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ORIGINAL ARTICLE

3-Dimensional Necklace Flower Constellations

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Abstract A new approach in satellite constellation design is presented in this paper, taking as a base the 3D Lattice Flower Constellation Theory and introducing the necklace problem in its formulation. This creates a further generalization of the Flower Constellation Theory, increasing the possibilities of constellation distribution while maintaining the characteristic symmetries of the original theory in the design.

Keywords Satellite constellations · Orbit design · Number theory

1 Introduction

The space industry has experienced great advances in the last decades due to the number of possibilities and benefits that the space environment brings. Satellites orbiting the Earth have a very advantageous position, since they are able to observe vast regions of the Earth in a small amount of time. This advantage can be improved even further with the use of satellite constellations, allowing the study of several regions of the Earth surface at the same time.

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Satellite constellations are groups of satellites that work cooperatively to achieve a common mission. They allow to optimize the performance of the system as a whole, reducing the costs of the mission. However, the study of several satellites at the same time, and more importantly, the relations that appear in the internal structure of the constellation, increases the complexity of the problem to solve, but also expands the possibilities in the design.

In the last decades, several satellite constellation design methodologies have appeared, such as the Walker Constellations (Walker 1984) for circular orbits, the design of Draim (1987) for elliptic orbits, or the Ground-track Constellations (Arnas et al. 2016a, 2017c) for any kind of configuration. In 2004, the Flower Constellation Theory (Mortari et al. 2004; Mortari and Wilkins 2008; Wilkins and Mortari 2008; Casanova et al. 2014b; Arnas et al. 2016b) was presented, including in its formulation circular and elliptic orbits. The theory was later improved by the 2D Lattice (Avendaño et al. 2013) and 3D Lattice (Davis et al. 2013) theories, which simplified the formulation and made the configuration independent of any reference frame.

In the 2D and 3D Lattice Flower Constellation theories, the configuration of the constellation presents symmetries and is highly uniform in the space, allowing to generate constellations where all satellites observe the same relative configuration. These properties have many advantages in missions such as global coverage or global positioning. Afterward, realizing that the amount of different configurations of a constellation for a certain number of satellites could be increased in the formulation, the concept of necklaces (Casanova et al. 2011, 2014a) was introduced for the 2D Lattice Flower Constellation theory. The theory of necklaces is based on the idea of generating a fictitious constellation with more satellites than required and then, selecting a subset of satellites from the fictitious constellation taking into account that the property of symmetry has to be maintained (Arnas et al. 2017b).

The solution of the necklace problem (as well as the Flower Constellation Theory) is related to Number Theory which implies working with integer numbers in the distribution of the orbital parameters of the constellation. This leads to interesting properties that are not presented with the use of real numbers.

The aim of this paper is to apply the necklace theory into the 3D Lattice Flower Constellations design methodology. This is done by the introduction of a new formulation, which constitutes a generalization of the 2D and 3D Lattice Flower Constellations, and that contains as a subset, all the former Lattice Flower Constellations. In this new formulation, it is possible to include necklaces in any of the variables of distribution: the right ascension of the ascending node, the argument of perigee and the mean anomaly. This allows to expand the possibilities of design, not limiting the generation of necklaces to the mean anomaly as done in previous works (Casanova et al. 2011, 2014a).

This manuscript is organized as follows: First, a short introduction on the 3D Lattice Flower Constellations and the Necklace Theory is performed. Second, a new formulation is introduced that includes necklaces directly into the formulation of the distribution. This provides a clearer formulation and moreover allows a faster computation of the real constellation, since only the real positions of the satellites are computed. Third, the expansion of the searching space is introduced, which allows to generate as many different possibilities in design as required. These two properties are especially interesting in optimization problems, where the time spent and the design possibilities are controlled using the size of the fictitious constellation. Fourth, the conditions to generate distributions that maintain the properties of symmetry and uniformity of the configuration (characteristic of the Lattice Flower Constellations) are presented. This allows to create structures in the constellation that are maintained during its movement. Finally, an example of application of this new formulation is presented, where the possibilities that this new methodology can provide in the design of satellite constellations are shown.

2 Preliminaries

In this section, a short introduction of the 3D Lattice Theory and the necklace problem is shown, in order to present the base of the problem treated in this paper and as a way to summarize the previous Flower Constellation Theory.

2.1 The 3D Lattice Flower Constellation Theory

The 3D Lattice Flower Constellation Theory is a satellite constellation design methodology in which the satellites are distributed in several inertial orbits, where each satellite has a different value of its mean anomaly and argument of perigee. Furthermore, the satellites of the constellation have the same semimajor axis, eccentricity and inclination. This design allows to generate constellations whose satellites present circular or elliptic orbits. The most important property of this constellation design is that the satellites are distributed generating a symmetric configuration in the lattice that is maintained over time.

As it can be seen in Avendaño et al. (2013), a 3D Lattice Flower Constellation can be described by the use of the Hermite Normal Form. The Hermite Normal Form is composed by six integers, three in the diagonal of the matrix and the other three in the lower triangular part of the matrix. The integers in the diagonal are the number of orbital planes of the constellation (L_Ω), the number of different argument of perigees in each orbital plane (L_ω) and the number of satellites in each orbit (L_M). The other three parameters are the configuration numbers ($L_{M\Omega}, L_{M\omega}, L_{\omega\Omega}$) defined as follows: $L_{M\Omega} \in [0, L_\Omega - 1]$, $L_{M\omega} \in [0, L_w - 1]$ and $L_{\omega\Omega} \in [0, L_\Omega - 1]$.

