

L^1 -optimality conditions for the circular restricted three-body problem

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Abstract In this paper, the L^1 -minimization for the translational motion of a spacecraft in the circular restricted three-body problem (CRTBP) is considered. Necessary conditions are derived by using the Pontryagin Maximum Principle (PMP), revealing the existence of bang-bang and singular controls. Singular extremals are analyzed, recalling the existence of the Fuller phenomenon according to the theories developed in (Marchal in *J Optim Theory Appl* 11(5):441–486, 1973; Zelikin and Borisov in *Theory of Chattering Control with Applications to Astronautics, Robotics, Economics, and Engineering*. Birkhäuser, Basel 1994; in *J Math Sci* 114(3):1227–1344, 2003). The sufficient optimality conditions for the L^1 -minimization problem with fixed endpoints have been developed in (Chen et al. in *SIAM J Control Optim* 54(3):1245–1265, 2016). In the current paper, we establish second-order conditions for optimal control problems with more general final conditions defined by a smooth submanifold target. In addition, the numerical implementation to check these optimality conditions is given. Finally, approximating the Earth-Moon-Spacecraft system by the CRTBP, an L^1 -minimization trajectory for the translational motion of a spacecraft is computed by combining a shooting method with a continuation method in (Caillau et al. in *Celest Mech Dyn Astron* 114:137–150, 2012; Caillau and Daoud in *SIAM J Control Optim* 50(6):3178–3202, 2012). The local optimality of the computed trajectory is asserted thanks to the second-order optimality conditions developed.

Keywords Circular restricted three-body problem · Low-thrust · L^1 -minimization · Conjugate points · Sufficient optimality conditions

Mathematics Subject Classification 49K15 · 70Q05

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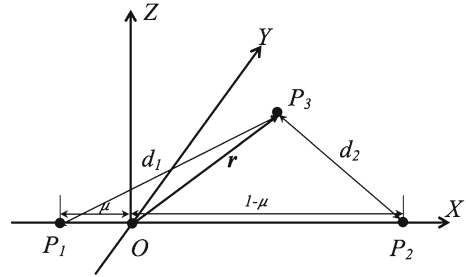
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1 Introduction

As an increasing number of artificial satellites or spacecrafts have been and are being launched into deeper space since the 1960s, the problem of controlling the translational motion of a spacecraft in the gravitational field of multiple celestial bodies such that some cost functionals are minimized or maximized arises in astronautics. The CRTBP that captures the chaotic property of n -body problem, has been widely used in the literature to study optimal trajectories in deeper space. The present paper is concerned with the L^1 -minimization problem of the motion of a spacecraft in the CRTBP. If the control is generated by propulsion systems which expel mass in a high speed to generate an opposite reaction force according to Newton's third law of motion, the L^1 -minimization problem is related to the important fuel-optimal control problem in astronautics. According to the controllability properties studied in (Caillau and Daoud 2012), there exist admissible controlled trajectories in an appropriate subregion of the state space. Then, the existence of L^1 -minimizing trajectories can be obtained by a combination of Filippov theorem in (Agrachev and Sachkov 2004) and the technique in (Gergaud and Haberkorn 2006), provided that the admissible controlled trajectories remain in a fixed compact set (see Caillau et al. 2012). While in the planar case the singular extremals and the corresponding chattering arcs were analyzed by Zelikin and Borisov (2003), the synthesis of the solutions of singular extremals in the 3-dimensional case has not yet been covered to the author's knowledge. Not considering singular and chattering controls, even the computation of a bang-bang control of the chaotic CRTBP is a challenging task in the framework of low thrust. To address this challenge, various numerical methods, *e.g.*, direct methods (Mingotti et al. 2009; Ross and Scheeres 2007), indirect methods (Caillau et al. 2012; Caillau and Daoud 2012), and hybrid methods (Ozimek and Howell 2010), have been developed recently. In this paper, the indirect method, proposed by Caillau et al. (2012), that combines a shooting method with a continuation method, is employed to compute the extremal trajectories of the L^1 -minimization problem. Based on this method, some fuel-optimal trajectories in the CRTBP were computed recently as well in (Zhang et al. 2015). Nevertheless, these extremal trajectories cannot be guaranteed to be at least locally optimal unless sufficient optimality conditions are satisfied. The sufficient conditions for optimal control problems have been widely studied in the literature in recent years (see Agrachev et al. 2002; Poggiolini and Stefani 2004; Schättler and Ledzewicz 2012; Noble and Schättler 2002; Chen et al. 2016; Agrachev and Sachkov 2004; Kupka 1987; Sarychev 1982; Flies and Hazewinkel 1987; Bonnard et al. 2007, and references therein). Through defining an accessory finite dimensional problem in (Agrachev et al. 2002; Poggiolini and Stefani 2004), some sufficient conditions were developed for optimal control problems with a polyhedral control set. In (Chen et al. 2016), no-fold conditions related to those of (Schättler and Ledzewicz 2012; Noble and Schättler 2002) were established for the L^1 -minimization problem. Assuming the endpoints are fixed, these conditions suffice to guarantee that a bang-bang extremal of the L^1 -minimization problem is a strong local optimizer. In addition to these no-fold conditions, an extra condition has to be taken into account whenever the target is a genuine submanifold (see Agrachev et al. 2002; Bruschi and Vincent 1970; Wood 1974). It is shown in this paper that the propagation of Jacobi fields is enough to test these sufficient optimality conditions (cf. Sect. 5).

The paper is organized as follows. In Sect. 2, the L^1 -minimization problem is formulated in the CRTBP. Then, the necessary conditions are derived with an emphasis on singular solutions in Sect. 3. In Sect. 4, a parameterized family of extremals is first constructed. Under some regularity assumptions, the sufficient conditions for the strong-local optimality of the nonsingular extremals with bang-bang controls are established. In Sect. 5, a numerical implementation

Fig. 1 Rotating frame $OXYZ$ of the CRTBP



for the optimality conditions is derived. In Sect. 6 a transfer trajectory of a spacecraft from a circular geosynchronous orbit of the Earth to a circular orbit around the Moon is computed, and its local optimality is tested thanks to the second-order optimality conditions developed.

2 Definitions and notations

The CRTBP in Celestial Mechanics is defined as an isolated dynamical system consisting of three gravitationally interacting bodies, P_1 , P_2 , and P_3 , whose masses are denoted by m_1 , m_2 , and m_3 , respectively, such that (1) the third mass m_3 is so smaller than the other two that its gravitational influence on the motions of the other two is negligible; (2) the two bodies, P_1 and P_2 , move on circular orbits around their common centre of mass. Without loss of generality, we assume $m_1 > m_2$ and consider a rotating frame $OXYZ$ such that its origin is located at the barycentre of the two bodies P_1 and P_2 (see Fig. 1). The unit vector of X -axis is oriented by the axis between the two primaries P_1 and P_2 and points toward P_2 ; the unit vector of Z -axis is defined as the unit vector of the momentum vector of the motion of P_1 and P_2 , and the Y -axis is defined to complete a right-hand coordinate system. It is advantageous to use non-dimensional parameters. Let d_* be the distance between P_1 and P_2 , and let m_* be the initial mass of the spacecraft, we denote by d_* and m_* the unit of length and mass, respectively. We also define the unit of time t_* in such a way that the gravitational constant $G > 0$ equals to one. Accordingly, one can obtain

$$t_* = \sqrt{\frac{d_*^3}{G(m_1 + m_2)}}$$

through the usage of Kepler’s third law. If $\mu = m_2/(m_1 + m_2)$, the two constant vectors $r_1 = [-\mu, 0, 0]^T$ and $r_2 = [1 - \mu, 0, 0]^T$ denote the position of P_1 and P_2 in the rotating frame $OXYZ$, respectively (the superscript “T” denotes the transpose of a matrix).

2.1 Dynamics

Throughout the paper, we denote the space of n -dimensional column vectors by \mathbb{R}^n and the space of n -dimensional row vectors by $(\mathbb{R}^n)^*$.

Let $t \in \mathbb{R}_+$ be the non-dimensional time and let $r \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ be the non-dimensional position and velocity vectors of P_3 in the rotating frame $OXYZ$. The spacecraft is the third mass point P_3 that is controlled by a finite-thrust propulsion system. Finally, if $m = m_3/m_*$, the state $x \in \mathbb{R}^n$ ($n = 7$) consists of r , v , and m . Denote by $r_{m_1} > 0$ and $r_{m_2} > 0$ the radiuses of the bodies P_1 and P_2 , and denote by $m_c > 0$ the mass of the spacecraft without any fuel.

