sum_{q=0}^s \binom{k+i}{s} \binom{s}{q} \left( \frac{e}{2} \right)^s (f - M) \delta_{l,0} \delta_{s+j,2q} \tag{39c}$$

Note, that there are poles in the denominator for certain combinations of indices. For the short-periodic solutions these correspond to long-periodic terms and for the long-periodic solutions they correspond to secular drifts, i.e. they have to be excluded from the sums.

**Semi-major axis  $a$**

$$\begin{aligned}
 (\Delta a)_{SP} = & \frac{\mu}{c^2} \left(\frac{R}{a}\right)^k \frac{J_k}{\eta^{2k+4}} \cdot \left[ \sum_{m=0}^k e \cdot F_{k,0,m}(I) \right. \\
 & \times \left( (14 + 10k) \cdot \left( \Phi_{1,1,k,k'}^* - \Phi_{1,-1,k,k'}^* \right) \right. \\
 & \left. \left. - 3(1+k)\eta^2 \cdot \left( \Phi_{0,1,k,k'}^* - \Phi_{0,-1,k,k'}^* \right) \right) \right. \\
 & - 2 \sin I \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} (-1)^k F_{k-2j-1,0,m}(I) \\
 & \times \left( 4\Phi_{3,0,k,k''}^* + 2\Phi_{2,0,k,k''}^* + \eta^2 \Phi_{1,0,k,k''}^* \right. \\
 & \left. \left. - e^2 \cdot \left( \Phi_{1,2,k,k''}^* - 2\Phi_{1,0,k,k''}^* + \Phi_{1,-2,k,k''}^* \right) \right) \right], \tag{40}
 \end{aligned}$$

$$\begin{aligned}
 (\Delta a)_{LP} = & J_k \frac{\mu}{c^2} \left(\frac{R}{a}\right)^k \frac{2n}{\eta} \cdot \left[ \sum_{m=0}^k e \cdot \frac{F_{k,0,m}(I)}{k' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k' \omega) \right. \\
 & \times \left( (14 + 10k) \cdot X_0^{-k-3,k'+1} - 3(1+k) \cdot X_0^{-k-2,k'+1} \right) \\
 & - (-1)^k \sin I \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \\
 & \times \left( 4\eta^4 X_0^{-k-5,k''} + 2\eta^2 X_0^{-k-4,k''} + \eta^2 X_0^{-k-3,k''} \right. \\
 & \left. \left. - e^2 \left[ X_0^{-k-3,k''+2} - 2X_0^{-k-3,k''} + X_0^{-k-3,k''-2} \right] \right) \right], \tag{41}
 \end{aligned}$$

where  $k' = k - 2m$  and  $k'' = k - 2j - 2m$ .

**Eccentricity  $e$**

$$\begin{aligned}
 (\Delta e)_{SP} = & \frac{\mu}{c^2 a} \left(\frac{R}{a}\right)^k \frac{J_k}{\eta^{2k+2}} \cdot \left[ \sum_{m=0}^k F_{k,0,m}(I) \right. \\
 & \times \left( (7 + 5k) \cdot \left( \Phi_{1,1,k,k'}^* - \Phi_{1,-1,k,k'}^* \right) - \frac{3}{2} (1+k)\eta^2 \cdot \left( \Phi_{0,1,k,k'}^* - \Phi_{0,-1,k,k'}^* \right) \right. \\
 & - 2(1+k) \cdot \left( \Phi_{2,1,k,k'}^* - \Phi_{2,-1,k,k'}^* \right) + 2e^2(1+k) \cdot \left( \Phi_{0,1,k,k'}^* - \Phi_{0,-1,k,k'}^* \right) \\
 & \left. \left. + e(1+k) \cdot \left( \Phi_{1,2,k,k'}^* - \Phi_{1,-2,k,k'}^* + \Phi_{0,2,k,k'}^* - \Phi_{0,-2,k,k'}^* \right) \right) \right]
 \end{aligned}$$



