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ORIGINAL ARTICLE

Dynamics of an isolated, viscoelastic, self-gravitating body

C. Ragazzo[1] · L. S. Ruiz[2]

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Abstract This paper is devoted to an alternative model for a rotating, isolated, self-gravitating, viscoelastic body. The initial approach is quite similar to the classical one, present in the works of Dirichlet, Riemann, Chandrasekhar, among others. Our main contribution is to present a simplified model for the motion of an almost spherical body. The Lagrangian function $\mathscr{L}$ and the dissipation function $\mathscr{D}$ of the simplified model are:

$$\mathscr{L} = \frac{\omega \cdot \mathrm{I}\omega}{2} + \frac{1}{36\,\mathrm{I}_\circ}(\|\dot{Q}\|^2 - \gamma\|Q\|^2)$$

and

$$\mathscr{D} = \frac{\nu}{36\,\mathrm{I}_\circ}\|\dot{Q}\|^2$$

where ω is the angular velocity vector, Q is the quadrupole moment tensor, $\mathrm{I} = \mathrm{I}_\circ \mathbb{I}\mathrm{d} - Q/3$ is the usual moment of inertia tensor with $\mathrm{I}_\circ$ equal to the moment of inertia of the spherical body at rest, γ is an elastic constant, and ν is a damping coefficient. The angular momentum $\mathrm{I}\omega$ transformed to an inertial reference frame is conserved. The constants γ and ν must be determined experimentally. We believe this to be the simplest model one can get without loosing the symmetries and the conserved quantities of the original problem. This model can be used as a building block for the study of many-body planetary systems.

Keywords Extended body · Dissipative forces · Rotation · Pseudo-rigid body

✉ L. S. Ruiz
lucasruiz@unifei.edu.br

C. Ragazzo
ragazzo@ime.usp.br

1 Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, SP 05508-090, Brazil

2 Instituto de Matemática Computação, Universidade Federal de Itajubá, Itajubá, MG 37.500-903, Brazil

1 Introduction

The problem of the equilibrium shapes that a rotating isolated, incompressible, ideal fluid can attain goes back to Newton in the *Principia Mathematica*. Generations of important scientists contributed to the understanding of this theme, which remains as a fruitful source of questions. For a brief historical review and general exposition, see Chandrasekhar (1987). Classical treatments can be found in Lamb (1932), Darwin (1886), Poincaré (1885) and Love (1944). The dynamics of the fluid is determined by a set of partial differential equations. Solutions to these equations that are steady in a rotating reference frame are called relative equilibria. They are important in the shape modeling of celestial bodies. Questions on stability of the known equilibria are still open in spite of the celebrated Poincaré's work on the subject (Poincaré 1885). The mathematical complexity of the equations is a challenge to numerical analysts and physicists. More realistic models for the shape of stars and planets may include compressibility, strain-forces, inhomogeneities, etc, which further increase the difficulty of the problem.

From the perspective of Celestial Mechanics, while a planet or a star is physically perceived as an object of finite size it is usually modeled as a point mass characterized only by its center of mass position. Although the point-mass assumption has been very successful in the study of planetary motion it precludes the analysis of some important phenomena like, for instance, dissipation of energy due to tides. An attempt to overcome the limitations of the point-mass model without introducing the infinitely many degrees of freedom of an extended body is provided by the so-called pseudo-rigid body: "a point to which is attached a measure of orientation and deformation", see Cohen and Muncaster (1988). Formally a pseudo-rigid body model is obtained in the following way.

Suppose that a body at rest has the shape of a ball $\mathscr{B} \subset \mathbb{R}^3$ with radius $R > 0$ (reference configuration). Let $x \in \mathscr{B}$ denote the initial position of a point in the body and $\phi(t, x) \in \mathbb{R}^3$ denote the position at time t of that point. The map $\phi(t, \cdot) : \mathscr{B} \longrightarrow \mathbb{R}^3$ determines the configuration of the body at time t. We remark that the description of continuum mechanics we are using in which the independent spatial variable is the material point $x \in \mathscr{B}$ is called the Lagrangian description (or material description). An alternative description of continuum mechanics in which the fixed point in space is the independent spatial variable is called the Eulerian description (or spatial description). In principle a configuration can be given by an arbitrary diffeomorphism $\phi : \mathscr{B} \longrightarrow \mathbb{R}^3$. The crucial hypothesis in the pseudo-rigid body formulation is that any configuration of the body is constrained to be of the form:

$$\phi(t, x) = G(t)x, \tag{1}$$

where $G(t)$ is an invertible matrix. In this paper we further assume that the body is incompressible, which means $\det G(t) = 1$. For simplicity, we suppose that the center of mass is fixed at the origin.

The assumption (1) was first proposed by Dirichlet. Using a suitable decomposition for G (see, for example, Chapter 4 of Chandrasekhar (1987)), Dirichlet was able to find some equilibrium shapes for a rotating isolated and incompressible fluid under self-gravity. Riemann, following Dirichlet, gave great contributions classifying new equilibria and deciding about their stability under the pseudo-rigid body constraint (1) (Riemann 1860) (see Riemann (2007) for a Portuguese translation of this paper or Oliva (2007) for an expository discussion on Riemann's work). See Borisov et al. (2009) or Kristiansen et al. (2012) for a modern Hamiltonian deduction of the equations of motion used by Riemann and for an application to a two-body problem. For figures of equilibrium of an inhomogeneous self-gravitating fluid see Bizyaev et al. (2015). For the Lagrangian mechanics setting, see Holm et al. (2009). For the introduction of elastic forces, see the references in Roberts and Sousa Dias (1999) and Cohen and Muncaster (1988).

The pseudo-rigid body approach raises the question: Are the main dynamical properties of the infinite-degree-of-freedom system preserved under the pseudo-rigid body constraint (1)? There are at least two works, Muncaster (1984a) and Muncaster (1984b), where these questions are analyzed from a mathematical point of view. In these papers it is proved the relationship between *fine* and *coarse* theories, where the former represents a "complete" theory and the later an approximation in which persist the "mean" characteristics of the fine one. In the second paper it is shown that the pseudo-rigid body is a coarse theory for the continuum theory of solids. In this sense, we suppose that results obtained from (1) represent a good approximation to the behavior in the continuous problem. Another comparison between the pseudo-rigid body approach and the *Cosserat point* theory is presented in Nordenholz and O'Reilly (1998).

The aims of this paper are: to introduce energy dissipation due to internal viscosity under the pseudo-rigid body hypothesis (1) and to further simplify the pseudo-rigid body model under the extra hypothesis of small deformations $G(t)^T G(t) \approx \mathbb{I}\mathrm{d} = \mathit{Identity}$. These goals will be accomplished using the Lagrangian formulation of mechanics with an additional Rayleigh dissipation function. The pseudo-rigid body constraint (1) will be imposed by means of D'Alembert's principle and Lagrange multipliers. The conserved quantities will be obtained from the symmetries of both the Lagrangian and the dissipation functions using a suitable generalization of Noether's theorem.

The remainder of the paper is organized as follows. In Sect. 2 the well-known Lagrangian formulation of continuum mechanics is presented. The Lagrangian and the Rayleigh dissipation functions are explicitly given under the usual hypotheses of linear elasticity. In Sect. 3 the pseudo-rigid body hypothesis (1) is used to constrain the continuum-Lagrangian function to a finite number of degrees of freedom. Then the usual polar decomposition $G = YA$, where Y is a rotation matrix and A is a symmetric positive matrix with $\det(A) = 1$, is used to obtain the Lagrangian as a function of A, Y, and their time derivatives. The same is done for the Rayleigh dissipation function. Finally the Euler–Lagrange equations for the constrained motion are obtained in terms of Y and A. Most of the results in Sect. 3 are similar to those found in the pseudo-rigid body literature (see, for instance, Cohen and Muncaster (1988)), except for the introduction of the Rayleigh dissipation function and the qualitative analysis of the dynamics of the system. Our main contribution is given in Sect. 4 and it is presented in the following.

In Sect. 4 we introduce the small-ellipticity hypothesis that stems from the almost round shape of most of the observed rotating celestial bodies. More precisely, let ε denote the ellipticity or flattening of the deformed body defined as:

$$\varepsilon = \frac{\text{equatorial} - \text{polar radius}}{\text{equatorial radius}}, \tag{2}$$

where the instantaneous polar radius is defined as the smallest semi-major axis of the ellipsoid $\{Ax : \|x\| \le R\}$ and the instantaneous equatorial radius is defined as the arithmetic mean of the two remaining semi-major axis. The hypothesis is that ε is much smaller than one. Then $A = \exp(B) \approx \mathbb{I}\mathrm{d} + B$ where B is a symmetric traceless matrix such that

$$\|B\| = \sqrt{B_{11}^2 + B_{12}^2 \cdots} = \sqrt{\mathrm{Tr}(BB^T)} \quad \text{is of the order of } \varepsilon$$

Let $\Omega = Y^T \dot{Y}$ denote the instantaneous angular velocity of the body. The analysis of the relative equilibria solutions to the equations of motion given in Sect. 3 shows that $\varepsilon \ll 1$

requires that $\|\Omega\|$ is of the order of $\sqrt{\varepsilon}$. Using these scalings we obtain our simplified Lagrangian function truncating the pseudo-rigid body Lagrangian function given in Sect. 3 at order ε^2. We remark that a truncation using the equations of motion instead of the Lagrangian function leads to a different result! The advantage of the Lagrangian truncation is that the symmetries of the original Lagrangian function are naturally preserved. The novelty of our work relies on this Lagrangian truncation. We must stress that in the pseudo-rigid body approach the imposed linear deformation (1) neither verifies the differential equations of linear elasticity nor the appropriate boundary conditions. At the end, we obtain an elastic rigidity of the body, with respect to the centrifugal force, that is not so different from that obtained by Love (1944) (see Sect. 4 and Appendices 2 and 3 for more details). We remark that Love obtains a nonlinear deformation as a solution of a linear equation and we propose a linear deformation as an approximation to the real solution.

Motivated by the ideas in the previous paragraph, we propose that under the small ellipticity hypothesis the equations for the motion of a body that is isolated, incompressible, and spherically symmetric at rest, are given by:

$$\ddot{B} + \nu\dot{B} + \gamma B = -\Omega^2 + \frac{1}{3}\mathrm{Tr}(\Omega^2)\mathbb{I}\mathrm{d}, \tag{3}$$

$$\dot{\Omega} + B\dot{\Omega} + \dot{\Omega}B = -\left(\Omega\dot{B} + \dot{B}\Omega + [\Omega^2, B]\right), \tag{4}$$

where

ν is an effective viscosity constant (1/s);
γ is an effective rigidity constant (1/s^2);
Ω is the average angular velocity matrix of the body (1/s)

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \text{with} \quad \|\Omega\|^2 = 2(\omega_1^2 + \omega_2^2 + \omega_3^2); \tag{5}$$

$B(t)$ is the deformation matrix ($dimensionless$) that is proportional to the "quadrupole moment tensor" $Q(t)$:

$$B_{ij} = \frac{1}{3\mathrm{I}_\circ}Q_{ij}, \qquad \text{where} \quad Q_{ij} = \int (3x_i x_j - |x|^2\delta_{ij})\rho(x,t)d^3x \quad \text{and}$$

$\mathrm{I}_\circ$ is the angular momentum (kg m^2) of the body at rest around an arbitrary axis passing through its center of mass.

For this model, Lagrangian, dissipation, and energy functions, and angular momentum with respect to an inertial frame, are given, respectively, by:

$$\mathscr{L}(B, \dot{B}, \Omega) = \frac{\mathrm{I}_\circ}{4}\left(\|\dot{B}\|^2 + \|\Omega\|^2 + 2\mathrm{Tr}(\Omega\Omega^T B)\right) - \frac{\mathrm{I}_\circ}{4}\gamma\|B\|^2, \tag{6}$$

$$\mathscr{D}(\dot{B}) = \nu\frac{\mathrm{I}_\circ}{4}\|\dot{B}\|^2, \tag{7}$$

$$E(B, \dot{B}, \Omega) = \frac{\mathrm{I}_\circ}{4}\left(\|\dot{B}\|^2 + \|\Omega\|^2 + 2\mathrm{Tr}(\Omega\Omega^T B)\right) + \frac{\mathrm{I}_\circ}{4}\gamma\|B\|^2 \tag{8}$$

$$L(Y, \Omega, B) = \mathrm{I}_\circ[Y(\Omega + \Omega B + B\Omega)Y^T]. \tag{9}$$

The angular momentum is conserved under time evolution. The energy function is nonnegative if $\|B\| < 1/2$ (Lemma 11), which is implied by our underlying hypothesis $\|B\| \ll 1$. Moreover, a simple computation gives $\dot{E} = -2\mathscr{D} \leq 0$ with equality being reached if and only if $\dot{B} = 0$. These facts imply (Theorem 4) that any solution to the equations of motion that is initially in the set

$$\left\{(B, \dot{B}, \Omega) : 0 \leq E < \mathrm{I}_\circ \gamma/20, \|B\| < 1/2\right\}$$

is attracted to a relative equilibrium solution where $\dot{B} = 0$ and $\dot{\Omega} = 0$. The eigenvalues of the linearized problem are easily computed from Eqs. (3) and (4) as it is done in Sect. 4. In this section we also use center-manifold arguments to show that the dynamics of the simplified Eqs. (3) and (4) is qualitatively the same, and quantitatively almost the same, as the dynamics of the pseudo-rigid body equations given in Sect. 3 provided that initially $\|B(0)\|$, $\|\dot{B}(0)\|$, $\|\Omega(0)\|$ are small.

