Celest Mech Dyn Astr (2018) 130:25
https://doi.org/10.1007/s10569-018-9818-8

ORIGINAL ARTICLE

Exact Delaunay normalization of the perturbed Keplerian Hamiltonian with tesseral harmonics

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Received: 23 September 2017 / Accepted: 31 January 2018 / Published online: 17 March 2018

Abstract A novel approach for the exact Delaunay normalization of the perturbed Keplerian Hamiltonian with tesseral and sectorial spherical harmonics is presented in this work. It is shown that the exact solution for the Delaunay normalization can be reduced to quadratures by the application of Deprit's Lie-transform-based perturbation method. Two different series representations of the quadratures, one in powers of the eccentricity and the other in powers of the ratio of the Earth's angular velocity to the satellite's mean motion, are derived. The latter series representation produces expressions for the short-period variations that are similar to those obtained from the conventional method of relegation. Alternatively, the quadratures can be evaluated numerically, resulting in more compact expressions for the short-period variations that are valid for an elliptic orbit with an arbitrary value of the eccentricity. Using the proposed methodology for the Delaunay normalization, generalized expressions for the short-period variations of the equinoctial orbital elements, valid for an arbitrary tesseral or sectorial harmonic, are derived. The result is a compact unified artificial satellite theory for the sub-synchronous and super-synchronous orbit regimes, which is nonsingular for the resonant orbits, and is closed-form in the eccentricity as well. The accuracy of the proposed theory is validated by comparison with numerical orbit propagations.

Keywords Artificial satellite theory · Tesseral harmonics · Canonical perturbation method · Delaunay normalization · The method of relegation

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1 Introduction

Artificial satellite theories have not lost their significance despite the availability of the off-the-shelf fast numerical orbit propagators. They are indispensable for space situational awareness, i.e., for catalog maintenance, orbit uncertainty predictions, conjunction analysis, etc., and more generally, for guidance and control, and perturbed relative motion solutions. The development of satellite theories that maintain long-term prediction accuracy under the effects of various perturbations such as nonspherical gravitational effects, atmospheric drag, solar radiation pressure, third-body perturbations, etc., still remains an active area of research. The current work specifically focuses on formulating an exact analytic solution, closed-form in the eccentricity, for the short-period perturbation effects of the tesseral and sectorial spherical harmonics (collectively referred to as tesserals henceforth).

Satellite theories are generally constructed by performing Delaunay normalizations of the perturbed Keplerian Hamiltonian in order to separate the short-period variations of the orbital elements from their secular and long-period variations (Deprit 1982). Brouwer's pioneering work constructed a satellite theory, closed-form in the eccentricity, with the first few zonal spherical harmonics included (Brouwer 1959). Prior to his work, the satellite theories relied on the expansions of the perturbation in powers of the eccentricity for eliminating the short-period terms from the perturbed Hamiltonian. Exact or closed-form in the eccentricity satellite theories are better suited for propagation of artificial satellites or space debris with moderate to high values of the orbital eccentricity. Additionally, avoiding the eccentricity expansions results in more compact theory. With regard to the zonal gravitational perturbations, a significant body of work exists for constructing the satellite theories using canonical perturbation methods up to second and higher orders (Aksnes 1970; Kaniecki 1979; Coffey and Deprit 1982; Breiter 1997; Mahajan et al. 2015, 2016). Forgoing the von Zeipel method originally used by Brouwer, modern theories use either Lie-series or Lie-transform-based canonical perturbation methods, in conjunction with the special techniques like elimination of the parallax and elimination of the perigee, which lead to more compact expressions at second and higher orders (Coffey and Deprit 1982; Coffey and Alfriend 1984). Recently, exact expressions for the secular, short-period and long-period variations of the equinoctial orbital elements due to an arbitrary zonal harmonic have been derived by the present authors using Deprit's Lie-transform-based perturbation method (Mahajan et al. 2016).

In case of the tesseral harmonics, however, an exact solution of the generating function for computing the short-period variations has yet to be found. Using the von Zeipel method, Garfinkel (1965) extended the theory of the main (or J_2) problem to include the long-period effects due to the tesserals. Vagners (1967) used a similar approach to compute a closed-form solution with the tesserals included for small-eccentricity orbits at or near resonance. Adopting a different approach, Kaula expressed the tesseral disturbing potential in terms of the classical orbital elements by expanding it in powers of the eccentricity (Kaula 1966). Using this expanded form of the potential, he computed the first-order short-period, m-daily, and long-period variations of the classical orbital elements by analytic integration of the Lagrange planetary equations. Due to the use of the eccentricity expansions, Kaula's tesseral theory has singularities for near-resonance orbits. A similar strategy was adopted by Cefola (1976) and Giacaglia (1977) by expressing the tesseral potential in terms of the equinoctial elements. McClain (1978) provided averaged equations of motion for tesseral harmonics in case of nonresonant as well as resonant orbits by using Cefola's tesseral potential. Cefola and McClain (1978) computed the short-period generating function in a recursive form using the generalized method of averaging. Their work was further extended by Proulx et al. (1981)

for computing a recursive semi-analytic theory including the first-order tesseral short-period variations of the equinoctial elements.

Coffey and Alfriend (1981) used two separate Lie-type transformations to construct a satellite theory for the tesseral harmonics. The first transformation, known as the elimination of the parallax, produced a simplified intermediate Hamiltonian and the second transformation completed the Delaunay normalization for each tesseral explicitly by first expanding the intermediate Hamiltonian in powers of the eccentricity. It was acknowledged by Coffey and Alfriend (1981) that no closed-form normalization of this problem is known to exist. Wnuk (1988) utilized Hori's Lie-series-based perturbation method (Hori 1966) to normalize an alternative form of Kaula's disturbing potential and computed analytic expressions for the short-period variations due to an arbitrary tesseral harmonic.

The methods discussed above for the Delaunay normalization of the tesseral Hamiltonian contain singularities in the form of small divisors near the resonance conditions. These singularities arise when the true anomaly in the tesseral Hamiltonian is expanded in powers of the eccentricity, which additionally degrades the accuracy of the theory for elliptic orbits of medium to high values of the eccentricity. The technique that comes close to providing a closed-form (in the eccentricity) normalization of the tesseral Hamiltonian is the method of relegation (Palacián 1992; Deprit et al. 2001). This method successively reduces the magnitude of the perturbation Hamiltonian by scaling it with a small multiplier with the help of canonical transformations. After a sufficient number of iterations, the perturbation Hamiltonian is deemed small enough to be safely ignored. Segerman and Coffey (2000) applied the method of relegation to the Delaunay normalization of the tesseral Hamiltonian. Their results showed that the method of relegation generated large number of terms in multiple transformations corresponding to each iteration for normalizing the tesseral Hamiltonian. In addition, two different flavors of the relegation method are required for sub-synchronous and super-synchronous orbit regimes, resulting in two separate theories. While the relegation method avoids the eccentricity expansions, its convergence properties restrict the method to small eccentricities in most cases as discussed by Lara et al. (2013, 2014). An alternative relegation approach by Lara et al. (2013) expressed the short-period generating function as a power series in the eccentricity, with convergence depending on the value of the eccentricity. Given a generic perturbed Hamiltonian system, Sansottera and Ceccaroni (2017) derived the asymptotic estimates for the relegation algorithm.

In the present work, a novel generalized approach to perform the exact Delaunay normalization of the tesseral Hamiltonian and compute the short-period variations of the orbital elements is presented. Using this approach, an exact solution for the tesseral short-period generating function is reduced to two quadratures. It is shown that these quadratures can be analytically evaluated by expanding their integrands in powers of either the eccentricity or the ratio of the Earth's angular velocity to the satellite's mean motion. The latter method produces the same expressions as generated by the conventional method of relegation. A reformulation of the relegation method in the Delaunay elements is also given in this work, which highlights its relationship to the proposed generalized approach. The major advantage of this generalized approach, however, lies in its capability of producing the exact short-period variations of the orbital elements with arbitrary precision (in theory) by evaluating the two quadratures numerically. To summarize, three different approaches to compute the tesseral short-period variations using the proposed exact solution for the generating function are presented. Additionally, the compact expressions for the short-period variations of the equinoctial elements valid for any arbitrary tesseral harmonic are incorporated into a satellite theory and its accuracy is verified by comparing the results with numerical integration.

This paper is organized into six major sections. In Sect. 2, the perturbed Hamiltonian for the motion of a satellite around a nonspherical central body is formulated. A review of Deprit's perturbation method and the relevant equations to normalize a time-dependent Hamiltonian are given in Sect. 3. Next, the relegation method is formulated in terms of the Delaunay elements to highlight the conditions for its convergence in Sect. 4. The derivation of the exact solution for the tesseral short-period generating function and its relationship with the relegation method is discussed in Sect. 5. Using this solution, the expressions for the short-period variations of the equinoctial elements valid for an arbitrary tesseral are provided in Sect. 6 along with the simulation results for an artificial satellite theory with spherical harmonics included up to degree and order two.