$$\begin{aligned}
 & - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} (-1)^k \alpha_{k,j} \cdot \sin I \cdot F_{k-2j-1,0,m}(I) \\
 & \times \left( \frac{\eta^2}{2} \cdot (\Phi_{-1,1,k,k''}^* + \Phi_{-1,-1,k,k''}^*) + \left( 1 + \frac{\eta^2}{2} \right) (\Phi_{0,1,k,k''}^* + \Phi_{0,-1,k,k''}^*) \right. \\
 & - e \cdot (\Phi_{1,2,k,k''}^* - 2\Phi_{1,0,k,k''}^* + \Phi_{1,-2,k,k''}^*) \\
 & + 3 \cdot (\Phi_{1,1,k,k''}^* + \Phi_{1,-1,k,k''}^*) + 2 \cdot (\Phi_{2,1,k,k''}^* + \Phi_{2,-1,k,k''}^*) \\
 & \left. + e \cdot (\eta^2 \Phi_{-1,0,k,k''}^* + 2\Phi_{0,0,k,k''}^* + 4\Phi_{1,0,k,k''}^*) \right), \tag{42}
 \end{aligned}$$

$$\begin{aligned}
 (\Delta e)_{LP} = & \frac{\mu^2}{c^2 a} \left( \frac{R}{a} \right)^k J_k \eta \cdot n \cdot \left[ \sum_{m=0}^k \frac{F_{k,0,m}(I)}{k' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k' \omega) \right. \\
 & \times \left( (14 + 10k) X_0^{-k-3,k'+1} - 3(1+k) X_0^{-k-2,k'+1} - 4\eta^2 (1+k) X_0^{-k-4,k'+1} \right. \\
 & \left. + 2e(1+k) \left[ X_0^{-k-3,k'+2} + \frac{1}{\eta^2} X_0^{-k-2,k'+2} \right] + \frac{4e^2}{\eta^2} (1+k) X_0^{-k-2,k'+1} \right) \\
 & - (-1)^k \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot \sin I \cdot \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \\
 & \times \left( \frac{1}{2\eta^2} \cdot [X_0^{-k-1,k''+1} + X_0^{-k-1,k''-1}] \right. \\
 & + \left( \frac{1}{2} + \frac{1}{\eta^2} \right) \cdot [X_0^{-k-2,k''+1} + X_0^{-k-2,k''-1}] \\
 & - e \cdot [X_0^{-k-3,k''+2} - 2X_0^{-k-3,k''} + X_0^{-k-3,k''-2}] \\
 & + 3 \cdot [X_0^{-k-3,k''+1} + X_0^{-k-3,k''-1}] + 2\eta^2 \cdot [X_0^{-k-4,k''+1} + X_0^{-k-4,k''-1}] \\
 & \left. \left. + \frac{e}{\eta^2} \cdot [X_0^{-k-1,k''} + 2X_0^{-k-2,k''} + 4\eta^2 X_0^{-k-3,k''}] \right) \right],
 \end{aligned}$$

where  $k' = k - 2m$  and  $k'' = k - 2j - 2m$ . (43)

**Inclination  $I$**

$$\begin{aligned}
 (\Delta I)_{SP} = & - \frac{\mu}{c^2 a} J_k \left( \frac{R}{a} \right)^k \frac{\cos I}{\eta^{2k}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} (-1)^k \alpha_{k,j} F_{k-2j-1,0,m}(I) \\
 & \times \left( \frac{2}{\eta^2} \Phi_{0,0,k,k''}^* + \Phi_{-1,0,k,k''}^* \right), \tag{44}
 \end{aligned}$$

$$\begin{aligned}
 (\Delta I)_{LP} = & -\frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{n \cos I}{\eta} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} (-1)^k \alpha_{k,j} \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \\
 & \times \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \cdot \left( 2X_0^{-k-2,k''} + X_0^{-k-1,k''} \right), \\
 \text{where } k'' = & k - 2j - 2m.
 \end{aligned} \tag{45}$$

**Argument of periapsis  $\omega$**

We give the results for  $\omega' = \omega + \Omega \cos I$ . As mentioned above, one has to pay attention to  $\omega$ , if one considers only odd multipoles, in which case the lowest odd multipole will not give rise to long-periodic perturbations.