We define the admissible subset for state x as

$$\mathcal{X} = \{(r, v, m) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+ \mid \|r - r_1\| > r_{m_1}, \|r - r_2\| > r_{m_2}, m \geq m_c\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Then, the differential equations for the controlled translational motion of the spacecraft in the CRTBP in the admissible set \mathcal{X} for positive times can be written as

$$\Sigma : \begin{cases} \dot{r}(t) = v(t), \\ \dot{v}(t) = h(v(t)) + g(r(t)) + \frac{\tau(t)}{m(t)}, \\ \dot{m}(t) = -\beta \|\tau(t)\|, \end{cases} \tag{1}$$

with

$$h(v) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v, \quad g(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} r - \frac{1 - \mu}{\|r - r_1\|^3} (r - r_1) - \frac{\mu}{\|r - r_2\|^3} (r - r_2),$$

where $\beta \geq 0$ is a scalar constant determined by the specific impulse of the engine and $\tau \in \mathbb{R}^3$ is the thrust vector taking values in

$$\mathcal{T} = \{\tau \in \mathbb{R}^3 \mid \|\tau\| \leq \tau_{max}\}.$$

The constant $\tau_{max} > 0$ (in unit of $m_* d_* / t_*^2$) denotes the maximum magnitude of the thrust of the engine. We denote by $\rho \in [0, 1]$ the normalized mass flow rate of the engine, $\rho = \|\tau\| / \tau_{max}$, and by $\omega \in \mathbb{S}^2$ the unit vector of the thrust direction, $\tau = \rho \tau_{max} \omega$; ρ and ω are control variables. Let $u = (\rho, \omega)$ and $\mathcal{U} = [0, 1] \times \mathbb{S}^2$, so \mathcal{U} is the admissible set for the control u . Let us define the controlled vector field f on $\mathcal{X} \times \mathcal{U}$ by

$$f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n, \quad f(x, \rho, \omega) = f_0(x) + \rho f_1(x, \omega),$$

where

$$f_0(x) = \begin{pmatrix} v \\ h(v) + g(r) \\ 0 \end{pmatrix}, \quad f_1(x, \omega) = \begin{pmatrix} 0 \\ \tau_{max} \omega / m \\ -\tau_{max} \beta \end{pmatrix}.$$

The dynamics in Eq. (1) can be rewritten in the control-affine form

$$\Sigma : \dot{x}(t) = f(x(t), \rho(t), \omega(t)) = f_0(x(t)) + \rho(t) f_1(x(t), \omega(t)). \tag{2}$$

2.2 L¹-minimization problem

Given $l \in \mathbb{N}$ such that $0 < l \leq n$, we define the l -codimensional target submanifold

$$\mathcal{M} = \{x \in \mathcal{X} \mid \phi(x) = 0\}, \tag{3}$$

where $\phi : \mathcal{X} \rightarrow \mathbb{R}^l$ denotes a twice continuously differentiable function of x whose expression depends on specific mission requirements. (See e.g., Eq. (30)). Given a fixed initial state $x_0 \in \mathcal{X}$ and a fixed final time $t_f > 0$, the L¹-minimization problem (Chen et al. 2016) for the translational motion in the CRTBP consists in steering the system Σ in \mathcal{X} by a measurable

control $(\rho(\cdot), \omega(\cdot)) \in \mathcal{U}$ on $[0, t_f]$ from the initial point $x_0 \in \mathcal{X}$ to a final point $x_f \in \mathcal{M}$ such that the L¹-norm of control is minimized:

$$\int_0^{t_f} \rho(t)dt \rightarrow \min. \tag{4}$$

Note that minimizing the cost in Eq. (4) is equivalent to maximizing the final mass once $\beta > 0$. Controllability of the CRTBP holds in an appropriate subregion of \mathcal{X} (see [Caillaud et al. 2012](#)). Let $t_m > 0$ be the minimum time to steer the system Σ by measurable controls $(\rho(\cdot), \omega(\cdot)) \in \mathcal{U}$ from the point $x_0 \in \mathcal{X}$ to a point $x_f \in \mathcal{M}$. Then, assuming $t_f > t_m$ and assuming that the admissible controlled trajectories of Σ remain in a fixed compact set, the existence of the L¹-minimization solutions can be obtained by combining Filippov theorem (see e.g., [Agrachev and Sachkov 2004](#)) and a suitable convexification procedure (see [Gergaud and Haberkorn 2006](#)).

3 Necessary conditions

3.1 Pontryagin maximum principle

According to the maximum principle in ([Pontryagin et al. 1962](#)), if a trajectory $x(\cdot) \in \mathcal{X}$ associated with a measurable control $u(\cdot) = (\rho(\cdot), \omega(\cdot))$ in \mathcal{U} on $[0, t_f]$ is optimal, there exists a nonpositive real number p^0 and an absolutely continuous mapping $t \mapsto p(\cdot) \in T_{x(\cdot)}^* \mathcal{X}$ on $[0, t_f]$, satisfying $(p(t), p^0) \neq 0$ for $t \in [0, t_f]$ such that, almost everywhere on $[0, t_f]$, there holds

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)), \end{cases} \tag{5}$$

and

$$H(x(t), p(t), p^0, u(t)) = \max_{\eta(t) \in \mathcal{U}} H(x(t), p(t), p^0, \eta(t)), \tag{6}$$

where the Hamiltonian H is defined by

$$H(x, p, p^0, u) = p[f_0(x) + \rho f_1(x, \omega)] + p^0 \rho. \tag{7}$$

Moreover, the following transversality condition holds:

$$p(t_f) = \nu d\phi(x(t_f)), \tag{8}$$

where $\nu \in (\mathbb{R}^l)^*$ is a constant vector whose elements are Lagrangian multipliers. The 4-tuple $t \mapsto (x(t), p(t), p^0, u(t))$ on $[0, t_f]$ is called an extremal. Furthermore, an extremal is called a normal one if $p^0 \neq 0$, and it is called an abnormal one if $p^0 = 0$. Abnormal extremals have been ruled out in ([Gergaud and Haberkorn 2006](#)). Thus, in this paper only normal extremals are considered and we set $p^0 = -1$. By virtue of the maximum condition in Eq. (6), for every extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ on $[0, t_f]$, the corresponding extremal control $u(\cdot)$ is a function of $(x(\cdot), p(\cdot))$ on $[0, t_f]$: $u(\cdot) = u(x(\cdot), p(\cdot))$ on $[0, t_f]$. In the rest of the paper, we denote by $(x(\cdot), p(\cdot)) \in T^* \mathcal{X}$ and $u(x(\cdot), p(\cdot)) \in \mathcal{U}$ on $[0, t_f]$ the normal extremal and the corresponding extremal control, respectively. We denote the maximized Hamiltonian by

$$H(x, p) := H_0(x, p) + \rho(x, p)H_1(x, p),$$

where $H_0(x, p) = pf_0(x)$ and $H_1(x, p) = pf_1(x, \omega(x, p)) - 1$. Given the scalar $p_m \in T_m \mathbb{R}_+$ and the column vectors $p_r \in T_r \mathbb{R}^3$ and $p_v \in T_v \mathbb{R}^3$ such that $p = [p_r^T, p_v^T, p_m]$, the maximum condition in Eq. (6) implies

$$\omega = p_v / \| p_v \|, \text{ if } \| p_v \| \neq 0, \tag{9}$$

and

$$\begin{cases} \rho = 1, & \text{if } H_1 > 0, \\ \rho = 0, & \text{if } H_1 < 0. \end{cases} \tag{10}$$

The optimal direction of the thrust vector τ is so collinear to p_v , a well known fact (“primer vector” theory of Lawden (1963)). If the switching function H_1 has only isolated zeros along an extremal $(x(\cdot), p(\cdot))$ on $[0, t_f]$, this extremal is called a bang-bang extremal.

Definition 1 Along a bang-bang extremal $(x(\cdot), p(\cdot))$ on $[0, t_f]$, an arc on a finite interval $[t_1, t_2] \subset [0, t_f]$ with $t_1 < t_2$ is called a maximum-thrust (or burn) arc if $\rho = 1$; otherwise it is called a zero-thrust (or coast) arc.

3.2 Singular solutions and chattering arcs

An arc $(x(\cdot), p(\cdot))$ on $[t_1, t_2] \subseteq [0, t_f]$ with $t_1 < t_2$ is said to be a singular one if $H_1(x(\cdot), p(\cdot)) \equiv 0$ on $[t_1, t_2]$. Note that the maximum condition in Eq. (6) is trivially satisfied for every $\rho \in [0, 1]$ if $H_1 \equiv 0$. One can compute the optimal value of ρ on singular arcs by repeatedly differentiating the identity $H_1 \equiv 0$ until ρ explicitly appears. It is known from (Kelley et al. 1966) that ρ appears in $d^q H_1/dt^q$ only if q is an even integer; the order of the singular arc is then defined as $q/2$.

Proposition 1 Given a singular extremal $(x(\cdot), p(\cdot))$ on $[t_1, t_2] \subseteq [0, t_f]$ with $t_1 < t_2$, assume $\|p_v(\cdot)\| \neq 0$ on $[t_1, t_2]$. Then, we have that the order of the singular extremal is at least two.

Proof Since $H_1 \equiv 0$ along a singular arc, differentiating H_1 with respect to time, one obtains

$$0 = H_{01} := \{H_0, H_1\} = -\tau_{max} \frac{p_v^T [p_r + dh(v)p_v]}{m \| p_v \|}, \tag{11}$$

where the $\{\cdot, \cdot\}$ denotes the Poisson bracket. Using Leibniz rule, Eq. (11) implies

$$\begin{aligned} H_{101} &:= \{H_1, H_{01}\} = 0, \\ H_{1001} &:= \{H_1, \{H_0, H_{01}\}\} \\ &= \{-H_{01}, H_{01}\} + \{H_0, H_{101}\} = 0. \end{aligned}$$

Then, the equality, $0 = H_{001} + \rho H_{101}$, implies $H_{001} = 0$, whose implicit equation is

$$H_{001} = \tau_{max} \frac{p_v^T dg(r)p_v + [p_r + 2dh(v)p_v]^T [p_r + dh(v)p_v]}{m \| p_v \|}.$$

A direct calculation on this equation yields

$$\begin{aligned} H_{0001} &:= \{H_0, H_{001}\} \\ &= \frac{\tau_{max}}{m \| p_v \|} \left\{ [p_v^T d^2g(r)p_v]v - p_v^T dg(r)[2p_r + 3dh(v)p_v] \right. \\ &\quad \left. - [2dg(r)p_v + 3dh(r)p_r + 4(dh(v))^2 p_v]^T [p_r + dh(v)p_v] \right\}. \end{aligned}$$

Eventually, one has $0 = \dot{H}_{0001} = H_{00001} + \rho H_{10001}$. Let α_i ($i = 1, 2$) be defined by

$$\cos(\alpha_i) = \frac{p_v^T (r - r_i)}{\|p_v\| \|r - r_i\|},$$

the explicit expression of $H_{10001} := \{H_1, H_{0001}\}$ is therefore

$$\begin{aligned} H_{10001} &= \tau_{max} \frac{[p_v^T d^2 g(r) p_v] p_v}{m^2 \|p_v\|^2} \\ &= 3\tau_{max} \frac{\|p_v\|}{m^2} \left[\mu \cos \alpha_2 \frac{3 - 5 \cos^2 \alpha_2}{\|r - r_2\|^4} + (1 - \mu) \cos \alpha_1 \frac{3 - 5 \cos^2 \alpha_1}{\|r - r_1\|^4} \right]. \end{aligned}$$

Note that the term H_{10001} does not vanish identically on a singular extremal. So the singular extremal is of order two according to Kelley’s definition in (Kelley et al. 1966), which proves the proposition. \square

This proposition for the 3-dimensional case expands the work in (Zelikin and Borisov 2003) where the motion of the spacecraft is restricted into a 2-dimensional plane and the work in (Robbins 1965) where the model of two-body problem ($\mu = 0$) is considered. Note that Kelley’s second-order necessary condition (Kelley et al. 1966) in terms of ρ on singular arcs is $H_{10001} \leq 0$. Let

$$\mathcal{S} = \{(x, p) \in T^* \mathcal{X} \mid H_1 = H_{01} = H_{001} = H_{0001} = 0, H_{10001} \leq 0\}$$

be the singular submanifold and denote by $\text{int}(\mathcal{S})$ the interior of \mathcal{S} . Note that $\text{int}(\mathcal{S})$ is not empty according to (Zelikin and Borisov 2003).