2 The Hamiltonian

The Hamiltonian for an artificial satellite orbiting around a nonspherical Earth-like central body can be represented as a sum of the Keplerian Hamiltonian and a disturbing function accounting for the nonuniform gravitational effects. The disturbing function or potential for the tesseral harmonics, when expressed in an inertial frame, depends on the longitude of the satellite. The Delaunay normalization of such a Hamiltonian can be performed using the canonical perturbation methods (Deprit 1969; Hori 1966). In this work, Kaula's form of the gravitational potential, which is expressed in the inertial frame using the classical orbital elements and the Greenwich Sidereal Time, is used as the disturbing potential (Kaula 1966). As shown in the next section, the Hamiltonian formulated in the classical orbital elements, instead of polar-nodal (also known as Hill or Whittaker variables), results in simpler perturbation equations. Kaula's gravitational potential, expressed in terms of the true anomaly of the satellite, is (Kaula 1966):

$$V_{nm} = \frac{\mu R_e^n}{r^{n+1}} \sum_{p=0}^{p=n} F_{nmp}(i)\, [CS_1 \cos(\Psi) + CS_2 \sin(\Psi)]\,, \tag{1}$$

where

$$\begin{aligned}
\Psi &\equiv (n-2p)(f+g) + m(h-\theta),\\
CS_1 &= \begin{cases} C_{nm} & \text{if } n-m \text{ is even} \\ -S_{nm} & \text{if } n-m \text{ is odd,} \end{cases}\\
CS_2 &= \begin{cases} S_{nm} & \text{if } n-m \text{ is even} \\ C_{nm} & \text{if } n-m \text{ is odd.} \end{cases}
\end{aligned}$$

This representation of the disturbing potential contains a mix of the classical and Delaunay elements $[l, g, h, L, G, H]$. The letters n and m represent the degree and order of the potential V_{nm}, respectively; μ, R_e and θ represent the gravitational parameter, the radius of Earth and the Greenwich Sidereal Time, respectively; and C_{nm} and S_{nm} are the spherical harmonic coefficients. Instead of using the original definition of the inclination function $F_{nmp}(i)$ given by Kaula, the following more computationally efficient definition is used in the present work (Yan et al. 2014; Emeljanov and Kanter 1989):

$$F_{nmp}(i) = (-1)^{E\left(\frac{n-m+1}{2}\right)} \frac{(n+m)!}{2^n p!(n-p)!} \sum_{j=\max(n-2p-m,0)}^{j=\min(n-m,2n-2p)} (-1)^j \times \binom{2n-2p}{j} \binom{2p}{n-m-j} \cos^{2n-s}\left(\frac{i}{2}\right) \sin^s\left(\frac{i}{2}\right), \tag{2}$$

where $s = m - n + 2p + 2j$ and $E()$ represents the Entier (also called integer part) function. The above definition of the inclination function has only one summation compared to three present in Kaula's inclination function. This representation of the disturbing potential is also more computationally efficient than the one using the polar-nodal variables given by Segerman and Coffey (2000).

Subsequent developments in this paper involve the integration of the disturbing potential with respect to f, a task made easier if the explicit occurrences of r in V_{nm} are eliminated by substituting the orbit equation. The powers of the cosine function introduced by the orbit equation are converted into the multiples of its argument using the following formulae (Gradshteyn and Ryzhik 2014):

$$(1 + e\cos f)^{n+1} = \sum_{s=0}^{n+1} \binom{n+1}{s} e^s \cos^s f, \tag{3}$$

$$\cos^s f = \frac{1}{2^{s-1}} \left\{ \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{k} \cos((s-2k)f) + \frac{\lambda_s}{2} \binom{s}{\frac{s}{2}} \right\}, \tag{4}$$

where λ_s is defined as

$$\lambda_s = \begin{cases} 1 & s \text{ is even} \\ 0 & s \text{ is odd.} \end{cases} \tag{5}$$

By using the trigonometric identities for converting the product of terms to sum of terms, the following expression for V_{nm} is obtained:

$$V_{nm} = \frac{\mu R_e^n}{a^{n+1}\eta^{2n+2}} \sum_{p=0}^{n} \sum_{s=0}^{n+1} F_{nmp}(i) e^s \binom{n+1}{s} \frac{1}{2^s} \left[\sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{k} \times [CS_1 (\cos(\Psi_1) + \cos(\Psi_2)) + CS_2 (\sin(\Psi_1) + \sin(\Psi_2))] + \lambda_s \binom{s}{\frac{s}{2}} [CS_1 \cos(\Psi) + CS_2 \sin(\Psi)] \right], \tag{6}$$

where

$$\Psi_1 \equiv (n - 2p + s - 2k)f + (n - 2p)g + m(h - \theta),$$
$$\Psi_2 \equiv (n - 2p - s + 2k)f + (n - 2p)g + m(h - \theta),$$

and η is the eccentricity function equal to $\sqrt{1-e^2}$ and $\lfloor\,\rfloor$ represents the floor function.

In case of Earth, the oblateness coefficient $C_{20} (= -J_2)$ is at least three orders of magnitude greater than any other zonal or tesseral coefficient. Therefore, C_{20} is considered as a first-order perturbation, while the rest of the harmonics are of second-order. In light of this, the complete Hamiltonian $\mathscr{H}$ is expressed as a power series in C_{20} with $\mathscr{H}_0$ being the unperturbed Keplerian Hamiltonian as shown below:

$$\mathscr{H} = \mathscr{H}_0 + \epsilon \mathscr{H}_1 + \frac{\epsilon^2}{2!}\mathscr{H}_2 , \tag{7}$$

where

$$\begin{aligned} \epsilon &= C_{20}, \\ \mathscr{H}_0 &= -\frac{\mu^2}{2L^2}, \\ \mathscr{H}_1 &= -\frac{V_{20}}{C_{20}}, \\ \mathscr{H}_2 &= -\frac{2!}{C_{20}^2}V_{nm}, \quad m \neq 0 \text{ if } n = 2. \end{aligned}$$

In the above equations, $\mathscr{H}_1$ is the oblateness disturbing potential, and $\mathscr{H}_2$ is the disturbing potential for the remaining spherical harmonics. It is noted that the negative signs in the above definitions of $\mathscr{H}_0$, $\mathscr{H}_1$, and $\mathscr{H}_2$ are the result of considering the angle variables (l, g, h) as the first three canonical variables and the action variables (L, G, H) as the last three.

3 Deprit's canonical perturbation method

Since the Hamiltonian is an explicit function of the Greenwich Sidereal Time θ in addition to the Delaunay elements, Deprit's perturbation method (Deprit 1969) with nonconservative transformations is used to transform the original Hamiltonian expressed in the osculating elements to an averaged Hamiltonian expressed in the mean elements. A brief summary and important equations of Deprit's perturbation method for nonconservative transformations is given in this section for completeness.

If a Hamiltonian $\mathscr{H}$, dependent on canonical variables (x, X) and time t, can be written as a power series using a small parameter ϵ as shown below:

$$\mathscr{H}(x, X; t; \epsilon) = \mathscr{H}_0 + \epsilon \mathscr{H}_1 + \frac{\epsilon^2}{2!}\mathscr{H}_2 + O(\epsilon^3), \tag{8}$$

with the associated canonical equations as:

$$\dot{x} = \frac{\partial \mathscr{H}(x, X; t; \epsilon)}{\partial X}, \quad \dot{X} = -\frac{\partial \mathscr{H}(x, X; t; \epsilon)}{\partial x}, \tag{9}$$

then a canonical transformation can be constructed to transform the original variables into a new set of the canonical variables (y, Y) such that

$$\dot{y} = \frac{\partial \mathscr{K}(y, Y; t; \epsilon)}{\partial Y}, \quad \dot{Y} = -\frac{\partial \mathscr{K}(y, Y; t; \epsilon)}{\partial y}, \tag{10}$$

where

$$\mathscr{K}(y, Y; t; \epsilon) = \mathscr{K}_0 + \epsilon \mathscr{K}_1 + \frac{\epsilon^2}{2!}\mathscr{K}_2 + O(\epsilon^3).$$

In the above equations, $\mathscr{K}$ represents the transformed Hamiltonian (sometimes referred to as Kamiltonian), which is a function of the new variables (y, Y) and time t. The old variables can be expressed in terms of the new variables and vice versa using the following set of nonconservative or time-dependent transformations computed by evaluating the Poisson brackets (_, _) of the new and old variables with a time-dependent generating function $\mathscr{W}$ as shown below:

$$x = y + \epsilon(y, \mathscr{W}_1) + \frac{\epsilon^2}{2!}\left[(y, \mathscr{W}_2) + ((y, \mathscr{W}_1), \mathscr{W}_1)\right] + O(\epsilon^3),$$
$$X = Y + \epsilon(Y, \mathscr{W}_1) + \frac{\epsilon^2}{2!}\left[(Y, \mathscr{W}_2) + ((Y, \mathscr{W}_1), \mathscr{W}_1)\right] + O(\epsilon^3), \tag{11}$$

and

$$y = x - \epsilon(x, \mathscr{W}_1) + \frac{\epsilon^2}{2!}\left[-(x, \mathscr{W}_2) + ((x, \mathscr{W}_1), \mathscr{W}_1)\right] + O(\epsilon^3),$$
$$Y = X - \epsilon(X, \mathscr{W}_1) + \frac{\epsilon^2}{2!}\left[-(X, \mathscr{W}_2) + ((X, \mathscr{W}_1), \mathscr{W}_1)\right] + O(\epsilon^3), \tag{12}$$

where

$$(\mathscr{E}, W) = \frac{\partial \mathscr{E}}{\partial l}\frac{\partial W}{\partial L} - \frac{\partial \mathscr{E}}{\partial L}\frac{\partial W}{\partial l} + \frac{\partial \mathscr{E}}{\partial g}\frac{\partial W}{\partial G} - \frac{\partial \mathscr{E}}{\partial G}\frac{\partial W}{\partial g} + \frac{\partial \mathscr{E}}{\partial h}\frac{\partial W}{\partial H} - \frac{\partial \mathscr{E}}{\partial H}\frac{\partial W}{\partial h},$$
$$\mathscr{W}(\_, \_; t; \epsilon) = \mathscr{W}_1 + \epsilon \mathscr{W}_2 + \frac{\epsilon^2}{2!}\mathscr{W}_3 + O(\epsilon^3).$$

