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ORIGINAL ARTICLE

No hanging out in neighborhoods of infinity in the three-body problem

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Abstract Consider the spatial Newtonian three-body problem at fixed negative energy and fixed angular momentum. The moment of inertia I provides a measure of the overall size of a three-body system. We will prove that there is a positive number I_0 depending on the energy and angular momentum levels as well as the masses such that every solution at these levels passes through $I \leq I_0$ at some instant of time. Motivation for this result comes from trying to prove the impossibility of realizing a certain syzygy sequence in the zero angular momentum problem.

Keywords Three-body problem · Lunar problem · Syzygy sequences · Perturbation methods

1 Introduction

The spatial three-body problem concerns three-point masses in space moving according to Newton's equations of gravitation. The point of this article is to prove that there exist no periodic solutions to this problem which "hang out near infinity."

The conserved quantities for the problem are the energy H, angular momentum J and linear momentum. As is standard, we may, without loss of generality, assume that the linear momentum is zero and the origin of space coincides with the center of mass of the three bodies. If m_i denotes the masses and $q_i \in \mathbb{R}^3$ the positions of the bodies, then the standard measure of size is $\|q\| = \sqrt{I(q)}$ where $q = (q_1, q_2, q_3)$ and $I = \sum m_i |q_i|^2$ is known as the total moment of inertia. Neighborhoods of infinity are regions of the form $\{q : I(q) \geq I_0\}$. As $I_0 \to \infty$ the neighborhood converges to infinity. Our main theorem is:

Theorem 1 *For $H < 0$ there exists $I_0(m_j, H, J) > 0$ such that any orbit at these energy and momentum levels beginning in the region $I > I_0$ enters the region $I \leq I_0$ in forward or backward time.*

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MOTIVATION The motivation behind our result came from the problem of which syzygy sequences are realized in the *zero angular momentum* planar three-body problem (see Moeckel and Montgomery 2015; Montgomery 2002, 2007). The term syzygy is from astronomy and refers to when the three bodies are in eclipse, that is collinear. Each syzygy has a 'type' 1, 2 or 3, according to the label of the mass in the middle. Then the syzygy sequence of an orbit is this list of syzygy types in temporal order. A first open problem is whether or not the periodic sequence of repeating 1212s is realized by a periodic solution to the zero angular momentum problem. One imagines such a motion as consisting of masses 1 and 2 going around each other in a near circular orbit, very far from mass 3, and the center of mass of the m_1 and m_2 orbit slowly going around mass 3, like the Earth–Moon–Sun system. The action over such solutions decreases as the distance of the Earth–Moon system to the Sun goes to infinity, i.e., minimizing the action forces the solution to slide off into a neighborhood of infinity, see Chenciner and Montgomery (2000). The theorem excludes the existence of such solutions "near infinity," i.e., in the region $I \geq I_0(m_j, H, 0)$.

REMARK In the theorem we may either exclude orbits having a binary collision singularity or pass through them using Levi–Civita regularization. Can we prove an analogous result to Theorem 1 for $N \geq 4$? The proof here breaks down in Proposition 1 where the neighborhoods of infinity fail to split into connected components characterized by a far body with suitable Jacobi coordinates. The connectedness of the neighborhoods of infinity due to these spread-out clusters of tight binaries is utilized for Jeff Xia's orbits realizing infinity in finite time singularities where $N \geq 5$. Can these infinities in finite time orbits provide counterexamples to Theorem 1 for $N \geq 5$?

REMARK In Meyer (1994) comet-like periodic orbits for the N-body problem are established in a region $I \geq I_C$ for I_C large. These orbits do not contradict our theorem because as $I_C \to \infty$ their orbits angular momentum $|J| \to \infty$.

2 Related results

The behavior of $I(t)$ has long been studied to gain some qualitative understanding of the N-body problem. Sundman (1912) showed for the three-body problem that nonzero angular momentum implies no orbits suffer triple collision, i.e., $I > 0$ for all orbits. Namely, there exists a positive lower bound, $I_S(m_i, H, J, I(0), \dot{I}(0))$, for orbits at such levels. That is $I(t) > I_S > 0$ over the solutions with energy H and angular momentum $J \neq 0$ and with initial conditions at $I(0), \dot{I}(0)$. Hadamard (1915), p. 259, gave an explicit formula for such an I_S, and Birkhoff (1927), Ch. IX §8, studied escape conditions in the nonzero angular momentum case by showing for example (p. 282) that I sufficiently small (near zero) at some instant, t_0, implies I becomes infinite as t goes to infinity. One might paraphrase Birkhoff's result as 'no hanging out in neighborhoods of triple collision.'

A great deal of analysis on I has followed these two tracks around small I values. See, for example, Laskar and Marchal (1984), Marchal and Yoshida (1984) on the greatest lower bound of I for bounded orbits and Marchal (1974), Marchal and Yoshida (1984), Pollard (1970) for efficient tests of escape in a variety of cases. The book Marchal (1990), Ch. 11, is a detailed reference for the qualitative study of I.

For each orbit let I_m be the minimum value of I over this orbit, sharpening Sundman leads one to seek a (greatest) lower bound of I_m over classes of orbits. An analogous question here is instead to seek a (least) upper bound of I_m over all orbits.