Remark 1 According to the theorems developed by Zelikin and Borisov (1994), given every point $(x, p) \in \text{int}(\mathcal{S})$, there exists a one parameter family of chattering solutions of Eqs. (5–7) passing through the point (x, p) and another one parameter family of chattering solutions of Eqs. (5–7) coming out from the point (x, p) .

Though the efficient computation of chattering solutions is an open problem (see e.g., Ghezzi et al. 2014; Park 2013), Remark 1 gives some insights on the control structure of the L^1 -minimization trajectory; there exists a chattering arc when concatenating a singular arc with a nonsingular arc if ρ is not saturate at the instant priori to the junction time.

4 Sufficient optimality conditions for bang-bang extremals

Definition 2 [*Local Optimality* (Poggiolini and Stefani 2004; Agrachev et al. 2002)] Given a fixed final time $t_f > 0$, an extremal trajectory $\bar{x}(\cdot) \in \mathcal{X}$ associated with the extremal control $\bar{u}(\cdot) = (\bar{\rho}(\cdot), \bar{\omega}(\cdot))$ in \mathcal{U} on $[0, t_f]$ is said to be a weak-local optimum in L^∞ -topology (resp. a strong-local optimum in C^0 -topology) if there exists an open neighborhood $\mathcal{W}_u \subseteq \mathcal{U}$ of $\bar{u}(\cdot)$ in L^∞ -topology (resp. an open neighborhood $\mathcal{W}_x \subseteq \mathcal{X}$ of $\bar{x}(\cdot)$ in C^0 -topology) such that for every admissible controlled trajectory $x(\cdot) \notin \bar{x}(\cdot)$ in \mathcal{X} associated with the measurable control $u(\cdot) = (\rho(\cdot), \omega(\cdot))$ in \mathcal{W}_u on $[0, t_f]$ (resp. for every admissible controlled trajectory $x(\cdot) \notin \bar{x}(\cdot)$ in \mathcal{W}_x associated with the measurable control $u(\cdot) = (\rho(\cdot), \omega(\cdot))$ in \mathcal{U} on $[0, t_f]$) with the boundary conditions $x(0) = \bar{x}(0)$ and $x(t_f) \in \mathcal{M}$, there holds

$$\int_0^{t_f} \rho(t) dt \geq \int_0^{t_f} \bar{\rho}(t) dt.$$

We say it is a strict weak-local (resp. strong-local) optimum if the strict inequality holds.

4.1 Parameterized family of extremals

Let us define by

$$\gamma : [0, t_f] \times T_{x_0}^* \mathcal{X} \rightarrow T^* \mathcal{X}, \quad \gamma(t, p_0) = (x(t), p(t)),$$

the solution trajectory of Eqs. (5–7) such that $(x_0, p_0) = \gamma(0, p_0)$. For every $p_0 \in T_{x_0}^* \mathcal{X}$, we say $\gamma(\cdot, p_0)$ on $[0, t_f]$ is an extremal.

Definition 3 We define $\bar{p}_0 \in T_{x_0}^* \mathcal{X}$ in such a way that the extremal $\gamma(\cdot, \bar{p}_0)$ at t_f satisfies the final condition in Eq. (3) and transversality condition in Eq. (8).

Definition 4 (*Parameterized family of extremals*) Given the extremal $\gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, let $\mathcal{D} \subset T_{x_0}^* \mathcal{X}$ be an open neighbourhood of \bar{p}_0 , we say that the subset

$$\mathcal{F} = \{(x(t), p(t)) \in T^* \mathcal{X} \mid (x(t), p(t)) = \gamma(t, p_0), t \in [0, t_f], p_0 \in \mathcal{D}\},$$

is a p_0 -parameterized family of extremals around the extremal $\gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$.

In the sequel, the mapping,

$$\Pi : T^* \mathcal{X} \rightarrow \mathcal{X}, \quad (x, p) \mapsto x,$$

will denote the canonical projection from the cotangent bundle to the state space. Local optimality of extremals is related to fold singularities of this projection through the notion of conjugate and focal point (see e.g., Agrachev and Sachkov 2004; Bonnard et al. 2007).

4.2 Sufficient conditions for the case of $l = n$

Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, let the positive integer $k \in \mathbb{N}$ be the number of switching times t_i ($i = 1, 2, \dots, k$) such that $0 < t_1 < t_2 < \dots < t_k < t_f$.

Assumption 1 Along the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, each switching point (at the switching time $t_i \in (0, t_f)$) is assumed to be a regular one, i.e., $H_1(\bar{x}(t_i), \bar{p}(t_i)) = 0$ and $H_{01}(\bar{x}(t_i), \bar{p}(t_i)) \neq 0$ for $i = 1, 2, \dots, k$.

As a result, if the subset \mathcal{D} is small enough, the number of switching times on each extremal $\gamma(\cdot, p_0) \in \mathcal{F}$ on $[0, t_f]$ remains equal to k and the i -th switching time of the extremals $\gamma(\cdot, p_0) \in \mathcal{F}$ on $[0, t_f]$ is a smooth function of p_0 , i.e., the function

$$t_i : \mathcal{D} \rightarrow \mathbb{R}_+, \quad p_0 \mapsto t_i(p_0).$$

is smooth by restricting \mathcal{D} . Let

$$\mathcal{F}_i = \{(x(t), p(t)) \in T^* \mathcal{X} \mid (x(t), p(t)) = \gamma(t, p_0), t \in (t_{i-1}(p_0), t_i(p_0)], p_0 \in \mathcal{D}\},$$

for $i = 1, 2, \dots, k, k + 1$ with $t_0 := 0$ and $t_{k+1} := t_f$. If the subset \mathcal{D} is small enough, there holds

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_k \cup \mathcal{F}_{k+1}.$$

Let $(x(\cdot, p_0), p(\cdot, p_0)) = \gamma(\cdot, p_0)$ on $[0, t_f]$ be extremals in \mathcal{F} . In order to avoid heavy notations, denote by $\delta(\cdot)$ the determinant of the matrix $\frac{\partial x}{\partial p_0}(\cdot, \bar{p}_0)$ on $[0, t_f]$, that is

$$\delta(t) = \det \left[\frac{\partial x}{\partial p_0}(t, \bar{p}_0) \right], \quad t \in [0, t_f].$$

Fig. 2 A typical picture for a fold singularity of the projection of \mathcal{F} onto the state space \mathcal{X} (see Agrachev and Sachkov 2004)

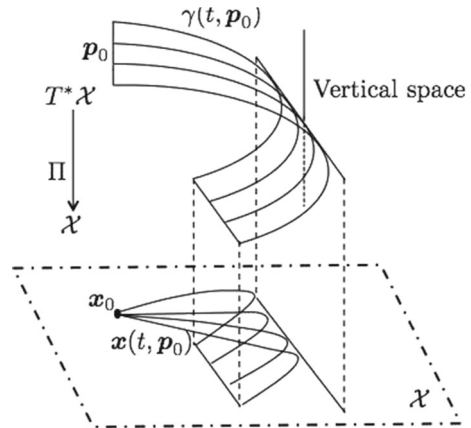
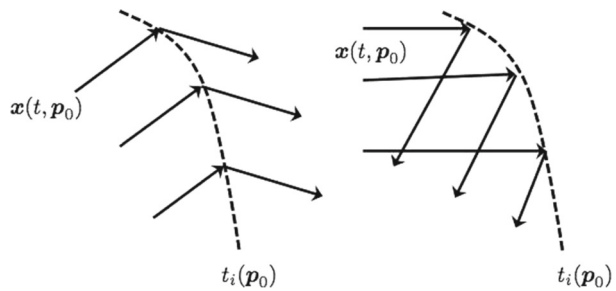


Fig. 3 Fold singularity for the projection around the switching time $t_i(p_0)$ on the right plot (see Noble and Schättler 2002; Schättler and Ledzewicz 2012)



Note that the projection of the subset \mathcal{F}_i at a time $t_c \in (t_i, t_{i+1})$ has a fold singularity if $\delta(t_c) = 0$, as is shown by the typical picture for the occurrence of a conjugate point in Fig. 2. If $\delta(\cdot) \neq 0$ on (t_i, t_{i+1}) , the projection of the subset \mathcal{F}_i restricted to the domain $(t_i, t_{i+1}) \times \mathcal{P}$ is a diffeomorphism (see Schättler and Ledzewicz 2012; Agrachev and Sachkov 2004). Let us define the following condition.