The generating function $\mathscr{W}$ depends on time t, the small parameter ϵ, the new variables in Eq. (11), and the old variables in Eq. (12). It is well known that for canonical transformations, the original Hamiltonian $\mathscr{H}$, the new Hamiltonian $\mathscr{K}$, and $\mathscr{W}$ are related by the following equation:

$$\mathscr{H}(x, X, t; \epsilon) = \mathscr{K}(y, Y, t; \epsilon) + \epsilon\,\frac{\partial \mathscr{W}}{\partial t}. \tag{13}$$

The original Hamiltonian $\mathscr{H}$ can be expanded in terms of the new variables (y, Y) by using the transformation equations given in Eq. (11). Subsequently, by equating the coefficients of ϵ, the so-called homological equations are found, which are given below up to order three (Deprit 1969):

$$\mathscr{H}_0 = \mathscr{K}_0,$$
$$(\mathscr{H}_0, \mathscr{W}_1) - \frac{\partial \mathscr{W}_1}{\partial t} + \mathscr{H}_1 = \mathscr{K}_1,$$
$$(\mathscr{H}_0, \mathscr{W}_2) - \frac{\partial \mathscr{W}_2}{\partial t} + (\mathscr{H}_1 + \mathscr{K}_1, \mathscr{W}_1) - \left(\frac{\partial \mathscr{W}_1}{\partial t}, \mathscr{W}_1\right) + \mathscr{H}_2 = \mathscr{K}_2,$$
$$(\mathscr{H}_0, \mathscr{W}_3) - \frac{\partial \mathscr{W}_3}{\partial t} + (2\mathscr{H}_1 + \mathscr{K}_1, \mathscr{W}_2) + (\mathscr{H}_2 + 2\mathscr{K}_2, \mathscr{W}_1) - ((\mathscr{K}_1, \mathscr{W}_1), \mathscr{W}_1)$$
$$- \left(\frac{\partial \mathscr{W}_2}{\partial t}, \mathscr{W}_1\right) + \mathscr{H}_3 = \mathscr{K}_3. \tag{14}$$

To perform the Delaunay normalization of $\mathscr{H}$ using the above equations, the terms in the series expansion of $\mathscr{K}$ can be chosen in a manner that makes one angle variable cyclic up to the desired order, thus making the corresponding action variable an integral of the transformed system. For constructing satellite theories, averaging is used to eliminate the short-period terms (that are a function of the true anomaly) from $\mathscr{H}$ in order to compute $\mathscr{K}$, in what is called a first Delaunay normalization. Specifically, at each order in Eq. (14), the series term of $\mathscr{K}$ is set equal to the average of each of the known terms with respect to the mean anomaly on the left-hand side of the same equation. The remaining short-period terms then become part of the partial differential equation (PDE) of the unknown series term of $\mathscr{W}$ at each order, which then must be solved to compute the short-period variations of the elements. In case of zonal harmonics, the first Delaunay normalization results in the first-order linear PDE with one independent variable for series terms of $\mathscr{W}$, which can be solved in a straightforward manner without any special techniques at least up to order two (Aksnes 1971; Mahajan et al. 2015).

The tesseral harmonics along with the zonal harmonics with degree higher than two enter the homological equations at second order through the Hamiltonian term $\mathscr{H}_2$. There is no coupling between these zonal and tesseral harmonics and therefore, the tesseral part of the second-order homological equation can be separated from all the zonal contributions as shown below:

$$(\mathscr{H}_0, \mathscr{W}_2^T) - \frac{\partial \mathscr{W}_2^T}{\partial t} + \mathscr{H}_2^T = \mathscr{K}_2^T, \tag{15}$$

where

$$\mathscr{H}_2^T = -\frac{2!}{C_{20}^2} V_{nm}\bigg|_{m \neq 0}.$$

The superscript T in the above equation indicates that these terms include only tesseral contributions. Through the disturbing function V_{nm}, there are short-period terms present in $\mathscr{H}_2^T$ that depend simultaneously on f and θ. The remaining terms that are only dependent on θ are called m-daily terms as they repeat m-times a day (Kaula 1966). If the resonance effects due to tesserals are ignored then there are no secular or long-period contributions (i.e., terms that neither depend on f nor on θ) to $\mathscr{K}_2^T$ from any of the terms in $\mathscr{H}_2^T$, i.e., $\mathscr{K}_2^T$ can be taken as zero. The resulting PDE for the second-order short-period tesseral generating function $\mathscr{W}_2^T$ takes the form:

$$-\tilde{n}\frac{\partial \mathscr{W}_2^T}{\partial l} - w_E \frac{\partial \mathscr{W}_2^T}{\partial \theta} + \mathscr{H}_2^T = 0, \tag{16}$$

where

$$\tilde{n} = \frac{\partial \mathscr{H}_0}{\partial L} = \frac{\mu^2}{L^3},$$
$$\frac{\partial}{\partial \theta} = \frac{1}{w_E}\frac{\partial}{\partial t}.$$

In the above equations, $\tilde{n}$ and w_E represent the mean motion and the Earth's angular velocity, respectively.

The advantage of formulating the Hamiltonian in the Delaunay elements is evident from the above equation, which is a linear first-order PDE in two independent variables l and θ as opposed to three in case of the similar PDE in the polar-nodal variables; the third independent variable being the radial distance r (Segerman and Coffey 2000). A closed-form solution of the above PDE, if feasible, completes the Delaunay normalization of the Hamiltonian up to order two. A closed-form analytic solution is not known to exist for this PDE without first expanding the true anomaly dependent terms in $\mathscr{H}_2$ in powers of the eccentricity (Coffey and Alfriend 1981). However, in the next two sections, two different approaches for solving this PDE without employing the eccentricity expansions are discussed.

4 Relegation of the tesseral harmonics

The method of relegation uses an iterative approach to normalize the tesseral Hamiltonian and compute the short-period generating function that is closed-form in the eccentricity. In this section, the method of relegation is formulated to normalize the Hamiltonian expressed in the Delaunay elements. Unlike the original formulation (Segerman and Coffey 2000) of this method in the polar-nodal variables, its convergence criteria are better illustrated using the Delaunay elements. An alternative formulation of this method in the Delaunay elements is also proposed by Lara et al. (2013) that produces a series representation of the tesseral short-period generating function in powers of the eccentricity.

In the relegation method, an approximate solution of Eq. (16) is chosen that only partially satisfies the homological equation. The residual produced by this approximate solution gets multiplied by a term having a value less than unity. This residual becomes the new perturbation Hamiltonian term for the next iteration and after a sufficient number of iterations, it becomes small enough to be safely relegated to the next higher order. To illustrate this method, a residual term represented by $\mathscr{R}_j^T$ is added to Eq. (16) as shown below, with the subscript j representing an iteration:

$$\tilde{n}\frac{\partial \mathscr{W}_{2,j}^T}{\partial l} + w_E \frac{\partial \mathscr{W}_{2,j}^T}{\partial \theta} - \mathscr{H}_{2,j}^T = \mathscr{R}_j^T. \tag{17}$$

It is noted that if an exact solution of Eq. (16) is known then $\mathscr{R}_j^T$ is equal to zero in the above equation, otherwise it will be nonzero. Two separate flavors of the relegation method, one applicable for the sub-synchronous and other for the super-synchronous orbit regimes, are needed and are discussed below.

4.1 Sub-synchronous relegation

If $\mathscr{W}_{2,j}^T$ is chosen to satisfy the following constraint:

$$\tilde{n}\frac{\partial \mathscr{W}_{2,j}^T}{\partial l} - \mathscr{H}_{2,j}^T = 0, \tag{18}$$

then the approximate solution of Eq. (16) is given by:

$$\mathscr{W}_{2,j}^T = \frac{1}{\tilde{n}} \int \gamma\, \mathscr{H}_{2,j}^T \, \mathrm{d}f, \tag{19}$$

where

$$\gamma = \frac{\eta^3}{(1 + e\, \cos(f))^2},$$
$$\mathrm{d}f = \frac{1}{\gamma}\mathrm{d}l.$$

This approximate solution is substituted back into Eq. (17) to obtain the residual as:

$$\mathscr{R}_j^T = \int \delta\, \gamma\, \frac{\partial \mathscr{H}_{2,j}^T}{\partial \theta} \, \mathrm{d}f, \tag{20}$$

where

$$\delta = \frac{w_E}{\tilde{n}}.$$

It is noted that for the sub-synchronous orbits, $\delta < 1$. To continue the iterations further, the above residual is collected in $\mathscr{H}^T_{2,j+1}$, i.e., $\mathscr{H}^T_{2,j+1} = \mathscr{R}^T_j$ and a new $\mathscr{W}^T_{2,j+1}$ is computed by repeating the same process. The iterations are started by choosing $\mathscr{H}^T_{2,0} = \mathscr{H}^T_2$. If the residual $\mathscr{R}^T_j$ is smaller in magnitude than the original perturbation $\mathscr{H}^T_{2,j}$, then the method of relegation is guaranteed to converge. In order to estimate an upper bound of the residual for any value of f, the following bounds on the function γ are noted for an elliptic orbit:

$$\frac{\eta^3}{(1+e)^2} = \gamma_{\min} \leq \gamma \leq \gamma_{\max} = \frac{\eta^3}{(1-e)^2}. \tag{21}$$

In the limit that the eccentricity approaches 1, the values of $\gamma_{\min}$ and $\gamma_{\max}$ approach 0 and ∞, respectively. The differentiation and integration operation in Eq. (20) produces the coefficients of θ and f in the numerator and denominator, respectively. The minimum and maximum coefficient of f can be determined using the expression for the Hamiltonian given in Eq. (6), which are 1 and $2n + 1$, respectively. As a result, the maximum magnitude that the residual corresponding to a single term in $\mathscr{H}^T_{2,j}$ can attain during one orbit is found using Eq. (20) to be:

$$\mathscr{R}^T_{j,\max} = m\,\delta\,\gamma_{\max}\,|\mathscr{H}^T_{2,j}|. \tag{22}$$

It is noted that the minimum coefficient value of f that appears in the denominator of right-hand side of the above equation is used in calculating the upper bound on the residual. Thus, a sufficient condition for the relegation method to converge is

$$m\,\delta\,\gamma_{\max} < 1. \tag{23}$$

Hence, for the sub-synchronous orbits of medium to high eccentricities, the method of relegation will converge slowly, especially for the tesserals with $m > 1$.