While most focus in the literature so far appears on the greatest lower bound and escape, this upper bound question has not entirely escaped notice. A statement similar to Theorem 1
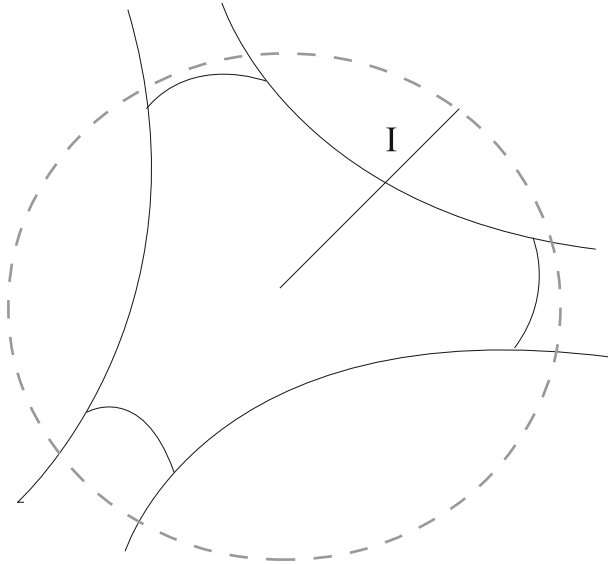
Fig. 1 For the planar three-body problem the shape space is $\mathbb{R}^3$ where I is the distance from the origin. The admissible configurations at fixed $H < 0$ are interior to a pair of pants where each leg of the pants is asymptotic to a cylinder around a binary collision ray. See Moeckel (1988) or Montgomery (2015) for details

appears in Marchal (1990), p. 468, where an upper bound is given in a remark about a class of equal mass cases (those with $H|J|^2 = -\frac{4}{3^5}$) and the least upper bound is conjectured to be attained over the Broucke–Henon orbit (Marchal 1990, p. 469). Here we give a new motivation to this question as to the existence of the 1212... solution in the *zero* angular momentum case and use a different method than that of Marchal (1990). Additionally, we observe that both methods give upper bounds in a general case rather than just treating an equal mass case (see also Marchal 1990, p. 483). Moreover, the method we use here offers hope of lowering the upper bound if the perturbation step (Propositions 3, 4, 5) is dealt with more effectively. In Appendix we give some comparison of the two methods.

3 Structure of proof

For $H < 0$ as we let I_0 increase, eventually the domain $\{I \geq I_0\}$ splits into three components each component characterized by the selection of one of the three masses. The two remaining masses stay close to each other, while this third selected mass stays relatively far away from either member of this pair (see Fig. 1). We fix attention on one of these regions, supposing, after relabeling, that the close masses are m_1 and m_2. In this region, we use the standard Jacobi coordinates ξ_1, ξ_2. See Fig. 2.

When written in these coordinates, Newton's differential equations become a perturbation of two uncoupled Kepler problems, one for each Jacobi vector, with the perturbation term getting arbitrarily small as $I_0 \to \infty$. We focus attention on the long Jacobi vector, which connects the center of mass of the m_1 and m_2 system to the third mass. When we drop the perturbation term of this perturbed Kepler system, we get an exact solvable Kepler problem whose solutions we call "the osculating solutions."

The Kepler parameters (energy, angular momentum, Laplace or Runge–Lenz vector) for the osculating system can be *bounded* using that H, J, the masses, are fixed and the fact
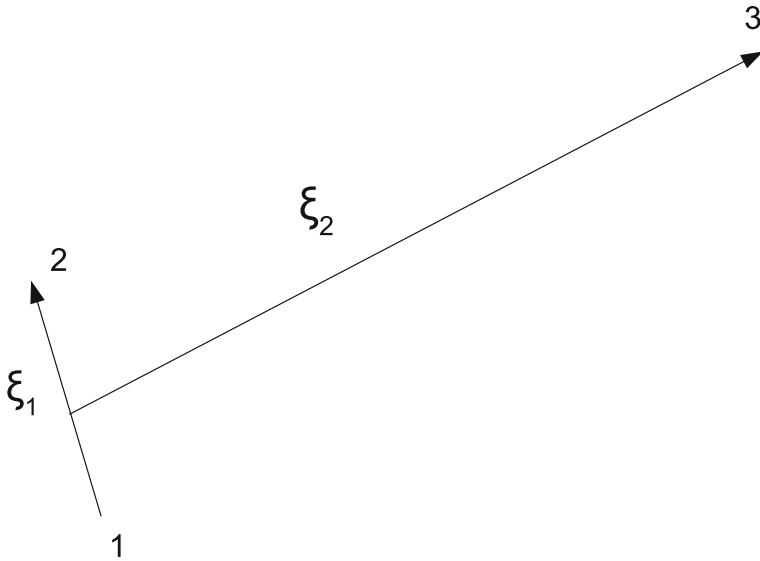
Fig. 2 A tight binary configuration, set $r := |\xi_1|$, $\rho := |\xi_2|$

that $I_0 \gg 0$. Now here comes the key observation, due to Chenciner. Consider a family of solutions to Kepler equation having fixed energy and bounded angular momentum. If, along the solutions of this family, the initial distance from the origin tends to infinity then these orbits become extremely eccentric and thus must come close to the origin. Thus, the osculating orbits cannot "hang out near infinity." Said slightly differently, since large circular orbits for the Kepler problem have large angular momentum and since our total angular momentum is fixed, large near circular motions for osculating system are excluded and this excludes orbits of the type of our Earth–Moon–Sun cartoon described above.