The expression that summarizes the distribution of the satellites in a 3D Lattice Flower Constellation is:

$$\begin{bmatrix} L_\Omega & 0 & 0 \\ L_{\omega\Omega} & L_\omega & 0 \\ L_{M\Omega} & L_{M\omega} & L_M \end{bmatrix} \begin{pmatrix} \Delta\Omega_{ijk} \\ \Delta\omega_{ijk} \\ \Delta M_{ijk} \end{pmatrix} = 2\pi \begin{pmatrix} i-1 \\ k-1 \\ j-1 \end{pmatrix}, \quad (1)$$

where $\Delta\Omega_{ijk}$ is the distribution in the right ascension of the ascending node of the constellation, $\Delta\omega_{ijk}$ is the distribution of the argument of perigee, and ΔM_{ijk} is the initial distribution of the mean anomaly with respect to a reference satellite of the constellation with orbital elements $\{\Omega_{000}, \omega_{000}, M_{000}\}$. Moreover, the list (i, j, k) represents the position of a satellite in the orbital plane $i \in [1, L_\Omega]$, with the argument of perigee $k \in [1, L_\omega]$ and the mean anomaly $j \in [1, L_M]$. Note also that the values of Ω_{ijk}, ω_{ijk} and M_{ijk} represent three angles, and thus, they are defined in the range $[0, 2\pi]$.

The distribution shown in Eq. (1) can be represented as a set of points that are situated over the surface of a three-dimensional torus in a four-dimensional space (a representation that is non-practical from a graphical point of view). However, the same distribution can also be represented by three different two-dimensional tori in a three-dimensional space.

As an example of that, a constellation with parameters: $L_\Omega = 7$, $L_\omega = 5$, $L_M = 10$, $L_{M\Omega} = 5$, $L_{M\omega} = 4$ and $L_{\omega\Omega} = 6$ is generated. Using Eq. (1), the distribution of the satellites is obtained, where the constellation is made by $L_\Omega L_\omega L_M = 350$ satellites. The tori representation of this constellation is shown in Fig. 1, where each point is represented by two coordinates, a polar longitude (toroidal direction), and the angle between the perpendicular

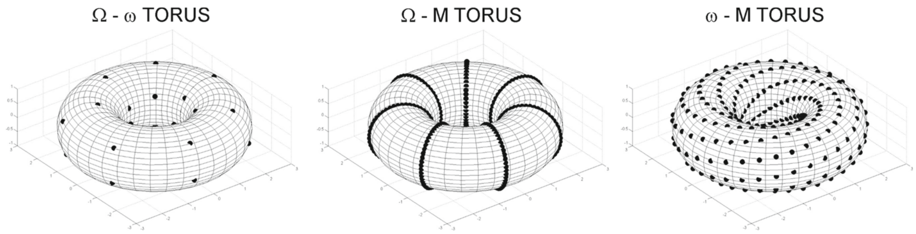

Fig. 1 Tori representation of the constellation distribution

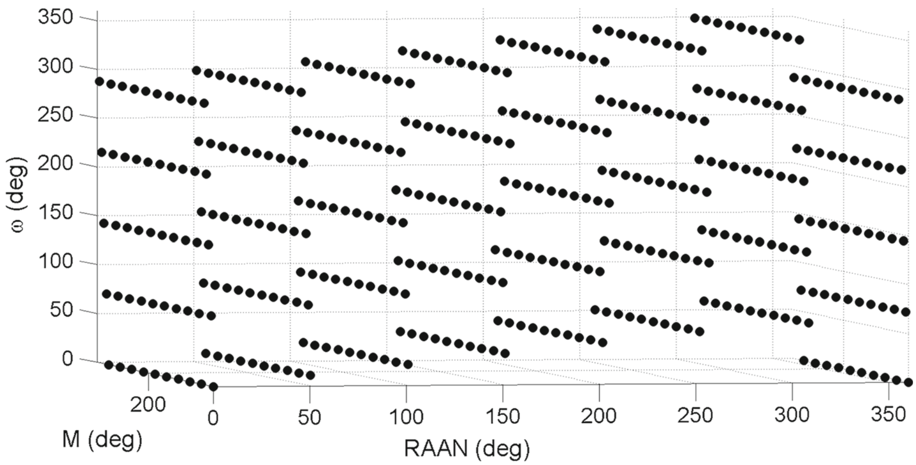

Fig. 2 (Ω, ω, M)-space representation of the constellation

to the torus surface in the point and the horizontal plane (poloidal direction). It is important to note that the figure represents all the satellites of the constellation, and as such, the points only show the different values of each variable in the constellation. That leads to $L_\Omega L_\omega = 35$ different combinations in the first torus, $L_\Omega L_\omega L_M = 350$ in the second and in the third one, since all the configuration numbers $(L_{M\Omega}, L_{M\omega}, L_{\omega\Omega})$ are different from zero, and L_Ω and L_ω are co-primes. On the other hand, the figure clearly shows that the points are situated generating closed lines in the tori, the lattice of the constellation.

Other useful representation is the (Ω, ω, M)-space, which can be observed for this example in Fig. 2. As it can be seen, the satellites are distributed in two sets of parallel planes, one vertical (the orbital planes) and the second inclined in the other axes. This is caused by the configuration numbers ($L_{M\Omega}$, $L_{M\omega}$ and $L_{\omega\Omega}$), which produce this effect in the distribution. As it will be seen later, this property has deep implications in the development of the necklace theory. Note also that the first and the last orbits are the same due to the modular nature of the problem.

On the other hand, it is important to remark the relation between both graphical representations. If a projection is performed over the different axes of Fig. 2, we can observe clearly the number of points that appear in the different tori of Fig. 1, as some points of the distribution will collapse in the same position of the tori during these projections. This provides an additional tool to study how the constellation is distributed in the configuration space.