If the above condition holds, then after a sufficient number of iterations d, the residual can be deemed small enough to be safely ignored at the current perturbation order and the tesseral short-period contributions are captured in the multiple $\mathscr{W}^T_{2,j}$ terms, where j runs from 0 to d. All these terms are summed together to get the complete short-period generating function for the tesserals.

4.2 Super-synchronous relegation

The relegation approach for super-synchronous orbits begins with the following choice of the constraint for $\mathscr{W}^T_{2,j}$ derived using Eq. (17):

$$w_E \frac{\partial \mathscr{W}^T_{2,j}}{\partial \theta} - \mathscr{H}^T_{2,j} = 0. \tag{24}$$

Solving the above equation, an approximate solution of Eq. (16) and the corresponding residual are obtained as:

$$\mathscr{W}^T_{2,j} = \frac{1}{w_E} \int \mathscr{H}^T_{2,j}\,\mathrm{d}\theta, \tag{25}$$

$$\mathscr{R}^T_j = \int \frac{1}{\delta\,\gamma} \frac{\partial \mathscr{H}^T_{2,j}}{\partial f}\,\mathrm{d}\theta. \tag{26}$$

The iterations are continued by using the residual $\mathscr{R}^T_j$ from the previous iteration as the new perturbation term $\mathscr{H}^T_{2,j+1}$. Using the same approach as for the sub-synchronous case, the maximum value of the residual corresponding to a single term in $\mathscr{H}_{2,j}^T$ over one complete orbit for the super-synchronous case is obtained as:

$$\mathscr{R}_{j,\max}^T = \frac{2n+1}{m\delta\,\gamma_{\min}} |\mathscr{H}_{2,j}^T|, \tag{27}$$

where $2n+1$ is the maximum coefficient of f in $\mathscr{H}_{2,j}^T$. The convergence condition for the super-synchronous obits is

$$\frac{2n+1}{m\delta\,\gamma_{\min}} < 1. \tag{28}$$

For the super-synchronous orbits, $\delta > 1$. For orbits with the periods approaching resonant values, the convergence degrades for both super-synchronous as well as sub-synchronous orbits due to the fact that δ approaches unity. Additionally, the convergence of the method of relegation depends on e, n, and m similar to the sub-synchronous case.

However, it is noted that the two sufficient conditions Eqs. (23) and (28) for the convergence of the relegation method in case of sub-synchronous and super-synchronous orbits, respectively, are derived using the upper bounds on the residuals in the two cases. It is possible that the method of relegation will converge, albeit slower, even if these conditions are not satisfied in certain cases. It should also be noted that there are no singularities present in the expressions for the generating functions $\mathscr{W}_{2,j}^T$ for resonant orbits.

4.3 *m*-Daily terms

To avoid secular terms in $\mathscr{W}_{2,j}^T$, the contributions from the m-daily terms are captured in a separate generating function $\mathscr{W}_2^{md}$. This generating function can be computed in closed-form without any iterations by directly integrating the m-daily part of $\mathscr{H}_2^T$ as shown below:

$$\mathscr{W}_2^{md} = \frac{1}{w_E} \int \mathscr{H}_2^{md}\, \mathrm{d}\theta, \tag{29}$$

where the superscript *md* is used to indicate that only the m-daily terms are included in the integrand.

Considering the above results, it can be concluded that the method of relegation is an iterative approach, with different expressions for the tesseral short-period generating functions for the sub-synchronous and super-synchronous orbit regimes. The generating function corresponding to each iteration captures only a part of the complete short-period contribution. It is known that the number of terms in the generating function increases significantly with each iteration. For instance, in one implementation of the relegation algorithm using the polar-nodal variables, the number of terms in the generating function easily exceeded one thousand after four iterations for C_{22} and S_{22} harmonics (Segerman and Coffey 2000). Another limitation of the relegation method is evident from the residual expressions given in Eqs. (22) and (27). For both sub-synchronous and super-synchronous cases, since γ is an unbounded monotonic increasing function of the eccentricity, the convergence criteria typically dictate small orbital eccentricities.

5 Exact Delaunay normalization of the tesseral harmonics

An exact solution for the PDE given in Eq. (16) is presented in this section, which accomplishes the Delaunay normalization of the tesseral Hamiltonian without using the eccentricity expansions and also avoids the convergence limitations of the relegation method discussed in the previous section. It is a linear first-order PDE with two independent variables and its exact solution can be reduced to quadrature using the method of characteristics to obtain the generating function for the short-periodic effects due to the tesseral harmonics.

The method of characteristics is a powerful technique for solving a linear or quasi-linear PDE such as the one-dimensional wave equation (Haberman 2003). In this method, a characteristic curve is found along which the PDE reduces to an ordinary differential equation (ODE). If the characteristic curves for the PDE given in Eq. (16) are chosen as the general solution of the equation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}l} = \frac{w_E}{\tilde{n}} = \delta, \tag{30}$$

then this PDE is transformed into the following ODE for $\mathscr{W}_2^T$:

$$\frac{\mathrm{d}\mathscr{W}_2^T}{\mathrm{d}l} - \frac{1}{\tilde{n}}\mathscr{H}_2^T = 0. \tag{31}$$

Before the above ODE can be integrated, all the appearances of θ in $\mathscr{H}_2^T$ have to be removed using the solution for the characteristic curves. These characteristic curves are found by integrating Eq. (30) to obtain:

$$\theta = \delta\, l + \theta_0, \tag{32}$$

where θ_0 is a constant. The above relation for θ is substituted in the expression for $\mathscr{H}_2^T$ given in Eq. (7) resulting in the following equation:

$$\begin{aligned} \mathscr{H}_2^T = & -\frac{2!}{C_{20}^2}\frac{\mu R_e^n}{r^{n+1}}\sum_{p=0}^{p=n} F_{nmp}(i)\,[C S_1\,\{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\} \\ & + C S_2\,\{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\}]\,, \end{aligned} \tag{33}$$

where

$$\alpha \equiv (n-2p) f - m\delta\, l, \tag{34}$$

$$\bar{\beta} \equiv (n-2p)g + m(h-\theta_0). \tag{35}$$

The ODE in Eq. (31) with $\mathscr{H}_2^T$ given above can now be integrated along the characteristic curves and the solution is given as follows:

$$\begin{aligned} \mathscr{W}_2^T = & -\frac{2!}{C_{20}^2}\frac{\mu R_e^n}{\tilde{n}}\sum_{p=0}^{p=n} F_{nmp}(i)\left[C S_1\left\{\cos(\bar{\beta})\int\frac{\cos(\alpha)}{r^{n+1}}\,\mathrm{d}l - \sin(\bar{\beta})\int\frac{\sin(\alpha)}{r^{n+1}}\,\mathrm{d}l\right\}\right. \\ & \left. + C S_2\left\{\cos(\bar{\beta})\int\frac{\sin(\alpha)}{r^{n+1}}\,\mathrm{d}l + \sin(\bar{\beta})\int\frac{\cos(\alpha)}{r^{n+1}}\,\mathrm{d}l\right\}\right] + F(\theta_0, a, e, i, h, g), \end{aligned} \tag{36}$$

where F is an arbitrary function. The final step to obtain the solution of the original PDE is to substitute back the value of the constant θ_0 from Eq. (32) in the above solution. After this substitution and the change of integration variable from l to f, the solution in the final form is obtained as follows:

$$\mathscr{W}_2^T = -\frac{2!}{C_{20}^2}\frac{\mu R_e^n}{\tilde{n}\, a^{n+1}\eta^{2n-1}}\sum_{p=0}^{p=n} F_{nmp}(i)\left[CS_1\left\{\cos(\beta)I_1^{nmp} - \sin(\beta)I_2^{nmp}\right\}\right.$$
$$\left. + CS_2\left\{\cos(\beta)I_2^{nmp} + \sin(\beta)I_1^{nmp}\right\}\right]$$
$$+ F(\theta - \delta\, l, a, e, i, h, g), \tag{37}$$

where

$$\beta \equiv (n-2p)g + m(h - \theta + \delta\, l), \tag{38}$$

$$I_1^{nmp} \equiv \int \cos((n-2p)f - m\delta\, l)(1 + e\cos f)^{n-1}\,\mathrm{d}f, \tag{39}$$

$$I_2^{nmp} \equiv \int \sin((n-2p)f - m\delta\, l)(1 + e\cos f)^{n-1}\,\mathrm{d}f. \tag{40}$$

The above result provides an exact and very succinct solution for the tesseral short-period generating function $\mathscr{W}_2^T$ valid for an arbitrary tesseral spherical harmonic. It satisfies the PDE given in Eq. (16) exactly, which can be easily verified by substituting this solution back into the PDE. In addition, F is an arbitrary function of the first five classical orbital elements and $\theta - \delta\, l$ and can be chosen to make the time-average of $\mathscr{W}_2^T$ zero, if required. Otherwise, it can be safely assumed to be equal to zero. A major advantage of this solution is that it remains valid for an arbitrary elliptic orbit in both the sub-synchronous as well as super-synchronous orbit regimes simultaneously, and avoids any singularities for the resonant orbits as well.