Condition 1 $\delta(\cdot) \neq 0$ on the open subintervals (t_i, t_{i+1}) for $i = 0, 1, \dots, k - 1$ as well as on the semi-open subinterval $(t_k, t_f]$.

Though this condition guarantees that both the restriction of $\Pi(\mathcal{F}_i)$ on $(t_{i-1}, t_i) \times \mathcal{P}$ for $i = 1, 2, \dots, k$ and the restriction of $\Pi(\mathcal{F}_{k+1})$ on $(t_k, t_f] \times \mathcal{P}$ are local diffeomorphisms, it is not sufficient to guarantee that the projection of the family \mathcal{F} restricted to the whole domain $(0, t_f] \times \mathcal{P}$ is a diffeomorphism as well, as Fig. 3 shows that the flows $x(t, p_0)$ may intersect with each other near a switching time $t_i(p_0)$.

Remark 2 The behavior that the projection of \mathcal{F} at a switching time t_i is a fold singularity can be excluded by a transversality condition established by Noble and Schättler (2002). This condition is reduced to $\delta(t_i-)\delta(t_i+) > 0$ by Chen et al. (2016).

Condition 2 $\delta(t_i-)\delta(t_i+) > 0$ for each switching time t_i for $i = 1, 2, \dots, k$.

If this condition is satisfied, the projection of the family \mathcal{F} around each switching time $t_i(p_0)$ is a diffeomorphism at least for a sufficiently small subset \mathcal{P} (see Chen et al. 2016).

Remark 3 Given an extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that every switching point is regular (cf. Assumption 1) and Conditions 1 and 2 are satisfied, if the subset \mathcal{P} is small enough, every extremal $\gamma(\cdot, p_0)$ on $[0, t_f]$ for $p_0 \in \mathcal{P}$ does not contain conjugate points. Then, for every $p_0 \in \mathcal{P}$, we are able to construct a perturbed Lagrangian submanifold $\mathcal{L}_{p_0} \subset T^*\mathcal{X}$ (see Chen et al. 2016, Appendix A) around the extremal $\gamma(\cdot, p_0)$ on $[0, t_f]$ such that

- (1) the projection of the Lagrangian submanifold \mathcal{L}_{p_0} onto its image is a diffeomorphism; and
- (2) the domain $\Pi(\mathcal{L}_{p_0})$ is a tubular neighborhood of the extremal trajectory $x(\cdot, p_0) = \Pi(\gamma(\cdot, p_0))$ on $[0, t_f]$.

Then, directly applying the theory of field of extremals (cf. Agrachev and Sachkov 2004, Theorem 17.1), one obtains the following result.

Theorem 1 [Agrachev and Sachkov (2004)] *Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular, let $(\rho(\cdot, p_0), \omega(\cdot, p_0)) \in \mathcal{U}$ be the optimal control function associated with the extremal $\gamma(\cdot, p_0) \in \mathcal{F}$ on $[0, t_f]$. Then, if Conditions 1 and 2 are satisfied and if the subset \mathcal{P} is small enough, every extremal trajectory $x(\cdot, p_0) = \Pi(\gamma(\cdot, p_0))$ on $[0, t_f]$ for $p_0 \in \mathcal{P}$ realizes a strict minimum cost with respect to every admissible controlled trajectory $x_*(\cdot) \in \Pi(\mathcal{L}_{p_0})$ associated with the measurable control $(\rho_*(\cdot), \omega_*(\cdot)) \in \mathcal{U}$ on $[0, t_f]$ with the same endpoints $x(0, p_0) = x_*(0)$ and $x(t_f, p_0) = x_*(t_f)$, i.e.,*

$$\int_0^{t_f} \rho(t, p_0)dt \leq \int_0^{t_f} \rho_*(t)dt,$$

where the equality holds if and only if $x_*(\cdot) \equiv \bar{x}(\cdot)$ on $[0, t_f]$.

Remark 4 As a consequence of Theorem 1, one obtains that Conditions 1 and 2 are sufficient to guarantee that the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$ is a strict strong-local optimum (cf. Definition 2) if $l = n$.

Under Assumption 1, the projection of the family \mathcal{F} near the switching time $t_i(p_0)$ is a fold singularity if the strict inequality $\delta(t_i-)\delta(t_i+) < 0$ is satisfied (Chen et al. 2016).

Remark 5 Given the extremal $\gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular (cf. Assumption 1), conjugate points can occur not only on each smooth bang arc at a time $t_c \in (t_{i-1}, t_i)$ if $\delta(t_c) = 0$ (see e.g., Agrachev and Sachkov 2004) but also at each switching time t_i if $\delta(t_i-)\delta(t_i+) < 0$ (see Noble and Schättler 2002).

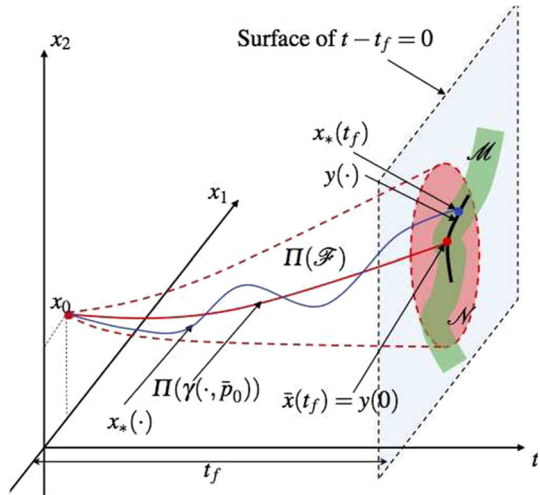
The fact that conjugate points can occur at switching times generalizes the conjugate point theory developed by the classical variational methods for totally smooth extremals (see Bryson and Ho 1969; Breakwell and Ho 1965; Mermay and Powers 1976; Wood 1974).

4.3 Sufficient conditions for the case of $l < n$

In this subsection, we establish the sufficient optimality conditions when the dimension of the final constraint submanifold \mathcal{M} is not zero.

Remark 6 If $l < n$, to ensure that the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$ is a strict strong-local optimum, in addition to Conditions 1 and 2, a further second-order condition (see e.g., Wood 1974; Brusch and Vincent 1970) is required to guarantee that every admissible controlled trajectory $x_*(\cdot) \in \Pi(\mathcal{L}_{\bar{p}_0})$ on $[0, t_f]$, also verifying the boundary conditions $\bar{x}(0) = x_*(0)$ and $x_*(t_f) \in \mathcal{M} \setminus \{\bar{x}(t_f)\}$, has a higher cost than the reference one.

Fig. 4 The relationship between \mathcal{N} and \mathcal{M}



Let $\mathcal{N} \subset \mathcal{X}$ be the restriction of $\Pi(\mathcal{F})$ on $\{t_f\} \times \mathcal{P}$, i.e.,

$$\mathcal{N} = \{x \in \mathcal{X} \mid x = \Pi(\gamma(t_f, p_0)), p_0 \in \mathcal{P}\}.$$

The mapping $p_0 \mapsto x(t_f, p_0)$ on a sufficiently small subset \mathcal{P} is a diffeomorphism if $\delta(t_f) \neq 0$, which indicates that the subset \mathcal{N} is an open neighborhood of $\bar{x}(t_f)$ if Condition 1 is satisfied. Thus, in the case of $l < n$, the subset $\mathcal{M} \cap \mathcal{N} \setminus \{\bar{x}(t_f)\}$ is not empty if $\delta(t_f) \neq 0$. (See Fig. 4.) For every sufficiently small subset \mathcal{P} , let us define by $\mathcal{Q} \subseteq \mathcal{P}$ a subset of all $p_0 \in \mathcal{P}$ satisfying $\Pi(\gamma(t_f, p_0)) \in \mathcal{M} \cap \mathcal{N}$, i.e.,

$$\mathcal{Q} = \{p_0 \in \mathcal{P} \mid \Pi(\gamma(t_f, p_0)) \in \mathcal{M} \cap \mathcal{N}\}.$$

Note that for every $p_0 \in \mathcal{Q}$ there holds $x_0 = \Pi(\gamma(0, p_0))$ and $\Pi(\gamma(t_f, p_0)) \in \mathcal{M}$.

Remark 7 For every $p_0 \in \mathcal{Q}$, the extremal trajectory $x(\cdot, p_0) = \Pi(\gamma(\cdot, p_0))$ on $[0, t_f]$ is an admissible controlled trajectory of the L¹-minimization problem.

Definition 5 Given the reference extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ and $\varepsilon > 0$ small enough, we define by $y : [-\varepsilon, \varepsilon] \rightarrow \mathcal{M} \cap \mathcal{N}$, $\eta \mapsto y(\eta)$ a twice continuously differentiable curve on $\mathcal{M} \cap \mathcal{N}$ such that $y(0) = \bar{x}(t_f)$.

Lemma 1 Given the reference extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular (cf. Assumption 1) and Conditions 1 and 2 are satisfied, let $l < n$. Then, if the subset \mathcal{P} is small enough, for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$, there exists a smooth path $\eta \mapsto p_0(\eta)$ on $[-\varepsilon, \varepsilon]$ in \mathcal{Q} such that $y(\cdot) = \Pi(\gamma(t_f, p_0(\cdot)))$ on $[-\varepsilon, \varepsilon]$.