$$\begin{aligned}
 (\Delta \omega')_{LP} = & \frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{n \eta}{e} \cdot \left[ \sum_{m=0}^k (-1)^k \cdot \frac{F_{k,0,m}(I)}{k' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k' \omega) \right. \\
 & \times \left( (14 + 10k) \cdot X_0^{-k-3,k'+1} - 3(1+k) \cdot X_0^{-k-2,k'+1} \right. \\
 & - 4\eta^2(1+k) \cdot X_0^{-k-4,k'+1} + 2e(1+k) \cdot \left[ \left( X_0^{-k-3,k'+2} - X_0^{-k-3,k'} \right) \right. \\
 & \left. \left. + \frac{1}{\eta^2} \left( X_0^{-k-2,k'+2} - X_0^{-k-2,k'} \right) \right] \right) \\
 & - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \sin I \cdot \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \\
 & \times \left( \frac{1}{2\eta^2} \cdot \left[ X_0^{-k-1,k''+1} - X_0^{-k-1,k''-1} \right] \right. \\
 & + \left( \frac{1}{2} + \frac{1}{\eta^2} \right) \cdot \left[ X_0^{-k-2,k''+1} - X_0^{-k-2,k''-1} \right] \\
 & - e \cdot \left[ X_0^{-k-3,k''+2} - X_0^{-k-3,k''-2} \right] \\
 & + 3 \cdot \left[ X_0^{-k-3,k''+1} - X_0^{-k-3,k''-1} \right] \\
 & \left. \left. + 2\eta^2 \cdot \left[ X_0^{-k-4,k''+1} - X_0^{-k-4,k''-1} \right] \right) \right]. \\
 \text{where } k' = & k - 2m \text{ and } k'' = k - 2j - 2m.
 \end{aligned} \tag{46}$$

$$\begin{aligned}
 (\Delta \omega')_{SP} = & \frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{1}{e \eta^{2k+2}} \cdot \left[ \sum_{m=0}^k (-1)^k F_{k,0,m}(I) \right. \\
 & \times \left( (14 + 10k) \cdot (\Phi_{1,1,k,k'} + \Psi_{1,1,k,k'}) \right. \\
 & - 3(1+k) \eta^2 \cdot (\Phi_{0,1,k,k'} + \Psi_{0,1,k,k'}) \\
 & - 4(1+k) \cdot (\Phi_{2,1,k,k'} + \Psi_{2,1,k,k'}) \\
 & \left. \left. + 2e(1+k) \cdot \left[ \Phi_{1,2,k,k'} - \Phi_{1,0,k,k'} + \Psi_{1,2,k,k'} - \Psi_{1,0,k,k'} \right] \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \Phi_{0,2,k,k'} - \Phi_{0,0,k,k'} + \Psi_{0,2,k,k'} - \Psi_{0,0,k,k'} \Big] \\
 & - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot \sin I \cdot F_{k-2j-1,0,m}(I) \\
 & \times \left( \frac{\eta^2}{2} \cdot (\Phi_{-1,1,k,k''} - \Phi_{-1,-1,k,k''}) \right. \\
 & + \left( 1 + \frac{\eta^2}{2} \right) \cdot (\Phi_{0,1,k,k''} - \Phi_{0,-1,k,k''}) \\
 & - e \cdot (\Phi_{1,2,k,k''} - \Phi_{1,-2,k,k''}) \\
 & + 3 \cdot (\Phi_{1,1,k,k''} - \Phi_{1,-1,k,k''}) \\
 & \left. + 2 \cdot (\Phi_{2,1,k,k''} - \Phi_{2,-1,k,k''}) \right) \Big],
 \end{aligned}$$

where  $k' = k - 2m$  and  $k'' = k - 2j - 2m$ . (47)