2.2 Necklace theory

The necklace problem is a combinatorial problem which answers how many different arrangements of n pearls in a circular loop can be produced, assuming that each pearl comes in one of k different colors (Arnas et al. 2017a, d). In the case of study, there are just two colors $k = 2$, representing an empty position or a satellite in the constellation (Casanova et al. 2014a). Thus, we can define a necklace as the subset of points selected from a set of available positions, that is, a necklace $\mathcal{G}$ is a subset of a ring of integers $\mathbb{Z}_n$:

$$\mathcal{G} \subseteq \mathbb{Z}_n = \{1, \ldots, n\}. \tag{2}$$

In this definition, two arrangements are considered to be identical if they only differ by a rotation inside the loop, that is:

$$\mathcal{G}_1 \cong \mathcal{G}_2 \Longleftrightarrow \exists s : \mathcal{G}_1 = \mathcal{G}_2 + s \mod (n), \tag{3}$$

where s is an integer that belongs to the ring $\mathbb{Z}_n$. In addition, another important concept to introduce is the symmetry of a necklace ($Sym(\mathcal{G})$), defined as:

$$Sym(\mathcal{G}) = \min\{1 \le r \le n : \mathcal{G} + r \equiv \mathcal{G} \mod (n)\}, \tag{4}$$

where $Sym(\mathcal{G})$ is the number of times that the configuration must be shifted in order to obtain a configuration identical to the initial.

3 The 3D necklace flower constellations theory

Equation (1) defines the distribution of a 3D Lattice Flower Constellation. This distribution has the particularity of presenting a symmetric configuration in the lattice of the constellation with respect to all its variables, the right ascension of the ascending node, the argument of perigee and the mean anomaly. The objective now is to introduce the concept of necklaces in the formulation, but preserving the symmetries of the initial configuration.

In order to introduce the necklaces, Eq. (1) must be expanded:

$$\begin{aligned}
\Delta\Omega_{ijk} &= \frac{2\pi}{L_\Omega}(i-1), \\
\Delta\omega_{ijk} &= \frac{2\pi}{L_\omega}(k-1) - \frac{2\pi}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}(i-1), \\
\Delta M_{ijk} &= \frac{2\pi}{L_M}(j-1) - \frac{2\pi}{L_M}\frac{L_{M\omega}}{L_\omega}(k-1) \\
&\quad - \frac{2\pi}{L_M}\left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1),
\end{aligned} \tag{5}$$

where this configuration corresponds to a fictitious constellation that is used to define the available positions in which the real satellites of the constellation are located.

From Eq. (5), it can be observed that the value of $\Delta\omega_{ijk}$ is different for $i = 1$ and $i = L_\Omega + 1$, and thus, moving in $i \in [1, L_\Omega + 1]$ does not close the configuration in the torus for a particular value of k. This means that in general $\Delta\omega_{ijk} \neq \Delta\omega_{(i+L_\Omega)jk}$. In the 3D Lattice formulation, this has no effect since all the positions are filled, and consequently, the configuration is complete. However, with the use of necklaces, this effect has to be taken into account in order to generate symmetric configurations. The same consideration has to be made in the expression of the mean anomaly. In that sense, a complete rotation in the right ascension of the ascending node or the argument of perigee does not generate in general the same value on the mean anomaly since $\Delta M_{ijk} \neq \Delta M_{(i+L_\Omega)jk}$ and $\Delta M_{ijk} \neq \Delta M_{ij(k+L_\omega)}$.

Two different necklaces can be defined in a 3D Lattice Flower Constellation, one in the mean anomaly and the other in the argument of perigee. It is possible to generate necklaces in the right ascension of the ascending node with the 3D Lattice Flower Constellation configuration. However, this is equivalent to generate the distribution and keeping just the orbital planes that we are interested in. For this reason, we do not consider this case, since the use of necklaces is not required in these kind of configurations.

Let $\mathcal{G}_M$ be a necklace defined in the mean anomaly with a number of elements equal to $N_M = |\mathcal{G}_M|$ and such that $\mathcal{G}_M \subseteq \mathbb{Z}_{L_M}$. This represents N_M satellites taken from a set of L_M available positions defined in a particular orbit. The necklace in the mean anomaly $\mathcal{G}_M$ is represented as a vector of dimension N_M:

$$\mathcal{G}_M = \left(\mathcal{G}_M(1), \dots, \mathcal{G}_M(j^*), \dots, \mathcal{G}_M(N_M)\right), \tag{6}$$

with

$$1 \leq \mathcal{G}_M(1) \leq \cdots \leq \mathcal{G}_M(j^*) \leq \cdots \leq \mathcal{G}_M(N_M) \leq L_M, \tag{7}$$

and where the index j^* represents an integer modulo N_M, that is, $j^* + N_M$ is the same index as j^*. This allows to define an application (T1) that points to the positions occupied by the necklace from the available positions:

$$\begin{aligned} \text{T1}: \quad \mathbb{Z}_{N_M} &\longrightarrow \mathbb{Z}_{L_M} \\ j^* &\longmapsto \mathcal{G}_M(j^*). \end{aligned} \tag{8}$$

Thus, it makes sense to refer to $\mathcal{G}_M(j^*)$, where the integer parameter $j^* \in \{1, \dots, N_M\}$ represents the position inside the necklace defined. In addition, and for simplicity of notation, we denote $\text{mod}(a, b) = a \mod (b)$. Thus, due to the modular arithmetic inside the necklace:

$$\mathcal{G}_M(j^*) = \mathcal{G}_M(\text{mod}(j^* + N_M, N_M)), \tag{9}$$

which corresponds to a complete loop in the available positions in the mean anomaly. It is important to note that this rotation is equivalent to a movement in the admissible locations defined by:

$$j = j + L_M \mod (L_M), \tag{10}$$

as both represent the same movement of the necklace, one using the parametrization of the necklace and the other using the parametrization of the fictitious constellation.