Sections 2, 3, and 4 are based on the assumption of an idealized homogeneous elastic body. For this idealized body it is possible to compute from first principles all the constants $I_\circ$, γ, and ν, and to establish the relation between B and Q. For the idealized body, let: M be the mass, R be the radius of the undeformed spherical body, $g = MG/R^2$ be the acceleration of gravity at the body surface, ρ be the density, μ be the elastic (shear) modulus of rigidity (kg/ms^2) (Landau and Lifshitz 1986), and η be the viscosity (shear) coefficient (kg/ms) (Landau and Lifshitz 1986, 1987). For the idealized body:

$$\begin{aligned} \mathrm{I}_\circ &= MR^2 2/5 \quad \text{is the moment of inertia of the solid ball,} \\ \gamma &= \frac{4}{5}\frac{GM}{R^3}\left(1 + \frac{25}{2}\frac{\mu}{g\rho R}\right), \\ \nu &= 40\pi\eta R/3M. \end{aligned} \tag{10}$$

These μ and η are "molecular constants" that in principle can be measured by means of simple laboratory experiments. Nevertheless, it is well-known that these molecular constants are inappropriate for use in most geophysical and astronomical models (see, for instance, Brito et al. (2004), for a discussion about η). So, even for an approximately homogeneous body "effective" constants γ and ν must replace μ and η.

Planets and particularly stars are not homogeneous bodies. Their density is almost radially symmetric with an increasing value towards the center. It is not possible to use the idealized homogeneous body hypothesis in a naive way to study real celestial bodies. For instance, the real moment of inertia of the Sun is $\mathrm{I}_\circ = 0.059\ \mathrm{MR}^2$ while the moment of inertia of the idealized homogeneous Sun is $0.4\ \mathrm{MR}^2$. So, in order to obtain the correct expression $(\mathrm{I}_\circ/2)\sum \omega_i^2$ for the rotational kinetic energy we replaced the idealized moment of inertia $0.4\ \mathrm{MR}^2$ in the Lagrangian function (47) in Sect. 4 by $\mathrm{I}_\circ$ in the Lagrangian function (6). This is equivalent to change the visual radius R of the body for an effective inertial radius (or "radius of gyration") R_g such that $0.4\ \mathrm{MR}_g^2 = \mathrm{I}_\circ$. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity vector associated to the angular velocity matrix Ω and, similarly, let $\boldsymbol{L}$ be the angular momentum vector associated to the angular momentum matrix L. Then the relation $\mathbf{L} = \mathrm{I}\omega$ (valid for inhomogeneous bodies) and Eq. (9) imply that the inertia matrix $\mathrm{I}(t)$ of the body at time t must satisfy

$$\mathrm{I}_{ij} = \mathrm{I}_\circ(\delta_{ij} - B_{ij})$$

This expression and the following relation between the moment of inertia tensor and the moment of quadrupole tensor (valid for non-homogeneous bodies),

$$Q_{ij} = -3\mathrm{I}_{ij} + (\mathrm{Tr\,I})\delta_{ij}$$

imply that $B_{ij} = Q_{ij}/(3\mathrm{I}_\circ)$ as stated above. Finally, the angular velocity matrix Ω in the Lagrangian function (6) must be interpreted as the average angular velocity of the body in

Table 1 Data from bodies of the solar system

Body	ω ($\times 10^{-5}\,\mathrm{s}^{-1}$)	$I_\circ/\mathrm{MR}^2$	J_2 ($\times 10^{-6}$)	γ ($\times 10^{-6}\mathrm{s}^{-2}$)	$\tilde{\gamma}$
Sun	0.338	0.059	0.218	3.092	0.5562
Mercury	0.1240	0.35	50.3	0.0107	0.0072
Venus	0.0299	0.33	4.458	0.0066	0.0042
Earth	7.2921	0.3308	1082.63	1.625	0.9909
Moon	0.26617	0.394	202.7	0.01377	0.0180
Mars	7.0882	0.366	1960.45	0.9380	0.9331
Jupiter	17.5852	0.254	14736.	0.5330	0.9093
Saturn	16.3788	0.21	16298.	0.3457	0.8556
Uranus	10.1237	0.225	3343.43	0.6897	1.024
Neptune	10.8338	0.2555	3411.	0.8792	1.225

the sense that instantaneously $\boldsymbol{L} = \mathrm{I}\omega$. Through this procedure $\varOmega$ is well-defined even for the Sun where the gas angular velocity is known to vary considerably with the latitude and with the distance to the center (see Rozelot and Damiani (2011) and Antia et al. (2008)). In this way we are able to extrapolate the results for an idealized homogeneous body given in Sect. 4 to the results for a inhomogeneous body given above. In the abstract the Lagrangian function (6) is rewritten in a different way.

For the Sun and for some planets of the solar system the value of γ can be obtained in the following way. Let $\varOmega$ and Q be the steady angular velocity and moment of quadrupole tensor of the body given by

$$\varOmega = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = 3\mathrm{I}_\circ B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}.$$

Defining the dynamic form factor $J_2 = \lambda/(MR^2)$, we get from equation (3) that

$$\gamma = \frac{\mathrm{I}_\circ}{MR^2}\frac{\omega^2}{J_2}. \tag{11}$$

The constant γ has the dimension $1/\mathrm{s}^2$. A hypothetical homogeneous body with radius R_g and with elastic modulus of rigidity $\mu = 0$ has $\gamma = (4/5)GM/R_g^3$ according to equation (10). This value can be used to define a "dimensionless γ" as:

$$\tilde{\gamma} = \frac{\gamma}{\frac{4}{5}\frac{GM}{R_g^3}} \quad \text{where} \quad R_g = \sqrt{\frac{5}{2}\frac{I_\circ}{M}} \tag{12}$$

In Table 1 the values of γ and $\tilde{\gamma}$ are given for several bodies of the Solar system. For the planets we used the data provided in (http://nssdc.gsfc.nasa.gov/planetary/factsheet/). For the Sun the value of $\mathrm{I}_\circ$ was taken from (http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html). Since the angular velocity of the gas in the Sun varies considerably with the position, the average angular velocity ω of the Sun was obtained from the formula $\omega = \|L\|/\mathrm{I}_\circ = 3.38 \times 10^{-6}\mathrm{s}^{-1}$ where the Sun angular momentum $\|L\| = 1.92 \times 10^{41}$ kg m^2 s^{-1} was taken from Iorio (2012). The $J_2 = 2.18 \times 10^{-7}$ of the Sun was taken from

Antia et al. (2008), see also Rozelot and Damiani (2011). The moment of inertia of Neptune ($I_\circ/(MR^2) = 0.2555$) was taken from Nettelmann et al. (2013).

As expected $\tilde{\gamma}$ is close to one for most of the celestial bodies in Table 1. The differences $\tilde{\gamma} - 1$ may be mostly explained by the lack of radial homogeneity of the bodies, especially in the case of the Sun. The low values of $\tilde{\gamma}$ found for Mercury, Venus, and Moon cannot be explained by the lack of radial homogeneity. In this case a possible explanation is that these bodies do not have a spherical equilibrium shape at rest, which violates one of our hypotheses. Notice that a small residual plastic deformation, and its consequent residual J_{2res}, combined with a small value of angular velocity ω yield a low value for $\omega^2/(J_{2res} + \Delta J_2)$, where ΔJ_2 is the part of J_2 that is caused by the rotation of the body. For large values of ω, and so of ΔJ_2, the residual value J_{2res} lacks importance in this ratio.

In order to estimate an effective value for the viscosity η using Eqs. (3) and (4) it is necessary to have measurements of non steady solutions to these equations. This is difficult. The value of η can be more easily estimated using tide measurements due to the gravitational interaction of the body with a second one. This is out of the scope of this paper.

Finally, Sect. 5 is a conclusion where we summarize and re-examine our work.

2 Euler–Lagrange equations for continuum mechanics

The derivation of all equations of motion in this paper are based upon the variational formulation of continuum mechanics, which we informally present below. Our main references on this classical subject are the two papers Baillieul and Levi (1987) and Baillieul and Levi (1991).

2.1 Euler–Lagrange equations with dissipation function

In this section we adopt a more abstract approach than in the remainder of the paper.

Let $M \subset \mathscr{V}$ be a differentiable manifold contained in a vector space $\mathscr{V}$, and $L, D : TM \longrightarrow \mathbb{R}$ be two smooth functions, the Lagrangian and the dissipation functions. We say that a smooth curve $\gamma : I \subset \mathbb{R} \longrightarrow M$, with $0 \in I$, $\gamma(0) = x$, solves the Euler–Lagrange equations with dissipation (henceforth ELD) on x (relative to L and D) if there exists a chart $\varphi : x \in U \longrightarrow \tilde{U} \subset \mathscr{W}$ such that

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{Q}}(d\varphi(\gamma(t), \gamma'(t)))\right) - \frac{\partial \mathscr{L}}{\partial Q}(d\varphi(\gamma(t), \gamma'(t))) + \frac{\partial \mathscr{D}}{\partial \dot{Q}}(d\varphi(\gamma(t), \gamma'(t))) = 0, \quad \forall t. \tag{13}$$

The functions $\mathscr{L}, \mathscr{D} : \tilde{U} \times \mathscr{W} \longrightarrow \mathbb{R}$ are given by $\mathscr{L}(Q, \dot{Q}) = L \circ d\varphi^{-1}(Q, \dot{Q})$, $\mathscr{D}(Q, \dot{Q}) = D \circ d\varphi^{-1}(Q, \dot{Q})$, the expressions for L, D in such chart. The derivatives occurring in the previous equation are the gradient of the functions, in the corresponding variables, relative to a fixed inner product on $\mathscr{V}$.

Remark that Eq. (13) is a local condition, and analogously to the Euler–Lagrange equations, is coordinate-free. Precisely, a curve solves ELD on a chart φ if and only if it does on another arbitrary chart around the same point. This calculation is presented in Baillieul and Levi (1987).

In this setting, we recall the Lagrange's Multipliers Theorem. The proof is the same as that of the conservative case.

Theorem 1 *Suppose* $\mathscr{V} = \mathbb{R}^n$ *and* $M \subset \mathbb{R}^n$ *is a submanifold given by the holonomic constraints* $f_1(Q) = 0, \ldots, f_N(Q) = 0$. *If we have smooth functions* $L, D : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ *and* $\gamma : I \longrightarrow M$ *solves the ELD on* $x \in M$ *(relative to* $L|_{TM}, D|_{TM}$*), then there are scalar functions* $\lambda_1, \cdots, \lambda_N$ *such that*

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{Q}}(\gamma(t), \gamma'(t))\right) - \frac{\partial \mathscr{L}}{\partial Q}(\gamma(t), \gamma'(t)) + \frac{\partial \mathscr{D}}{\partial \dot{Q}}(\gamma(t), \gamma'(t)) + \sum_{k=1}^{N} \lambda_k \frac{\partial f_k}{\partial Q}(\gamma(t)) = 0, \tag{14}$$

where the derivatives denote the gradient relative to the Euclidean metric.

As in the Lagrangian case, we would not need a metric to get equations of motion, but this form is appropriate to our purposes. The essential advantages of this formalism rest on the independence on coordinate systems and the behavior about restrictions to submanifolds (constraints).

In the modeling, it is imposed linearity of dissipative forces on the velocities. Usually this leads to a two-degree homogeneous dissipation function (on the velocities, of course), i.e. $\forall \lambda \in \mathbb{R}, \mathscr{D}(Q, \lambda \dot{Q}) = \lambda^2 \mathscr{D}(Q, \dot{Q})$. In this case, defining the "energy function" $E : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$,

$$E(Q, \dot{Q}) := \left\langle \frac{\partial \mathscr{L}}{\partial \dot{Q}}(Q, \dot{Q}), \dot{Q} \right\rangle - \mathscr{L}(Q, \dot{Q}) \tag{15}$$

a straightforward calculation shows that along a solution γ,

$$\frac{d}{dt} E(\gamma(t), \gamma'(t)) = -\left\langle \frac{\partial \mathscr{D}}{\partial \dot{Q}}(\gamma(t), \gamma'(t)), \gamma'(t) \right\rangle = -2\mathscr{D}(\gamma(t), \gamma'(t)), \tag{16}$$

where the inner product $<, >$ is that fixed in Theorem 1. In the last equality we used Euler's theorem.

We remark that for natural Lagrangians $\mathscr{L}(Q, \dot{Q}) = T(Q, \dot{Q}) - V(Q)$, with T two-degree homogeneous on velocities, the energy is

$$E(Q, \dot{Q}) = \left.\frac{d}{d\lambda}\right|_{\lambda=1} T(Q, \lambda \dot{Q}) - T(Q, \dot{Q}) + V(Q) = T(Q, \dot{Q}) + V(Q).$$

Note that this homogeneity condition is also coordinate-free.

2.2 Dissipative Noether's theorem

We need to adapt Noether's theorem of Lagrangian mechanics to these modified systems.