The above general solution contains two quadratures I_1^{nmp} and I_2^{nmp} and they need to be evaluated for computing $\mathscr{W}_2^T$ and its partial derivatives. It is noted that these integrals are functions of the true and mean anomaly simultaneously and as a consequence, a closed-form solution for I_1^{nmp} or I_2^{nmp} is not known to exist. Additionally, their integrands are not periodic functions because of the presence of δ, which is generally an irrational number, except for some special cases. Nevertheless, these two integrals represent only a part of the complete generating function and a series representation of them can be obtained by expanding their integrands either by using the typical eccentricity expansions of the true anomaly or by using the integration by parts method. They can also be computed numerically to arbitrary precision (in theory). These three different ways to compute a solution for I_1^{nmp} and I_2^{nmp} and their partial derivatives are discussed in the following subsections.

It has been brought to authors' attention that Palacián (2002) proposed a similar solution for the generating function that accomplishes the normalization of a perturbed Keplerian system using the method of characteristics. The solution obtained by Palacián (2002) applies to a generic form of the perturbed Keplerian Hamiltonian [see Eq. (14) in Palacián (2002)] that represents only a part of the complete tesseral Hamiltonian used in this work. Palacián (2002) acknowledged that the integrals present in his solution cannot be evaluated analytically except in some special cases, and a numerical quadrature must be performed for computing a solution. However, he arbitrarily chose the lower limit of integration for these integrals as zero, and ignored the computation of the integration constant (see Proposition 4.1.) It is seen through simulations in this work that ignoring the integration constant while evaluating the integrals using the numerical quadrature affects the accuracy of the proposed theory significantly (see Sect. 5.3).

5.1 Evaluation of I_1^{nmp} and I_2^{nmp} using eccentricity expansion

The well-known elliptic expansion of f in terms of e and l up to $\mathcal{O}(e^7)$ is (Battin 1987):

$$
\begin{aligned}
f = l &+ \left(2\,e - \frac{1}{4}\,e^3 + \frac{5\,e^5}{96}\right)\sin(l) + \left(\frac{5}{4}\,e^2 - \frac{11\,e^4}{24} + \frac{17\,e^6}{192}\right)\sin(2\,l)\\
&+ \left(\frac{13\,e^3}{12} - \frac{43\,e^5}{64}\right)\sin(3\,l) + \left(\frac{103\,e^4}{96} - \frac{451\,e^6}{480}\right)\sin(4\,l)\\
&+ \frac{1097\,e^5\sin(5\,l)}{960} + \frac{1223\,e^6\sin(6\,l)}{960}.
\end{aligned}
\tag{41}
$$

A series solution of I_1^{nmp} and I_2^{nmp} can be obtained for an arbitrary tesseral harmonic by substituting the above relation in Eqs. (39) and (40), and expanding the integrands using Taylor series about $e = 0$. These series solutions are known to be convergent for low to moderate values of the eccentricity (approx. $e < 0.7$), therefore a term-by-term differentiation can be performed to compute the partial derivatives of I_1^{nmp} and I_2^{nmp}. The results can be substituted in Eq. (37) to obtain $\mathscr{W}_2^T$ and its partial derivatives. All of these operations can easily be mechanized using any off-the-shelf symbolic algebra package.

Evaluation of I_1^{nmp} and I_2^{nmp} using the eccentricity expansions introduces singularities for the orbits with periods close to the resonant values because of the appearance of the term $(n - 2p - m\delta)$ in their denominators. This drawback can be avoided by finding a series representation of these two integrals using the integration by parts method as discussed in the following subsection.

5.2 Evaluation of I_1^{nmp} and I_2^{nmp} using integration by parts

Since δ appears as a coefficient of the mean anomaly in I_1^{nmp} and I_2^{nmp}, the repeated differentiation or integration of their integrands can produce powers of δ multiplying the numerator or denominator of the same integrands. This observation motivates the application of the integration by parts method to produce a series representation of I_1^{nmp} and I_2^{nmp}. The repeated application of the integration by parts method is equivalent to the iterations performed in the method of relegation in Sect. 4, for the two methods produce the same solution for the generating function. As for the method of relegation, the series solutions of ${I_1}^{nmp}$ and ${I_2}^{nmp}$ using integration by parts are also different for the sub-synchronous and super-synchronous orbit regimes.

5.2.1 Sub-synchronous case

Before applying integration by parts to ${I_1}^{nmp}$ and I_2^{nmp}, the integration variable is changed to l and the terms containing f and l are separated from each other. Using trigonometric identities, Eqs. (39) and (40) can be written as shown below:

$$
I_1^{nmp} = \frac{1}{\eta^3}\int \left(\cos(m\delta l)A_0(f) + \sin(m\delta l)B_0(f)\right)\,\mathrm{d}l, \tag{42}
$$

$$
I_2^{nmp} = \frac{1}{\eta^3}\int \left(\cos(m\delta l)B_0(f) - \sin(m\delta l)A_0(f)\right)\,\mathrm{d}l, \tag{43}
$$

where

$$A_0(f) = \cos\left((n-2p)f\right)\ (1+e\cos(f))^{n+1},$$
$$B_0(f) = \sin\left((n-2p)f\right)\ (1+e\cos(f))^{n+1}.$$

Repeated application of the integration by parts, with each term containing l considered as the first function, produces a series expansion of the two integrals in the powers of δ. These series solutions of I_1^{nmp} and I_2^{nmp} up to $\mathscr{O}(\delta^4)$ are obtained as:

$$I_1^{nmp} = \frac{1}{\eta^3}\Big[\cos(m\delta l)\left(A_1 - (m\delta)B_2 - (m\delta)^2 A_3 + (m\delta)^3 B_4 + (m\delta)^4 A_5 - \cdots\right) + \sin(m\delta l)\left(B_1 + (m\delta)A_2 - (m\delta)^2 B_3 - (m\delta)^3 A_4 + (m\delta)^4 B_5 + \cdots\right)\Big], \tag{44}$$

$$I_2^{nmp} = \frac{1}{\eta^3}\Big[-\sin(m\delta l)\left(A_1 + (m\delta)B_2 - (m\delta)^2 A_3 - (m\delta)^3 B_4 + (m\delta)^4 A_5 + \cdots\right) + \cos(m\delta l)\left(B_1 - (m\delta)A_2 + (m\delta)^2 B_3 + (m\delta)^3 A_4 - (m\delta)^4 B_5 - \cdots\right)\Big], \tag{45}$$

where for $k \geq 1$

$$A_k(f) = \int \gamma\ A_{k-1}(f)\,\mathrm{d}f,$$
$$B_k(f) = \int \gamma\ B_{k-1}(f)\,\mathrm{d}f.$$

The functions A_k and B_k appearing in the above equations only depend on the eccentricity and the true anomaly and as a result, they are integrable. As k increases, the magnitudes of A_k and B_k are guaranteed to decrease if $\gamma_{\max} < 1$, where $\gamma_{\max}$ is the maximum value of γ for any f given in Eq. (21). Therefore, the sufficient condition for the series solutions of I_1^{nmp} and I_2^{nmp} to converge is:

$$m\,\delta\,\gamma_{\max} < 1. \tag{46}$$

It is noted that the coefficients of f that are produced while evaluating A_k and B_k appear in the denominator of the left-hand side of the above inequality condition. Using formulae given in Eqs. (3) and (4), it can be verified that these coefficients are lower bounded by unity. Therefore, the above convergence criterion is valid for any tesseral harmonic.

5.2.2 *Super-synchronous case*

For super-synchronous orbits, $\delta > 1$. Therefore, each term containing f is treated as the first function in the application of integration by parts to Eqs. (42) and (43), resulting in the following series solutions of I_1^{nmp} and I_2^{nmp} up to $\mathscr{O}(\delta^{-4})$:

$$I_1^{nmp} = \frac{1}{\eta^3}\left[\frac{\cos(m\delta l)}{m\delta}\left(-B_0 + \frac{A^1}{(m\delta)} + \frac{B^2}{(m\delta)^2} - \frac{A^3}{(m\delta)^3} - \frac{B^4}{(m\delta)^4} + \cdots\right) + \frac{\sin(m\delta l)}{m\delta}\left(A_0 + \frac{B^1}{(m\delta)} - \frac{A^2}{(m\delta)^2} - \frac{B^3}{(m\delta)^3} + \frac{A^4}{(m\delta)^4} + \cdots\right)\right], \tag{47}$$

$$I_2^{nmp} = \frac{1}{\eta^3}\left[\frac{\cos(m\delta l)}{m\delta}\left(-A_0 + \frac{B^1}{(m\delta)} + \frac{A^2}{(m\delta)^2} - \frac{B^3}{(m\delta)^3} - \frac{A^4}{(m\delta)^4} + \cdots\right)\right.$$
$$\left. + \frac{\sin(m\delta l)}{m\delta}\left(B_0 + \frac{A^1}{(m\delta)} - \frac{B^2}{(m\delta)^2} - \frac{A^3}{(m\delta)^3} + \frac{B^4}{(m\delta)^4} + \cdots\right)\right], \tag{48}$$

where for $k \geq 1$

$$A^k = \frac{1}{\gamma}\frac{\partial A^{k-1}}{\partial f},$$
$$B^k = \frac{1}{\gamma}\frac{\partial B^{k-1}}{\partial f}.$$

Since the absolute values of the coefficients of f in A_0 and B_0 are bounded by $2n+1$, the sufficient condition for the convergence of the above series solutions of I_1^{nmp} and I_2^{nmp} is:

$$\frac{2n+1}{m\delta\gamma_{\min}} < 1. \tag{49}$$

It is not a coincidence that the convergence criteria for sub-synchronous and super-synchronous orbits given in this section are, respectively, identical to criteria in Eqs. (23) and (28) obtained in Sect. 4 for the method of relegation. In fact, the above series solutions of I_1^{nmp} and I_2^{nmp} if substituted in Eq. (37), gives the same solution for the tesseral short-period generating function $\mathscr{W}_{2,j}^T$ as produced by the method of relegation. Each expansion order in the series solutions of I_1^{nmp} and I_2^{nmp} corresponds to that from an iteration in the method of relegation. The approach presented in this section is, therefore, can be considered as a more general method, of which the method of relegation is a special case.