Here is the strategy of proof then. Show that for sufficiently large I_0 all of the osculating solutions starting in $\{I \geq I_0\}$ are extremely eccentric, enough so to enter the region $\{I \leq I_0\}$ (see Proposition 2). Next show that the real solutions do not vary too much from these osculating solutions, as long as they stay in the region $I \geq I_0$, and for bounded times (indeed for times of order $O(I_0^{3/2})$, Proposition 3). It follows that if the osculating orbit enters the region $I \leq I_0$ within the time $O(I_0^{3/2})$ (which we expect by Kepler third law), then the true orbit must also enter into that region. Finally, (Proposition 5) we verify that there is indeed sufficient time: the timescale over which the approximation of the true motion by the osculating motion is valid is long enough that the true motions must follow their osculating leads into a region $I \leq I_0$.

4 Setup and notation

In the spatial three-body problem, we consider the motion of three-point masses m_1, m_2, m_3 under Newton's gravitational attraction. We will denote the configurations by

$$q = (q_1, q_2, q_3) \in (\mathbb{R}^3)^3 \setminus \{(x_1, x_2, x_3) : x_i = x_j \text{ some } i \neq j\}.$$

As is standard, we may take the center of mass zero coordinates ($\sum m_i q_i = 0$) and will now define the Jacobi coordinates in which the splitting into two perturbed Kepler problems will be clear (see Fig. 2 as well as Pollard 1966 2.7; Féjoz 2002, or Kaplan et al. 2008):

$$\xi_1 = q_2 - q_1,$$
$$\xi_2 = q_3 - (m_1 + m_2)^{-1}(m_1 q_1 + m_2 q_2) = \frac{m_1 + m_2 + m_3}{m_1 + m_2} q_3.$$

We set

$$r = |\xi_1| \text{ and } \rho = |\xi_2|.$$

For reference we record here in one place the mass constants that will be used throughout:
MASS CONSTANTS:

$$\mu = m_1 + m_2$$
$$M = m_1 + m_2 + m_3$$
$$\alpha_1 = m_1 m_2 \mu^{-1}$$
$$\alpha_2 = m_3 \mu M^{-1}$$
$$\beta_1 = \mu \alpha_1$$
$$\beta_2 = M \alpha_2$$

Then in these coordinates we find:

$$I := \sum m_i |q_i|^2 = \alpha_1 r^2 + \alpha_2 \rho^2 \tag{1}$$

$$J := \sum m_i (q_i \times \dot{q}_i) = \alpha_1 \xi_1 \times \dot{\xi}_1 + \alpha_2 \xi_2 \times \dot{\xi}_2 = J_1 + J_2 \tag{2}$$

for the moment of inertia and angular momentum, respectively. Also the energy splits into

$$H = H_{kep} + g,$$

where

$$H_{kep} = \frac{1}{2}\alpha_1 |\dot{\xi}_1|^2 - \frac{\beta_1}{r} + \frac{1}{2}\alpha_2 |\dot{\xi}_2|^2 - \frac{\beta_2}{\rho} = H_1 + H_2$$

is an energy for two uncoupled Kepler problems and

$$g = \frac{\beta_2}{\rho} - \frac{m_1 m_3}{|\xi_2 + m_2 \mu^{-1} \xi_1|} - \frac{m_2 m_3}{|\xi_2 - m_1 \mu^{-1} \xi_1|}$$

is a perturbation term with $g = O(r^2/\rho^3)$, $g_{\xi_1} = O(r/\rho^3)$ and $g_{\xi_2} = O(r^2/\rho^4)$.
The equations of motion are then the two perturbed Kepler problems

$$\alpha_i \ddot{\xi}_i = -\frac{\beta_i \xi_i}{|\xi_i|^3} - g_{\xi_i}. \tag{3}$$

Definition 1 A solution to the unperturbed Kepler problems satisfying the same initial conditions as a solution to these perturbed Kepler problems (Eq. 3) will be called an *osculating orbit* (see Pollard 1966, 1.16).

5 Proof of main theorem

Fix the masses, angular momentum, negative energy $H < 0$, linear momentum zero and a parameter $\lambda > 0$ and only consider orbits at these energy and momentum levels in appropriate Jacobi coordinates. We will use $\overline{I}$ for a placeholder constant.

Proposition 1 *For $H < 0$, there exists $I^*(m_i, H, J) > 0$ such that the region $I > I^*$ consists of three connected components B_1, B_2, B_3. Moreover, relabeling if necessary to fix our attention to B_3 (where q_3 is the far body) with appropriate Jacobi coordinates we have the bounds:*

$$|g| \leq c_g (r^2/\rho^3), \quad |g_{\xi_2}| \leq c_{g_2} (r^2/\rho^4) \tag{4}$$
$$|J_2| \leq \alpha_2 c_{J_2} \tag{5}$$
$$r \leq c_r \tag{6}$$

on the perturbation term g, angular momentum J_2 and short Jacobi vector r throughout B_3 for some constants $c_g, c_{g_2}, c_{J_2}, c_r$ depending on masses, energy and angular momentum.