Let $M \subset \mathbb{R}^n$ be a submanifold, G a Lie group and $\Phi : G \times M \longrightarrow M$ a smooth action. We say a dissipative system $(\mathbb{R}^n, \mathscr{L}, \mathscr{D})$ *admits a* $1-$*parameter symmetry* $\gamma : I \longrightarrow G$, on M, if:

$$\begin{aligned} &\mathscr{L}\left(d\Phi_{\gamma(s)}(x, \dot{x})\right) = \mathscr{L}(x, \dot{x}), \qquad \partial_{\dot{x}} \mathscr{D}(x, \dot{x}) \left(\left.\frac{\partial}{\partial s}\right|_{s=0} \Phi_{\gamma(s)}(x) \right) = 0, \\ &\forall s \in I, (x, \dot{x}) \in TM, \end{aligned} \tag{17}$$

with the notation $\Phi_g(x) := \Phi(g, x)$, and $d\Phi_g(x, \dot{x}) = (\Phi_g(x), D\Phi_g(x)[\dot{x}])$.

Theorem 2 *If a dissipative system* $(\mathbb{R}^n, \mathscr{L}, \mathscr{D})$ *admits a symmetry* $\gamma(s)$ *(*$\gamma(0) = e$*) on* M*, then it has the first integral:*

$$h(x, \dot{x}) := \partial_{\dot{x}} \mathscr{L}(x, \dot{x}) \left(\frac{\partial}{\partial s}\bigg|_{s=0} \Phi_{\gamma(s)}(x) \right). \tag{18}$$

Proof Using the chain rule and the equations of motion we see,

$$\frac{d}{dt} h(x, \dot{x}) = \frac{d}{dt} \frac{\partial}{\partial s}\bigg|_{s=0} \mathscr{L}\left(d\Phi_{\gamma(s)}(x, \dot{x})\right) - \partial_{\dot{x}} \mathscr{D}(x, \dot{x}) \left(\frac{\partial}{\partial s}\bigg|_{s=0} \Phi_{\gamma(s)}(x) \right) \equiv 0.$$

□

Here the partial derivative symbol does not denote the gradient but the corresponding functional.

2.3 Lagrangian and dissipation function

Following Baillieul and Levi (1987), let $\mathscr{B} \subset \mathbb{R}^3$ be a reference configuration for the body. The motion of a point $x \in \mathscr{B}$ is given by $\phi(t, x) := Y(t)u(t, x)$, where $Y \in \mathrm{SO}(3)$ represents a "rotation" of the body and $u : \mathscr{B} \longrightarrow \mathbb{R}^3$ the particles positions relative to a "body frame" at time t. In Baillieul and Levi (1991), the space of all $u : \mathscr{B} \longrightarrow \mathbb{R}^3$ is denoted by $\mathscr{C}(\mathscr{B}, \mathbb{R}^3)$ and is the set of diffeomorphisms from $\mathscr{B}$ onto their images containing the identity in its interior. Since here we are only interested on isolated bodies the center of mass of the body can be considered at rest. Then $u - x$ represents the body deformation. Our configuration space is $\mathrm{SO}(3) \times \mathscr{C}$.

The kinetic energy is given by

$$T(Y, \dot{Y}, u, u_t) := \frac{1}{2} \int_{\mathscr{B}} \left\| \frac{\partial}{\partial t} \phi(t, x) \right\|^2 \rho(x) dx = \frac{1}{2} \int_{\mathscr{B}} \| \Omega u + u_t \|^2 \rho(x) dx, \tag{19}$$

where $\Omega = Y^T \dot{Y}$ is the skew-symmetric matrix representing the angular velocity in the body frame.

The Lagrangian is given by $\mathscr{L}(Y, \dot{Y}, u, u_t) := T(Y, \dot{Y}, u, u_t) - V(u)$, where $V : \mathscr{C} \longrightarrow \mathbb{R}$ is the potential energy. The potential energy can be split into an internal energy term plus a term due to the interaction with external agencies. In this paper only the internal energy will be considered. The dissipation function $\mathscr{D}$ is supposed to depend only on the deformation $u - x$ and its time derivative u_t and neither on the body attitude Y nor on its time derivative $\dot{Y}$. The derivation of the equations of motion from the Lagrangian and the dissipation functions is given, for instance, in Baillieul and Levi (1987). These equations couple angular velocity and deformation. The equation for the deformations is obtained from (13)

$$\rho(u_{tt} + 2\omega \times u_t + \dot{\omega} \times u + \omega \times (\omega \times u)) = -\frac{\delta V}{\delta u} - \frac{\delta \mathscr{D}}{\delta u_t}, \tag{20}$$

where $\frac{\delta V}{\delta u}$ means the gradient of V with respect to the perturbed coordinate u and ω is the angular velocity vector associated to the angular velocity matrix Ω as in Eq. (5). Notice that the multiplication of a vector by a matrix Ω is equivalent to applying a cross-product by ω. Equation (20) is nothing but the Second Law of Newton for the continuum body. As we will see below, the first term on the right-hand side of the equation comprises three forces: the one generated by the elastic part of the stress, the gravity force due to the exterior perturber (when such a perturber exists), and the self-gravitation force of the distorted body (which exists, no matter whether an external perturber is brought in or not). The second term on the right-hand side is the force emerging due to the viscous part of the stress (see Efroimsky (2012), in particular equation (158), for more details).

2.3.1 Elastic potential and dissipation function

Following Landau and Lifshitz (1986) and Efroimsky (2000), suppose we are dealing with small deformations (u close to the identity) of an isotropic, Hookean, viscoelastic body. Quoting Efroimsky (2000), we adopt the Kelvin–Voigt model that is obtained by the following choices

$$V_{el} = \int_{\mathscr{B}} \mu \,\mathrm{Tr}\,(\varepsilon^2) + \left(\frac{K}{2} - \frac{\mu}{3}\right) \mathrm{Tr}\,(\varepsilon)^2 dx, \tag{21}$$

$$\mathscr{D} = \int_{\mathscr{B}} \eta \,\mathrm{Tr}\,(\dot{\varepsilon}^2) + \left(\frac{\zeta}{2} - \frac{\eta}{3}\right) \mathrm{Tr}\,(\dot{\varepsilon})^2 dx \tag{22}$$

where μ, K are the (adiabatic) shear and bulk moduli respectively, and η, ζ are the viscous shear and viscous bulk moduli, respectively. Also we have the strain tensor and the strain rate tensor

$$\varepsilon = \frac{1}{2}(du + du^T) - \mathbb{I}\mathrm{d}, \quad \dot{\varepsilon} = \frac{1}{2}(du_t + du_t^T). \tag{23}$$

with components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}, \quad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right).$$

The stress tensor obtained from the elastic potential V_{el} (21) and from the dissipation function $\mathscr{D}$ (22) is Efroimsky (2000)

$$\sigma = \sigma^e + \sigma^v$$

where σ^e is the elastic part

$$\sigma^e = K(\mathrm{Tr}\,\varepsilon)\,\mathbb{I}\mathrm{d} + 2\mu\left(\varepsilon - \frac{1}{3}(\mathrm{Tr}\,\varepsilon)\,\mathbb{I}\mathrm{d}\right)$$

and σ^v is the viscous part

$$\sigma^v = \zeta(\mathrm{Tr}\,\dot{\varepsilon})\,\mathbb{I}\mathrm{d} + 2\eta\left(\dot{\varepsilon} - \frac{1}{3}(\mathrm{Tr}\,\dot{\varepsilon})\,\mathbb{I}\mathrm{d}\right).$$

Attach to the space of deformations $\mathscr{C}$ the condition that the stress tensor vanishes when applied to the normal exterior vector, on the boundary, $\sigma n(x) = 0$, $\forall x \in \partial\mathscr{B}$. In this case, a straightforward calculation shows:

$$\frac{\delta V_{el}}{\delta u} = -\left(K + \frac{\mu}{3}\right)\nabla(\mathrm{div}\,u) - \mu\Delta u, \tag{24}$$

$$\frac{\delta \mathscr{D}}{\delta u_t} = -\left(\zeta + \frac{\eta}{3}\right)\nabla(\mathrm{div}\,u_t) - \eta\Delta u_t, \tag{25}$$

where $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ and Δu_i is the Laplacian of u_i.

The first term is the elastic force for a (compressible) solid, see chapter V, equation (19) of Love (1944). The second is *analogous* to the viscosity expressed in the Navier-Stokes equations for compressible, isotropic fluids [see paragraph 15 of Landau and Lifshitz (1987)].

2.3.2 Gravitational energy

Since we are interested in isolated bodies, the gravitational energy is given only by the self-gravitational potential

$$V_g(u) = \frac{G}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{\rho(x)\rho(y)}{\|u(x)-u(y)\|}dxdy, \tag{26}$$

whose associated force is

$$\frac{\delta V_g}{\delta u}(x) = -G\rho(x)\int_{\mathscr{B}} \frac{u(x)-u(y)}{\|u(x)-u(y)\|^3}\rho(y)dy.$$

3 Pseudo-rigid bodies

A crucial point in the kinematics of a moving body with fixed center of mass is the factorization of the motion $\phi(t,x)$ into a rotation $Y(t)$ and a deformation $u(t,x)$, $\phi(t,x) = Y(t)u(t,x)$. If the body is rigid, $u(t,x) = x$, then an orthonormal reference frame can be fixed to the body. The motion of the body $\phi(t,x) = Y(t)x$ is determined by the "moving frame". There is no way to fix an orthonormal reference frame to a deformable body. The choice of a moving frame that captures the motion of a deformable body may be a difficult task. Though not unique, this choice is easier for the motion $\phi(t,x) = G(t)x$ of a pseudo-rigid body. In this case, a standard method (see Borisov et al. 2009; Chandrasekhar 1987; Fassó and Lewis 2001; Kristiansen et al. 2012) is to use the *singular value decomposition*

$$G = R\tilde{A}S^T, \quad R, S \in \mathrm{SO}(3), \quad \tilde{A} = \mathrm{diag}(a_1, a_2, a_3) > 0, \tag{27}$$

where $\mathrm{SO}(3)$ is the group of 3×3 orthogonal matrices. The matrix R represents the rotation of the body (shape) and S the circulation of the matter in its interior. This decomposition is not unique and there exists a smooth curve $t \to G(t)$ that does not admit any smooth singular decomposition (if $t \to G(t)$ is analytic, there is always an analytic singular decomposition (see Fassó and Lewis 2001). For pseudo-rigid body motions another choice of moving frame is provided by the polar decomposition. Let $\mathrm{SL}(3)$ be the group of 3×3 matrices with determinant one and $\mathrm{SSym}_+(3)$ be the subset of $\mathrm{SL}(3)$ of symmetric and positive matrices. Then given $G \in \mathrm{SL}(3)$, there are unique matrices $Y \in \mathrm{SO}(3)$ and $A \in \mathrm{SSym}_+(3)$ such that $G = YA$. Therefore, the mapping

$$\Phi : \mathrm{SSym}_+(3) \times \mathrm{SO}(3) \longrightarrow \mathrm{SL}(3),$$

$\Phi(A, Y) := YA$, is bijective.

Let $\mathrm{Sym}(3)$ be the set of symmetric 3×3 matrices. This set is diffeomorphic to $\mathbb{R}^6$. We claim that $\mathrm{SSym}_+(3)$ is a $5-$dimensional submanifold of $\mathrm{Sym}(3)$. Indeed, $\mathrm{SSym}_+(3)$ is the preimage of one by the function $\det : \mathrm{Sym}(3) \longrightarrow \mathbb{R}$. Moreover, the derivative of det is nonsingular on $\mathrm{SSym}_+(3)$ because if $M \in \mathrm{SSym}_+(3)$

$$D\det_M(M) = \det(M)\,\mathrm{Tr}\,(\mathbb{I}\mathrm{d}) = 3.$$

Therefore one is a regular value of $\det : \mathrm{Sym}(3) \longrightarrow \mathbb{R}$, which ensures that $\mathrm{SSym}_+(3)$ is a submanifold of $\mathrm{Sym}(3)$. Moreover, it can be shown that the mapping Φ is a diffeomorphism (see Ferraz-Mello et al. (2015)) and therefore to each smooth motion $G(t) \in \mathrm{SL}(3)$ corresponds a unique smooth motion $(A(t), Y(t)) \in \mathrm{SSym}_+(3) \times \mathrm{SO}(3)$ (see Dieci and Eirola 1999 for an algorithm to find $(A(t), Y(t))$). So, in the following we use the polar decomposition to investigate the dynamics of an incompressible homogeneous body ($\rho(x) = \rho =$ constant, $\mu(x) = \mu =$ constant, $\eta(x) = \eta =$ constant, $\cdots$) under the pseudo-rigid body hypothesis (1).