5.3 Evaluation of I_1^{nmp} and I_2^{nmp} by numerical quadrature

The two approximate approaches for evaluating the indefinite integrals I_1^{nmp} and I_2^{nmp}, which are needed to compute the tesseral short-period generating function, have different limitations. The approach using the eccentricity expansion has inherent singularities for resonant elliptic orbits and convergence issues for medium to high values of the orbital eccentricity. In contrast, the series solutions found in the previous subsection using integration by parts produce large expressions for each term in the series. Also, the domain of convergence is limited by the convergence criteria. Another approach to evaluate these two integrals to arbitrary precision (in theory) is by numerical quadrature. The numerical approach not only produces very compact expressions for the short-period variations of the orbital elements due to tesserals, it also avoids the singularities and convergence limitations of the previous two methods.

The integrands of I_1^{nmp} and I_2^{nmp} are not periodic functions due to the presence of the irrational number δ. As a consequence, the current value of the true anomaly, which is used as the upper limit for the numerical quadrature, must take into account the number of orbital periods passed since the initial time, i.e., it must not be wrapped between 0 and 2π. The lower limit for the quadrature is arbitrary and can be taken as 0. However, the values of the two indefinite integrals at the lower limit, i.e., $f = 0$ are required, which can be computed by using either of the previous two approaches discussed. This hybrid approach uses the eccentricity expansions or the integration by parts method only to evaluate I_1^{nmp} and I_2^{nmp} at a single point, $f = 0$. With this discussion, the solutions of the two integrals are now written as follows:

$$I_1^{nmp} = \int_0^f \cos((n-2p)f - m\delta\, l)(1+e\cos f)^{n-1}\,\mathrm{d}f, \tag{50}$$

$$I_2^{nmp} = I_2^{nmp}|_{f=0} + \int_0^f \sin((n-2p)f - m\delta\, l)(1+e\cos f)^{n-1}\,\mathrm{d}f. \tag{51}$$

It can be easily verified that $I_1^{nmp} = 0$ at $f = 0$. It is noted that by using the true anomaly as the integration variable in the two quadratures, the need to solve Kepler's equation at each function evaluation is avoided. The minimum step size needed for the numerical quadrature should take into account the highest frequency component present in the integrands of I_1^{nmp} and I_2^{nmp}. Using formulae in Eqs. (3) and (4), the largest numerical coefficient of f in the integrands is obtained as $2n-1$. Therefore, the highest frequency component in the two integrands has the time-period as $2\pi/(2n-1)$.

From an operational point of view, the numerical quadrature approach has many advantages. The exact solution for $\mathscr{W}_2^T$, which is closed-form in the eccentricity, is very compact in size compared to the approximate solutions generated by the method of relegation. This stems from the fact that the term $(1+e\cos(f))^{n-1}$ need not be expanded for evaluating the two quadratures I_1^{nmp} and I_2^{nmp}. The numerical quadrature approach is also well suited for parallel computations for faster execution. Moreover, unlike the method of relegation, no separate treatment for m-daily terms is required and their contributions are included in the proposed solution without any approximation.

6 Artificial satellite theory with tesseral harmonics

The developments in the previous section have been implemented into an artificial satellite theory, which includes tesseral short-period effects. The expressions for the second-order short-period variations of the equinoctial orbital elements valid for an arbitrary tesseral are provided below. The tesseral short-period contributions are required in addition to the secular and periodic contributions of the zonal harmonics to formulate a complete satellite theory valid for an Earth-like nonspherical central body. The readers are referred to previous work (Mahajan et al. 2016) on computing the expressions for secular, short-period and long-period variations of the equinoctial elements due to the zonal harmonics.

Provided a generating function for the periodic variations computed using the Deprit's perturbation method, the expressions for the periodic variations of any set of orbital elements can be computed by evaluating the Poisson brackets in Eqs. (11) and (12). The second-order tesseral contributions in the mean to osculating transformation include the Poisson bracket $(\mathscr{E}, \mathscr{W}_2^T)$, where $\mathscr{E}$ is any orbital element. In the present work, the equinoctial elements are chosen since they are free of singularities for circular and equatorial orbits. The equinoctial elements are defined in terms of the classical and Delaunay elements as given below:

$$a = a, \qquad \Lambda = l + g + h, \qquad p_1 = \tan\left(\frac{i}{2}\right)\cos(h),$$
$$p_2 = \tan\left(\frac{i}{2}\right)\sin(h), \qquad q_1 = e\ \cos(g+h), \qquad q_2 = e\ \sin(g+h).$$

Using the above definitions of the equinoctial elements, the following expressions for the variations of equinoctial elements valid for any tesseral harmonic are derived:

$$(a, \mathscr{W}_2^T) = -2\sqrt{\frac{a}{\mu}}\frac{\partial f}{\partial l}\frac{\partial \mathscr{W}_2^T}{\partial f}, \tag{52}$$

$$(\Lambda, \mathscr{W}_2^T) = 2\sqrt{\frac{a}{\mu}}\frac{\partial \mathscr{W}_2^T}{\partial a} - \frac{\eta}{L}\sqrt{\frac{1-\eta}{1+\eta}}\frac{\partial \mathscr{W}_2^T}{\partial e} - \frac{\partial \mathscr{W}_2^T}{\partial i}\left(\frac{\tan(\frac{i}{2})}{G}\right), \tag{53}$$

$$(p_1, \mathscr{W}_2^T) = \frac{\sin(h)}{G(1+\cos i)}\frac{\partial \mathscr{W}_2^T}{\partial i} - \frac{\cos h}{(1+\cos i)}\frac{1}{G}\mathscr{W}_{2,gh}^T, \tag{54}$$

$$(p_2, \mathscr{W}_2^T) = \frac{-\cos(h)}{G(1+\cos i)}\frac{\partial \mathscr{W}_2^T}{\partial i} - \frac{\sin h}{(1+\cos i)}\frac{1}{G}\mathscr{W}_{2,gh}^T, \tag{55}$$

$$(q_1, \mathscr{W}_2^T) = -\sin(g+h)\left(\frac{-\eta}{L}\frac{\partial \mathscr{W}_2^T}{\partial e} - e\frac{\partial \mathscr{W}_2^T}{\partial i}\frac{\tan(\frac{i}{2})}{G}\right) - \frac{\eta^2}{L}\cos(g+h)\mathscr{W}_{2,lg}^T, \tag{56}$$

$$(q_2, \mathscr{W}_2^T) = \cos(g+h)\left(\frac{-\eta}{L}\frac{\partial \mathscr{W}_2^T}{\partial e} - e\frac{\partial \mathscr{W}_2^T}{\partial i}\frac{\tan(\frac{i}{2})}{G}\right) - \frac{\eta^2}{L}\sin(g+h)\mathscr{W}_{2,lg}^T, \tag{57}$$

where

$$\mathscr{W}_{2,gh}^T \equiv \frac{1}{\sin i}\left(\cos i\frac{\partial \mathscr{W}_2^T}{\partial g} - \frac{\partial \mathscr{W}_2^T}{\partial h}\right), \tag{58}$$

$$\mathscr{W}_{2,lg}^T \equiv \frac{1}{e}\left(\frac{\partial \mathscr{W}_2^T}{\partial l} - \frac{1}{\eta}\frac{\partial \mathscr{W}_2^T}{\partial g}\right). \tag{59}$$

The apparent singularities in the above expressions for $\mathscr{W}_{2,gh}^T$ and $\mathscr{W}_{2,lg}^T$ are only artificial and they are eliminated when the expression for $\mathscr{W}_2^T$ is substituted into these equations. The final expressions for these terms after canceling out the singular terms along with the partial derivatives of I_1^{nmp} and I_2^{nmp} needed to compute the above expressions are provided in appendix. It is noted that as for I_1^{nmp} and I_2^{nmp}, their partial derivatives can also be computed using any of the three methods discussed in the previous section: the eccentricity expansion, integration by parts, or numerical quadrature.

By incorporating the above expressions for the tesseral short-period variations into a semi-analytic satellite theory with zonal short-period variations (Mahajan et al. 2016), a complete semi-analytic satellite theory for a nonspherical Earth-like planetary body is implemented in MATLAB. It is emphasized that the resulting satellite theory is valid for any tesseral of arbitrary degree and order. The simulation results in this section, however, are provided only for the 2×2 geopotential for comparison with those already published in the literature. The zonal semi-analytic theory includes secular and long periodic effects of J_2 up to $\mathscr{O}(J_2^3)$, and short-period effects up to $\mathscr{O}(J_2^2)$. The tesseral short-period variations are included by evaluating I_1^{nmp}, I_2^{nmp}, and their partial derivatives. The adaptive quadrature algorithm (*integral* command) available in MATLAB with tolerance set to 10^{-9} is used to generate the results. The accuracy of this theory is ascertained by comparison with the results obtained from a variable-step, variable-order Adams–Bashforth–Moulton integrator (MATLAB *ode113* command) with a 2×2 JGM-3 Earth's gravity model and relative tolerances set to 10^{-11}.

