Celest Mech Dyn Astr (2014) 119:27–44
DOI 10.1007/s10569-014-9545-8

ORIGINAL ARTICLE

Solving Kepler's equation via Smale's α-theory

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Received: 20 January 2014 / Revised: 7 March 2014 / Accepted: 21 March 2014 /

Abstract We obtain an approximate solution $\tilde{E} = \tilde{E}(e, M)$ of Kepler's equation $E - e\sin(E) = M$ for any $e \in [0, 1)$ and $M \in [0, \pi]$. Our solution is guaranteed, via Smale's α-theory, to converge to the actual solution E through Newton's method at quadratic speed, i.e. the n-th iteration produces a value E_n such that $|E_n - E| \leq (\frac{1}{2})^{2^n-1}|\tilde{E} - E|$. The formula provided for $\tilde{E}$ is a piecewise rational function with conditions defined by polynomial inequalities, except for a small region near $e = 1$ and $M = 0$, where a single cubic root is used. We also show that the root operation is unavoidable, by proving that no approximate solution can be computed in the entire region $[0, 1) \times [0, \pi]$ if only rational functions are allowed in each branch.

Keywords Kepler's equation · Smale's α-theory · Newton's method · Optimal starter

1 Introduction

Kepler's laws describe the way planets move in their orbits about the Sun. Geometrically, they state that the planets move in planar elliptical orbits with eccentricity $e \in [0, 1)$ and that the area swept by the line joining the planet and the Sun increases linearly with time, which leads immediately to Kepler's equation $E - e\sin(E) = M$, relating mean and eccentric anomalies. The mean anomaly is a fictitious angle M that increases linearly with time at a rate $M = 2\pi t/T$, where T is the orbital period, and the eccentric anomaly E gives

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the coordinates of the planet in its orbit plane as $(x, y) = (a\cos(E), b\sin(E))$. Here, the xy-plane has origin at the center of the ellipse with the x-axis pointing to the perihelion, and the values a and b are the semi-major and semi-minor axis of the ellipse. Therefore, finding the exact location of a planet at a given time requires solving an instance of Kepler's equation for some M, assuming that the values a, b, e and T are known (actually, only a and e are needed, since $b = a\sqrt{1-e^2}$ and T can be obtained from a using the third law). For a derivation of these formulas and a detailed introduction to Kepler's equation, see Battin (1987).

By a symmetry argument, the equation can be easily reduced to the case $M \in [0, \pi]$. The existence and uniqueness of solution $E \in [0, \pi]$ follows from the fact that the function $f_{e,M} : [0, \pi] \to [0, \pi]$ given by $f_{e,M}(E) = E - e\sin(E) - M$ is strictly increasing.

Several solutions to the problem have been proposed since it was stated 400 years ago. Some authors have tried non-iterative methods to solve the equation up to a fixed predetermined accuracy (Markley 1995; Mortari and Clochiatti 2007). However, we want to calculate the solution with arbitrary precision, hence our interest in iterative techniques.

Kepler himself proposed to use a fixed-point iteration to solve the equation (Chap. 1 of Colwell 1993), i.e. guess E_0, an approximation of the exact solution E, and then iterate $E_{n+1} = M + e\sin(E_n)$. This sequence converges to E, since $|E_{n+1} - E| = |M + e\sin(E_n) - E| = e|\sin(E_n) - \sin(E)| \leq e|E_n - E|$, which implies that $|E_n - E| \leq e^n|E_0 - E| \longrightarrow 0$ as $n \to \infty$. The problem with this approach is that the convergence is slow for values of e near 1. For the orbit of Mercury, which has $e \approx 0.2$, about five iterations are needed to reduce the error by a factor of 10^{-3}, while for values of eccentricity $e > 0.5$ the fixed-point iteration is even slower than a bisection method.

Although the fixed-point iteration does not provide an efficient solution to Kepler's equation, it exhibits the structure of most of the current methods to solve it: first, guess an approximation $\tilde{E}$ of the solution (called *starter*), and then use some iterative technique to produce a sequence quickly converging to the actual solution (see Danby 1987; Danby and Burkardt 1983; Mortari and Elipe 2014; Palacios 2002). For the second part, Newton's method seems to be the most used iteration, mainly due to its conceptual simplicity, generality and fast convergence. The guessing part, however, requires some specific understanding on the equation and has been the subject of many recent papers (Calvo et al. 2013; Mikkola 1987; Ng 1979; Nijenhuis 1991; Odell and Gooding 1986; Taff and Brennan 1989).