Proof Note that the mapping $p_0 \mapsto x(t_f, p_0)$ restricted to the subset \mathcal{Q} is a diffeomorphism under the hypotheses of the lemma. One concludes using the inverse function theorem. \square

Definition 6 Define a path $\lambda : [-\varepsilon, \varepsilon] \rightarrow T_{y(\cdot)}^* \mathcal{X}$, $\eta \mapsto \lambda(\eta)$ in such a way that $(y(\cdot), \lambda(\cdot)) = \gamma(t_f, p_0(\cdot))$ on $[-\varepsilon, \varepsilon]$. Then, for every $\xi \in [-\varepsilon, \varepsilon]$, we define by $J :$

$[-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$, $\xi \mapsto J(\xi)$ the integrand of the Poincaré-Cartan form $pdx - Hdt$ along the extremal lift $(y(\cdot), \lambda(\cdot))$ on $[0, \xi]$, i.e.,

$$J(\xi) = \int_0^\xi \lambda(\eta)y'(\eta) - H(y(\eta), \lambda(\eta)) \frac{dt_f}{d\eta} d\eta, \quad \xi \in [-\varepsilon, \varepsilon]. \tag{12}$$

Proposition 2 *In the case of $l < n$, given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular (cf. Assumption 1) and Conditions 1 and 2 are satisfied, assume $\varepsilon > 0$ is small enough. Then, the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$ is a strict strong-local optimum if and only if there holds*

$$J(\xi) > J(0), \quad \xi \in [-\varepsilon, \varepsilon] \setminus \{0\}, \tag{13}$$

for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$.

Proof Let us first prove that, under the hypotheses of this proposition, Eq. (13) is a sufficient condition for strict strong-local optimality. Denote by $x_*(\cdot)$ in $\Pi(\mathcal{L}_{\bar{p}_0})$ on $[0, t_f]$ an admissible controlled trajectory with boundary conditions $x_*(0) = \bar{x}(0)$ and $x_*(t_f) \in \mathcal{M} \cap \mathcal{N} \setminus \{\bar{x}(t_f)\}$. Let $(\rho_*(\cdot), \omega_*(\cdot)) \in \mathcal{U}$ and $(\rho(\cdot, p_0), \omega(\cdot, p_0)) \in \mathcal{U}$ on $[0, t_f]$ be the measurable control and the optimal control associated with $x_*(\cdot)$ and $x(\cdot, p_0)$ on $[0, t_f]$, respectively. According to Definition 5 and Lemma 1, for every final point $x_*(t_f) \in \mathcal{M} \cap \mathcal{N} \setminus \{\bar{x}(t_f)\}$, there must exist a $\xi \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and a smooth path $p_0(\cdot) \in \mathcal{Q}$ associated with the smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$ such that $y(0) = \bar{x}(t_f) = \Pi(\gamma(t_f, p_0(0)))$ and $y(\xi) = x_*(t_f) = \Pi(\gamma(t_f, p_0(\xi)))$. Since the trajectory $x_*(\cdot)$ on $[0, t_f]$ has the same endpoints as the extremal trajectory $x(\cdot, p_0(\xi)) = \Pi(\gamma(\cdot, p_0(\xi)))$ on $[0, t_f]$, according to Theorem 1, one obtains

$$\int_0^{t_f} \rho_*(t)dt \geq \int_0^{t_f} \rho(t, p_0(\xi))dt, \tag{14}$$

where the equality holds if and only if $x_*(\cdot) \equiv x(\cdot, p_0(\xi))$ on $[0, t_f]$. Note that the four paths $(x_0, p_0(\cdot))$ on $[0, \xi]$, $\gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, $(x(\cdot, p_0(\xi)), p(\cdot, p_0(\xi))) = \gamma(\cdot, p_0(\xi))$ on $[0, t_f]$, and $(y(\cdot), \lambda(\cdot))$ on $[0, \xi]$ constitute a closed curve on the family \mathcal{F} . Since the integrand of the Poincaré-Cartan form $pdx - Hdt$ is closed on \mathcal{F} (see Agrachev and Sachkov 2004; Schättler and Ledzewicz 2012; Chen et al. 2016), one obtains

$$\begin{aligned} J(\xi) &+ \int_0^{t_f} [\bar{p}(t)\dot{\bar{x}}(t) - H(\bar{x}(t), \bar{p}(t))]dt \\ &= \int_0^{t_f} [p(t, p_0(\xi))\dot{x}(t, p_0(\xi)) - H(x(t, p_0(\xi)), p(t, p_0(\xi)))]dt \\ &+ \int_0^\xi \left[p_0(\eta) \frac{dx_0}{d\eta} - H(x_0, p_0(\eta)) \frac{dt_0}{d\eta} \right] d\eta, \end{aligned} \tag{15}$$

where $t_0 = 0$. Since x_0 is fixed, one obtains

$$\int_0^\xi \left[p_0(\eta) \frac{dx_0}{d\eta} - H(x_0, p_0(\eta)) \frac{dt_0}{d\eta} \right] d\eta = 0$$

for every $\xi \in [-\varepsilon, \varepsilon]$. Then, taking into account Eq. (7), a combination of Eq. (15) with Eq. (14) leads to

$$\begin{aligned} \int_0^{t_f} \bar{\rho}(t)dt &= \int_0^{t_f} [\bar{p}(t)\dot{\bar{x}}(t) - H(\bar{x}(t), \bar{p}(t))]dt \\ &= -J(\xi) + \int_0^{t_f} [p(t, p_0(\xi))\dot{x}(t, p_0(\xi)) - H(x(t, p_0(\xi)), p(t, p_0(\xi)))]dt \\ &= -J(\xi) + \int_0^{t_f} \rho(t, p_0(\xi))dt \\ &\leq -J(\xi) + \int_0^{t_f} \rho_*(t)dt. \end{aligned} \tag{16}$$

Since $J(0) = 0$, Eq. (13) implies that the strict inequality

$$\int_0^{t_f} \bar{\rho}(t)dt < \int_0^{t_f} \rho_*(t)dt, \tag{17}$$

holds if $\xi \neq 0$ or $x_*(t_f) \neq \bar{x}(t_f)$. For the case of $x_*(t_f) = \bar{x}(t_f)$, Eq. (17) is satisfied as well according to Theorem 1, which proves that Eq. (13) is a sufficient condition. Let us finally prove that Eq. (13) is a necessary condition. Assume Eq. (13) is not satisfied; there exists a smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$ and a $\xi \in [-\varepsilon, \varepsilon] \setminus \{0\}$ such that $J(\xi) \leq J(0) = 0$. Then, according to Eq. (16), one obtains

$$\int_0^{t_f} \bar{\rho}(t)dt \geq \int_0^{t_f} \rho(t, p_0(\xi))dt.$$

By restricting \mathcal{P} , we have $\Pi(\mathcal{P}) \subset \Pi(\mathcal{L}_{\bar{p}_0})$. Therefore, the extremal trajectory $\Pi(\gamma(\cdot, p_0(\xi)))$ in $\Pi(\mathcal{L}_{\bar{p}_0})$ is an admissible trajectory of the L¹-minimization problem (cf. Remark 7), which proves the proposition. \square

Proposition 3 *Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular (cf. Assumption 1) and Conditions 1 and 2 are satisfied, let $l < n$. Then, if $\varepsilon > 0$ is small enough, the inequality $J''(0) \geq 0$ (resp. the strict inequality $J''(0) > 0$) for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$ is a necessary condition (resp. a sufficient condition) for the strict strong-local optimality of the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$.*

Proof Since the final time t_f is fixed, Eq. (12) is reduced as

$$J(\xi) = \int_0^\xi \lambda(\eta)y'(\eta)d\eta.$$

Taking derivative of $J(\xi)$ with respect to ξ yields

$$J'(\xi) = \lambda(\xi) \cdot y'(\xi). \tag{18}$$

Note that $\lambda(0) = \bar{p}(t_f)$. Taking into account Eq. (8), for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$, we have $J'(0) = \lambda(0)y'(0) = 0$ since $y'(0)$ is a tangent vector of the submanifold \mathcal{M} at $\bar{x}(t_f)$. Then, according to Proposition 2, this proposition is proved. \square

Definition 7 Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, denote by $\bar{v} \in (\mathbb{R}^l)^*$ the vector of the Lagrangian multipliers of this extremal such that

$$\bar{p}(t_f) = \bar{v}d\phi(\bar{x}(t_f)).$$

Proposition 4 *In the case of $l < n$, given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that each switching point is regular (cf. Assumption 1), assume Conditions 1 and 2 are satisfied. Then, the inequality $J''(0) \geq 0$ (resp. strict inequality $J''(0) > 0$) is satisfied for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$ if and only if there holds*

$$\zeta^T \left\{ \frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v} d^2\phi(\bar{x}(t_f)) \right\} \zeta \geq 0 \text{ (resp. } > 0),$$

for every tangent vector $\zeta \in T_{\bar{x}(t_f)}\mathcal{M} \setminus \{0\}$.

Proof Differentiating $J'(\xi)$ in Eq. (18) with respect to ξ yields

$$J''(\xi) = \lambda'(\xi)y'(\xi) + \lambda(\xi)y''(\xi). \tag{19}$$

Then, differentiating $\phi(y(\xi))$ with respect to ξ yields

$$\begin{aligned} \frac{d}{d\xi}\phi(y(\xi)) &= d\phi(y(\xi))y'(\xi) = 0, \\ \frac{d^2}{d\xi^2}\phi(y(\xi)) &= [d^2\phi(y(\xi))y'(\xi)]y'(\xi) + d\phi(y(\xi))y''(\xi) = 0. \end{aligned} \tag{20}$$

Since $(\bar{x}(t_f), \bar{p}(t_f)) = (y(0), \lambda(0))$, according to the definition of the vector \bar{v} in Definition 7, one immediately has $\lambda(0) = \bar{v}d\phi(y(0))$. Thus, multiplying \bar{v} on both sides of Eq. (20) and fixing $\xi = 0$, we obtain

$$\begin{aligned} \bar{v} \frac{d^2\phi(y(0))}{d\xi^2} &= \lambda(0)y''(0) + \bar{v} [d^2\phi(y(0))y'(0)]y'(0) \\ &= \lambda(0)y''(0) + [y'(0)]^T [\bar{v}d^2\phi(y(0))]y'(0) \\ &= 0. \end{aligned}$$