Two different initial conditions are chosen, one with a small and the other with a moderate value of the eccentricity to showcase the accuracy improvements obtained by using the proposed approach. These initial osculating orbital elements, given in Table 1, are taken from Lara et al. (2013) in order to compare the results with the Lara's implementation of the method of relegation. A step-by-step algorithm for the semi-analytic integration of the 2×2 geopotential using the proposed approach is as follows:

1. Choose the osculating initial elements;

Table 1 Initial orbital elements for the two test orbits

IC	a (km)	e	i	Ω	ω	M
Orbit-1	12159.596	0.01	5°	0°	270°	0°
Orbit-2	18520	0.35	100°	0°	270°	0°

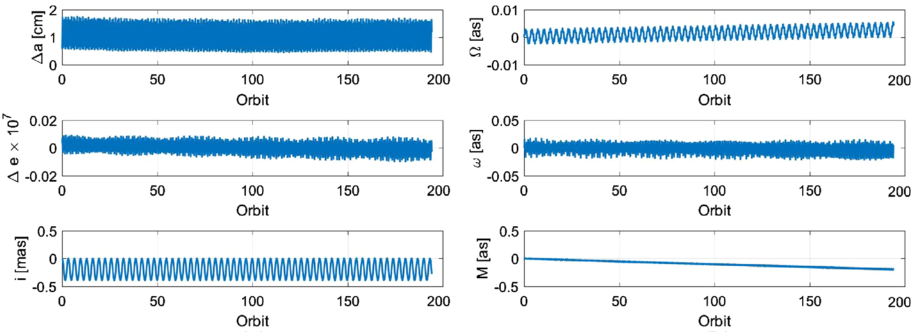

Fig. 1 Propagation errors for Orbit-1 in the classical orbital elements for 2 × 2 semi-analytic satellite theory with the tesseral short-period variations computed by numerical quadrature of I_1^{nmp} and I_2^{nmp}. The propagation time corresponds to 30 days with data points plotted at 30 min intervals

2. Compute the integrals I_1^{nmp} and I_2^{nmp}, and their partials by numerical quadrature with the lower limit $f = 0$. Compute integrals values at $f = 0$ using either the eccentricity expansion or by integration by parts method;
3. An alternative to the previous step is to use the eccentricity expansion or the integration by parts method to directly evaluate the two indefinite integrals and their partials for a given value of f;
4. Using the results from Step 2 or 3, compute the tesseral short-period variations using Eqs. (52–57);
5. Compute the zonal short-period variations (Mahajan et al. 2016) and using the inverse transformation given in Eq. (12), compute the initial mean elements;
6. Numerically integrate the mean elements using the equations capturing secular and long-period effects of the zonal harmonics (Mahajan et al. 2016);
7. Convert the mean elements at the desired time back to the osculating elements by evaluating the transformation equations given in Eq. (11). Use the same approach as in Steps 2 or 3 to compute the tesseral short-period contributions.

The initial conditions from Table 1 are propagated numerically as well as semi-analytically using the above algorithm and the differences in the classical orbital elements over a period of 30 days are plotted in Figs. 1, 2, 3 and 4. The numerical quadrature-based approach (as described in Step 2 of the above algorithm) was used for Orbit-1 and Orbit-2 in Figs. 1 and 3, respectively. The eccentricity expansion-based approach is used to generate the results shown in Figs. 2 and 4. Comparing the plots for Orbit-1 in Figs. 1 and 2, it is noted that the exact solution for tesseral short-period variations using numerical quadrature has the same accuracy as the eccentricity expansion approach as expected for this near-circular orbit. However, significant difference in the accuracy of the two approaches became apparent for Orbit-2, which has a higher eccentricity of 0.35, as seen in Figs. 3 and 4. Here, the numerical quadrature-based exact solution for tesseral short-period variations outperforms the eccen-

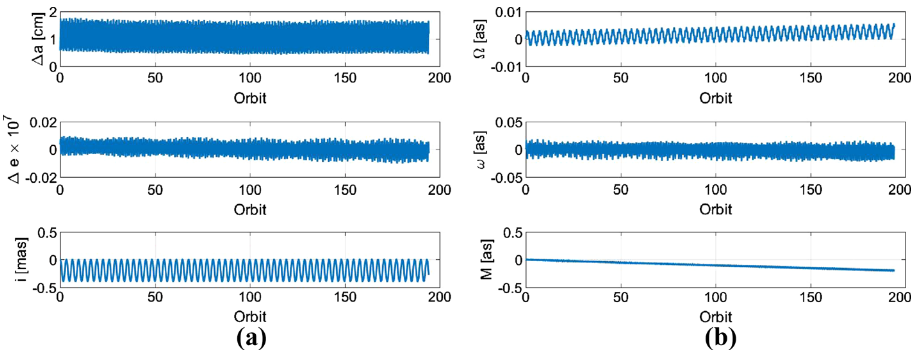

Fig. 2 Propagation errors for Orbit-1 in the classical orbital elements for 2 × 2 semi-analytic satellite theory with the tesseral short-period variations computed using the series solution of I_1^{nmp} and I_2^{nmp} in powers of the eccentricity. The propagation time corresponds to 30 days with data points plotted at 30 min intervals

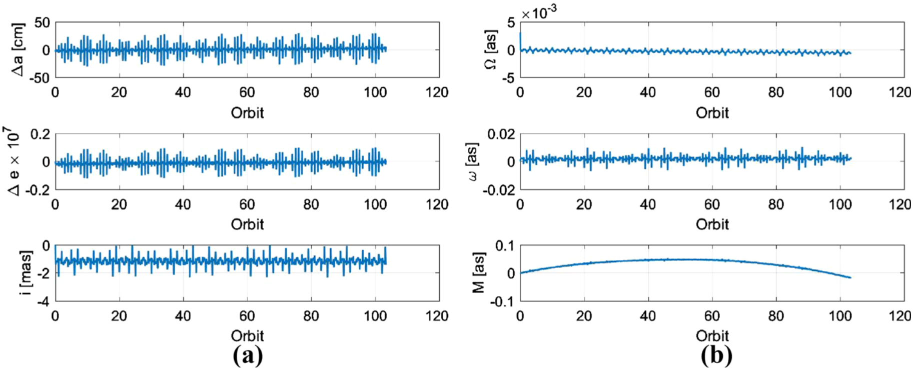

Fig. 3 Propagation errors for Orbit-2 in the classical orbital elements for 2 × 2 semi-analytic satellite theory with the tesseral short-period effects computed by numerical quadrature of I_1^{nmp} and I_2^{nmp}. The propagation time corresponds to 30 days with data points plotted at 30 min intervals

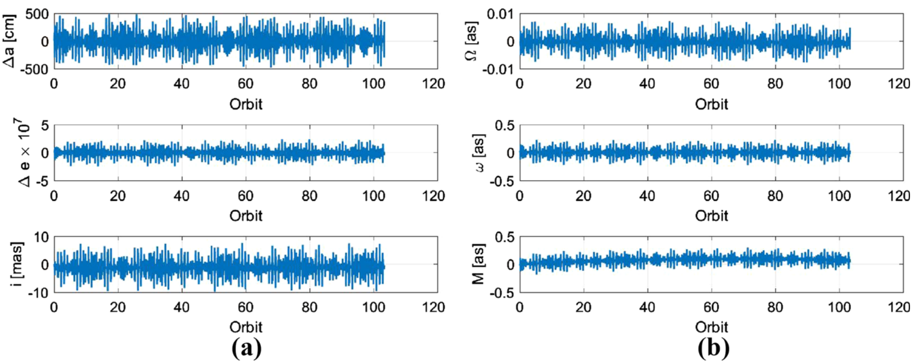

Fig. 4 Propagation errors for Orbit-2 in the classical orbital elements for 2 × 2 semi-analytic satellite theory with the tesseral short-period effects computed using the series solution of I_1^{nmp} and I_2^{nmp} in powers of the eccentricity. The propagation time corresponds to 30 days with data points plotted at 30 min intervals

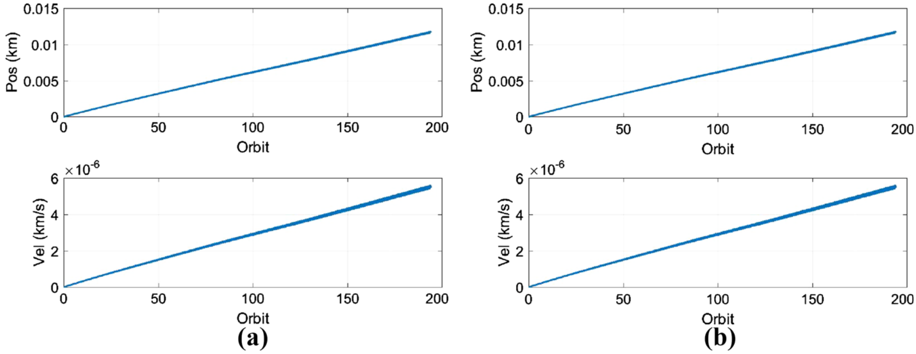

Fig. 5 Position and velocity errors for Orbit-1 using 2 × 2 semi-analytic satellite theory including tesseral short-period effects with integrals computed using: **a** numerical quadrature, and **b** eccentricity expansion. The propagation time corresponds to 30 days with data points plotted at 30 min intervals. Neglecting the tesseral corrections in the semi-analytic theory for this orbit results in position and velocity errors of, respectively, 80 km and 0.038 km/s in 30 days