On the other hand, let $\mathcal{G}_\omega$ be a necklace defined in the argument of perigee with a number of elements equal to $N_\omega = |\mathcal{G}_\omega|$, the number of real orbits per plane and a number of available positions equal to L_ω, which correspond to the size of the space of this variable in the fictitious constellation. This necklace is defined as a vector in the same way as $\mathcal{G}_M$:

$$\mathcal{G}_\omega = \left(\mathcal{G}_\omega(1), \dots, \mathcal{G}_\omega(k^*), \dots, \mathcal{G}_\omega(N_\omega)\right), \tag{11}$$

with

$$1 \leq \mathcal{G}_\omega(1) \leq \cdots \leq \mathcal{G}_\omega(k^*) \leq \cdots \leq \mathcal{G}_M(N_\omega) \leq L_\omega, \tag{12}$$

where the index k^* is an integer modulo N_ω. This allows to define an application (T2) that points to the positions occupied by the necklace from the available positions:

$$\begin{aligned} \text{T2}: \quad \mathbb{Z}_{N_\omega} &\longrightarrow \mathbb{Z}_{L_\omega} \\ k^* &\longmapsto \mathcal{G}_\omega(k^*), \end{aligned} \tag{13}$$

which is used to refer to $\mathcal{G}_\omega(k^*)$, where the integer parameter $k^* \in \{1, \ldots, N_\omega\}$ represents the movement inside the necklace defined. Moreover, the necklace represents a ring of integers; thus, there exists a modular arithmetic inside the necklace:

$$\mathcal{G}_\omega(k^*) = \mathcal{G}_\omega(\text{mod}(k^* + N_\omega, N_\omega)), \tag{14}$$

which is equivalent to a complete loop in the available positions in the argument of perigee:

$$k = k + L_\omega \mod (L_\omega), \tag{15}$$

as both are two formulations for the same movement, one using the parametrization of the necklace and the other using the parametrization of the fictitious constellation.

Now, an application (T3) has to be defined which relates the distribution indexes (i, j^*, k^*) from the necklace, to the indexes of the available positions (i, j, k):

$$\begin{aligned} \text{T3}: \quad \mathbb{Z}_{L_\Omega} \times \mathbb{Z}_{N_M} \times \mathbb{Z}_{N_\omega} &\longrightarrow \mathbb{Z}_{L_\Omega} \times \mathbb{Z}_{L_M} \times \mathbb{Z}_{L_\omega} \\ (i, j^*, k^*) &\longmapsto (i, j, k), \end{aligned} \tag{16}$$

where the effects of the possible movement with respect to the right ascension of the ascending node and the argument of perigee are introduced in the formulation by the use of the three shifting parameters, $S_{\omega\Omega}$ the shifting parameter that relates the argument of perigee to the right ascension of the ascending node, $S_{M\Omega}$ the shifting parameter that relates the mean anomaly to the right ascension of the ascending node, and $S_{M\omega}$ the shifting parameter that relates the mean anomaly to the argument of perigee. That way, the possible movements of the integers k and j are described, respectively, by:

$$\begin{aligned} k &= \mathcal{G}_\omega(k^*) + S_{\omega\Omega}(i-1), \\ j &= \mathcal{G}_M(j^*) + S_{M\omega}(k-1) + S_{M\Omega}(i-1). \end{aligned} \tag{17}$$

We now subtract one unit of each expression to relate to the original formulation provided by Eq. (5), obtaining:

$$\begin{aligned} k-1 &= \mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), \\ j-1 &= \mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1). \end{aligned} \tag{18}$$

Both expressions present modular arithmetic with respect to the symmetries of their necklaces, thus:

$$\begin{aligned} k-1 &= \mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) \mod Sym(\mathcal{G}_\omega), \\ j-1 &= \mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1) \mod Sym(\mathcal{G}_M). \end{aligned} \tag{19}$$

However, j depends on k, and we require a dependency over k^*; consequently, a substitution of k is performed in the second expression, leading to:

$$\begin{aligned} j-1 = \mathcal{G}_M(j^*) - 1 &+ S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &+ S_{M\Omega}(i-1) \mod Sym(\mathcal{G}_M), \end{aligned} \tag{20}$$

where it can be seen that the movement in j depends also on the necklace in the argument of perigee.

Once the distribution over each index is performed, we introduce Eqs. (19) and (20) into Eq. (5), resulting in:

$$\begin{aligned}
\Delta\Omega_{ij^*k^*} &= \frac{2\pi}{L_\Omega}(i-1),\\
\Delta\omega_{ij^*k^*} &= \frac{2\pi}{L_\omega}\left[\mathrm{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right)-\frac{L_{\omega\Omega}}{L_\Omega}(i-1)\right],\\
\Delta M_{ij^*k^*} &= \frac{2\pi}{L_M}\Bigg[\mathrm{mod}\Big(\mathcal{G}_M(j^*)-1+S_{M\omega}\mathrm{mod}\Big(\mathcal{G}_\omega(k^*)-1\\
&\quad+S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\Big)+S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big)\\
&\quad-\frac{L_{M\omega}}{L_\omega}\mathrm{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right)\\
&\quad-\left(\frac{L_{M\Omega}}{L_\Omega}-\frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1)\Bigg],
\end{aligned} \tag{21}$$

which describes the possible movements of the two necklaces defined ($\mathcal{G}_M$ and $\mathcal{G}_\omega$) inside the distribution created in the fictitious constellation.

Equation (21) allows not only to make the distribution of the satellites in the lattice, but also to find all symmetric configurations using the necklace theory. Note that, in the expression for $\Delta M_{ij^*k^*}$, the necklace in the argument of perigee appears, which means that properties in this necklace are affecting the distribution of the constellation in the mean anomaly. This effect is also seen in the conditions for the shifting parameters of the configuration as it will be seen later.

One important thing to notice regarding Eq. (21) is that, since the shifting parameters ($S_{\omega\Omega}$, $S_{M\omega}$, $S_{M\Omega}$) are subjected to a modular arithmetic in the symmetry of the necklaces, duplicities can appear if no boundaries are defined. In that sense, and in order to avoid these duplicities in the formulation, we impose:

$$\begin{aligned}
S_{\omega\Omega} &\in [0, Sym(\mathcal{G}_\omega)-1],\\
S_{M\omega} &\in [0, Sym(\mathcal{G}_M)-1],\\
S_{M\Omega} &\in [0, Sym(\mathcal{G}_M)-1],
\end{aligned} \tag{22}$$

to the shifting parameters. That way, we can assure that all combinations of parameters generate different constellation configurations, while we are still able to create all the different distributions that this formulation can provide.