Starters have been compared (and optimized) using different criteria, such as the number of iterations needed to reach certain precision, the distance to the actual solution, the number of floating point operations needed for its computation, etc. For this purpose, we adopt a criterion which is very specific to Newton's method and guarantees that the iterations reduce the error at quadratic speed. More precisely, we will only accept an approximate solution $\tilde{E}$ of the equation $f_{e,M}(E) = 0$ if Newton's method starting at $E_0 = \tilde{E}$ produces a sequence E_n such that $|E_n - E| \leq (\frac{1}{2})^{2^n-1}|\tilde{E} - E|$ for all $n \geq 0$.

Taking one of these starters satisfying $\tilde{E} \in [0, \pi]$, the initial error is at most π, so we obtain an accuracy 10^{-N} after only $n = \lceil \log_2\left(1 + \log_2(\pi) + \log_2(10)N\right)\rceil$ iterations. In particular, ten iterations of Newton's method starting from $\tilde{E}$ give an error less than 10^{-307} for any input value of e and M.

We will use a simple test, due to Smale (1986) and later improved by Wang and Han (1990), which depends only on the starter $\tilde{E}$ and guarantees the speed of convergence that we claim.

Table 1 Classical starters

Starter	Formula
S_1	M
S_2	$M + e\sin(M)$
S_3	$M + e\sin(M)(1 + e\cos(M))$
S_4	$M + e$
S_5	$M + \frac{e\sin(M)}{1-\sin(M+e)+\sin(M)}$
S_6	$M + \frac{e(\pi-M)}{1+e}$
S_7	$\min\left\{\frac{M}{1-e}, S_4, S_6\right\}$
S_8	$S_3 + \frac{e^4(\pi-S_3)}{20\pi}$
S_9	$M + e\sin(M)(1 - 2e\cos(M) + e^2)^{-\frac{1}{2}}$
S_{10}	$s - \frac{q}{s}$, where $r = \frac{3M}{e}$, $q = \frac{2(1-e)}{e}$ and $s = [(r^2 + q^3)^{\frac{1}{2}} + r]^{\frac{1}{3}}$

Definition 1.1 (*Smale's α-test*) We say that $\widetilde{E}$ is an *approximate zero* of $f_{e,M}$ if it satisfies the following condition

$$\alpha(f_{e,M}, \widetilde{E}) = \beta(f_{e,M}, \widetilde{E}) \cdot \gamma(f_{e,M}, \widetilde{E}) < \alpha_0,$$

where

$$\beta(f_{e,M}, \widetilde{E}) = \left|\frac{f_{e,M}(\widetilde{E})}{f'_{e,M}(\widetilde{E})}\right|, \quad \gamma(f_{e,M}, \widetilde{E}) = \sup_{k\geq 2}\left|\frac{f^{(k)}_{e,M}(\widetilde{E})}{k!f'_{e,M}(\widetilde{E})}\right|^{\frac{1}{k-1}}$$

and $\alpha_0 = 3 - 2\sqrt{2} \approx 0.1715728$.

Odell and Gooding (1986) compiled a list of starters that have been proposed in the literature by many authors. Table 1 provides a formula for those that will be studied in this paper.

In Sect. 2 we present an analytical study of the starters $\tilde{E} = 0, \pi, M, \frac{M}{1-e}$ using the notion of approximate zero. More precisely, for each of these starters, we obtain in Theorems 2.2, 2.3, 2.4 and 2.5 regions where they satisfy Smale's α-test, thus providing approximate solutions. We also show in Theorem 2.6 that Ng's starter S_{10} (Eq. 9 of Ng 1979), which is obtained by solving a cubic equation, gives an approximate solution on the entire domain.

Similarly, in Sect. 3 we compare the remaining starters $S_2, \ldots, S_9$, and the improved S_7 starter obtained by Calvo et al. (Prop. 1, 2013). More precisely, we check numerically where those starters satisfy Smale's α-test on a very fine grid of points in $[0, 1) \times [0, \pi]$.