Substituting this equation into Eq. (19) yields

$$J''(0) = \lambda'(0)y'(0) - [y'(0)]^T [\bar{v}d^2\phi(y(0))]y'(0). \tag{21}$$

Note that we have

$$\begin{aligned} y'(\xi) &= \frac{dx(t_f, p_0(\xi))}{d\xi} = \frac{\partial x(t_f, p_0(\xi))}{\partial p_0} [p'_0(\xi)]^T, \\ [y'(\xi)]^T &= \frac{dp^T(t_f, p_0(\xi))}{d\xi} = \frac{\partial p^T(t_f, p_0(\xi))}{\partial p_0} [p'_0(\xi)]^T. \end{aligned} \tag{22}$$

Since the matrix $\frac{\partial x(t_f, p_0(\xi))}{\partial p_0}$ is nonsingular if Condition 1 is satisfied, we have

$$[p'_0(\xi)]^T = \left[\frac{\partial x(t_f, p_0(\xi))}{\partial p_0} \right]^{-1} y'(\xi).$$

Substituting this equation into Eq. (22) yields

$$[\lambda'(\xi)]^T = \frac{\partial p^T(t_f, p_0(\xi))}{\partial p_0} \left[\frac{\partial x(t_f, p_0(\xi))}{\partial p_0} \right]^{-1} y'(\xi).$$

Substituting again this equation into Eq. (21) and taking into account $\bar{p}_0 = p_0(0)$ and $\bar{x}(t_f) = y(0)$, we eventually get that for every smooth curve $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ on $[-\varepsilon, \varepsilon]$ there holds

$$J''(0) = [y'(0)]^T \left\{ \frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v}d^2\phi(\bar{x}(t_f)) \right\} y'(0).$$

Note that the vector $y'(0)$ can be an arbitrary vector in the tangent space $T_{\bar{x}(t_f)}\mathcal{X} \setminus \{0\}$, which proves the proposition. □

Condition 3 Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, let

$$\zeta^T \left\{ \frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v}d^2\phi(\bar{x}(t_f)) \right\} \zeta > 0,$$

be satisfied for every vector $\zeta \in T_{\bar{x}(t_f)}\mathcal{M} \setminus \{0\}$.

Then, as a combination of Propositions 3 and 4, we obtain the following result.

Theorem 2 Given the extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ such that every switching point is regular (cf. Assumption 1), let $l < n$. Then, if Conditions 1, 2, and 3 are satisfied, the extremal trajectory $\bar{x}(\cdot)$ on $[0, t_f]$ realizes a strict strong-local optimum.

Consequently, in the case of $l < n$, Conditions 1, 2, and 3 are sufficient to guarantee a bang-bang extremal with regular switching points to be a strict strong-local optimum. In next section, the numerical implementation for these three conditions will be derived.

5 Numerical implementation for sufficient optimality conditions

In this section, we assume that the reference extremal $(\bar{x}(\cdot), \bar{p}(\cdot)) = \gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$ is computed by applying necessary conditions, which means that the final time t_f and the switching time t_i are known before testing sufficient conditions. In accordance with Definition 7, the vector \bar{v} of Lagrangian multipliers in Condition 3 can be computed by

$$\bar{v} = \bar{p}(t_f)d\phi^T(\bar{x}(t_f)) \left[d\phi(\bar{x}(t_f))d\phi^T(\bar{x}(t_f)) \right]^{-1}. \tag{23}$$

Definition 8 We define by $C \in \mathbb{R}^{n \times (n-l)}$ a full-rank matrix such that its columns form a basis of the tangent space $T_{\bar{x}(t_f)}\mathcal{M}$.

One immediately gets that Condition 3 is satisfied if and only if there holds

$$C^T \left\{ \frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v}d^2\phi(\bar{x}(t_f)) \right\} C > 0. \tag{24}$$

The matrix C can be computed by a simple Gram-Schmidt process once the explicit expression of the matrix $d\phi(\bar{x}(t_f))$ is derived. Thus, it amounts to compute the matrix $\frac{\partial x}{\partial p_0}(\cdot, \bar{p}_0)$ on $[0, t_f]$ and the matrix $\frac{\partial p^T}{\partial p_0}(\cdot, \bar{p}_0)$ at t_f in order to test Conditions 1, 2, and 3.

It follows from the classical results about solutions to ODEs that the extremal trajectory $(x(t, p_0), p(t, p_0))$ and its time derivative are continuously differentiable with respect to p_0

on $[0, t_f]$. Thus, taking derivative of Eq. (5) with respect to p_0 on each segment (t_i, t_{i+1}) , we obtain

$$\begin{bmatrix} \frac{d}{dt} \frac{\partial x}{\partial p_0}(t, \bar{p}_0) \\ \frac{d}{dt} \frac{\partial p^T}{\partial p_0}(t, \bar{p}_0) \end{bmatrix} = \begin{bmatrix} H_{px}(\bar{x}(t), \bar{p}(t)) & H_{pp}(\bar{x}(t), \bar{p}(t)) \\ -H_{xx}(\bar{x}(t), \bar{p}(t)) & -H_{xp}(\bar{x}(t), \bar{p}(t)) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial p_0}(t, \bar{p}_0) \\ \frac{\partial p^T}{\partial p_0}(t, \bar{p}_0) \end{bmatrix}. \tag{25}$$

Since the initial point x_0 is fixed, one can obtain the initial conditions as

$$\frac{\partial x}{\partial p_0}(0, \bar{p}_0) = 0_n, \quad \frac{\partial p^T}{\partial p_0}(0, \bar{p}_0) = I_n, \tag{26}$$

where 0_n and I_n denote the zero and identity matrix of $\mathbb{R}^{n \times n}$, respectively. Note that the two matrices $\frac{\partial x}{\partial p_0}(\cdot, \bar{p}_0)$ and $\frac{\partial p^T}{\partial p_0}(\cdot, \bar{p}_0)$ are discontinuous at each switching time t_i . Comparing with (Schättler and Ledzewicz 2012; Noble and Schättler 2002; Chen et al. 2016), the update formulas for the two matrices $\frac{\partial x}{\partial p_0}(\cdot, \bar{p}_0)$ and $\frac{\partial p^T}{\partial p_0}(\cdot, \bar{p}_0)$ at each switching time t_i can be written as

$$\frac{\partial x}{\partial p_0}(t_i+, \bar{p}_0) = \frac{\partial x}{\partial p_0}(t_i-, \bar{p}_0) - \Delta\rho_i f_1(x(t_i), \omega(t_i))dt_i(\bar{p}_0), \tag{27}$$

$$\frac{\partial p^T}{\partial p_0}(t_i+, \bar{p}_0) = \frac{\partial p^T}{\partial p_0}(t_i-, \bar{p}_0) + \Delta\rho_i \frac{\partial f_1}{\partial x}(x(t_i), \omega(t_i))p^T(t_i)dt_i(\bar{p}_0), \tag{28}$$

where $\Delta\rho_i = \rho(t_i+) - \rho(t_i-)$. Up to now, every requested quantity but $dt_i(\bar{p}_0)$ can be explicitly computed. For every $p_0 \in \mathcal{P}$ there holds

$$H_1(x(t_i(p_0), p_0), p(t_i(p_0), p_0)) = 0. \tag{29}$$

Taking into account $\dot{H}_1(x(t), p(t)) = H_{01}(x(t), p(t))$ and differentiating Eq. (29) with respect to p_0 yields

$$\begin{aligned} 0 &= H_{01}(x(t_i, p_0), p(t_i, p_0))dt_i(p_0) + p(t_i, p_0) \frac{\partial f_1}{\partial x}(x(t_i, p_0), \omega(t_i, p_0)) \frac{\partial x(t_i, p_0)}{\partial p_0} \\ &\quad + f_1^T(x(t_i, p_0), \omega(t_i, p_0)) \frac{\partial p^T(t_i, p_0)}{\partial p_0}. \end{aligned}$$

By virtue of Assumption 1, there holds $H_{01}(\bar{x}(t_i), \bar{p}(t_i)) \neq 0$ for $i = 1, 2, \dots, k$. Thus, we obtain

$$\begin{aligned} dt_i(\bar{p}_0) &= - \left[p(t_i, \bar{p}_0) \frac{\partial f_1}{\partial x}(x(t_i, p_0), \omega(t_i, p_0)) \frac{\partial x(t_i, \bar{p}_0)}{\partial p_0} \right. \\ &\quad \left. + f_1^T(x(t_i, \bar{p}_0), \omega(t_i, \bar{p}_0)) \frac{\partial p^T(t_i, \bar{p}_0)}{\partial p_0} \right] / H_{01}(\bar{x}(t_i), \bar{p}(t_i)). \end{aligned}$$

Therefore, in order to compute the two matrices $\frac{\partial x}{\partial p_0}(\cdot, \bar{p}_0)$ and $\frac{\partial p^T}{\partial p_0}(\cdot, \bar{p}_0)$ on $[0, t_f]$, it is sufficient to choose the initial condition in Eq. (26), then to numerically integrate Eq. (25) and to use Eqs. (27) and (28) when a switching point is encountered.

According to (Chen et al. 2016), given every bang-bang extremal $\gamma(\cdot, \bar{p}_0)$ on $[0, t_f]$, $\delta(\cdot)$ is a constant on zero-thrust arcs. Hence, to check optimality, it suffices to test the zero of $\delta(\cdot)$ on each maximum-thrust arc and to test the non-positivity of $\delta(t_i-)\delta(t_i+)$ at each switching time t_i .