3.1 Symmetry in the 3D Lattice Flower Constellations

In this section, we impose the conditions of symmetry to the constellation configurations that can be obtained using Eq. (21). That way, a relation between the distribution and the shifting parameters is obtained, which allows to define all the possible symmetric configurations that can be generated inside a given fictitious constellation.

3.1.1 Symmetry with respect to the mean anomaly

The conditions for symmetry in the three variables when a complete rotation in the mean anomaly is performed are:

$$\begin{aligned}
\Delta\Omega_{ij^*k^*} &= \Delta\Omega_{i(j^*+N_M)k^*},\\
\Delta\omega_{ij^*k^*} &= \Delta\omega_{i(j^*+N_M)k^*},\\
\Delta M_{ij^*k^*} &= \Delta M_{i(j^*+N_M)k^*},
\end{aligned} \tag{23}$$

where all expressions are automatically fulfilled as $\Delta\Omega_{ij^*k^*}$ and $\Delta\omega_{ij^*k^*}$ do not depend on the movement of the mean anomaly, while $\Delta M_{ij^*k^*}$ is also achieved due to the modular arithmetic nature of the problem seen in Eq. (9).

3.1.2 Symmetry with respect to the argument of perigee

In order to have symmetry in the argument of perigee, the configuration of the constellation has to fulfill the following conditions:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \Delta\Omega_{ij(k^*+N_\omega)}, \\ \Delta\omega_{ij^*k^*} &= \Delta\omega_{ij^*(k^*+N_\omega)}, \\ \Delta M_{ij^*k^*} &= \Delta M_{ij^*(k^*+N_\omega)}, \end{aligned} \tag{24}$$

where the first equation is always true as it does not depend on the movement in the argument of perigee. On the other hand, the other two equations depend on k^*, and as such, they have to be studied.

Taking the condition in $\Delta\omega_{ij^*k^*}$, and from the equivalences in the definition between Eqs. (14) and (15), we can conclude that the operation $k^* + N_\omega$ is equivalent to a full rotation in the argument of perigee, that is:

$$\Delta\omega_{ij^*k^*} + 2\pi = \Delta\omega_{ij^*(k^*+N_\omega)}, \tag{25}$$

which applied to the expression of the argument of perigee, leads to:

$$\begin{aligned} &\frac{2\pi}{L_\omega}\left[\operatorname{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{L_{\omega\Omega}}{L_\Omega}(i-1)\right] + 2\pi \\ &\quad = \frac{2\pi}{L_\omega}\left[\operatorname{mod}\left(\mathcal{G}_\omega(k^* + N_\omega) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{L_{\omega\Omega}}{L_\Omega}(i-1)\right], \end{aligned} \tag{26}$$

from where a relation between the two modular operators can be established:

$$\begin{aligned} &\operatorname{mod}\left(\mathcal{G}_\omega(k^* + N_\omega) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &\quad - \operatorname{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) = L_\omega, \end{aligned} \tag{27}$$

where this equation will be used later in order to impose the condition of symmetry in the mean anomaly with respect to the argument of perigee.

On the other hand, regarding the condition in $\Delta M_{ij^*k^*}$ from the system of Eqs. (24), and using (21), the following expression can be derived:

$$\begin{aligned} &\frac{2\pi}{L_M}\Bigg[\operatorname{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\operatorname{mod}\big(\mathcal{G}_\omega(k^*) - 1 \\ &\quad + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\big) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) \\ &\quad - \frac{L_{M\omega}}{L_\omega}\operatorname{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &\quad - \left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1)\Bigg] \\ &= \frac{2\pi}{L_M}\Bigg[\operatorname{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\operatorname{mod}\big(\mathcal{G}_\omega(k^* + N_\omega) - 1 \\ &\quad + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\big) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) \end{aligned}$$

$$- \frac{L_{M\omega}}{L_\omega} \mathrm{mod}\left(\mathcal{G}_\omega(k^* + N_\omega) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1)\Bigg], \quad (28)$$

which can be simplified to:

$$\begin{aligned} &\mathrm{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &\quad + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) \\ &\quad - \frac{L_{M\omega}}{L_\omega}\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &= \mathrm{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\mathrm{mod}\left(\mathcal{G}_\omega(k^* + N_\omega) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\ &\quad + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) \\ &\quad - \frac{L_{M\omega}}{L_\omega}\mathrm{mod}\left(\mathcal{G}_\omega(k^* + N_\omega) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right). \end{aligned} \quad (29)$$

Moreover, expanding the modular arithmetic in $Sym(\mathcal{G}_M)$ and using Eq. (27) lead to:

$$A Sym(\mathcal{G}_M) = S_{M\omega}L_\omega - L_{M\omega}, \quad (30)$$

where A is an unknown integer number. This equation can be also represented with the following expression:

$$Sym(\mathcal{G}_M) \mid S_{M\omega}L_\omega - L_{M\omega}, \quad (31)$$

which reads, $Sym(\mathcal{G}_M)$ divides $(S_{M\omega}L_\omega - L_{M\omega})$.

Equation (31) shows the first condition for the shifting parameters of the configuration. As it can be seen, it depends on the symmetry of the necklace and some elements from the Hermite Normal Form. Note that the shifting parameter of the mean anomaly with respect to the argument of perigee ($S_{M\omega}$) depends on the number of fictitious orbits per orbital plane and not the real number, a property that increases the number of possibilities in the configuration.

3.1.3 Symmetry with respect to the right ascension of the ascending node

The conditions of symmetry that we have to impose with respect to the right ascension of the ascending node are the following:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \Delta\Omega_{(i+L_\Omega)j^*k^*}, \\ \Delta\omega_{ij^*k^*} &= \Delta\omega_{(i+L_\Omega)j^*k^*}, \\ \Delta M_{ij^*k^*} &= \Delta M_{(i+L_\Omega)j^*k^*}, \end{aligned} \quad (32)$$

where each one of these conditions is treated separately.