In Sect. 4 we prove Theorem 1.2, showing a simple starter $\tilde{E} = \tilde{E}(e, M)$ that satisfies the α-test for all $e \in [0, 1)$ and $M \in [0, \pi]$. The starter is a piecewise-defined function that requires a single cubic root in a small part of the region close to the corner $e = 1$, $M = 0$. Apart from that root, the rest of the expressions involved are constant or rational functions that can be computed with at most two arithmetic operations. The highlights of this starter are its computational simplicity and the fact that it is formally proven to converge at quadratic speed since the first iteration, thus providing arbitrary precision with a very few Newton's method steps. It should be noted that reducing the initial error (i.e. the distance from the starter to the exact solution) is not our design goal.

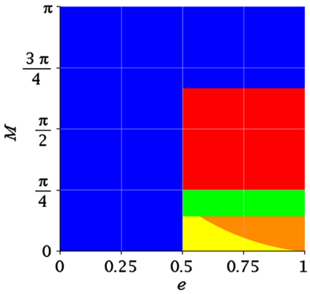

Fig. 1 The points where $\tilde{E} = M$, $\tilde{E} = \frac{3\pi}{2}$ and $\tilde{E} = \frac{\pi}{2}$ satisfy the α-test for $f_{e,M}(E)$ are shown in *blue*, *red* and *green* respectively. The ones of $\tilde{E} = \frac{M}{1-e}$ and $\tilde{E} = \frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}}$ appear in *yellow* and *orange*

Theorem 1.2 *The starter*

$$\tilde{E}(e, M) = \begin{cases} M & \text{if } e \leq \frac{1}{2} \text{ or } M \geq \frac{2\pi}{3} \\ \frac{2\pi}{3} & \text{if } e \geq \frac{1}{2} \text{ and } \frac{\pi}{4} \leq M \leq \frac{2\pi}{3} \\ \frac{\pi}{2} & \text{if } e \geq \frac{1}{2} \text{ and } \frac{\pi}{7} \leq M \leq \frac{\pi}{4} \\ \frac{M}{1-e} & \text{if } e \geq \frac{1}{2},\ M \leq \frac{\pi}{7} \text{ and } M < \frac{\sqrt[4]{12\alpha}(1-e)^{\frac{3}{2}}}{\sqrt{e}} \\ \frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}} & \text{otherwise} \end{cases}$$

is an approximate zero of $f_{e,M}$ *for all* $e \in [0, 1)$ *and* $M \in [0, \pi]$.

This way of constructing an approximate solution by a piecewise function (see Fig. 1) can be compared to Ng's approach (see Figure 2 of Ng 1979). However, our function is computationally simpler because Ng's formula outside the corner uses rational functions involving many terms and near the corner uses S_{10}, which requires at least a cubic and a square root for its computation.

The region near the $(1, 0)$ corner where a cubic root is needed can be reduced as much as desired but cannot be completely avoided, as can be seen in Theorems 1.3 and 1.4, which will be proven in Sect. 5. Other authors have found similar obstructions in handling values of the eccentricity near 1 (Mikkola 1987; Ng 1979; Nijenhuis 1991).

Theorem 1.3 *For any* $\varepsilon > 0$, *there is a piecewise constant function* $\tilde{E}$ *defined in* $([0, 1) \times [0, \pi]) \setminus ([1 - \varepsilon, 1] \times [0, \arccos(1 - \varepsilon)])$ *that satisfies the* α*-test.*

Theorem 1.4 *Let* $\tilde{E}$ *be a piecewise rational function in* $[0, 1) \times [0, \pi]$ *with a finite number of branches defined by polynomial inequalities. Then there exists* (e_0, M_0) *such that* $\tilde{E}(e_0, M_0)$ *is not an approximate zero of* f_{e_0,M_0}.

The starter defined in Theorem 1.3 can be extended if $\varepsilon < 1 - \cos(\frac{\pi}{7})$ to the whole region by using $\frac{M}{1-e}$ and $\frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}}$ in the corner, as in Theorem 1.2. This result is the basis for constructing lookup tables of starters.

Finally, Theorem 1.4 and Remark 5.1 show that the classical starters $S_1, \ldots, S_8$ and the improved S_7 of Calvo et al. (2013) will necessarily fail near the corner $(1, 0)$, as Figs. 3, 4, 5 and 6 will later illustrate. Our theorem also excludes the possibility of using truncated power series (with integer exponents) for approximate zeros near the corner.