Let $\phi(t, x) = Y(t)A(t)x,\ x \in \mathscr{B}$, be the function that describes the motion of a body that at rest has the shape of a ball $\mathscr{B} = \{x \in \mathbb{R}^3 : \|x\| \leq R\}$. The motion is determined by the rotation matrix $Y(t)$ and the deformation $A(t)$ matrix. Using that the total mass of the body is $M = 4\pi R^3\rho/3$ and that for every 3×3 matrix

$$\int_{\mathscr{B}} \langle x, Cx\rangle\, dx = \frac{4\pi R^5}{15}\mathrm{Tr}(C), \tag{28}$$

we obtain that the kinetic energy (19) of the body is given by

$$T(Y, A, \dot{Y}, \dot{A}) = \frac{MR^2}{10}(\mathrm{Tr}(\dot{A}^2) - \mathrm{Tr}(\Omega^2 A^2) + 2\mathrm{Tr}(\dot{A}\Omega A)). \tag{29}$$

Using that the pseudo-rigid body deformation is $u(t, x) = A(t)x$, the strain tensors (23) become:

$$\varepsilon = \frac{1}{2}(A + A^T) - \mathbb{I}\mathrm{d} = A - \mathbb{I}\mathrm{d}, \quad \dot{\varepsilon} = \dot{A}.$$

So, the elastic potential (21) and the dissipation function (22) become

$$V_{el}(A) = \frac{4\pi R^3}{3}\left(\mu\left(\mathrm{Tr}(A^2) - \frac{1}{3}\mathrm{Tr}(A)^2\right) + \frac{K}{2}(\mathrm{Tr}(A) - 3)^2\right), \tag{30}$$

$$\mathscr{D}(\dot{A}) = \frac{4\pi R^3}{3}\left(\eta\left(\mathrm{Tr}(\dot{A}^2) - \frac{1}{3}\mathrm{Tr}(\dot{A})^2\right) + \frac{\zeta}{2}\mathrm{Tr}(\dot{A})^2\right), \tag{31}$$

and the self-gravitational potential (26) becomes

$$V_g(A) = -\frac{\rho^2 G}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{1}{\|A(x - y)\|}dxdy. \tag{32}$$

It is easy to check (since $A > 0$) that $V_g(A)$ is differentiable. The function $V_g(A)$ can also be written as

$$V_g(A) = -\frac{3M^2 G}{10R}\int_0^\infty \frac{1}{\sqrt{\det(A^2 + \lambda\mathbb{I}\mathrm{d})}}d\lambda, \quad \forall A \in \mathrm{SSym}_+(3), \tag{33}$$

which is a formula known to Dirichlet. For a proof, see Chandrasekhar (1987) or Thomson and Tait (2009) and for a recent discussion see Khavinson and Lundberg (2013). The next three lemmas show that the potentials above have some natural properties. The first shows that the elastic stresses generated by (30) tend to restore the body to the relaxed shape, like a spring.

Lemma 1 *Take any $\mu, K \geq 0$, with at least one positive, and $A \in \mathrm{SSym}_+(3)$. Then we have*

(i) $\mathrm{Tr}(A)^2 \leq 3\mathrm{Tr}(A^2)$;
(ii) $\mathrm{Tr}(A) \geq 3$;
(iii) $V_{el}(A) \geq 0$.

Moreover, the equalities are reached if and only if $A = \mathbb{I}\mathrm{d}$.

Proof The Cauchy–Schwarz inequality for the inner product $\langle H_1, H_2\rangle := \mathrm{Tr}\,(H_1^T H_2)$ gives: $\mathrm{Tr}\,(A) = \langle A, \mathbb{I}\mathrm{d}\rangle \leq \|A\|\|\mathbb{I}\mathrm{d}\| = \sqrt{3}\sqrt{\mathrm{Tr}\,(A^2)}$. The equality holds if and only if $A = \lambda\mathbb{I}\mathrm{d}$, i.e., $A = \mathbb{I}\mathrm{d}$. Let $x, c > 0$ be such that x^{-1}, c^{-1}, xc are the eigenvalues of A. Defining $f_c(x) := cx + x^{-1} + c^{-1}$, $f_c'(x) = cx^{-2}(x^2 - c^{-1})$. So $f_c(x)$, for $x > 0$, has a global minimum at $x = c^{-\frac{1}{2}}$. Hence, $f_c(x) \geq f_c(c^{-\frac{1}{2}}) = h(c) := 2c^{\frac{1}{2}} + c^{-1}\ \forall x, c > 0$. But, $h'(c) = c^{-2}(c^{\frac{3}{2}} - 1)$, then $h(c) \geq h(1) = 3$. Thus, $\mathrm{Tr}\,(A) = f_c(x) \geq h(c) \geq 3$, and since the minima are strict, the equality holds only for $A = \mathbb{I}\mathrm{d}$. The last assertion follows from (i), (ii) and (30). □

The next lemma shows that the self-gravitational potential (33) has a global minimum exactly when the body has the shape of a ball.

Lemma 2 *For all* $A \in \mathrm{SSym}_+(3)$,

$$-\frac{3M^2G}{5R} \leq V_g(A) < 0.$$

The equality holds if and only if $A = \mathbb{I}\mathrm{d}$.

Proof Let $a_1, a_2, a_3 > 0$ be the eigenvalues of A. Thus, $\det(A^2 + \lambda\mathbb{I}\mathrm{d}) = \lambda^3 + (a_1^2 + a_2^2 + a_3^2)\lambda^2 + (a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2)\lambda + (a_1a_2a_3)^2 = \lambda^3 + \mathrm{Tr}\,(A^2)\lambda^2 + \mathrm{Tr}\,(\tilde{A})\lambda + 1$, where $\tilde{A} = \mathrm{diag}(a_1^2a_2^2, a_1^2a_3^2, a_2^2a_3^2) \in \mathrm{SSym}_+(3)$. Hence, from (ii) of Lemma 1, follows $\det(A^2 + \lambda\mathbb{I}\mathrm{d}) \geq \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$. So,

$$V_g(A) \geq -\frac{3M^2G}{10R}\int_0^\infty (\lambda + 1)^{-\frac{3}{2}}d\lambda = -\frac{3M^2G}{5R}.$$

□

The next lemma shows that energy dissipation ceases along with the internal motion.

Lemma 3 *Take any* $\eta > 0, \zeta \geq 0$ *and* $\dot{A} \in T_A\mathrm{SSym}_+(3)$. *Therefore* $\mathscr{D}(\dot{A}) \geq 0$, *and equality holds if and only if* $\dot{A} = 0$.

Proof Note that the arguments used in Lemma 1, item (i), can be applied here, ensuring that $\mathscr{D} \geq 0$. Thus, if $\mathscr{D}(\dot{A}) = 0$, then $\dot{A} = \lambda\mathbb{I}\mathrm{d}$. But, $0 = \mathrm{Tr}\,(A^{-1}\dot{A}) = \lambda\mathrm{Tr}\,(A^{-1})$. So $\dot{A} = 0$. □

In order to obtain the equations of motion for the deformation A, we use Theorem 1. On the nine-dimensional vector-space of 3×3 matrices consider the (Euclidean) metric $< H_1, H_2 >:= \mathrm{Tr}\,(H_1^T H_2)$. The symmetric matrices with determinant one is the subset of matrices of this vector-space that satisfy the constraints:

$$g_{12}(A) = 0,\quad g_{13}(A) = 0,\quad g_{23}(A) = 0,\quad g(A) = 0, \tag{34}$$

where $g_{ij}(A) = a_{ij} - a_{ji}$ and $g(A) = \det(A) - 1$. Let χ_{ij} and χ be the Lagrangian multipliers associated to g_{ij} and g, respectively. If we write

$$Z = \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{A}}\right) - \frac{\partial\mathscr{L}}{\partial A} + \frac{\partial\mathscr{D}}{\partial\dot{A}},$$

then the constrained Euler–Lagrange equations can be written as $Z + X + \chi A^{-1} = 0$ where

$$\sum_{ij}\chi_{ij}\frac{\partial g_{ij}}{\partial A} + \chi\frac{\partial g}{\partial A} = \begin{pmatrix} 0 & \chi_{12} & \chi_{13} \\ -\chi_{12} & 0 & \chi_{23} \\ -\chi_{13} & -\chi_{23} & 0 \end{pmatrix} + \chi A^{-1} = X + \chi A^{-1}. \tag{35}$$

In order to determine the Lagrange multipliers X we take the skew-symmetric part of $Z + X + \chi A^{-1} = 0$ to get $X = -(Z - Z^T)/2$. In order to determine χ we multiply both sides of $Z + X + \chi A^{-1} = 0$ by A^{-1} and take the trace to get $\chi = -\mathrm{Tr}(A^{-1}Z)/\mathrm{Tr}(A^{-2})$. To eliminate the term $\mathrm{Tr}(A^{-1}\ddot{A})$ from $\mathrm{Tr}(A^{-1}Z)$ we use that $\det(A) - 1 = 0$ implies $\mathrm{Tr}(A^{-1}\ddot{A}) = \mathrm{Tr}(A^{-1}\dot{A}A^{-1}\dot{A})$. Finally, using that $\mathscr{L} = T - V_{el} - V_g$ we obtain the equations of motion relative to the deformations:

$$\ddot{A} + \frac{1}{2}[\dot{\Omega}, A] + [\Omega, \dot{A}] + \frac{1}{2}(\Omega^2 A + A\Omega^2) + \frac{20\pi R}{3M}\left[2\mu A + \left(K - \frac{2\mu}{3}\right)\mathrm{Tr}(A)\mathbb{I}\mathrm{d}\right.$$
$$\left. -3K\,\mathbb{I}\mathrm{d} + 2\eta\dot{A} + \left(\zeta - \frac{2\eta}{3}\right)\mathrm{Tr}(\dot{A})\mathbb{I}\mathrm{d}\right] + \frac{3MG}{2R^3}\int_0^\infty \frac{A(A^2 + \lambda\mathbb{I}\mathrm{d})^{-1}}{\sqrt{\det(A^2 + \lambda\mathbb{I}\mathrm{d})}}d\lambda + \chi A^{-1} = 0 \tag{36}$$

where,

$$\chi = \frac{1}{\mathrm{Tr}(A^{-2})}\left\{2\mathrm{Tr}(A^{-1}\dot{A}\Omega) - \mathrm{Tr}(A^{-1}\dot{A}A^{-1}\dot{A}) - \mathrm{Tr}(\Omega^2)\right.$$
$$-\frac{20\pi R}{3M}\left[6\mu + \left(K - \frac{2\mu}{3}\right)\mathrm{Tr}(A)\mathrm{Tr}(A^{-1}) - 3K\,\mathrm{Tr}(A^{-1})\right.$$
$$\left. + \left(\zeta - \frac{2\eta}{3}\right)\mathrm{Tr}(\dot{A})\mathrm{Tr}(A^{-1})\right]$$
$$\left. -\frac{3MG}{2R^3}\int_0^\infty \frac{\mathrm{Tr}((A^2 + \lambda\mathbb{I}\mathrm{d})^{-1})}{\sqrt{\det(A^2 + \lambda\mathbb{I}\mathrm{d})}}d\lambda\right\}.$$

Notice that the initial condition $(A_0, \dot{A}_0)$ must satisfy the constraints: $A_0 \in \mathrm{SSym}_+(3)$, $\dot{A}_0$ symmetric, and $\mathrm{Tr}(A_0^{-1}\dot{A}_0) = 0$. Since $\det A(t) = 1$ the positiveness of $A(t)$ is ensured for all t.

The equations of motion for the rotation Y are obtained in the same way as those for A. The set of matrices Y is considered as a subset of the vector-space of 3×3 matrices that satisfy $Y^T Y = \mathbb{I}\mathrm{d}$. These constraints can be written as:

$$f_{km}(Y) = \mathrm{Tr}(Y^T Y \sigma_{km}) = \begin{cases} 1, & \textit{if } k = m \\ 0, & \textit{if } k \neq m \end{cases},$$

where σ_{km} are the matrices

$$(\sigma_{km})_{ij} = \begin{cases} 1, & \textit{if } \{k, m\} = \{i, j\} \\ 0, & \textit{otherwise.} \end{cases}$$

Let χ_{km} denote the Lagrange multiplier associated to f_{km}. Then, using

$$\sum_{k,m} \chi_{km}\frac{\partial f_{km}}{\partial Y} = 2Y^T \sum_{k,m} \chi_{km}\sigma_{km} = 2Y^T \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{12} & \chi_{22} & \chi_{23} \\ \chi_{13} & \chi_{23} & \chi_{33} \end{pmatrix} = 2Y^T X,$$

we get that the constrained Euler–Lagrange equations associated to Y are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right) - \frac{\partial T}{\partial Y} + 2Y^T X = 0.$$

Since X is a symmetric matrix, in order to eliminate the Lagrangian multipliers of this equation it is enough to multiply it by Y and to take its skew-symmetric part

$$\left(Y\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right)-\frac{\partial T}{\partial Y}\right)\right)-\left(Y\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right)-\frac{\partial T}{\partial Y}\right)\right)^T=0, \tag{37}$$

to obtain

$$A^2\dot{\Omega}+\dot{\Omega}A^2+2\Omega\dot{A}A+2A\dot{A}\Omega+[\Omega^2,A^2]=[A,\ddot{A}]. \tag{38}$$

Finally we use (36) to eliminate $\ddot{A}$ from equation (38) to obtain

$$\Psi_A(\dot{\Omega})+2\Omega\dot{A}A+2A\dot{A}\Omega+2A\Omega\dot{A}+2\dot{A}\Omega A+[\Omega^2,A^2]=0, \tag{39}$$

where $\Psi_A:\mathrm{skew}(3)\longrightarrow\mathrm{skew}(3)$ is the linear operator $\Psi_A(H):=A^2H+HA^2+2AHA$.

Lemma 4 *For every $A\in\mathrm{SSym}_+(3)$, the operator Ψ_A is invertible.*

Proof Take $H\in\mathrm{skew}(3)$ such that $\Psi_A(H)=0$. In a basis such that $A=\mathrm{diag}(a_1,a_2,a_3)$,

$$H=\begin{pmatrix}0&-h_3&h_2\\h_3&0&-h_1\\-h_2&h_1&0\end{pmatrix}$$

we see that H must satisfy

$$\begin{pmatrix}0&-(a_1+a_2)^2h_3&(a_1+a_3)^2h_2\\(a_1+a_2)^2h_3&0&-(a_2+a_3)^2h_1\\-(a_1+a_3)^2h_2&(a_2+a_3)^2h_1&0\end{pmatrix}=0.$$

So, $H=0$. □

Therefore, we have a well-posed problem, since we can write the system of equations (36), (39) and $\dot{Y}=Y\Omega$ in an explicit form.