The condition in the right ascension of the ascending node is automatically fulfilled as:

$$\Delta\Omega_{ij^*k^*} = \frac{2\pi}{L_\Omega}(i-1) = \frac{2\pi}{L_\Omega}(i-1) + 2\pi \quad \mathrm{mod}\ (2\pi), \quad (33)$$

which is independent of any of the shifting parameters of the problem.

From the condition in the argument of perigee:

$$\frac{L_\omega}{2\pi}\Delta\omega_{ij^*k^*} = \frac{L_\omega}{2\pi}\Delta\omega_{(i+L_\Omega)j^*k^*}, \quad (34)$$

that can be used to obtain the following expression:

$$\begin{aligned}
&\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{L_{\omega\Omega}}{L_\Omega}(i-1) \\
&= \text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}L_\Omega, Sym(\mathcal{G}_\omega)\right) \\
&- \frac{L_{\omega\Omega}}{L_\Omega}(i-1) - L_{\omega\Omega},
\end{aligned} \tag{35}$$

which can be simplified, leading to:

$$\begin{aligned}
&\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}L_\Omega, Sym(\mathcal{G}_\omega)\right) \\
&- \text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) = L_{\omega\Omega},
\end{aligned} \tag{36}$$

where Eq. (36) is used later to solve the symmetries in the mean anomaly.

Expanding now the modular arithmetic in $Sym(\mathcal{G}_\omega)$ from Eq. (36) and simplifying, we obtain:

$$B\,Sym(\mathcal{G}_\omega) = S_{\omega\Omega}L_\Omega - L_{\omega\Omega}, \tag{37}$$

where B is an unknown integer. This expression is equivalent to:

$$Sym(\mathcal{G}_\omega) \mid S_{\omega\Omega}L_\Omega - L_{\omega\Omega}. \tag{38}$$

Equation (38) shows the second condition for the shifting parameters. As it can be observed, it relates the shifting of the argument of perigee with respect to the right ascension of the ascending node $S_{\omega\Omega}$, to the symmetries of the necklace in the argument of perigee $Sym(\mathcal{G}_\omega)$ and some elements of the Hermite Normal Form (L_Ω and $L_{\omega\Omega}$).

Once the problem of symmetry in the argument of perigee is solved, we impose the condition of symmetry in the mean anomaly by the use of its condition from Eq. (32):

$$\frac{L_M}{2\pi}\Delta M_{ij^*k^*} = \frac{L_M}{2\pi}\Delta M_{(i+L_\Omega)j^*k^*}, \tag{39}$$

from where we can derive:

$$\begin{aligned}
&\text{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\
&+ S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) - \frac{L_{M\omega}}{L_\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) \\
&- \left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1) \\
&= \text{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1)\right. \\
&+ \left. S_{\omega\Omega}L_\Omega, Sym(\mathcal{G}_\omega)\right) + S_{M\Omega}(i-1) + S_{M\Omega}L_\Omega, Sym(\mathcal{G}_M)\Big) \\
&- \frac{L_{M\omega}}{L_\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}L_\Omega, Sym(\mathcal{G}_\omega)\right) \\
&- \left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1) - \left(L_{M\Omega}\frac{L_{M\omega}L_{\omega\Omega}}{L_\omega}\right),
\end{aligned} \tag{40}$$

which, using Eq. (36), can be simplified to:

$$\begin{aligned}
&\text{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1)\right. \\
&+ \left. S_{\omega\Omega}L_\Omega, Sym(\mathcal{G}_\omega)\right) + S_{M\Omega}(i-1) + S_{M\Omega}L_\Omega, Sym(\mathcal{G}_M)\Big)
\end{aligned}$$

$$- \text{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right)$$
$$+ S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big) = L_{M\Omega}. \tag{41}$$

Now, we expand the modular arithmetic in $Sym(\mathcal{G}_\omega)$ and apply again the relation from Eq. (36) in order to obtain:

$$C Sym(\mathcal{G}_M) = S_{M\Omega}L_\Omega - (L_{M\Omega} - S_{M\omega}L_{\omega\Omega}), \tag{42}$$

where C is an unknown integer. The former expression can also be written as:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega}L_\Omega - (L_{M\Omega} - S_{M\omega}L_{\omega\Omega}). \tag{43}$$

Equation (43) shows the third condition for the shifting parameters. As we can see, this relation has a particularity, $S_{M\Omega}$ depends also on other shifting parameter, $S_{M\omega}$, which generates a logical order in the generation of the shifting parameters.

3.1.4 Symmetric configurations

In this subsection, the formulation of the theory is summarized in order to present all the methodology in a more compact and clear way. All possible distributions of a particular set of necklaces can be described by these expressions:

$$\begin{aligned}
\Delta\Omega_{ij^*k^*} &= \frac{2\pi}{L_\Omega}(i-1),\\
\Delta\omega_{ij^*k^*} &= \frac{2\pi}{L_\omega}\left[\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{L_{\omega\Omega}}{L_\Omega}(i-1)\right],\\
\Delta M_{ij^*k^*} &= \frac{2\pi}{L_M}\Bigg[\text{mod}\Big(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\big(\mathcal{G}_\omega(k^*) - 1\\
&\quad + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\big) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\Big)\\
&\quad - \frac{L_{M\omega}}{L_\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right)\\
&\quad - \left(\frac{L_{M\Omega}}{L_\Omega} - \frac{L_{M\omega}}{L_\omega}\frac{L_{\omega\Omega}}{L_\Omega}\right)(i-1)\Bigg],
\end{aligned} \tag{44}$$

where the values of the shifting parameters $S_{\omega\Omega}$, $S_{M\omega}$ and $S_{M\omega}$ have to fulfill the following relations in order to obtain symmetric configurations:

$$\begin{aligned}
Sym(\mathcal{G}_\omega) &\mid S_{\omega\Omega}L_\Omega - L_{\omega\Omega},\\
Sym(\mathcal{G}_M) &\mid S_{M\omega}L_\omega - L_{M\omega},\\
Sym(\mathcal{G}_M) &\mid S_{M\Omega}L_\Omega - (L_{M\Omega} - S_{M\omega}L_{\omega\Omega}).
\end{aligned} \tag{45}$$