**Longitude of the ascending node  $\Omega$**

$$\begin{aligned}
 (\Delta\Omega)_{SP} = & -\frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{\cot I}{\eta^{2k}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot F_{k-2j-1,0,m}(I) \\
 & \times \left( \Phi_{-1,0,k,k''} + \frac{2}{\eta^2} \Phi_{0,0,k,k''} + \Psi_{-1,0,k,k''} + \frac{2}{\eta^2} \Psi_{0,0,k,k''} \right),
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 (\Delta\Omega)_{LP} = & -\frac{\mu}{c^2 a} J_k \left(\frac{R}{a}\right)^k \frac{n \cot I}{\eta} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \\
 & \times \begin{bmatrix} \cos \\ \sin \end{bmatrix}_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \cdot \left( 2X_0^{-k-2,k''} + X_0^{-k-1,k''} \right),
 \end{aligned}$$

where  $k'' = k - 2j - 2m$ . (49)

**Mean anomaly  $M$**

We give results for  $M' = M - n \cdot t + \omega \cdot \eta + \Omega \eta \cos I$ . For  $k$  odd there are no secular perturbations in the mean anomaly.

$$\begin{aligned}
 (\Delta M')_S = & \left(\frac{\mu}{ac^2}\right) J_k \left(\frac{R}{a}\right)^k \cdot F_{k,0,k/2}(I) \\
 & \times \left[ (28 + 20k) X_0^{-k-2,0} - 6(1+k) X_0^{-k-1,0} - 8(1+k) \eta^2 X_0^{-k-3,0} \right] \cdot nt,
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 (\Delta M')_{\text{SP}} = & \frac{J_k}{\eta^{2k+1}} \frac{\mu}{ac^2} \left(\frac{R}{a}\right)^k \cdot \left[ \sum_{m=0}^k (-1)^k F_{k,0,m}(I) \cdot \left( (28 + 20k) (\Phi_{0,0,k,k'} + \Psi_{0,0,k,k'}) \right. \right. \\
 & \left. \left. - 6(1+k)\eta^2 (\Phi_{-1,0,k,k'} + \Psi_{-1,0,k,k'}) - 8(1+k) (\Phi_{1,0,k,k'} + \Psi_{1,0,k,k'}) \right) \right. \\
 & \left. + 4e \sin I \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot F_{k-2j-1,0,m}(I) \cdot \left( \Phi_{0,1,k,k''} - \Phi_{0,-1,k,k''} \right) \right], \tag{51}
 \end{aligned}$$

$$\begin{aligned}
 (\Delta M')_{\text{LP}} = & \frac{\mu n}{c^2 a} J_k \left(\frac{R}{a}\right)^k \cdot \left[ (-1)^k \sum_{m=0}^k \frac{F_{k,0,m}(I)}{k' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k' \omega) \right. \\
 & \times \left( (28 + 20k) X_0^{-k-2,k'} - 6(1+k) X_0^{-k-1,k'} - 8(1+k)\eta^2 X_0^{-k-3,k'} \right) \\
 & + 4e \sin I \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=0}^{k-2j-1} \alpha_{k,j} \cdot \frac{F_{k-2j-1,0,m}(I)}{k'' \dot{\omega}} \cdot \left[ \frac{\cos}{\sin} \right]_{k \text{ odd}}^{k \text{ even}} (k'' \omega) \\
 & \left. \times \left( X_0^{-k-2,k''+1} - X_0^{-k-2,k''-1} \right) \right],
 \end{aligned}$$

where  $k' = k - 2m$  and  $k'' = k - 2j - 2m$ . (52)

### C. Expressions in the canonical approach

First, we list two remaining expressions from the calculation process in the canonical approach. From these the (quasi-)Newtonian results for arbitrary  $J_k$  can be reproduced. Finally we list the expressions for the second-order drifts in  $h = \Omega$  and  $g = \omega$ .