6 Orbital transfer computation

In this section, we consider the three-body problem defined by Earth, Moon and a spacecraft. Since the orbits of the Earth and the Moon around their common centre of mass are nearly circular (the eccentricity is around 5.49×10^{-2}), and since the mass of a spacecraft is negligible compared with that of the two celestial bodies, the CRTBP is valid (see [Szebehely 1967](#)). The physical parameters corresponding to the Earth-Moon system are $\mu = 1.2153 \times 10^{-2}$, $d_* = 384,400.00$ km, and $t_* = 3.7521 \times 10^5$ seconds (or 4.3427 days). The initial mass of the spacecraft is specified as $m_* = 500$ kg, the maximum thrust of the spacecraft engine is 1.0 N,

$$\tau_{max} = 1.0 \frac{t_*^2}{m_* d_*},$$

so that the initial maximum acceleration is $2.0 \times 10^{-3} \text{ m}^2/\text{s}$. The spacecraft initially evolves on a circular geosynchronous Earth orbit (GEO) lying on the XY -plane such that the radius of the initial orbit is $r_g = 42,165.00$ km. When the spacecraft moves to the point on X -axis between the Earth and the Moon, i.e., $\|r(0)\| = r_g/d_* - \mu$, we start to control the spacecraft to reach a circular orbit around the Moon with radius $r_m = 13,069.60$ km such that the L¹-norm of control is minimized at the fixed final time $t_f = 38.46$ days. Accordingly, the initial state $x_0 = (r_0, v_0, m_0)$ is given as

$$r_0 = (r_g/d_* - \mu, 0, 0)^T, \quad v_0 = (0, v_g, 0)^T, \quad \text{and } m_0 = 500/m_*,$$

where v_g is the non-dimensional velocity taking the value such that, without any control, the spacecraft moves freely on the GEO, and an explicit expression of the function ϕ in Eq. (3) is

$$\phi(x_f) = \begin{bmatrix} \frac{1}{2} \|r(t_f) - [1 - \mu, 0, 0]^T\|^2 - \frac{1}{2}(r_m/d_*)^2 \\ \frac{1}{2} \|v(t_f)\|^2 - \frac{1}{2}v_m^2 \\ v^T(t_f) \cdot (r(t_f) - [1 - \mu, 0, 0]^T) \\ r^T(t_f) \cdot 1_Z \\ v^T(t_f) \cdot 1_Z \end{bmatrix}, \tag{30}$$

where $1_z = [0, 0, 1]^T$ denotes the unit vector of the Z -axis of the rotating frame $OXYZ$ and v_m is the non-dimensional velocity taking the value such that the spacecraft, once steered to a point x_f with $\phi(x_f) = 0$, will freely move on the final circular orbit around the Moon with radius r_m .

We consider the constant mass model, $\beta = 0$, since this model captures the main features of the original problem (see [Chen et al. 2016](#); [Caillau et al. 2012](#); [Caillau and Daoud 2012](#)). In this case, the mass m is a constant parameter instead of a state in the system Σ , so $x = (r, v)$ and $p = (p_r, p_v)$. First, we compute the extremal $(\bar{x}(\cdot), \bar{p}(\cdot))$ on $[0, t_f]$. We search a zero of the shooting function corresponding to a two-point boundary value problem ([Pan et al. 2013](#)). A simple shooting method does not allow one to solve this problem because one does not know *a priori* the structure of the optimal control. Moreover, the numerical computation of the shooting function and its differential may be intricate, as the function may not even be differentiable (typically at points corresponding to a change in the structure of the control strategy, that is a change in the number of switchings, here). We use a regularization procedure in ([Caillau et al. 2012](#)) that smoothes the controls discontinuities and get an energy-optimal

trajectory first, then use a homotopy method to find the real trajectory with a bang-bang control. Note that both the initial point x_0 and the final constraint submanifold \mathcal{M} lie on the XY -plane, in order that the whole trajectory lies on the XY -plane as well. Fig. 5 illustrates the (non-dimensional) profile of the position vector r along the computed extremal trajectory. The profiles of ρ , $\|p_v\|$, and H_1 with respect to time are shown in Fig. 6; we can see that the number of maximum-thrust arcs is 15 with 29 switching points and that the regularity condition in Assumption 1 at every switching point is satisfied. Since the extremal trajectory is computed thanks to necessary conditions, one has to check sufficient optimality conditions to make sure that it is at least locally optimal. According to what has been developed in Sect. 4, it suffices to check if Conditions 1, 2, and 3 are satisfied. Using Eqs. (25–28), one can compute $\delta(\cdot)$ on $[0, t_f]$. In order to have a clear view, the profile of $\delta(\cdot)$ on $[0, t_f]$ is rescaled by $\text{sgn}(\delta(\cdot)) * |\delta(\cdot)|^{1/12}$ (see Fig. 7). We can see that there exist no sign changes at switching points, and no zeros on smooth bang subarcs. Thus, Conditions 1 and 2 are satisfied along

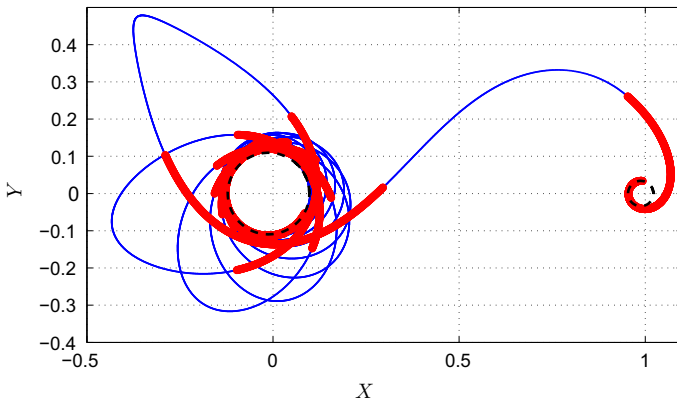


Fig. 5 Non-dimensional profile of the position vector r of the L^1 -minimization trajectory in the rotating frame $OXYZ$. The thick curves are the maximum-thrust arcs, while the thin curves are the zero-thrust ones. The bigger dashed circle and the smaller one are the initial and final circular orbits around the Earth and the Moon, respectively

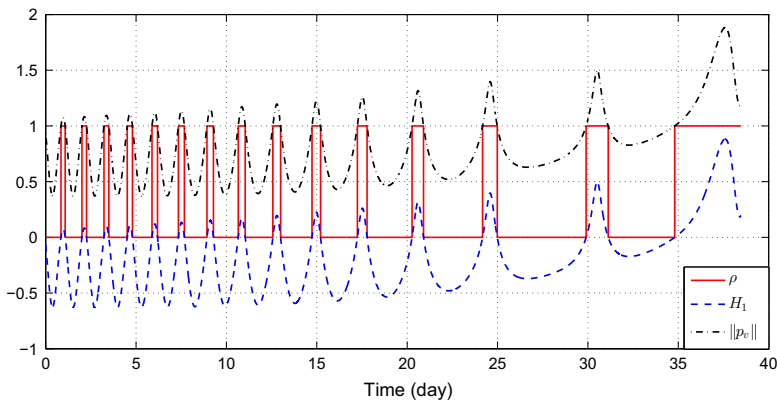


Fig. 6 Profiles of ρ , $\|p_v\|$, and H_1 with respect to time along the L^1 -minimization trajectory

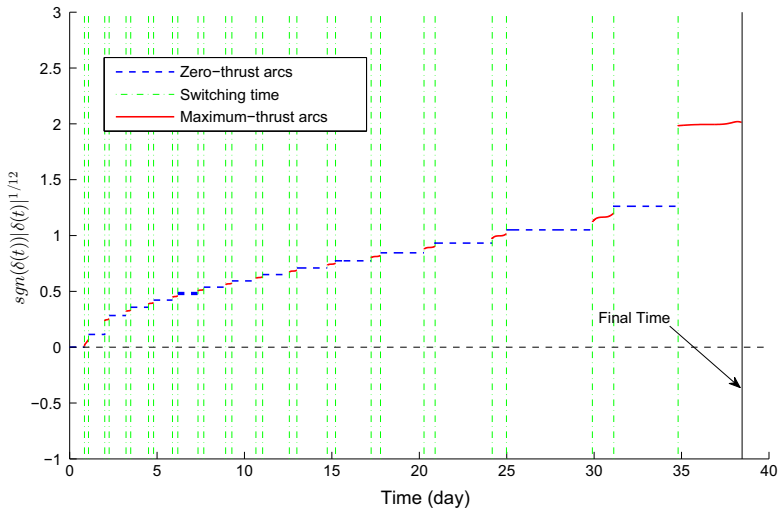


Fig. 7 The profile of $sgn(\delta(t))|\delta(t)|^{1/12}$ with respect to time along the L¹-minimization extremal

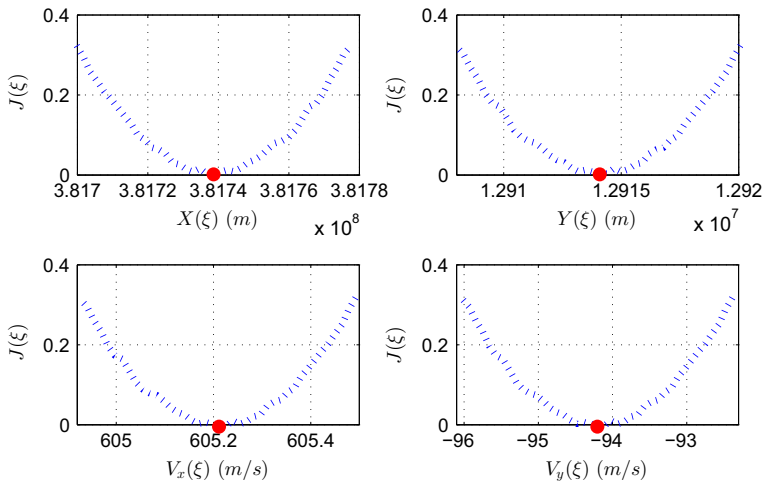


Fig. 8 Let $X(\xi)$ and $Y(\xi)$ be the projection of the position vector $r(\xi)$ on X - and Y -axis of the rotating frame $OXYZ$, respectively, and let $V_x(\xi)$ and $V_y(\xi)$ be the projection of the velocity vector $v(\xi)$ on X - and Y -axis of the rotating frame $OXYZ$, respectively. The figure plots the profiles $J(\xi)$ with respect to $X(\xi)$, $Y(\xi)$, $V_x(\xi)$, and $V_y(\xi)$. The dots on each plot denote $(J(0), y(0))$