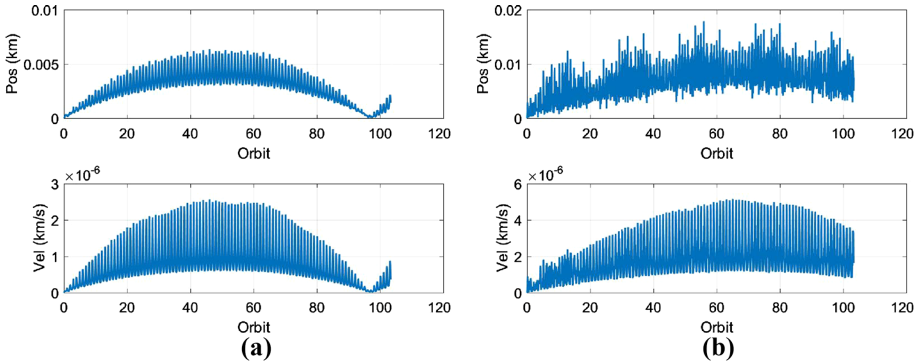

Fig. 6 Position and velocity errors for Orbit-2 using 2 × 2 semi-analytic satellite theory including the tesseral short-period effects with integrals computed using: **a** numerical quadrature, and **b** eccentricity expansion. The propagation time corresponds to 30 days with data points plotted at 30 min intervals. Neglecting the tesseral contributions in the semi-analytic theory for this orbit results in position and velocity errors of, respectively, 9.6 km and 0.004 km/s in 30 days. Smaller propagation errors for Orbit-2 compared to Orbit-1 are attributed to higher inclination of the former orbit

tricity expansion-based approach significantly for all the elements. The peak semimajor axis error for the numerical quadrature approach is smaller than that for the eccentricity expansion approach by a factor of 20.

The position and velocity errors for Orbit-1 and Orbit-2 are shown in Figs. 5 and 6. The proposed theory for tesseral short-period variations outperforms Lara's version of the method of relegation for both Orbit-1 and Orbit-2, when I_1^{nmp} and I_2^{nmp}, and their partial derivatives are computed using numerical quadrature. The results in Figs. 1, 2, 3 and 4 should be compared with Figs. 2 and 4 of Lara et al. (2013). For the near-circular Orbit-1, the exact solution with I_1^{nmp} and I_2^{nmp} computed using eccentricity expansions, produces more accurate results than those from Lara's method of relegation after one iteration. In case of Orbit-2, Lara's method with five iterations produces better results than the proposed solution with integrals computed using the eccentricity expansions, for all the elements except for the mean anomaly. It is noted that for the mean anomaly, Lara's method produces a secular growth in the error, which causes significant contributions to position and velocity errors compared to the other elements.

7 Conclusion

The proposed approach for the exact Delaunay normalization of the tesseral Hamiltonian provides a general methodology for constructing a very compact artificial satellite theory, valid for perturbed elliptic orbits with an arbitrary value of the eccentricity. It is a unified theory for sub-synchronous and super-synchronous orbit regimes and remains nonsingular for resonant orbits. It is shown that the more conventional methods of Delaunay normalization of the tesseral harmonics such as the method of relegation or Kaula's eccentricity expansion-based approach, can be derived from the proposed more general methodology. Simulation results showed significant improvements in long-term prediction accuracies due to the proposed theory when compared with the method of relegation.

Acknowledgements This research was partially supported by AFRL Contract FA9453-13-C-0202 with Dr. Alan T. Lovell serving as the Technical Project Monitor. The authors also acknowledge Dr. Martin Lara for private communications and for providing the code for his implementation of the method of relegation.

Appendix: Required partial derivatives for computing short-period variations due to tesseral harmonics

$$
\begin{aligned}
\mathscr{W}^T_{2,gh} &= \frac{1}{\sin i}\left(\cos i \frac{\partial \mathscr{W}^T_2}{\partial g} - \frac{\partial \mathscr{W}^T_2}{\partial h}\right) \\
&= -\frac{2!}{C^2_{20}} \frac{\mu R^n_e}{\tilde{n}\, a^{n+1} \eta^{2n-1}} \sum_{p=0}^{p=n} F'_{nmp}(i) \left[CS_1 \left\{-\cos(\beta) I^{nm}_2 - \sin(\beta) I^{nmp}_1\right\}\right. \\
&\quad \left. + CS_2 \left\{\cos(\beta) I^{nmp}_1 - \sin(\beta) I^{nmp}_2\right\}\right]
\end{aligned}
$$

$$
\text{where } F'_{nmp}(i) = \begin{cases} \frac{-m}{G} \tan(i/2)\; F_{nmp}(i) & \text{if } (n-2p) = m \\ \frac{1}{2} \frac{\cos(i)(n-2p)-m}{G\cos(i/2)} \left(\frac{1}{\sin(i/2)} F_{nmp}(i)\right) & \text{if } (n-2p) \neq m \end{cases}
$$

$$
\begin{aligned}
\mathscr{W}^T_{2,lg} &= \frac{1}{e}\left(\frac{\partial \mathscr{W}^T_2}{\partial l} - \frac{1}{\eta} \frac{\partial \mathscr{W}^T_2}{\partial g}\right) \\
&= -\frac{2!}{C^2_{20}} \frac{\mu R^n_e}{\tilde{n}\, a^{n+1} \eta^{2n-1}} \sum_{p=0}^{p=n} F_{nmp}(i) \\
&\quad \left[CS_1 \left\{\cos(\beta)\left(-(n-2p) I^{nmp}_{2,lga} - \frac{n+1}{\eta^3} I^{nmp}_{1,lgb}\right)\right.\right. \\
&\quad \left. + \sin(\beta)\left(-(n-2p) I^{nmp}_{1,lga} + \frac{n+1}{\eta^3} I^{nmp}_{2,lgb}\right)\right\} \\
&\quad + CS_2 \left\{\cos(\beta)\left((n-2p) I^{nmp}_{1,lga} - \frac{n+1}{\eta^3} I^{nmp}_{2,lgb}\right)\right.
\end{aligned}
$$

$$+\sin(\beta)\left(-(n-2p)I_{2,lga}^{nmp}-\frac{n+1}{\eta^3}I_{1,lgb}^{nmp}\right)\Big\}\Big]$$

$$\frac{\partial I_1^{nmp}}{\partial a}=m\frac{\mathrm{d}\delta}{\mathrm{d}a}\int l\,\sin((n-2p)f-m\delta\,l)(1+e\cos f)^{n-1}\,\mathrm{d}f$$

$$\frac{\partial I_2^{nmp}}{\partial a}=-m\frac{\mathrm{d}\delta}{\mathrm{d}a}\int l\,\cos((n-2p)f-m\delta\,l)(1+e\cos f)^{n-1}\,\mathrm{d}f$$

$$\begin{aligned}\frac{\partial I_1^{nmp}}{\partial e}=&\frac{1}{\eta^2}\int\left(\frac{-3e(n-1)}{2}\cos(\alpha(f))+\frac{(-(n-1)+4p)}{2}\cos(\alpha(f)-f)\right.\\&+\frac{((3n+1)-4p)}{2}\cos(\alpha(f)+f)+\frac{e(2p+1)}{4}\cos(\alpha(f)-2f)\\&\left.+\frac{e(2n-2p+1)}{4}\cos(\alpha(f)+2f)\right)(1+e\cos f)^{n-1}\,\mathrm{d}f\end{aligned}$$

$$\begin{aligned}\frac{\partial I_2^{nmp}}{\partial e}=&\frac{1}{\eta^2}\int\left(\frac{-3e(n-1)}{2}\sin(\alpha(f))+\frac{(-(n-1)+4p)}{2}\sin(\alpha(f)-f)\right.\\&+\frac{((3n+1)-4p)}{2}\sin(\alpha(f)+f)+\frac{e(2p+1)}{4}\sin(\alpha(f)-2f)\\&\left.+\frac{e(2n-2p+1)}{4}\sin(\alpha(f)+2f)\right)(1+e\cos f)^{n-1}\,\mathrm{d}f,\end{aligned}$$

where $\alpha(f)\equiv(n-2p)f-m\delta\,l$

$$\frac{\partial I_1^{nmp}}{\partial f}=\cos((n-2p)f-m\delta\,l)(1+e\cos f)^{n-1}$$

$$\frac{\partial I_2^{nmp}}{\partial f}=\sin((n-2p)f-m\delta\,l)(1+e\cos f)^{n-1}$$

$$\begin{aligned}I_{1,lga}^{nmp}=&\int\frac{1}{\eta^3}\left(e+2\cos(f)+e\cos^2(f)\right)\cos((n-2p)f-m\delta\,l)\\&(1+e\cos f)^{n-1}\,\mathrm{d}f\end{aligned}$$

$$\begin{aligned}I_{2,lga}^{nmp}=&\int\frac{1}{\eta^3}\left(e+2\cos(f)+e\cos^2(f)\right)\sin((n-2p)f-m\delta\,l)\\&(1+e\cos f)^{n-1}\,\mathrm{d}f\end{aligned}$$

$$I_{1,lgb}^{nmp}=\int\sin(f)\,\cos((n-2p)f-m\delta\,l)(1+e\cos f)^{n}\,\mathrm{d}f$$

$$I_{2,lgb}^{nmp}=\int\sin(f)\,\sin((n-2p)f-m\delta\,l)(1+e\cos f)^{n}\,\mathrm{d}f.$$

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