$$\begin{aligned}
 \mathcal{H}_1^* = & \frac{J_k R^k \mu^{k+2}}{L^3 G^{2k-1}} \sum_{m=0}^{\lfloor k/2 \rfloor} (2 - \delta_{k,2m}) \cdot b_{k-2m} \cdot \sum_{s=0}^{k-1} \sum_{q=0}^s \left(\frac{e'}{2}\right)^s \binom{k-1}{s} \binom{s}{q} \\
 & \times \delta_{s-2q,k-2m} \cdot \cos((k-2m) \cdot (g' - \frac{\pi}{2})) \\
 & - \frac{\mu^4}{c^2 L'^4} \left[ 3 \frac{L'}{G'} - \frac{15}{8} \right] \tag{53}
 \end{aligned}$$

$$\begin{aligned}
 W_1 = & \frac{J_k R^k \mu^k}{G^{2k-1}} \sum_{m=0}^{\lfloor k/2 \rfloor} (2 - \delta_{k,2m}) \cdot b_{k-2m} \sum_{s=0}^{k-1} \sum_{q=0}^s \left(\frac{e'}{2}\right)^s \binom{k-1}{s} \binom{s}{q} \\
 & \times \left[ \delta_{s-2q,k-2m}(f' - l') \cdot \cos((k-2m) \cdot (g' - \frac{\pi}{2})) \right. \\
 & \left. + (1 - \delta_{s-2q,k-2m}) \frac{\sin((s-2q+k-2m)f' + (k-2m) \cdot (g' - \frac{\pi}{2}))}{s-2q+k-2m} \right] \\
 & - \frac{\mu^2}{c^2 L'} \left[ 3 \frac{L'}{G'} (f' - l') - 2(E' - l') \right] \tag{54}
 \end{aligned}$$

Since the two Lie-transformations contain the short-periodic and long-periodic perturbations only, the secular drifts in  $h = \Omega$  and  $g = \omega$  are given by the drifts in  $h''$  and  $g''$ , respectively. For these we find to second order

$$\begin{aligned}
 g'' &= g_0'' + \frac{3\mu^4}{c^2 L'^5} \frac{L'^2}{G'^2} \cdot t \\
 &\quad - \frac{J_k R^k \mu^{k+2}}{L'^3 G'^{2k}} \cdot \sum_{\substack{s=0 \\ s \text{ even}}}^{k-1} \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{s/2} \\
 &\quad \cdot \left[ (2k-1)b_0 + \tilde{b}_0 \frac{H''}{G''} + b_0 \cdot G'' \frac{s}{e''} \frac{\partial e''}{\partial G''} \right] \cdot t \\
 &\quad + \frac{1}{2} \left( \frac{\partial \langle \mathcal{H}_2 \rangle_l}{\partial G''} \Big|_{g'} + \frac{\partial \langle \mathcal{T}_2 \rangle_l}{\partial G''} \Big|_{g'} \right) \cdot t, \tag{55}
 \end{aligned}$$

$$\begin{aligned}
 h'' &= h_0'' + \frac{J_k R^k \mu^{k+2}}{L'^3 G'^{2k}} \cdot \tilde{b}_0 \cdot \sum_{\substack{s=0 \\ s \text{ even}}}^{k-1} \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{s/2} \cdot t \\
 &\quad + \frac{1}{2} \left( \frac{\partial \langle \mathcal{H}_2 \rangle_l}{\partial H''} \Big|_{g'} + \frac{\partial \langle \mathcal{T}_2 \rangle_l}{\partial H''} \Big|_{g'} \right) \cdot t,
 \end{aligned}$$

where  $\tilde{b}_m = \frac{(k-m)!}{(k+m)!} \cdot P_k^m(0) \cdot \frac{\partial P_k^m(z)}{\partial z} \Big|_{z=H'/G'}$  (56)