See Moeckel (1988), Féjoz (2002), Kaplan et al. (2008), Marchal (1990) regarding these well-known *lunar regions*.

Proposition 2 *Take $I^{**} = \max\{I^*, \alpha_1 c_r^2 + \alpha_2 c_{J_2}^4/M^2\}$ where I^*, c_r, c_{J_2} are from Proposition 1. Then any osculating orbit with initial condition in $I > I^{**}$ falls in forward or backward time into the region $I \leq I^{**}$. Moreover, the time to fall into the region $I \leq I^{**}$ is less than or equal to the time to reach pericenter.*

Proof By Eqs. (1, 6) in the region $I > I^{**}$ we have $\rho^2 > c_{J_2}^4/M^2$.
The 'ρ' component of the osculating orbit of an initial condition in $I > I^{**}$ is a solution to the Kepler problem

$$\ddot{\xi}_{osc} = -M\xi_{osc}/|\xi_{osc}|^3$$

with $\rho_{osc}^2(0) = |\xi_{osc}(0)|^2 > c_{J_2}^4/M^2$ and the restriction from Eq. (5)

$$|\xi_{osc} \times \dot{\xi}_{osc}| = \alpha_2^{-1}|J_2(0)| \leq c_{J_2}$$

on the angular momentum. Also from Proposition 1, we have the r component satisfying $r \leq c_r$ as long as we remain in the region $I > I^*$.

We now verify that for all such orbits, ξ_{osc}, the pericenter distance, ρ_{osc}^{pc} is bounded.

Case 1 $J_2 \neq 0$.
In polar coordinates, any non-collision osculating orbit is (for some $e \geq 0$):

$$\rho_{osc} = \frac{\alpha_2^{-2}|J_2(0)|^2}{M(1 + e\cos\theta)},$$

where $\theta = 0$ corresponds to the pericenter.

Then as $e \geq 0$ and by Eq. (5),

$$\rho_{osc}^{pc} = \frac{\alpha_2^{-2}|J_2(0)|^2}{M(1 + e)} \leq \frac{c_{J_2}^2}{M}.$$

Case 2 $J_2 = 0$.
Collision! So the pericenter distance in this case is zero.

Now an osculating orbit starting in $I > I^{**}$ either reaches pericenter or leaves I^* before it reaches pericenter. If it reaches pericenter before leaving $I > I^*$ then we have $I_{pc} \leq \alpha_1 c_r^2 + \alpha_2 c_{J_2}^4/M^2 \leq I^{**}$, so in either case we fall into the region $I \leq \max\{I^*, \alpha_1 c_r^2 + \alpha_2 c_{J_2}^4/M^2\} = I^{**}$ in forward or backward time which is no more than t_{pc}, the time to pericenter.   □

Proposition 3 *Let* $\overline{I} \geq \max\{I^*, \alpha_1 c_r^2 + \max\{1, (\frac{3c_{J_2}^2}{2M})^2\}\alpha_2\} = \overline{R}$. *Set* $\overline{\rho} = \sqrt{\alpha_2^{-1}(\overline{I} - \alpha_1 c_r^2)}$
and $\varepsilon = 1/\overline{\rho}$. *Then any orbit with initial condition in* $I \geq \overline{I}$ *satisfies:*

$$|\rho(t) - \rho_{osc}(t)| < A_1 \varepsilon \tag{7}$$

for time

$$|t| \leq B_1 \varepsilon^{-3/2} \tag{8}$$

throughout the region $I \geq \overline{I}$.

Here we may pick the constant $B_1 > 0$ *and then define* $A_1 = \frac{a}{M}(2 + e^{\sqrt{2M+3c_{J_2}^2}B_1})$ *where*
$a = \alpha_2^{-1}((c_{g_2}c_r^2 B_1)^2 + 2c_{J_2}c_{g_2}c_r^2 B_1 + c_{g_2}c_r^2)$.

Proof First, from Eq. (6) any configuration with $I \geq \overline{I}$ has $\rho \geq \overline{\rho} \geq \max\{1, \frac{3c_{J_2}^2}{2M}\} \geq \max\{1, \frac{3\alpha_2^{-2}|J_2|^2}{2M}\}$, in particular our initial condition.