As it can be seen, the set of Eqs. (44) and (45) leads to the 3D Lattice Flower Constellations distributions if no necklace is defined and to the 2D Lattice Flower Constellations (Davis et al. 2013) if additionally no distribution is performed in the argument of perigee. Regarding the 2D Lattice Flower Constellations using necklaces (Casanova et al. 2014a), the shifting parameter in the mean anomaly was defined as:

$$Sym(\mathcal{G}) \mid S_{M\Omega}L_\Omega - N_c, \tag{46}$$

where $\mathcal{G}$ is a necklace in the mean anomaly and N_c is the configuration number for the 2D Lattice Flower Constellations which corresponds to the $L_{M\Omega}$ parameter in the 3D Lattice Flower Constellations. This relation is equivalent to the last condition in Eq. (45) when the argument of perigee is not a variable of the configuration; thus, the 3D Necklace Flower Constellations also includes the 2D Lattice Flower Constellations using necklaces.

Therefore, Eqs. (44) and (45) constitute the generalization of the necklace theory for the 3D Lattice Flower Constellations, which include all the former Lattice Flower Constellations: 2D Lattice Flower Constellations, 2D Lattice Flower Constellations using necklaces, 3D Lattice Flower Constellations and now 3D Lattice Flower Constellations using necklaces.

In the next section, a detailed example is presented in order to show, in a clear manner, the methodology to generate 3D Necklace Flower Constellations.

3.2 Example of application

For this example, we assume that a constellation made of 42 satellites is chosen. Let suppose that the constellation is required to be built in 7 orbital planes; thus, $L_\Omega = 7$, and each plane contains two orbits, that is, the number of real orbits per plane is $N_\omega = 2$. Moreover, the number of real satellites per orbit is $N_M = 3$.

Now, an expansion of the search space is done, choosing a fictitious constellation with parameters $L_\omega = 6$ and $L_M = 9$. This means that we are generating two different necklaces, one in the argument of perigee and the other in the mean anomaly. Moreover, as it can be seen, the available positions both in mean anomaly and in the argument of perigee have been trebled, being just the ninth part of all available real positions of satellites in the constellation.

Applying the 3D Necklace Flower Constellations to these parameters, we obtain $|\mathcal{G}_M| = 10$ different necklaces in the mean anomaly and $|\mathcal{G}_\omega| = 3$ in the argument of perigee (Arnas et al. 2017a; Cattell et al. 2000; Sawada 2003), generating a total of $|\mathcal{G}_M||\mathcal{G}_\omega|L_\Omega^2 L_\omega = 8820$ different symmetrical configurations (compared to the $L_\Omega^2 N_\omega = 98$ configurations obtained using just the 3D Lattice Flower Constellations theory due to the boundaries in the configuration numbers). Note that the number of configurations using necklaces can be increased even further by expanding the fictitious constellation or generating other fictitious constellations.

As there are too many configurations to analyze, we choose, without losing generality, $L_{M\Omega} = 4$, $L_{M\omega} = 3$ and $L_{\omega\Omega} = 6$ as combination numbers of the constellation, and $G_M = \{1, 4, 7\}$ and $G_\omega = \{1, 4\}$ as the necklaces in the mean anomaly and the argument of perigee, respectively. Applying the definition of symmetry of a necklace from Eq. (4), these results are obtained: $Sym(\mathcal{G}_M) = 3$ and $Sym(\mathcal{G}_\omega) = 3$.

With these parameters, we can use Eq. (38) to obtain the shifting of the argument of perigee with respect to the right ascension of the ascending node:

$$Sym(\mathcal{G}_\omega) \mid S_{\omega\Omega} L_\Omega - L_{\omega\Omega} \quad \Rightarrow \quad 3 \mid 7S_{\omega\Omega} - 6, \tag{47}$$

which leads to $S_{\omega\Omega} = 0$. On the other hand, the shifting parameter of the mean anomaly with respect to the argument of perigee can be computed using Eq. (31):

$$Sym(\mathcal{G}_M) \mid S_{M\omega} L_\omega - L_{M\omega} \quad \Rightarrow \quad 3 \mid 6S_{M\omega} - 3, \tag{48}$$

which has three solutions, $S_{M\omega} = 0, 1, 2$. Now, with this result, we apply Eq. (43) to obtain the shifting parameter of the mean anomaly with respect to the right ascension of the ascending node:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega} L_\Omega - (L_{M\Omega} - S_{M\omega} L_{\omega\Omega}) \quad \Rightarrow \quad 3 \mid 7S_{M\Omega} - (4 - 6S_{M\omega}), \tag{49}$$