and the derivatives, giving the second-order secular drifts, are

$$\begin{aligned}
 \frac{\partial \langle \mathcal{H}_2 \rangle_l}{\partial H''} \Big|_{g'} &= \frac{J_k R^k \mu^{k+4}}{c^2 L'^5 G'^{2k}} \cdot \tilde{b}_0 \\
 &\quad \times \left[ \frac{4}{\eta^2} \sum_{\substack{s=0 \\ s \text{ even}}}^k \left(\frac{e''}{2}\right)^s \binom{k}{s} \binom{s}{s/2} - 3 \sum_{\substack{s=0 \\ s \text{ even}}}^{k-1} \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{s/2} \right], \tag{57}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \langle \mathcal{H}_2 \rangle_l}{\partial G''} \Big|_{g'} &= - \frac{J_k R^k \mu^{k+4}}{c^2 L'^5 G'^{2k}} \cdot \left( \left[ (2k-1)b_0 + \tilde{b}_0 \cdot \frac{H''}{G''} \right] \right. \\
 &\quad \times \left[ 4 \frac{L'^2}{G'^2} \sum_{\substack{s=0 \\ s \text{ even}}}^k \left(\frac{e''}{2}\right)^s \binom{k}{s} \binom{s}{s/2} - 3 \sum_{\substack{s=0 \\ s \text{ even}}}^{k-1} \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{s/2} \right] \\
 &\quad + b_0 \left[ \sum_{\substack{s=0 \\ s \text{ even}}}^k \left(\frac{e''}{2}\right)^s \binom{k}{s} \binom{s}{s/2} \left( \frac{4s}{e''^2} + 8 \frac{L'^2}{G'^2} \right) \right. \\
 &\quad \left. - 3 \sum_{\substack{s=0 \\ s \text{ even}}}^{k-1} \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{s/2} \frac{s}{e''^2} \frac{G'^2}{L'^2} \right] \Big), \tag{58}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \langle \langle \mathcal{T}_2 \rangle \rangle_{g'}}{\partial H''} &= \frac{J_k R^k \mu^{k+4}}{c^2 L'^5 G'^{2k}} \cdot \tilde{b}_0 \\
 &\times \left( (k+1) \sum_{s=0}^{k+2} \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k+2}{s} \binom{s}{q} \delta_{s,2q-1} \frac{3}{e''} \frac{L'^4}{G'^4} \right. \\
 &- (k+1) \sum_{s=0}^{k+1} \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k+1}{s} \binom{s}{q} \left[ \delta_{s,2q} \frac{9}{2} \frac{L'^4}{G'^4} \right. \\
 &+ \delta_{s,2q-1} \frac{2}{e''} \frac{L'^2}{G'^2} + \delta_{s,2q-2} \frac{3}{2} \frac{L'^4}{G'^4} \left. \right] \\
 &+ (k+1) \sum_{s=0}^k \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k}{s} \binom{s}{q} \left[ \delta_{s,2q} \left( 3 \frac{L'^2}{G'^2} + \frac{3}{2} \frac{L'^4}{G'^4} \right) \right. \\
 &+ \delta_{s,2q-1} \left( \frac{2}{e''} - \frac{3}{e''} \frac{L''}{G''} \right) - \delta_{s,2q-2} \left( \frac{3}{2} \frac{L'^4}{G'^4} - \frac{L'^2}{G'^2} \right) \left. \right] \\
 &+ \sum_{s=0}^{k-1} \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k-1}{s} \binom{s}{q} \\
 &\times \left( \delta_{s,2q} \left[ -4k - 12 + 6k \frac{L''}{G''} + 15 \frac{L''}{G''} + (k+1)e'' \frac{L'^2}{G'^2} + 7 \frac{L'^2}{G'^2} \right] \right. \\
 &+ \delta_{s,2q-1} \left[ 6e'' \frac{L'^2}{G'^2} + \frac{4}{e''} \frac{L'^2}{G'^2} \right] \\
 &+ \delta_{s,2q-2} \left[ 3 \frac{L'^2}{G'^2} - (k+1)e'' \frac{L'^2}{G'^2} \right] \\
 &- s \delta_{s,2q} \cdot \left[ 2 + \frac{3}{16} e''^4 \frac{L'^2}{G'^2} - 6 \frac{L'^2}{G'^2} \right] \\
 &\left. + \frac{s}{s-2q} \left[ \delta_{s,2q-1} \left( \frac{2}{e''} - \frac{6}{e''} \frac{L'^2}{G'^2} \right) - \delta_{s,2q-2} 3 \frac{L'^2}{G'^2} \right] \right), \tag{59}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \langle \langle \mathcal{T}_2 \rangle \rangle_{g'}}{\partial G''} &= - \frac{J_k R^k \mu^{k+4}}{c^2 L'^5 G'^{2k}} \cdot \left( \left[ (2k-1)b_0 + \tilde{b}_0 \cdot \frac{H''}{G''} \right] \cdot \left( \dots \right) \right. \\
 &+ b_0 \cdot \left( (k+1) \sum_{s=0}^{k+2} \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k+2}{s} \binom{s}{q} \right. \\
 &\times \delta_{s,2q-1} \left( \frac{12}{e''} \frac{L'^4}{G'^4} - \frac{3}{e''^3} \frac{L'^2}{G'^2} + \frac{3s}{e''^3} \frac{L'^2}{G'^2} \right) \\
 &- (k+1) \sum_{s=0}^{k+1} \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k+1}{s} \binom{s}{q} \left[ \delta_{s,2q} \left( 18 \frac{L'^4}{G'^4} + \frac{9s}{2e''^2} \frac{L'^2}{G'^2} \right) \right. \\
 &+ \delta_{s,2q-1} \left( \frac{4}{e''} \frac{L'^2}{G'^2} - \frac{2}{e''^3} + \frac{2s}{e''^3} \right) + \delta_{s,2q-2} \left( 6 \frac{L'^4}{G'^4} + \frac{3s}{2e''^2} \frac{L'^2}{G'^2} \right) \left. \right] \\
 &+ (k+1) \sum_{s=0}^k \sum_{q=0}^s \left(\frac{e''}{2}\right)^s \binom{k}{s} \binom{s}{q} \left[ \delta_{s,2q} \left( 6 \frac{L'^2}{G'^2} + 6 \frac{L'^4}{G'^4} + \frac{3s}{e''^2} + \frac{3s}{2e''^2} \frac{L'^2}{G'^2} \right) \right.
 \end{aligned}$$