Now, we describe the qualitative aspects of the motion.

Lemma 5 *The angular momentum on an inertial frame*

$$L(\Omega,A,\dot{A}):=\frac{1}{2}Y(A^2\Omega+\Omega A^2+[\dot{A},A])Y^T \tag{40}$$

is conserved by the motion.

Proof Consider the action $\Phi:\mathrm{SO}(3)\times(\mathrm{SO}(3)\times\mathrm{SSym}_+(3))\longrightarrow\mathrm{SO}(3)\times\mathrm{SSym}_+(3)$, $\Phi(U,Y,A):=(UY,A)$. For every $\xi\in\mathrm{skew}(3)$, we have $\mathscr{L}(d\Phi_{e^{s\xi}}((Y,A),(\dot{Y},\dot{A})))=\mathscr{L}((e^{s\xi}Y,A),(e^{s\xi}\dot{Y},\dot{A}))=\mathscr{L}((Y,A),(\dot{Y},\dot{A}))$. Note that $\mathscr{D}$ is independent of $Y,\dot{Y}$, then the second condition of (17) is fulfilled. Hence, by Theorem 2,

$$\mathrm{Tr}\left((YA^2\Omega Y^T+Y\Omega A^2Y^T-2YA\dot{A}Y^T)\xi^T\right)$$

is a first integral, for all $\xi\in\mathrm{skew}(3)$. Then, its skew-symmetric part is conserved. □

Now, consider the action $\Phi(U,Y,A):=(YU^T,UAU^T)$. Note that

$$\begin{aligned}\mathscr{L}(d\Phi_{e^{s\xi}}((Y,A),(\dot{Y},\dot{A})))&=\mathscr{L}((Ye^{-s\xi},e^{s\xi}Ae^{-s\xi}),(\dot{Y}e^{-s\xi},e^{s\xi}\dot{A}e^{-s\xi}))\\&=\mathscr{L}((Y,A),(\dot{Y},\dot{A})).\end{aligned}$$

However,

$$\partial_{\dot{x}}\mathscr{D}\left(\frac{\partial}{\partial s}\bigg|_{s=0}\Phi_{e^{s\xi}}(x)\right)=\frac{16\pi R^3\eta}{3}\mathrm{Tr}\,(\dot{A}\xi A) \tag{41}$$

does not vanish, except for $\eta = 0$. In the case $\eta = 0$, given any $\xi \in \mathrm{skew}(3)$ we have the additional first integral

$$h = \partial_{\dot{x}}\mathscr{L}\,(-Y\xi, \xi A - A\xi) = \mathrm{Tr}\,(\xi^T(A\dot{A} + A\varOmega A)).$$

This implies that, for $\eta = 0$, $\Sigma := [A, \dot{A}] + 2A\varOmega A$ is conserved. This Σ is called *vorticity* in Borisov et al. (2009) (equation (29)) and *circulation* in Chandrasekhar (1987).

3.1 Equilibria

One easily checks that both $\mathscr{L}(Y, A, \dot{Y}, \dot{A})$, $\mathscr{D}(Y, A, \dot{Y}, \dot{A})$ are two-degree homogeneous functions in the velocities. So, from (16) we see that the energy $E := T + V_g + V_{el} + 3M^2G/(5R)$ is such that

$$\dot{E} = -2\mathscr{D} \le 0,$$

where the last inequality comes from Lemma 3. In other words, $E : T(\mathrm{SO}(3) \times \mathrm{SSym}_+(3)) \longrightarrow \mathbb{R}$ is a Lyapunov function. Note that the kinetic energy (29) can be rewritten as

$$T(Y, A, \dot{Y}, \dot{A}) = \frac{MR^2}{10}\|\dot{A} + \varOmega A\|^2 \ge 0.$$

This and Lemmas 1 and 2 imply that $E \ge 0$.

Lemma 6 *Suppose that* $\mu > 0$. *Then given* $E_0 \ge 0$, *the set* $E^{-1}([0, E_0]) \subset T\,\mathrm{SSym}_+(3) \times \mathrm{skew}(3)$ *is compact where* $\mathrm{skew}(3)$ *denotes the set of* 3×3 *skew-symmetric matrices.*

Proof Notice that $E^{-1}([0, E_0])$ is closed in $T\,\mathrm{SSym}_+(3) \times \mathrm{skew}(3)$ that is closed in the set in $W = M \times M \times \mathrm{skew}(3)$ where M is the vector space of 3×3 matrices. We will show that there is no sequence $(A_n, \dot{A}_n, \varOmega_n) \in E^{-1}([0, E_0])$ that is unbounded in W. The definition of E, Lemma 2, and the positivity of T imply:

$$\frac{4\pi\mu R^3}{3}\left(\mathrm{Tr}\,(A_n^2) - \frac{1}{3}\mathrm{Tr}\,(A_n)^2\right) \le E \le E_0, \quad \forall n, \tag{42}$$

$$\frac{MR^2}{10}\left(\|\dot{A}_n + \varOmega_n A_n\|^2\right) \le E \le E_0. \quad \forall n. \tag{43}$$

From (42) we see that the norms of the vectors

$$A_n - \frac{< A_n, \mathbb{I}\mathrm{d} >}{\|\mathbb{I}\mathrm{d}\|^2}\mathbb{I}\mathrm{d}$$

are bounded. So, the projection of the sequence A_n on any vector orthogonal to $\mathbb{I}\mathrm{d}$ is bounded. Then, writing $A_n = s_n\mathbb{I}\mathrm{d} + \alpha_n$, where $< \alpha_n, \mathbb{I}\mathrm{d} >= 0$, we get that the $\|\alpha_n\|$ is bounded and the sequence $\|A_n\|$ is unbounded if, and only if, the sequence $|s_n|$ is unbounded. But if $|s_n|$ is sufficiently large $s_n^{-1}\alpha_n$ is close to zero and $\det(A_n) = s_n^3\det(\mathbb{I}\mathrm{d} + s_n^{-1}\alpha_n) > 1$ that is impossible because $\det(A_n) = 1$. So A_n is bounded. Denote $(A_n)_{ij} = a_{ij}^n$ and $(\dot{A}_n)_{ij} = b_{ij}^n$. Equation (43) shows that $\dot{A}_n + \varOmega_n A_n$ is bounded, as well as its skew-symmetric part $(A_n\varOmega_n + \varOmega_n A_n)/2$. Taking A_n diagonal again, we see that the vector associated to

$A_n\Omega_n + \Omega_n A_n$ (as in Eq. 5) is $((a_{22}^n + a_{33}^n)\omega_1^n, (a_{11}^n + a_{33}^n)\omega_2^n, (a_{11}^n + a_{22}^n)\omega_3^n)$. Since none of the coefficients $(a_{ii}^n + a_{jj}^n)$ can accumulate at zero, because $a_{11}^n a_{22}^n a_{33}^n = 1$, we conclude that the sequence Ω_n is bounded. Hence $\dot{A}_n$ must also be bounded. □

Therefore, we are able to apply the LaSalle invariance principle more or less as in Bambusi and Haus (2012).

Lemma 7 (LaSalle's Invariance Principle) *Let $\gamma^+(x_0)$ be the positive orbit of the initial condition x_0 in the phase space M. If $\gamma^+(x_0)$ is bounded and V is a Lyapunov function on M, then the $\omega-$limit set of this solution is contained in the largest invariant subset (by the flow) of $\{x \in M : \dot{V}(x) = 0\}$.*

For a proof, consult Hale (1980). We denote the largest invariant subset under the flow contained in $\{X = (Y, \Omega, A, \dot{A}) : \dot{E}(X) = 0\}$ by $\mathscr{A}$. So, by Lemma 6, the $\omega-$limit set of *every* initial condition is contained in $\mathscr{A}$. Taking an initial condition on $\mathscr{A}$, since it is invariant, its flow is such that $\dot{E}(X(t)) = -2\mathscr{D}(X(t)) \equiv 0$. Then, by Lemma 3, $A(t) = A, \forall t \geq 0$. The next lemma shows that any point in the attracting set $\mathscr{A}$ is a relative equilibrium.

Lemma 8 $\dot{\Omega} = 0$ *on the set* $\mathscr{A}$.

Proof Since for all $R_0 \in \mathrm{SO}(3)$, $(R_0 Y(t) R_0^T, R_0\Omega(t)R_0^T, R_0 A R_0^T, 0)$ is also a solution, without loss of generality, we may assume $A = \mathrm{diag}(a_1, a_2, a_3)$. Since $\dot{A} = 0$ on $\mathscr{A}$, Eqs. (36) and (39) imply

$$\begin{gathered} A^2\dot{\Omega} + \dot{\Omega}A^2 + 2A\dot{\Omega}A + [\Omega^2, A^2] = 0, \\ \frac{1}{2}[\dot{\Omega}, A] + \frac{1}{2}(\Omega^2 A + A\Omega^2) + D = 0, \end{gathered}$$

where D is a diagonal matrix. The first of these equations imply:

$$\begin{cases} (a_1^2 + a_2^2)\dot{\omega}_3 + 2a_1a_2\dot{\omega}_3 - (a_2^2 - a_1^2)\omega_1\omega_2 = 0 \\ (a_1^2 + a_3^2)\dot{\omega}_2 + 2a_1a_3\dot{\omega}_2 + (a_3^2 - a_1^2)\omega_1\omega_3 = 0 \\ (a_2^2 + a_3^2)\dot{\omega}_1 + 2a_2a_3\dot{\omega}_1 - (a_3^2 - a_2^2)\omega_2\omega_3 = 0 \end{cases} \Rightarrow \begin{cases} (a_1 + a_2)\dot{\omega}_3 + (a_1 - a_2)\omega_1\omega_2 = 0 \\ (a_1 + a_3)\dot{\omega}_2 + (a_3 - a_1)\omega_1\omega_3 = 0 \\ (a_2 + a_3)\dot{\omega}_1 + (a_2 - a_3)\omega_2\omega_3 = 0 \end{cases}$$

The off diagonal terms of the second of those equations imply:

$$\begin{cases} (a_2 - a_1)\dot{\omega}_3 + (a_1 + a_2)\omega_1\omega_2 = 0 \\ (a_1 - a_3)\dot{\omega}_2 + (a_1 + a_3)\omega_1\omega_3 = 0 \\ (a_3 - a_2)\dot{\omega}_1 + (a_2 + a_3)\omega_2\omega_3 = 0 \end{cases} \Rightarrow \begin{cases} ((a_1 + a_2)^2 + (a_2 - a_1)^2)\dot{\omega}_3 = 0 \\ ((a_1 + a_3)^2 + (a_1 - a_3)^2)\dot{\omega}_2 = 0 \\ ((a_2 + a_3)^2 + (a_3 - a_2)^2)\dot{\omega}_1 = 0 \end{cases}$$

and the conclusion follows. □

Equations (36) and (39) are invariant under rotations. So, the system can be initially rotated such that the angular momentum vector, Eq. (40), has the form $(0, 0, \ell)$. We aim to understand qualitatively the set of all relative equilibria of Eqs. (36) and (39) for a fixed value of ℓ. Since $\dot{\Omega} = 0$ and $\dot{Y}(t) = Y(t)\Omega$ we conclude that Ω commutes with $Y(t)$. Differentiating with respect to t the angular momentum equation $2Y^T LY = A^2\Omega + \Omega A^2$ we conclude that L commutes with Ω and therefore the vector Ω is also of the form $(0, 0, \omega_3)$. Equation (39) with $\dot{A} = 0$ and $\dot{\Omega} = 0$ gives

$$0 = [\Omega^2, A^2] = A[\Omega^2, A] + [\Omega^2, A]A$$

that implies $[\Omega^2, A] = 0$ because A is positive definite. This equation implies that A has the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \tilde{A} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \tag{44}$$

The relative equilibria can be of three types depending on the equilibrium shape of the body being: a sphere, an ellipsoid of revolution or a triaxial ellipsoid. The spherical shape, $A = \mathbb{I}\mathrm{d}$, can only occur for $\ell = 0$ and it is the only relative equilibrium in this case. If the equilibrium shape is an ellipsoid then one of the semi-major axis is in the vertical direction. For a given L, if there exists one relative equilibrium with the shape of a triaxial ellipsoid, then there are infinitely many others: one for each angle of rotation ($\mod \pi$) around the vertical axis. For one of these relative equilibria $a_{12} = a_{21} = 0$ and $a_{11} > a_{22}$. Notice that the invariance of ellipsoids of revolution under rotations around the vertical axis imply that there may exist a unique relative equilibrium of this kind for a fixed value of ℓ (as it happens to the spherical shape for $\ell = 0$). Our quantitative study of the relative equilibria is restricted to the case of small angular momentum.