We consider our perturbed Kepler problem for the 'ρ' motion:

$$\ddot{\xi}_2 = -\frac{M\xi_2}{\rho^3} + F(\xi_2, t),$$

where the time dependence in the perturbation term $F = -\alpha_2^{-1}g_{\xi_2}$ is due to the interaction of the motion of masses 1 and 2.
In the region $I \geq \overline{I}$, we have $|F| \leq \alpha_2^{-1}c_{g_2}c_r^2\rho^{-4} \leq \alpha_2^{-1}c_{g_2}c_r^2\varepsilon^4$. We will set

$$A = \alpha_2^{-1}c_{g_2}c_r^2.$$

An estimate for the variation of $c_t^2 := |\xi_2 \times \dot{\xi}_2|^2 = \alpha_2^{-2}|J_2(t)|^2$ will be needed. Since $|\dot{c}| \leq |\alpha_2^{-1}\dot{J}_2| = |\xi_2 \times F| \leq A\rho^{-3}$, we have

$$|\dot{c}| \leq A\varepsilon^3,$$

so that

$$|c_t - c_0| \leq A\varepsilon^3|t|.$$

Hence,

$$|c_t^2 - c_0^2| \leq A\varepsilon^3|t|(A\varepsilon^3|t| + 2c_0) \leq A\varepsilon^3|t|(A\varepsilon^3|t| + 2c_{J_2}),$$

so that for $|t| \leq B_1\varepsilon^{-3/2}$ and $I \geq \overline{I}$ with $b = (AB_1)^2 + 2c_{J_2}AB_1$ we have

$$|c_t^2 - c_0^2| \leq b\varepsilon^{3/2}, \tag{9}$$

provided $\varepsilon \leq 1$ which is guaranteed so long as $\overline{I} \geq \alpha_1 c_r^2 + \alpha_2$ as is indeed the case since $\overline{I} \geq \overline{R}$.

To prove the proposition we will use the Sandwich Lemma (see Montgomery 2007, p. 1942). Note that in Montgomery (2007) there is an unneeded assumption requiring that $F_+ < 0$):

Sandwich Lemma: Given $\ddot{x}_- = F_-(x_-)$, $\ddot{x} = F(x, t)$ and $\ddot{x}_+ = F_+(x_+)$ satisfying $F_-(x) \leq F(x, t) \leq F_+(x)$ and $\frac{\partial F_\pm}{\partial x_\pm} \geq 0$ over some time interval, then over this same time interval the solutions to $F_\pm, F$ satisfying the same initial conditions have:

$$x_-(t) \leq x(t) \leq x_+(t).$$

Now:

$$\rho_{osc}\ddot{\rho}_{osc} + \dot{\rho}_{osc}^2 = \frac{d}{dt}\rho_{osc}\dot{\rho}_{osc} = \frac{d}{dt}\xi_{osc}\cdot\dot{\xi}_{osc} = -M\rho_{osc}^2\rho_{osc}^{-3} + |\dot{\xi}_{osc}|^2 = -M\rho_{osc}^{-1}$$
$$+ \dot{\rho}_{osc}^2 + c_0^2\rho_{osc}^{-2},$$

so

$$\ddot{\rho}_{osc} = c_0^2\rho_{osc}^{-3} - M\rho_{osc}^{-2}.$$

And likewise:

$$\ddot{\rho} = c_t^2\rho^{-3} - M\rho^{-2} + f(t), \tag{10}$$

where $|f(t)| = |\rho(t)^{-1}(\xi_2(t)\cdot F(\xi_2(t), t))| \leq A\rho(t)^{-4}$.

Take $v_1(\rho) = c_0^2\rho^{-3} - M\rho^{-2}$ and $v_2(\rho, t) = c_t^2\rho^{-3} - M\rho^{-2} + f$. We view f here as $f(t)$ by plugging the true solutions $\xi_1(t), \xi_2(t)$ into F, ρ.

Now using our $|c_t^2 - c_0^2|$ estimate Eq. (9) and our bound on f we get:

$$|v_1 - v_2| \leq b\varepsilon^{9/2} + A\varepsilon^4 \leq a\varepsilon^4$$

for $a = b + A$, or

$$v_1 - a\varepsilon^4 \leq v_2 \leq v_1 + a\varepsilon^4$$

for time $|t| \leq B_1\varepsilon^{-3/2}$ and $I \geq \overline{I}$.

Now ρ is a solution to $\ddot{\rho} = v_2$ and let $\rho_\pm$ be solutions to:

$$\ddot{\rho}_\pm = v_1(\rho_\pm) \pm a\varepsilon^4 =: F_\pm(\rho_\pm)$$

satisfying the same initial conditions as ρ. Throughout the region $I \geq \overline{I}$ we have $\rho_\pm \geq \frac{3c_0^2}{2M}$ which implies $\frac{\partial F_\pm}{\partial \rho_\pm} \geq 0$, so that we may apply the Sandwich Lemma throughout the region $I \geq \overline{I}$ yielding:

$$\rho_- \leq \rho \leq \rho_+$$

for time $|t| \leq B_1\varepsilon^{-3/2}$ as long as we remain in the region $I \geq \overline{I}$.

Likewise since $v_1 - a\varepsilon^4 \leq v_1 \leq v_1 + a\varepsilon^4$, we have for $|t| \leq B_1\varepsilon^{-3/2}$ and throughout $I \geq \overline{I}$ that

$$\rho_- \leq \rho_{osc} \leq \rho_+$$

holds.