the computed extremal. To check Condition 3, differentiating $\phi(\cdot)$ in Eq. (30) yields

$$d\phi(\bar{x}(t_f)) = \begin{bmatrix} r(t_f) - [1 - \mu, 0, 0]^T & 0_{3 \times 1} & v(t_f) & 1_Z & 0_{3 \times 1} \\ 0_{3 \times 1} & v(t_f) & r(t_f) - [1 - \mu, 0, 0]^T & 0_{3 \times 1} & 1_Z \end{bmatrix}^T, \quad (31)$$

and

$$d^2\phi_1(\bar{x}(t_f)) = \begin{pmatrix} I_3 & 0_3 \\ 0_3 & 0_3 \end{pmatrix}, \quad d^2\phi_2(\bar{x}(t_f)) = \begin{pmatrix} 0_3 & 0_3 \\ 0_3 & I_3 \end{pmatrix},$$

$$d^2\phi_3(\bar{x}(t_f)) = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}, \quad d^2\phi_4(\bar{x}(t_f)) = d^2\phi_5(\bar{x}(t_f)) = 0_6,$$

where $\phi_i(\cdot) : \mathcal{X} \rightarrow \mathbb{R}$, $x \mapsto \phi_i(x)$ for $i = 1, 2, \dots, l$ are the elements of the vector-valued function $\phi(x)$. Then, substituting the values of $\bar{x}(t_f)$ and $\bar{p}(t_f)$ into Eq. (23), the vector \bar{v} can be computed. With the exception of the matrix C , all quantities in Eq. (24) are obtained. One can use a Gram-Schmidt process to compute the matrix C associated with the matrix in Eq. (31). Substituting numerical values into Eq. (24), we eventually obtain

$$C^T \left\{ \frac{\partial p^T(t_f, \bar{p}_0)}{\partial p_0} \left[\frac{\partial x(t_f, \bar{p}_0)}{\partial p_0} \right]^{-1} - \bar{v} d^2\phi(\bar{x}(t_f)) \right\} C \approx 0.5292 \succ 0.$$

Thus, Condition 3 is satisfied. Fig. 8 shows the profile of $J(\cdot)$ with respect to $y(\cdot) \in \mathcal{M} \cap \mathcal{N}$ in a small neighbourhood of $\bar{x}(t_f)$. One can clearly see that $J(\cdot) > J(0)$ on $[-\varepsilon, \varepsilon] \setminus \{0\}$. All the conditions in Theorem 2 are satisfied, So the computed L^1 -minimization trajectory realizes a strict strong-local optimum in C^0 -topology.

7 Conclusion

In this paper, the Pontryagin maximum principle is first used to derive the Hamiltonian system associated with the L^1 -minimization problem for the translational motion of a spacecraft in the CRTBP, showing that the optimal control can be bang-bang or singular. The singular extremals are of at least order two, revealing the existence of the Fuller (or chattering) phenomenon. To establish sufficient optimality conditions in the bang-bang case, a parameterized family of extremals is constructed. We obtain that conjugate points (focal points if one considers a genuine submanifold target) may occur not only on maximum-thrust arcs between switching times but also at switching times. Directly applying the theory of field of extremals, we obtain that the disconjugacy conditions (cf. Conditions 1 and 2) are sufficient to guarantee that an extremal is locally optimal if the endpoints are fixed. When the dimension of the final constraint submanifold is not zero, we establish a further second-order condition (cf. Condition 3) that is sufficient for the strict strong-local optimality of a bang-bang extremal if the disconjugacy conditions are satisfied. In addition, the numerical implementation of these optimality conditions is derived. Finally, an example of transferring a spacecraft from a circular orbit around the Earth to an orbit around the Moon is computed in the CRTBP model. The second-order sufficient optimality conditions developed in this paper are tested to show that the computed extremal realizes a strict strong-local optimum. The sufficient optimality conditions for problems with free final time will be considered in a future work.

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References

Agrachev, A.A., Sachkov, Y.L.: Control Theory from the Geometric Viewpoint, Encyclopedia of Mathematical Sciences, vol. 87. Springer, Berlin (2004)

- Agrachev, A.A., Stefani, G., Zezza, P.: Strong optimality for a bang-bang trajectory. *SIAM J. Control Optim.* **41**(4), 1991–2041 (2002)
- Bonnard, B., Caillaud, J.B., Trélat, E.: Second-order optimality conditions in the smooth case and applications in optimal control. *ESAIM Control Optim. Calc. Var.* **13**(2), 207–236 (2007)
- Breakwell, J.V., Ho, Y.C.: On the conjugate point condition for the control problem. *Int. J. Eng. Sci.* **2**, 565–579 (1965)
- Brusch, R.G., Vincent, T.L.: Numerical implementation of a second-order variable endpoint condition. *AIAA J.* **8**(12), 2230–2235 (1970)
- Bryson, A., Ho, Y.: *Applied Optimal Control*. Blaisdell, Waltham (1969)
- Caillaud, J.B., Daoud, B.: Minimum time control of the restricted three-body problem. *SIAM J. Control Optim.* **50**(6), 3178–3202 (2012)
- Caillaud, J.B., Daoud, B., Gergaud, J.: Minimum fuel control of the planar circular restricted three-body problem. *Celest. Mech. Dyn. Astron.* **114**, 137–150 (2012)
- Chen, Z., Caillaud, J.B., Chitour, Y.: L^1 -minimization for mechanical systems. *SIAM J. Control Optim.* **54**(3), 1245–1265 (2016)
- Flies, M., Hazewinkel, M. (eds.): *Envelopes, Conjugate Points and Optimal Bang-Bang Extremals*. In: *Proceedings 1985 Paris Conference on Nonlinear Systems*, Reidel Publishers, Dordrecht, the Netherlands (1987)
- Gergaud, J., Haberkorn, T.: Homotopy method for minimum consumption orbital transfer problem. *ESAIM Control Optim. Calc. Var.* **12**, 294–310 (2006)
- Ghezzi, R., Caponigro, M., Piccoli, B., Trélat, E.: Regularization of chattering phenomena via bounded variation controls. In: *NetCo 2014*, Tours, France (2014)
- Kelley, H.J., Kopp, R.E., Moyer, A.G.: *Singular Extremals, Optimization-Theory and Applications*, vol. 1. Academic Press, New York (1966)
- Kupka, I.: Geometric theory of extremals in optimal control problems i: the fold and Maxwell case. *Trans. Am. Math. Soc.* **299**(1), 225–243 (1987)
- Lawden, D.F.: *Optimal Trajectories for Space Navigation*. Butterworth, London (1963)
- Marchal, C.: Chattering arcs and chattering controls. *J. Optim. Theory Appl.* **11**(5), 441–486 (1973)
- Mermau, P.M., Powers, W.F.: Conjugate point properties for linear quadratic problems. *J. Math. Anal Appl.* **55**, 418–433 (1976)
- Mingotti, G., Toppoto, F., Bernelli-Zazzera, F.: Low-energy, low-thrust transfers to the Moon. *Celest. Mech. Dyn. Astron.* **105**(1–3), 61–74 (2009)
- Noble, J., Schättler, H.: Sufficient conditions for relative minima of broken extremals in optimal control theory. *J. Math. Anal. Appl.* **269**, 98–128 (2002)
- Ozimek, M.T., Howell, K.C.: Low-thrust transfers in the earth-moon system, including applications to libration point orbits. *J. Guid. Control Dyn.* **33**(2), 533–549 (2010)
- Pan, B., Chen, Z., Lu, P., Gao, B.: Reduced transversality conditions for optimal space trajectories. *J. Guid. Control Dyn.* **36**(5), 1289–1300 (2013)
- Park, C.: Necessary conditions for the optimality of singular arcs of spacecraft trajectories subject to multiple gravitational bodies. *Adv. Space Res.* **51**, 2125–2135 (2013)
- Poggiolini, L., Stefani, G.: State-local optimality of a bang-bang trajectory: a hamiltonian approach. *Syst. Control Lett.* **53**, 269–279 (2004)
- Pontryagin, L.S., Boltyanski, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: *The mathematical theory of optimal processes* (K. N. Trifogoff, Trans. from the Russian). In: Neustadt, L. W. (ed.). Interscience Publishers, London (1962)
- Robbins, H.M.: Optimality of intermediate-thrust arcs of rocket trajectories. *AIAA J.* **3**(6), 1094–1098 (1965)
- Ross, S.D., Scheeres, D.J.: Multiple gravity assists, capture, and escape in the restricted three-body problem. *SIAM J. Appl. Dyn. Syst.* **6**(3), 576–596 (2007)
- Sarychev, A.V.: The index of second variation of a control system. *Matematicheskii Sb.* **41**, 338–401 (1982)
- Schättler, H., Ledzewicz, U.: *Geometric Optimal Control: Theory, Methods, and Examples*. Springer, Berlin (2012)
- Szebehely, V.: *Theory of Orbits: The Restricted Problem of Three Bodies*. Academic Press, Cambridge (1967)
- Wood, L.J.: Second-order optimality conditions for the Bolza problem with endpoints variable. *J. Aircr.* **11**(4), 212–221 (1974)
- Zelikin, M.I., Borisov, V.F.: *Theory of Chattering Control with Applications to Astronautics, Robotics, Economics, and Engineering*. Birkhäuser, Basel (1994)
- Zelikin, M.I., Borisov, V.F.: Optimal chattering feedback control. *J. Math. Sci.* **114**(3), 1227–1344 (2003)
- Zhang, C., Toppoto, F., Bernelli-Zazzera, F., Zhao, Y.: Low-thrust minimum-fuel optimization in the circular restricted three-body problem. *J. Guid. Control Dyn.* **38**(8), 1501–1510 (2015)