Lemma 9 *There exists* $|\ell_0| > 0$ *such that for each angular momentum* L *with* $0 < \|L\| = |\ell| < |\ell_0|$ *there exists a unique relative equilibrium with the shape of an ellipsoid of revolution and with angular momentum* L. *For this equilibrium,* $A = \exp(B)$ *where* B *is a symmetric traceless matrix given approximately by*

$$B = -\frac{1}{\gamma}\left(\Omega^2 - \frac{\mathrm{Tr}\,(\Omega^2)}{3}\mathbb{I}\mathrm{d}\right) + \mathscr{O}(\|\Omega\|^{3/2}) = \frac{\ell^2}{12\gamma}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \mathscr{O}(|\ell|^{3/2})$$

Proof The angular momentum definition (40) and the above discussion imply that $\ell = \omega_3\,\mathrm{Tr}\,(\tilde{A}^2)$, where $\tilde{A}$ is the matrix defined in Eq. (44). We use this equation to replace ω_3 for ℓ in Eq. (36) after imposing $\dot{A} = 0$, $\dot{\Omega} = 0$. As a result we obtain a set of three scalar equations that can be represented as $F(\ell^2, A) = 0$, where A has the form in Eq. (44). For a given ℓ^2, the three scalar equations $F(\ell^2, A) = 0$ must be solved for the four unknowns in A under the additional constraint $\det(A) - 1 = 0$. Notice that $F(0, \mathbb{I}\mathrm{d}) = 0$. So, to finish the proof it is enough to show that the matrix $\partial_A F(0, \mathbb{I}\mathrm{d})$ is nonsingular when restricted to the tangent space to $\det(A) - 1$ at $A = \mathbb{I}\mathrm{d}$. The lemma follows from the implicit function theorem. The best way to compute $\partial_A F(0, \mathbb{I}\mathrm{d})$ is to write $A = \mathbb{I}\mathrm{d} + \varepsilon B$ and to expand $F(0, \mathbb{I}\mathrm{d} + \varepsilon B) = \varepsilon \partial_A F(0, \mathbb{I}\mathrm{d}) + \mathscr{O}(\varepsilon^2)$. The constraint $0 = \det(\mathbb{I}\mathrm{d} + \varepsilon B) - 1 = \varepsilon\,\mathrm{Tr}\,(B) + \mathscr{O}(\varepsilon^2)$ implies that $\mathrm{Tr}\,(B) = 0$. So substituting $A = \mathbb{I}\mathrm{d} + \varepsilon B$ into Eq. (36) with $\dot{\Omega} = 0$, $\dot{A} = 0$, and expanding up to order ε we obtain:

$$0 = \left(\Omega^2 - \frac{\mathrm{Tr}\,(\Omega^2)}{3}\mathbb{I}\mathrm{d}\right) + \varepsilon\left(\gamma B + \frac{\Omega^2 B + B\Omega^2}{2} + \frac{\mathrm{Tr}\,(\Omega^2)}{3} B\right) + \mathscr{O}(\varepsilon^2) \tag{45}$$

where $\gamma > 0$ is given in Eq. (10). Using that $\ell = 0 \Leftrightarrow \Omega = 0$ we get $\partial_A F(0, \mathbb{I}\mathrm{d})B = \gamma\mu B$, so the lemma is proved. The stated form of B for ℓ small follows directly from Eq. (45) and $\ell = 2\omega_3 + \mathscr{O}(|\omega_3|^{3/2})$. □

For $\mu > 0$, we know that the ω-limit set of any solution to the equations of motion (36) and (39) is contained in the set $\mathscr{A}$, which is the set of relative equilibria. Lemma 9 states that there is exactly one point in $\mathscr{A}$ with a given angular momentum L if $\|L\| < \ell_0$. Therefore we get the following.

Theorem 3 *For $\mu > 0$, every solution to equations (36) and (39) with small angular momentum L, $\|L\| < \ell_0$, is attracted to the unique relative equilibrium given in Lemma 9. The asymptotic shape is an oblate ellipsoid of revolution.*

We remark that a theorem similar to Theorem 3 holds in the case $\mu = 0$. In this case some additional bound on the energy of the initial condition must be imposed. The gravitational force is not strong enough to restrain the growth of very energetic initial conditions.

4 Small deformations regime

Our goal in this section is to simplify the pseudo-rigid body equations of motion (36) and (39) under the small ellipticity hypothesis $A \approx \mathbb{I}\mathrm{d}$. In this case, it is convenient to write $A = \exp(B) = \mathbb{I}\mathrm{d} + B + B^2/2\ldots$ where B is a symmetric traceless matrix. Clearly the map

$$\exp : \mathrm{ssym}(3) = \{B \in \mathrm{Sym}(3) : \mathrm{Tr}(B) = 0\} \longrightarrow \mathrm{SSym}_+(3) \tag{46}$$

is a diffeomorphism near the origin. It is indeed a global diffeomorphism, see Ferral-Mello et al. (to apper), but this fact will not be used in this paper.

We wish to insert a small scaling parameter $\varepsilon > 0$ in order to express the small amplitude of the internal vibrations of $A = \exp(\varepsilon B)$. This requires also an scaling for Ω, since small deformations can only exist for small angular velocities. In order to balance the scalings for B and Ω we use the relative equilibrium expression given in Lemma 9. Replacing B by εB in this expression we obtain that the correct scaling for Ω is $\sqrt{\varepsilon}\Omega$. So, we introduce the modified coordinates $A = \exp(\varepsilon B)$ and $\sqrt{\varepsilon}\Omega$ into the Lagrangian and dissipation functions presented in the last section. Performing their Taylor expansions up to order $\varepsilon^{\frac{5}{2}}$, we obtain:

$$\begin{aligned}\mathscr{L}(B, \dot{B}, \Omega) = \frac{MR^2}{10}(\varepsilon^2 \mathrm{Tr}(\dot{B}^2) - \varepsilon \mathrm{Tr}(\Omega^2) - 2\varepsilon^2 \mathrm{Tr}(\Omega^2 B))\\ -\varepsilon^2 \frac{MR^2}{10}\gamma \mathrm{Tr}(B^2) + \mathscr{O}(\varepsilon^{\frac{5}{2}}),\end{aligned}$$

where γ is given in Eq. (10) in the Introduction. See Eq. (61) from Appendix 1 for further details. We also have,

$$\mathscr{D}(\dot{B}) = \varepsilon^2 \frac{4\pi\eta R^3}{3} \mathrm{Tr}(\dot{B}^2) + \mathscr{O}(\varepsilon^3).$$

The parameter ε was introduced by means of a change of variables only to understand the relative scale between B and Ω. So, we neglect all terms in $\mathscr{O}(\varepsilon^{\frac{5}{2}})$ from the above Lagrangian function and reverse the change of variables, or equivalently take $\varepsilon = 1$.

So, we get the following functions

$$\mathscr{L}(B, \dot{B}, \Omega) = \frac{MR^2}{10}\left(\|\dot{B}\|^2 + \|\Omega\|^2 + 2\mathrm{Tr}(\Omega\Omega^{TB})\right) - \frac{MR^2}{10}\gamma\|B\|^2, \tag{47}$$

$$\mathscr{D}(\dot{B}) = \frac{4\pi\eta R^3}{3}\|\dot{B}\|^2. \tag{48}$$

To obtain the equations of motion, we consider the functions g_{12}, g_{13} and g_{23} in Eq. (34), but now we replace $g(A) = \det(A) - 1$ by $g(B) = \mathrm{Tr}\,(B) = 0$. So, similarly to (35), we get

$$\sum_{ij} \chi_{ij} \frac{\partial g_{ij}}{\partial B} + \chi \frac{\partial g}{\partial B} = \begin{pmatrix} 0 & \chi_{12} & \chi_{13} \\ -\chi_{12} & 0 & \chi_{23} \\ -\chi_{13} & -\chi_{23} & 0 \end{pmatrix} + \chi\, \mathbb{I}\mathrm{d} = X + \chi\, \mathbb{I}\mathrm{d}.$$

The same type of analysis used to obtain Eq. (36) gives the equation of motion

$$\ddot{B} + \nu \dot{B} + \gamma B = -\Omega^2 + \frac{1}{3}\mathrm{Tr}\,(\Omega^2)\mathbb{I}\mathrm{d}, \tag{49}$$

where the constants ν and γ are given by (10).

Using the same procedure we used to get Eq. (39), we obtain the equations of motion for Ω

$$\Phi_B(\dot{\Omega}) = -\left(\Omega \dot{B} + \dot{B}\Omega + [\Omega^2, B]\right), \tag{50}$$

where $\Phi_B : \mathrm{skew}(3) \longrightarrow \mathrm{skew}(3)$ is the linear operator (inertia tensor)

$$\Phi_B(H) := H + BH + HB, \qquad \forall H \in \mathrm{skew}(3).$$

We remark that the deformations behave like a damped harmonic oscillator, externally forced by the noninertial effects of the rotation. Notice that all the variables are coupled.

The existence of solutions is the first problem we must deal with. Likewise Lemma 4, we state the following.

Lemma 10 *For every $B \in \mathrm{ssym}(3)$, $\|B\| < 1/2$, the operator Φ_B is symmetric, positive definite and, therefore, invertible.*

Proof Taking arbitrary $H_1, H_2 \in \mathrm{skew}(3)$,

$$\langle \Phi_B(H_1), H_2 \rangle = -\mathrm{Tr}\,(H_1 H_2) - \mathrm{Tr}\,(H_1 B H_2) - \mathrm{Tr}\,(B H_1 H_2) = \langle H_1, \Phi_B(H_2) \rangle\,.$$

Now, take $H \in \ker(\Phi_B)$. So, $(\mathbb{I}\mathrm{d} + 2B)H + H(\mathbb{I}\mathrm{d} + 2B) = 0$. Let $v \in \mathbb{R}^3$ be an unitary eigenvector of $(\mathbb{I}\mathrm{d} + 2B)$. Then, $\exists \lambda \in \mathbb{R}$:

$$v + 2Bv = \lambda v \Rightarrow Bv = \frac{\lambda - 1}{2} v \Rightarrow \frac{|\lambda - 1|}{2} < \frac{1}{2}.$$

Hence, $\lambda > 0$, i.e, $A = (\mathbb{I}\mathrm{d} + 2B)$ is symmetric, positive definite. By taking $A = \mathrm{diag}(a_1, a_2, a_3)$, and $S(H) = (h_1, h_2, h_3)$, the previous condition implies:

$$\begin{pmatrix} 0 & -(a_1 + a_2)h_3 & (a_1 + a_3)h_2 \\ (a_1 + a_2)h_3 & 0 & -(a_2 + a_3)h_1 \\ -(a_1 + a_3)h_2 & (a_2 + a_3)h_1 & 0 \end{pmatrix} = 0.$$

So, $H = 0$. Let β and H be eigenvalue and eigenvector of Φ_B. So, we have

$$(\beta - 1)H = (BH + HB) \Rightarrow |\beta - 1| \leq 2\|B\| \leq 1.$$

Therefore, since $\beta \neq 0$, $\beta \in (0, 2)$. □

This lemma shows that the equation for $\dot{\Omega}$ can be written in explicit form for $\|B\| < 1/2$ and therefore in this region the standard existence and uniqueness theorems for ordinary differential equations hold.

The functions $\mathscr{L}(Y, B, \dot{Y}, \dot{B})$, $\mathscr{D}(Y, B, \dot{Y}, \dot{B})$ are two-degree homogeneous functions in the velocities. So, the energy function associated to $\mathscr{L}$ is

$$E(B, \dot{B}, \Omega) = \frac{MR^2}{10}\left(\|\dot{B}\|^2 + \|\Omega\|^2 + 2\,\mathrm{Tr}(\Omega\Omega^T B)\right) + \frac{MR^2}{10}\gamma\|B\|^2 \tag{51}$$

and, from Eq. (16), we get $\dot{E} = -2\mathscr{D} \leq 0$, where the equality is reached if and only if $\dot{B} = 0$. The next lemma establishes a region where the solutions to the equations of motion are defined for all time.

Lemma 11 *Let $\beta = MR^2\gamma/50$ and $\mathscr{V}$ be the connected component of*

$$\left\{(B, U, \Omega) \in E^{-1}([0, \beta)) : \|B\| < 1/2\right\} \tag{52}$$

that contains the origin. The set $\mathscr{V}$ is bounded and any solution to the equations (49) and (50) initially in $\mathscr{V}$ remains in $\mathscr{V}$ for all $t > 0$.

Proof Recall that the energy is given by:

$$E(B, \dot{B}, \Omega) = \frac{MR^2}{10}\left(\mathrm{Tr}(\dot{B}^2) - \mathrm{Tr}((\mathbb{I}\mathrm{d} + 2B)\Omega^2) + \gamma\,\mathrm{Tr}(B^2)\right).$$

Inside $\mathscr{V}$, $A := \mathbb{I}\mathrm{d} + 2B > 0$, so $-\mathrm{Tr}((\mathbb{I}\mathrm{d}+2B)\Omega^2) = \mathrm{Tr}((\sqrt{A}\Omega)^T(\sqrt{A}\Omega)) \geq 0$. Lemma 10 ensures the existence and uniqueness. Taking one such solution, we know that $E(t) < \beta$. Suppose that exists a sequence $\{t_n\}$ such that $\|B(t_n)\| < 1/2$, $\forall n$, and $\lim\limits_{n\to+\infty} \|B(t_n)\| = 1/2$. But, in this case,

$$E(0) \geq E(t_n) \geq \frac{MR^2\gamma}{10}\|B(t_n)\|^2 \Rightarrow E(0) \geq \frac{MR^2\gamma}{40},$$

which is a contradiction. So, $\mathscr{V}$ is invariant. The set is bounded because, $\forall(B, \dot{B}, \Omega) \in \mathscr{V}$, $\|B\| < 1/2$, $\|\dot{B}\|^2 < 10\beta/MR^2$, $\|\sqrt{A}\Omega\|^2 < 10\beta/MR^2$. □

Again, we can apply the LaSalle's invariance principle (Lemma 7). Let $\mathscr{A}$ be the largest invariant subset under the flow contained in the set $\{(Y, \Omega, B, \dot{B}) : \dot{B} = 0\} \cap \mathscr{V}$. The $\omega-$limit set of any solution in $\mathscr{V}$ is contained in $\mathscr{A}$. Since $\mathscr{A}$ is invariant, for an initial condition in $\mathscr{A}$, $\dot{B}(t) = 0$ for all t. Then, by (49),

$$B = -\gamma^{-1}(\Omega^2 - \frac{1}{3}\mathrm{Tr}(\Omega^2)\mathbb{I}\mathrm{d}) \Rightarrow [B, \Omega^2] = 0$$

that implies $\dot{\Omega} \equiv 0$. So, every point in the attracting set $\mathscr{A}$ is a relative equilibrium.