Now we will show that ρ_+ and ρ_- remain close to finish the proof. Set $\eta = \rho_+ - \rho_- \geq 0$. Note that v_1 is Lipschitz in the region $\rho \geq \overline{\rho}$ with

$$|v_1(x) - v_1(y)| \leq \omega|x - y| \text{ for } x, y \geq \overline{\rho} \text{ and } \omega = (2M + 3c_0^2)\varepsilon^3 = k\varepsilon^3,$$

Then $\ddot{\eta} = v_1(\rho_+) - v_1(\rho_-) + 2a\varepsilon^4 \Rightarrow |\ddot{\eta}| \leq \omega|\eta| + 2a\varepsilon^4 = \omega\eta + 2a\varepsilon^4$, so

$$|\ddot{\eta}| \leq \omega\eta + 2a\varepsilon^4.$$

Let $F = v_1(\rho_+) - v_1(\rho_-) + 2a\varepsilon^4$ then we have $0 \leq F \leq \omega\eta + 2a\varepsilon^4$ provided $\rho_- \leq \rho_+$ and $\frac{\partial v_1}{\partial \rho} > 0$, which indeed holds throughout the region $I \geq \overline{I}$ for time $|t| \leq B_1\varepsilon^{-3/2}$. Now the Sandwich Lemma with $F_+(\eta) = \omega\eta + 2a\varepsilon^4$ and $F_- = 0$ gives:

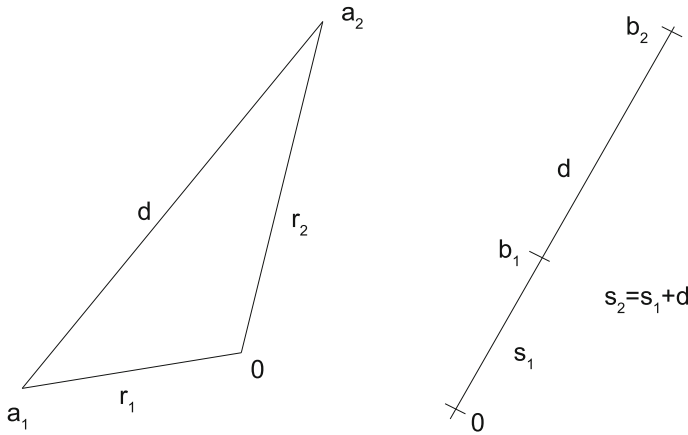$$0 \leq \eta(t) \leq \frac{2a\varepsilon^4}{\omega}(\cosh\sqrt{\omega}t - 1)$$

Fig. 3 Two equivalent configurations

and since $\omega = k\varepsilon^3$ where $2M \leq k \leq 2M + 3c_{J_2}^2$ we have

$$|\rho(t) - \rho_{osc}(t)| \leq \rho_+(t) - \rho_-(t) = \eta(t) \leq \frac{2a\varepsilon}{k}(2 + e^{\sqrt{\omega}|t|}) \leq A_1\varepsilon$$

for time $|t| \leq B_1\varepsilon^{-3/2}$ as long as we are in the region $I \geq \overline{I}$ and where we set $A_1 = \frac{a}{M}(2 + e^{\sqrt{2M+3c_{J_2}^2}\,B_1})$. □

Proposition 4 *Set* $R = \max\{\overline{R}, I^{**}, 4\alpha_1 c_r^2\}$ *where* $\overline{R}$ *is from Proposition 3. For* $\overline{I} \geq R$ *set*

$$\overline{I}^+ = 4(\overline{I} - \alpha_1 c_r^2) > \overline{I}.$$

Then for any orbit with an initial condition in the strip

$$\overline{I} \leq I \leq \overline{I}^+,$$

we have that Eq. (7) holds with $B_1 = 2^{3/2}\pi\sqrt{M}$ *until the osculating orbit enters the region* $I \leq \overline{I}$.

Proof First consider orbits with initial condition in $I \geq \overline{I}$ for some $\overline{I} \geq \max\{I^{**}, \overline{R}\}$ and with $\varepsilon = 1/\overline{\rho}$ defined as in Proposition 3 and recall that $I \geq \overline{I}$ implies that $\rho \geq \overline{\rho}$. For osculating collision orbits with $J_2(0) = 0$, some energy H_2 and $\rho(0) = \rho_{osc}(0) > \overline{\rho}$ the time to collision in forward time (or time from expulsion in backward time) t_c satisfies:

$$t_c \leq \pi(8M)^{-1/2}\rho_{osc}(0)^{3/2}. \tag{11}$$

We will use Lambert's theorem (see Albouy 2002) to compare time to pericenter for general osculating orbits to these collision times. Lambert says that for Kepler orbits, the time of travel between two points, a_1, a_2 on the orbit is a function of the energy, chord length $d = |a_1 - a_2|$ and $|a_1| + |a_2| = r_1 + r_2$ (where the origin is at the focus, see Fig. 3). Namely, for equivalent configurations (those having the same energy, same chord length d, and $r_1 + r_2 = s_1 + s_2$) the time of travel from a_2 to a_1 is the same as the time of travel from b_2 to b_1. Figure 3 shows how we will choose our equivalent configurations.