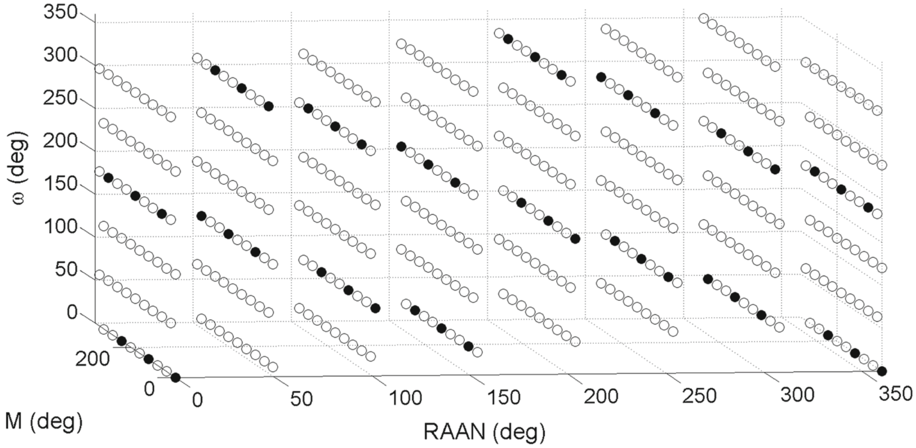

Fig. 3 (Ω, ω, M)-space representation of the constellation distribution

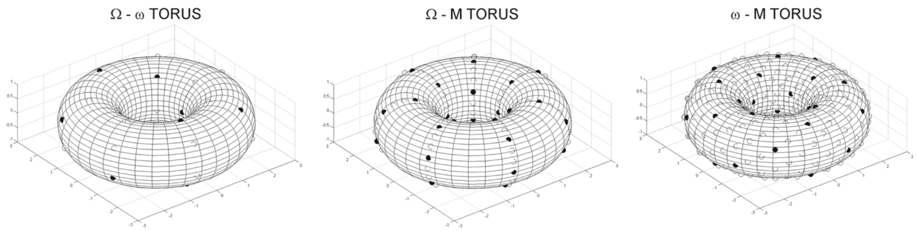

Fig. 4 Tori representation of the constellation distribution

which is $S_{M\Omega} = 1$ no matter the value of $S_{M\omega} = 0, 1, 2$ used. Note that in other examples, different values of $S_{M\omega}$ require different $S_{M\Omega}$.

As it can be seen, three configurations can be generated due to the multiple solutions of $S_{M\omega}$. In particular, we choose $S_{\omega\Omega} = 0$, $S_{M\omega} = 2$ and $S_{M\Omega} = 1$ as the selected configuration. The lattice obtained from this configuration is shown in Fig. 3 where the (Ω, ω, M)-space of the distribution selected is shown. The circles represent available positions, while the colored ones are the real satellites of the configuration.

Moreover, it is interesting to study the representation of this lattice using tori. This is shown in Fig. 4 where the three tori that define the distribution are shown. As shown in Figs. 3 and 4, the distribution is symmetrical in all three orbital parameters: the right ascension of the ascending node, the argument of perigee and the mean anomaly.

Now, this configuration is applied to a satellite constellation. Without losing generality, we choose an eccentricity of $e = 0.3$, an inclination equal to the critical inclination $i = 63.43°$ and a semimajor axis equal to $a = 12,770\,\text{km}$. With these orbital parameters, an inertial configuration as shown in Fig. 5 is obtained.

This constellation is just an example of the possibilities that the application of necklaces into the 3D Lattice Flower Constellations theory can bring. As it has been said, the number of possibilities can be increased indefinitely, being the only constraint the computational power available.

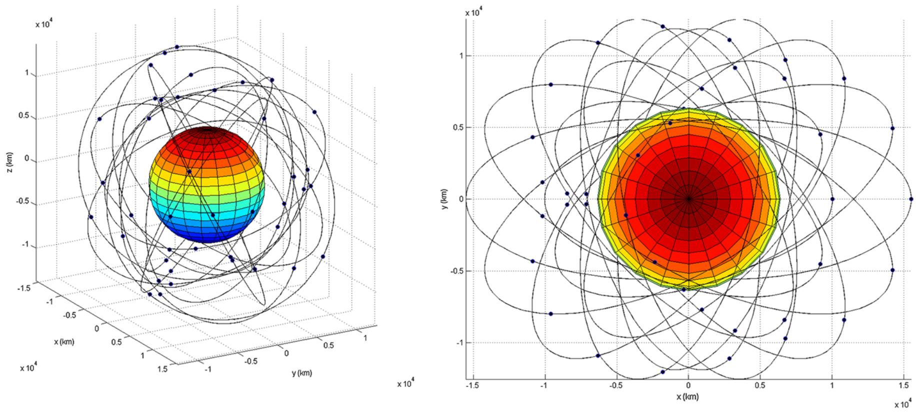

Fig. 5 Inertial orbits of the constellation

4 Conclusions

3D Lattice Flower Constellations is a powerful tool that allows the generation of constellations with symmetric configurations and minimum parametrization. The distribution obtained with this methodology is fixed to certain positions which is a constraint in the number of possible configurations that the theory can generate.

This paper introduces the concept of necklaces in the formulation of 3D Lattice Flower Constellations, increasing the number of possible symmetric configurations, being the only limitation the computational power available. This is achieved by an expansion of the searching space of the constellation and applying the necklace to fit the configuration again to the one sought. Moreover, all the configurations obtained by this methodology maintain the properties of the former Flower Constellations, presenting symmetry in the lattice of the right ascension of the ascending node, the argument of perigee and the mean anomaly of all the satellites in the constellation.

In addition, this new design framework can be used to introduce non-uniformities in the distribution while maintaining a structure in the configuration. This is done by defining necklaces adapted to the mission requirements, which provides a powerful tool during the initial constellation design process. Other applications of this methodology include the study of constellation reconfiguration problems, the assessment of satellite failure in a distribution, or the definition of the launching schedule for a constellation made of a large number of satellites.

Furthermore, the 3D Necklace Flower Constellations includes all the former Lattice Flower Constellation designs, being as such, a generalization of the Lattice Flower Constellation theory. This means that the 3D Necklace Flower Constellation theory is able to generate all former configurations (2D, 3D Lattice Flower Constellations and 2D Lattice Flower Constellations using necklaces) and create new distributions using the necklace theory.

Finally, it is important to note that the expansion of the search space can be increased as much as desired, providing more possibilities of design as the size of the fictitious constellation becomes larger. Moreover, this expansion can also be done in an n-dimensional Lattice instead of just a 2D or a 3D Lattice. This further generalization will be treated in future works.

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