As in the previous section the equations of motion are invariant under rotations. Therefore we may assume that the angular velocity vector of a relative equilibrium has the form $(0, 0, \omega_3)$. The relative equilibrium obtained from equations (49) and (50) is that given by the first term in the expression in Lemma 9. Let ε be the ellipticity as defined in equation (2). Using that $A = \exp(B)$, we obtain that the ellipticity associated to this relative equilibrium is:

$$\varepsilon = e^{\frac{\omega^2}{3\gamma}} - e^{-\frac{2\omega^2}{3\gamma}} \approx \frac{\omega^2}{\gamma} = \frac{1}{2}\frac{\omega^2 R^3}{GM}h_2, \quad h_2 = \frac{5}{2}\frac{1}{\left(1 + \frac{25}{2}\frac{\mu}{\rho g R}\right)},$$

where h_2 is analogous to the *second Love number* that is a long-standing known value, appearing in the works of Thomson and Tait (2009) and Love (1944), given by

$$h_2 = \frac{5}{2} \frac{1}{\left(1 + \frac{19}{2} \frac{\mu}{\rho g R}\right)}.$$

The factor $25/2$ appearing in our formula is different from the $19/2$ appearing in the second Love number because the pseudo-rigid body assumption (1) implies an overestimation of the stresses in the body (see a detailed explanation for this difference in Appendices 2 and 3). Since in both models, Love's and ours, the effective constant μ must be estimated using the model itself this difference is irrelevant. We remark that as the radius R becomes larger, we may neglect the term corresponding to the elasticity. So, we obtain the standard flattening

$$\varepsilon_g \approx \frac{5}{4} \frac{\omega^2 R^3}{GM}, \tag{53}$$

see for instance paragraph 374 of Lamb (1932).

By defining the same action as in Lemma 5, a calculation similar to that in Lemma 5 gives that the angular momentum given in Eq. (9) is conserved by this new system. We remark that the angular momentum in Eq. (9) is an approximation to the original angular momentum L given in Eq. (40) (for small values of $\|L\|$).

Finally the same arguments used to prove Theorem 3 can be used to prove the following.

Theorem 4 *Every solution to Equations (49) and (50) initially in the set $\mathscr{V}$, given in Lemma 11, is attracted to the unique relative equilibrium that has the same angular momentum as the solution. The asymptotic shape is an oblate ellipsoid of revolution.*

So, for small angular momentum the asymptotic behavior of the system studied in this section an that from the previous section is essentially the same.

4.1 Quantitative analysis

We recall from Carr (1981) some basic results which we apply to this specific model. In the general case, take $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and let $A_1 \in M(n)$, $A_2 \in M(m)$ be square matrices, $f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ be smooth functions such that $f(0,0) = 0$, $Df(0,0) = 0$, $g(0,0) = 0$ and $Dg(0,0) = 0$. Construct the system

$$\begin{cases} \dot{x} = A_1 x + f(x, y) \\ \dot{y} = A_2 y + g(x, y) \end{cases}, \tag{54}$$

supposing that all the eigenvalues of A_1 have zero real parts and all the eigenvalues of A_2 have negative real parts.

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $h(0) = 0$ and $Dh(0) = 0$, be a smooth function. If its graph $y = h(x)$ is an invariant manifold for the flow of (54), it is called a *center manifold* for (54). In this case the flow on this manifold is given by $(u(t), h(u(t)))$, where $u(t) \in \mathbb{R}^n$ is solution of

$$\dot{u} = A_1 u + f(u, h(u)). \tag{55}$$

The following theorem is proved in Carr (1981).

Theorem 5 *Suppose that the zero solution of (55) is stable. Then*

(i) *The zero solution of (54) is stable.*

(ii) *Let* $(x(t), y(t))$ *be solution of (54), with* $(x(0), y(0))$ *sufficiently small, and* $\sigma := \min\{|Re(\lambda)| : \lambda \in Spectrum(A_2)\}$. *So, exist constants* $C_1, C_2 > 0$ *and a solution* $u(t)$ *of (55) such that*

$$\|x(t) - u(t)\| \leq C_1 e^{-\sigma t}, \quad \|y(t) - h(u(t))\| \leq C_2 e^{-\sigma t} \tag{56}$$

Now, we apply the rescaling $B \to \varepsilon B$ and $\Omega \to \sqrt{\varepsilon}\Omega$ directly at the equations of motion (36), (39) and (49), (50). The first system becomes

$$\begin{aligned} \ddot{B} + \nu\dot{B} + \gamma B &= -\Omega^2 + \frac{1}{3}\operatorname{Tr}(\Omega^2)\mathbb{I}\mathrm{d} + \mathscr{O}(\varepsilon^{\frac{1}{2}}) \\ \dot{\Omega} &= -\varepsilon\Omega\dot{B} - \varepsilon\dot{B}\Omega - \varepsilon^{\frac{3}{2}}\frac{1}{2}[\Omega^2, B] + \mathscr{O}(\varepsilon^2), \end{aligned} \tag{57}$$

and the second

$$\begin{aligned} \ddot{B} + \nu\dot{B} + \gamma B &= -\Omega^2 + \frac{1}{3}\operatorname{Tr}(\Omega^2)\mathbb{I}\mathrm{d} \\ \dot{\Omega} &= -\varepsilon\Omega\dot{B} - \varepsilon\dot{B}\Omega - \varepsilon^{\frac{3}{2}}[\Omega^2, B] + \mathscr{O}(\varepsilon^2), \end{aligned} \tag{58}$$

So, we see that the systems coincide when $\varepsilon = 0$ and have the form

$$\begin{aligned} \dot{\Omega} &= 0 \\ \begin{pmatrix} \dot{B} \\ \dot{U} \end{pmatrix} &= A_2 \begin{pmatrix} B \\ U \end{pmatrix} + \begin{pmatrix} 0 \\ -\Omega^2 + 1/3\operatorname{Tr}(\Omega^2)\mathbb{I}\mathrm{d} \end{pmatrix}, \end{aligned} \tag{59}$$

where $A_2(B, U) := (U, -\nu U - \gamma B)$. Note that this system satisfies the hypotheses of Theorem 5, with $x = \Omega$, $y = (B, U)$ and $A_1(\Omega) = 0$. If $A_2(B, U) = \lambda(B, U)$, then $\lambda^2 + \nu\lambda + \gamma = 0$, so

$$\lambda = -\frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu^2 - 4\gamma}$$

whose real part is always negative.

We remark that the graph of the function $(B, U) = h(\Omega) = (\gamma^{-1}(-\Omega^2 + 1/3\operatorname{Tr}(\Omega^2)\mathbb{I}\mathrm{d}), 0)$ defines a global center manifold. Indeed, each point on this graph is a stable relative equilibrium. So, the whole graph is invariant under the flow. We can see explicitly from (59) that for each $(\Omega(0), B(0), U(0))$ in this graph there exist constants $C_1, C_2 > 0$ such that

$$\|\Omega(t) - \Omega(0)\| \leq C_1 e^{-\sigma t}, \quad \left\|\gamma B(t) - \left(-\Omega(0)^2 + \frac{1}{3}\operatorname{Tr}(\Omega(0)^2)\mathbb{I}\mathrm{d}\right)\right\| \leq C_2 e^{-\sigma t}, \tag{60}$$

where $\sigma = Re\left(-\nu/2 + \sqrt{\nu^2 - 4\gamma}/2\right)$. Quoting the Section 9 from Ferraz-Mello (2013), we see that for a wide range of examples in the Solar System, such eigenvalues are real, i.e., the corresponding harmonic oscillator is overdamped. Remark that the constant denoted by γ in Ferraz-Mello (2013) is obtained in the present paper by the quotient $25\gamma/(4\nu)$.

Therefore, we see that this graph is a normally hyperbolic invariant manifold and using the main theorems from Fenichel (1971) we see that for each $\varepsilon > 0$ sufficiently small, there is an invariant manifold, diffeomorphic to this one, for each one of the systems [(49), (50)] and [(36), (39)]. Such manifolds are also attractive in the sense of (56). For both systems [(49), (50)] and [(36), (39)] these are manifolds of equilibria.

5 Conclusion

The main contribution given in this paper is a mathematical model for the motion of a body that is isolated, incompressible, and spherically symmetric at rest. We were lead to this model by means of a series of simplifications that started with Newton's equation for a continuum (Sect. 2), passed the pseudo-rigid body approach of Dirichlet, Riemann, etc... (Sect. 3), and ended with an even simpler low ellipticity approximation to the pseudo-rigid body (Sect. 4). During the computations the body was supposed linearly viscoelastic, homogeneous and isotropic. At a first moment these strong hypotheses suggested that the final model could not be applied to any realistic physical celestial body. Indeed, most real bodies are far from being homogeneous which becomes evident when we compare the moment of inertia of a real body with the moment of inertia of a homogeneous ball with the same mass and geometric radius of the real body. This deficiency lead us to abandon the original meaning of the geometric quantities of the original model (as, for instance, the geometric meaning of B) and to group the model parameters into measurable physical quantities. In this way we were lead to the model presented in the Introduction that, as in the Abstract, is given by the following Lagrangian and dissipation functions

$$\mathscr{L} = \frac{\omega \cdot \mathrm{I}\omega}{2} + \frac{1}{36\mathrm{I}_\circ}(\|\dot{Q}\|^2 - \gamma\|Q\|^2)$$

and

$$\mathscr{D} = \frac{\nu}{36\mathrm{I}_\circ}\|\dot{Q}\|^2$$

Our model has the following short a posteriori explanation. From the point of view of Celestial Mechanics the gravitational field of a body can be approximated by its monopole and quadrupole gravitational moment. Due to self-gravitation a non-rotating and non-rigid isolated body must have a null quadrupole moment. On the contrary, a rotating non-rigid body must have a non-null quadrupole moment which has an intensity that depends on the angular speed. Regardless on how complicated the quadrupole moment varies as a function of the angular speed it is sensible to suppose that this dependence can be linearized. This reasoning brings up the term

$$\frac{1}{36\mathrm{I}_\circ}(\|\dot{Q}\|^2 - \gamma\|Q\|^2)$$

that attributes an inertia and a rigidity to the variations of the gravitational quadrupole tensor Q of the body. The quadrupole moment is naturally coupled to the angular velocity ω by means of the rotational kinetic energy

$$\frac{\omega \cdot \mathrm{I}\omega}{2}$$

and the moment of inertia tensor

$$\mathrm{I} = \mathrm{I}_\circ\left(\mathbb{I}\mathrm{d} - \frac{Q}{3}\right)$$

Notice that $\mathrm{I}_\circ = \mathrm{Tr}\,\mathrm{I}/3$ is constant and equal to the moment of inertia of the spherical body at rest The total mass M and $\mathrm{I}_\circ$ are the inertial physical properties of the body. Finally, linear dissipation is introduced by means of an additional Rayleigh dissipation function to the conservative linear quadrupole oscillator:

$$\mathscr{D} = \frac{\nu}{36\mathrm{I}_\circ}\|\dot{Q}\|^2$$

The effective viscosity coefficient ν and the elastic constant γ must be estimated from direct observation. Therefore, within this model the body is completely determined by the physical constants: M, $\mathrm{I}_\circ$, γ, and ν. The total mass M is necessary for the characterization of the translational energy of the body, which has been completely neglected in this work since the center of mass is supposed at rest.

We believe that our model is the simplest one can get without loosing the symmetries and the conserved quantities of the original system. With respect to the coupling between the quadrupole moment tensor and the body rotation our model provides the same answers as those more sophisticated models, at least for low angular momentum. In contrast to more sophisticated models, it does not answer, and it is not its purpose, many interesting questions about the geometry of the deformed body, as for instance, about the flattening. When compared to simpler models it has several advantages. At first there is no hypothesis of alignement of principal moments of inertia and angular velocity. This is a common hypothesis in many simplified models. For our model the initial angular velocity and the initial quadrupole moment tensor is arbitrary. The alignment is a natural consequence of the dynamics. Of course our model shares many features and dynamical properties with other simplified models. This is more evident in the study of two-body systems, which we will pursue in a future work. Finally, due to its Lagrangian formulation our model can be used as a building block in the study of many-body systems without any further assumptions (compare to the study of a planar two body system in Zlenko (2015).