For a general osculating orbit ρ_{osc}, take $r_1 = \rho_{osc}^{pc}$, $r_2 = \rho_{osc}(0) = \rho(0) > \overline{\rho}$ and then s_1, s_2 are determined by $s_2 - s_1 = d = |a_2 - a_1|$ and $s_1 + s_2 = r_1 + r_2$. By Lambert's theorem and Eq. (11) we have that the time to pericenter, t_{pc}, satisfies

$$t_{pc} \leq \pi (8M)^{-1/2} s_2^{3/2}. \tag{12}$$

And since $r_2 \geq r_1$ (as we are in $I \geq I^{**}$) we have:

$$2s_2 - (r_1 + r_2) = s_2 - s_1 = d \leq r_1 + r_2 \Rightarrow$$
$$s_2 \leq r_1 + r_2 \leq 2r_2.$$

So continuing with Eq. (12), we have

$$t_{pc} \leq \pi M^{-1/2} r_2^{3/2}.$$

To compare t_{pc} with our estimates Eq. (8) we want $t_{pc} \leq B_1 \varepsilon^{-3/2} = B_1 \overline{\rho}^{3/2}$, which holds when:

$$\pi M^{-1/2} r_2^{3/2} \leq B_1 \overline{\rho}^{3/2} \Rightarrow$$
$$r_2^{3/2} \leq \pi^{-1} M^{1/2} B_1 \overline{\rho}^{3/2}.$$

Take $B_1 = 2^{3/2} \pi / \sqrt{M}$ so that we will be working in the strip:

$$\overline{\rho}^{3/2} \leq r_2^{3/2} \leq 2^{3/2} \overline{\rho}^{3/2}$$

i.e., (recall that $r_2 = \rho(0) = \rho_{osc}(0)$)

$$\overline{\rho} \leq \rho \leq 2\overline{\rho}.$$

The condition $\rho \leq 2\overline{\rho}$ is ensured (Eqs. 1, 6) when $I \leq \overline{I}^+ := 4\alpha_2 \overline{\rho}^2 = 4(\overline{I} - \alpha_1 c_r^2)$. Also, we ensure $\overline{I} < \overline{I}^+ = 4(\overline{I} - \alpha_1 c_r^2)$ provided $\overline{I} \geq 4\alpha_1 c_r^2 > \frac{4}{3}\alpha_1 c_r^2$. $\qquad\square$

Proposition 5 (Main Theorem) *Fix a parameter $\lambda > 0$. Then there exists $R_\lambda(m_i, H, J) > 0$ such that any orbit with initial condition satisfying $I(0) \geq R_\lambda$ comes in forward or backward time into the region $I \leq R_\lambda$.*

Explicitly, take $\overline{R}_\lambda = \max\{R, \alpha_1 c_r^2 + \alpha_2(\frac{\alpha_2 A_1^2}{\lambda}), 2\alpha_2 A_1 + 4\alpha_1 c_r^2 + \lambda\}$ and $R_\lambda = \overline{R}_\lambda + 2\alpha_2 A_1 + \lambda$ where R is from Proposition 4.

Proof Take $\overline{I} \geq \overline{R}_\lambda$ and $\varepsilon^{-1} = \overline{\rho} = \sqrt{\alpha_2^{-1}(\overline{I} - \alpha_1 c_r^2)}$ and consider an orbit with initial condition in $\overline{I}^+ \geq I \geq \overline{I}$ as in Proposition 4. By Proposition 2 we can let t^* be the time the osculating orbit hits $\alpha_1 c_r^2 + \alpha_2 \rho_{osc}^2(t^*) = \overline{I}$, i.e., $\rho_{osc}(t^*) = \overline{\rho} = \varepsilon^{-1}$.
Along the true motion then at t^* we have by Proposition 4 (Eqs. 6, 7) that

$$I(t^*) \leq \alpha_1 c_r^2 + \alpha_2 \rho(t^*)^2 \leq \alpha_1 c_r^2 + \alpha_2(\rho_{osc}(t^*) + A_1\varepsilon)^2 = \overline{I} + 2\alpha_2 A_1 + \alpha_2 A_1^2 \varepsilon^2$$

holds. Moreover, due to the condition $\overline{I} \geq \overline{R}_\lambda \geq \alpha_1 c_r^2 + \alpha_2(\frac{\alpha_2 A_1^2}{\lambda})$ we have $\varepsilon^2 \leq \frac{\lambda}{\alpha_2 A_1^2}$, so that

$$I(t^*) \leq \overline{I} + 2\alpha_2 A_1 + \lambda.$$

Also the condition $\overline{I} \geq \overline{R}_\lambda \geq 2\alpha_2 A_1 + 4\alpha_1 c_r^2 + \lambda > \frac{1}{3}(2\alpha_2 A_1 + 4\alpha_1 c_r^2 + \lambda)$ ensures that $\overline{I} + 2\alpha_2 A_1 + \lambda < \overline{I}^+$.
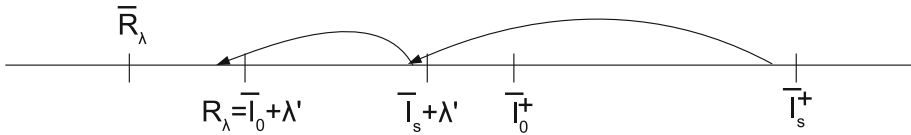
$$\overline{R}_\lambda$$

$$R_\lambda = \overline{I}_0 + \lambda' \qquad \overline{I}_s + \lambda' \quad \overline{I}_0^+ \qquad\qquad \overline{I}_s^+$$