- Huang, C., Liu, L.: Analytical solutions to the four post-Newtonian effects in a near earth satellite orbit. *Celes. Mech. Dyn. Astron.* **53**, 293–307 (1992)
- Hughes, S.: The computation of tables of Hansen coefficients. *Celest. Mech.* **29**, 101–107 (1981)
- Iorio, L.: A critical analysis of a recent test of the Lense–Thirring effect with the LAGEOS satellites. *J. Geod.* **80**, 128–136 (2006)
- Iorio, L.: Post-Newtonian direct and mixed orbital effects due to the oblateness of the central body. *Int. J. Mod. Phys. D* **24**, 1550067 (2015)
- Kaula, W.M.: *Theory of Satellite Geodesy*. Blaisdell Publishing Company, Waltham (1966)
- Kopeikin, S., Efroimsky, M., Kaplan, G.: *Relativistic Celestial Mechanics of the Solar System*. Wiley, Weinheim (2011)
- Kozai, Y.: The motion of a close earth satellite. *Astron. J.* **64**, 367–377 (1959)
- Lucchesi, D., Anselmo, L., Bassan, M., Pardini, C., Peron, R., Pucacco, G., Visco, M.: Testing the gravitational interaction in the field of the Earth via satellite laser ranging and the laser ranged satellites experiment (LARASE). *Class. Quant. Grav.* **32**, 155012 (2015)
- Mielnik, B., Plebanski, J.: A study of geodesic motion in the field of Schwarzschild's solution. *Acta Phys. Polonica* **21**, 239268 (1962)
- Milani, A., Nobili, A., Farinella, P.: *Non-gravitational Perturbations and Satellite Geodesy*. Adam Hilger, Bristol (1987)
- Schanner M (2017) Master thesis. unpublished, Dresden
- Soffel, M.H.: *Relativity in Astrometry, Celestial Mechanics and Geodesy*. Springer, Berlin (1989)
- Soffel, M., Frutos, F.: On the usefulness of relativistic space-times for the description of the Earth's gravitational field. *J. Geod.* **90**, 1345–1357 (2016)
- Soffel, M., Ruder, H., Schneider, M.: The two-body problem in the (truncated) PPN-theory. *Celest. Mech.* **40**, 77–85 (1987)
- Soffel, M., Wirrer, R., Schastok, J., Ruder, H., Schneider, M.: Relativistic effects in the motion of artificial satellites: the oblateness of the central body I. *Celest. Mech.* **42**, 81–89 (1988)
- Weinberg, S.: *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. Wiley, New York (1972)