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Appendix 1

We perform the Taylor expansion of (32) up to order ε^2. If $A = \exp(\varepsilon B) \in \mathrm{SSym}_+(3)$ then:

$$\begin{aligned}\phi(\varepsilon B) &= -\frac{G\rho^2}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{1}{\|\exp(\varepsilon B)(x-y)\|}dxdy + C\\ &= -\gamma\int_0^\infty \frac{1}{\sqrt{\det(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})}}d\lambda + C.\end{aligned}$$

Thus, (choosing $\phi(0) = 0$),

$$\phi(\varepsilon B) = \varepsilon D\phi(0)B + \frac{1}{2}\varepsilon^2 D^2\phi(0)B^2 + \mathscr{O}(\varepsilon^3).$$

Since the derivatives of the integrand are continuous and bounded, we may perform the following calculations

$$D\phi(\varepsilon B)B = \frac{d}{d\varepsilon}\phi(\varepsilon B) = \gamma\int_0^\infty \frac{\mathrm{Tr}\left((\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})^{-1}\exp(2\varepsilon B)B\right)}{\sqrt{\det(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})}}d\lambda,$$

$$\begin{aligned}D^2\phi(\varepsilon B)B^2 &= \frac{d}{d\varepsilon}D\phi(\varepsilon B)B = -\gamma\int_0^\infty \frac{\mathrm{Tr}\left((\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})^{-1}\exp(2\varepsilon B)B\right)^2}{4\sqrt{\det(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})}}\\ &\quad -\frac{\mathrm{Tr}\left(-4(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})^{-2}\exp(4\varepsilon B)B^2 + 4(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})^{-1}\exp(2\varepsilon B)B^2\right)}{2\sqrt{\det(\exp(2\varepsilon B)+\lambda\mathbb{I}\mathrm{d})}}d\lambda.\end{aligned}$$

Hence,

$$D\phi(0)B = 0,$$
$$D^2\phi(0)B^2 = -4\gamma \int_0^\infty \frac{1}{2(1+\lambda)^{\frac{3}{2}}} \left((1+\lambda)^{-2} - (1+\lambda)^{-1}\right) d\lambda \,\mathrm{Tr}\,(B^2) = \frac{8}{15}\gamma\,\mathrm{Tr}\,(B^2).$$

Recalling that $\gamma = 3M^2G/(10R)$, we get

$$\phi(\varepsilon B) = \frac{2}{25}\frac{M^2G}{R}\varepsilon^2\,\mathrm{Tr}\,(B^2) + \mathscr{O}(\varepsilon^3). \tag{61}$$

Appendix 2

Love (1944) (chapter XI) studied the relative equilibria of Eq. (20) for an elastic, incompressible, homogeneous, and isotropic body. Love assumed, as we did, that the body angular velocity was small in such a way that he could linearize the problem around the spherical shape of equilibrium. Then he used spherical coordinates (r, θ, ϕ) to write the deformed surface of the body as $r = R + \varepsilon S$, where $S = S(\theta, \phi)$ and R is the radius of the sphere of reference. Finally, he supposed εS to be expanded in a series $\sum \varepsilon_n S_n$ of surface spherical harmonics and following this expansion he also expanded all other quantities in the problem: pressure, deformation, and boundary conditions, in spherical harmonics and powers of r. Equating powers of r he obtained recurrence relations for the coefficients of the spherical harmonics and solved the linear problem. Up to second order spherical harmonics the parametrization for the deformed surface becomes

$$r = R\left(1 - \frac{2}{3}\varepsilon\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right), \tag{62}$$

where ε is the flattening of the body up to first order in ε. The deformation u of the body (here we follow the notation of Love and writes $u(x)$ for what we had previously written as $u(x) - x$) up to the second order, is

$$\mathbf{u}(x_1, x_2, x_3) = -A_2 r^2 \nabla p_2 - B_2 p_2 \mathbf{r} - \nabla\phi_2$$

where p_2, ϕ_2 are spherical solid harmonics:

$$p_2 = \frac{2}{3}\left(\frac{r}{R}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = \frac{1}{3R^2}\left(2x_3^2 - x_1^2 - x_2^2\right)$$

and ϕ_2 is proportional to p_2 (we are following Love's notation). The equations of motion, boundary conditions and div $\mathbf{u} = 0$ imply (Love (1944), chapter XI, paragraph 177, equations (22) to (28)) $A_2 = -5B_2/4$ and $2\phi_2 = -R^2(B_2 + 4A_2)p_2$. So,

$$\mathbf{u} = -\frac{B_2}{6R^2}\left(3x_1^3 + 3x_1x_2^2 + 9x_1x_3^2 - 8R^2x_1, 3x_2^3 + 3x_2x_1^2 + 9x_2x_3^2 - 8R^2x_2,\right.$$
$$\left. -6x_3^3 - 12x_3x_1^2 - 12x_3x_2^2 + 16R^2x_3\right).$$

Using the definition of flattening in Eq. (2), Eq. (62), and that the surface of the deformed body is also given by $x + u(x)$ with $\|x\| = R$ we get a relation between the flattening ε and the coefficient B_2: $\varepsilon = 5B_2/(2(1 + 5B_2/6)) \approx 5B_2/2$. Now, using the elastic energy formula (21) and the elastic strain tensor in Eq. (23) we obtain that the elastic energy of Love's deformation as a function of ε is:

$$E_{Love} = \mu \int_{\mathscr{B}} \mathrm{Tr}\,(D\mathbf{u}^2)(\mathbf{x})d\mathbf{x} = \left(\frac{4\pi R^3}{3}\right)\left(\frac{2\cdot 19}{3\cdot 25}\mu\right)\varepsilon^2 = \frac{k_{Love}}{2}\varepsilon^2. \tag{63}$$

In our approach, the deformation of the body is a priori imposed as $e^{\varepsilon B}x - x \approx \varepsilon Bx$ where ε is the flattening of the ellipsoid up to first order in ε, and

$$B = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The elastic energy associated to this deformation is

$$E = \mu \int_{\mathscr{B}} \mathrm{Tr}\,((\varepsilon B)^2)d\mathbf{x} = \left(\frac{4\pi R^3}{3}\right)\left(\frac{2}{3}\mu\right)\varepsilon^2 = \frac{k}{2}\varepsilon^2. \tag{64}$$

So, the elastic rigidity with respect to ellipticity ε-deformations found by Love and ours have a ratio $k_{Love}/k = 19/25$. This is exactly the ratio between the elastic contribution to the Love numbers of each model as it should be. Notice that Love solves the Euler–Lagrange equations associated to the elastic energy functional up to first order in ε. Indeed he is minimizing this energy functional under the boundary conditions of linear elasticity. Our deformation neither satisfies the elastic boundary conditions nor minimizes the elastic functional energy. So, our a priori imposed deformation overestimates both the elastic energy and the stress while Love's deformation gives the correct value up to order ε. Since both models lead to a deformed surface represented by the same ellipsoid up to first order in ε, the gravitational energy of both models coincide.

Appendix 3

In Appendix 2 we presented a quantitative comparison between our results and some of those obtained by Love for the equilibrium of a rotating elastic body. The computations of Love are involved which makes the comparison not easy. In this appendix we use a simple example to explain the qualitative difference between our approach and that by Love.

Consider a homogeneous rod of natural length ℓ, elastic modulus λ, and linear density ρ rotating with constant angular velocity ω around its center point. Let $s \in [-\ell/2, \ell/2]$ be a point in the undeformed rod and $p(s)$ be the position of this point after deformation (Lagrangian description). If $u(s) = p(s) - s$ denotes the deformation of the point originally at s then the equilibrium equation is:

$$-\lambda u''(s) = \rho\omega^2 s, \qquad u'(-\ell/2) = u'(\ell/2) = 0 \tag{65}$$

This equation is analogous to the equation solved by Love in the sense that both are linear equations that were obtained under the assumption of small deformations and linear constitutive relations. The solution to problem (65) is:

$$u'(s) = -\frac{cs^2}{2} + \frac{c\ell^2}{8}, \quad u(s) = -\frac{cs^3}{6} + \frac{c\ell^2 s}{8}, \quad \text{where} \quad c = \frac{\rho\omega^2}{\lambda}$$

Since the problem treated by Love is three-dimensional, he was not able to obtain a simple solution as this one. He obtained a solution in the form of an infinite series in powers of a scalar parameter ε that essentially measures the flattening of the deformed body. His solution satisfies the boundary conditions accordingly. Notice that the solution to equation (65), which is linear, is a nonlinear function (a cubic polynomial). The same happens for the solution found by Love. Now, the solution to problem (65) is the critical point of the functional

$$u \to E(u) = \int_{-\ell/2}^{\ell/2} \left[\frac{\lambda}{2} u'^2 - \rho\omega^2 su \right] ds, \quad \text{with} \quad u'(-\ell/2) = u'(\ell/2) = 0 \tag{66}$$

The first term in E

$$E_e(u) = \int_{-\ell/2}^{\ell/2} \frac{\lambda}{2} u'^2 ds$$

is the elastic energy of the rod. The second term

$$E_c(u) = -\int_{-\ell/2}^{\ell/2} \rho\omega^2 suds$$

is the centrifugal energy of the rod in the following sense. The energy spent to move a point particle of mass m from the origin to a point s in the centrifugal field $\omega^2 s$ is $-m\omega^2 s^2/2$. So the centrifugal energy of the rod after deformation minus the centrifugal energy before deformation is

$$-\int_{-\ell/2}^{\ell/2} \rho \frac{\omega^2}{2} \underbrace{[s+u]}_{=p(s)}{}^2 ds + \int_{-\ell/2}^{\ell/2} \rho \frac{\omega^2 s^2}{2} ds = -\int_{-\ell/2}^{\ell/2} \rho\omega^2 suds - \int_{-\ell/2}^{\ell/2} \rho \frac{\omega^2}{2} u^2 ds$$

that is equal to $E_c(u)$ except for the negligible term which is quadratic in u. It is important to say that we are disregarding variations of density after the deformation of the rod. This can be done when the elastic constant is large compared to the centripetal force, namely, if

$$\frac{\rho\omega^2\ell^2}{\lambda} \ll 1$$

The length of the deformed rod minus its natural length is

$$\delta = 2p(\ell/2) - \ell = 2u(\ell/2) = 2\left(\frac{\ell}{2} + \frac{c\ell^3}{24}\right) - \ell = \frac{c\ell^3}{12} \tag{67}$$

The elastic energy stored in the rod is

$$E_e = \int_{-\ell/2}^{\ell/2} \frac{\lambda}{2} u'^2 ds = \frac{1}{240} \lambda \ell^5 c^2 = \frac{1}{2} \frac{144}{120} \frac{\lambda}{\ell} \delta^2 \tag{68}$$

The centrifugal energy of the rod is

$$E_c(u) = -\int_{-\ell/2}^{\ell/2} \rho\omega^2 suds = -\frac{\rho\omega^2\ell^5 c}{120} = -\frac{\rho\omega^2\ell^2}{10}\delta \tag{69}$$

The length of the rod as a function of ω^2 is given by the critical point of the total energy of the rod as a function of δ:

$$0 = \frac{d}{d\delta} E = \frac{d}{d\delta} \left\{ \frac{1}{2} \frac{144}{120} \frac{\lambda}{\ell} \delta^2 - \frac{\rho\omega^2\ell^2}{10} \delta \right\} \Longrightarrow \delta = \frac{\rho\ell^3}{12\lambda} \omega^2$$

that gives exactly the value in Eq. (67), as expected.

Our approach to problem (65) is the following. We a priori assume that the deformation is linear, $u(s) = bs$, where b is a free parameter (Love obtains a nonlinear deformation as a solution of a linear equation and we propose a linear deformation as an "approximation" to the real solution). The linear deformation neither solves the equation nor the boundary conditions of problem (65). In order to determine b we use the variational characterization (66) of problem (65) to obtain the function:

$$b \to \int_{-\ell/2}^{\ell/2} \left[\frac{\lambda}{2} b^2 - \rho\omega^2 s^2 b \right] ds = \frac{\lambda b^2 \ell}{2} - \frac{\rho\omega^2 \ell^3 b}{12}.$$

The critical point of this function is

$$b = \frac{\rho\omega^2\ell^2}{12\lambda} = \frac{c\ell^2}{12}$$

In this case the length of the deformed rod minus its natural length is

$$\delta = 2p(\ell/2) - \ell = 2u(\ell/2) = b\ell = \frac{c\ell^3}{12}, \tag{70}$$

the elastic energy stored in the rod is

$$E_e = \int_{-\ell/2}^{\ell/2} \frac{\lambda}{2} u'^2 ds = \frac{\lambda b^2 \ell}{2} = \frac{1}{2}\frac{\lambda}{\ell}\delta^2$$

and the centrifugal energy of the rod is

$$E_c(u) = -\int_{-\ell/2}^{\ell/2} \rho\omega^2 s u ds = -\frac{\rho\omega^2\ell^3 b}{12} = -\frac{\rho\omega^2\ell^2}{12}\delta$$

As above, the length of the rod as a function of ω^2 is given by the critical point of the total energy of the rod as a function of δ:

$$0 = \frac{d}{d\delta}E = \frac{d}{d\delta}\left\{ \frac{1}{2}\frac{\lambda}{\ell}\delta^2 - \frac{\rho\omega^2\ell^2}{12}\delta \right\} \Longrightarrow \delta = \frac{\rho\ell^3}{12\lambda}\omega^2$$

that gives exactly the value in Eq. (70), as expected, and also the same value in Eq. (67), as not expected. Therefore in this case the approximated method, which is similar to that used in the paper, gives exactly the same rigidity with respect to variations of ω^2 as the exact solution. In general this does not happen. For instance, the elastic rigidity we obtain in our paper is close to (but different from) that obtained by Love with a rigorous analysis.

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