Fig. 4 Jumping back along the strips

That is taking any $\overline{I} \geq \overline{R}_\lambda$ and setting $\lambda' = 2\alpha_2 A_1 + \lambda$, then all orbits with initial condition in the strip

$$\overline{I} + \lambda' \leq I(0) \leq \overline{I}^+$$

come in forward or backward time into the region

$$I \leq \overline{I} + \lambda'.$$

In particular by setting

$$\overline{I}_s = \overline{R}_\lambda + s$$

for $s \geq 0$, we may exhaust the region $I \geq R_\lambda = \overline{R}_\lambda + \lambda' = \overline{I}_0 + \lambda'$ with the strips

$$\overline{I}_s + \lambda' \leq I \leq \overline{I}_s^+.$$

Note that $s > s'$ implies $\overline{I}_s^+ - \overline{I}_s > \overline{I}_{s'}^+ - \overline{I}_{s'}$. Hence, any orbit with initial condition $I(0) \geq R_\lambda$ will be forced to jump back along the strips (see Fig. 4).

Finally in Theorem 1 we can take $I_0 = R_\lambda$ for any choice of λ (for instance, $I_0 = \min_{\lambda \in (0,1)} R_\lambda$). □

Appendix: Comparison to Marchal's equal mass case

We shall now compare our results to Marchal (1990) by examining the case:

$$m_i = \frac{1}{3}, \ H = -\frac{1}{6}, \ |J| = \frac{\sqrt{8}}{9}.$$

As I^* corresponds to the apocenter of the collinear Euler motion (where $U|_{I=1}$ has a saddle point), we have

$$I^* = \frac{32}{27}.$$

Here Marchal observed in Marchal (1990), p. 468, that $\ddot{\rho} < 0$ for $I > I_M$ where

$$I_M = \frac{1}{3}(2.709629\ldots)^2 = 2.447363\ldots,$$

so that every orbit will enter the region $I \leq I_M$ at some instant (of course excluding or passing through any binary collision). Moreover, it was conjectured that in fact all orbits pass

below the minimal inertia of the Henon–Brouke orbit (see Marchal 1990, p. 469), which is approximately 2.402035... and resembles an $e = 0$ Earth–Moon–Sun cartoon-type orbit.

Applying our final result (Proposition 5) in this case we obtain an $I_0 > I_M$. However, our bound only becomes larger when we apply our perturbation arguments; in this case and in general we obtain a lower pre-perturbation $I^{**} < I_M$ (of Proposition 2). This gives hope that if our perturbation methods are improved (Propositions 3, 4, 5), then the Marchal's bound of I_M could be lowered. Now we outline how $I^{**} < I_M$ in general.

Note that Marchal's observation on the negativity of $\ddot{\rho}$ works not just in this equal mass case but lends to a shorter proof of Theorem 1 by using Eq. 10. We set $\mu_i = m_i \mu^{-1}, \lambda = r/\rho$, $\gamma = \angle(\xi_1, \xi_2)$ and rewrite Eq. 10 as

$$\rho^3 \ddot{\rho} = c^2 - \rho M \phi(\lambda, \gamma),$$

where

$$\phi(\lambda, \gamma) = \mu_1 \frac{1 + \mu_2 \cos(\gamma)\lambda}{(1 + 2\mu_2 \cos(\gamma)\lambda + \mu_2^2 \lambda^2)^{3/2}} + \mu_2 \frac{1 - \mu_1 \cos(\gamma)\lambda}{(1 - 2\mu_1 \cos(\gamma)\lambda + \mu_1^2 \lambda^2)^{3/2}}.$$

Note that $\phi \geq \delta > 0$ throughout $I \geq I^*$ for a $\delta \leq \phi(\lambda, \pi/2) < 1$ dependent on the masses. Hence, a ρ_M corresponding to Marchal's I_M is (in general)

$$\rho_M = \frac{c_{J_2}^2}{M\delta}.$$

However although Eq. (10) leads to a simpler proof, following an orbit to pericenter rather than over the region where $\ddot{\rho} < 0$ has the potential to yield lower upper bounds; as for Keplerian orbits we have

$$\rho^{pc} = \frac{c^2}{M(1+e)} \leq \frac{c_{J_2}^2}{M} < \rho_M.$$

Thus, the I^{**} of Proposition 2 satisfies

$$I^{**} < I_M.$$

So our strategy of proof provides hope of lowering the bound toward Marchal's conjectured Henon–Broucke value in this case and a lower upper bound in general *provided* the techniques in the perturbation steps Propositions 3, 4, 5 are improved to follow the orbits past the $\ddot{\rho} < 0$ regime. Can they be improved? Perhaps in some non-equal mass cases or for some (outer) eccentricity $e > 0$ orbits above Henon–Broucke? I am optimistic that taking advantage of the sharper bounds and techniques of the literature they can be improved at least for large classes of orbits. This is especially so as many bounds of the perturbation steps here are not the sharpest (as the original motivation here was merely the existence of some upper bound in general specifically the zero angular momentum case).

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