2 Analytical study of classical starters via α-theory

In this section we find regions where the starters $\tilde{E} = 0, \pi, M, \frac{M}{1-e}$ are approximate zeros of Kepler's equation in Theorems 2.2, 2.3, 2.4 and 2.5. We compare these with the regions computed numerically on a fine grid in Figs. 2 and 3. We also show that Ng's starter S_{10} works in the entire region in Theorem 2.6.

Throughout the paper, we will need the following technical result.

Lemma 2.1 *Let* $n \geq 2$ *and* $x \geq \frac{n!}{(n+1)^{n-1}}$. *Then, the sequence* $\{(\frac{x}{k!})^{\frac{1}{k-1}}\}_{k \geq n}$ *is decreasing.*

Proof It is enough to show that $(\frac{x}{k!})^{\frac{1}{k-1}} \geq (\frac{x}{(k+1)!})^{\frac{1}{k}}$ for all $k \geq n$, which is equivalent to the inequality $(\frac{x}{k!})^k \geq (\frac{x}{(k+1)!})^{k-1}$, or more simply $x \geq \frac{k!}{(k+1)^{k-1}}$. Note that the sequence $\frac{k!}{(k+1)^{k-1}}$ is decreasing, since

$$\frac{(k+1)!(k+1)^{k-1}}{k!(k+2)^k} = \frac{(k+1)^k}{(k+2)^k} < 1.$$

In particular, $x \geq \frac{n!}{(n+1)^{n-1}} \geq \frac{k!}{(k+1)^{k-1}}$ for all $k \geq n$, as we needed. □

Theorem 2.2 $\tilde{E} = 0$ *is an approximate zero of* $f_{e,M}(E)$ *in the region* $R_1 \cup R_2$*, where*

$$R_1 = \left\{0 \leq M \leq 4\alpha_0(1-e),\ 0 \leq e \leq \frac{3}{11}\right\},$$

$$R_2 = \left\{0 \leq M \leq \frac{\sqrt{6}\alpha_0(1-e)^{\frac{3}{2}}}{\sqrt{e}},\ \frac{3}{11} \leq e < 1\right\}.$$

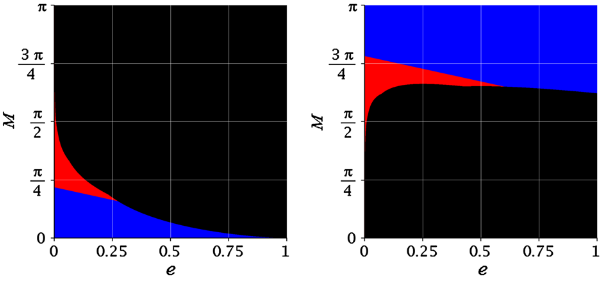

Fig. 2 The regions of Theorems 2.2 and 2.3 are shown in blue. *Red color* shows the points where $\tilde{E} = 0$ and $\tilde{E} = \pi$ satisfy the α-test for $f_{e,M}(E)$ that are not in the *blue region*

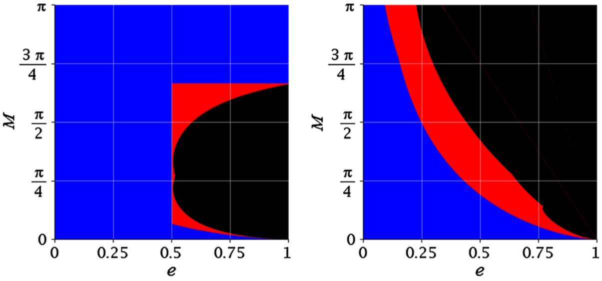

Fig. 3 The regions of Theorems 2.4, and 2.5 are shown in *blue*. *Red color* shows the points where $\tilde{E} = M$ and $\tilde{E} = \frac{M}{1-e}$ satisfy the α-test for $f_{e,M}(E)$ that are not in the *blue region*

Proof It is enough to show that $\alpha(f_{e,M},0) < \alpha_0$, which is equivalent to

$$\frac{M}{1-e} \sup_{\substack{k\geq 3\\ k \text{ odd}}} \left(\frac{e}{k!(1-e)}\right)^{\frac{1}{k-1}} < \alpha_0,$$

since $f(0) = -M$, $f'(0) = 1-e$, $f^{(\text{even})}(0) = 0$ and $f^{(\text{odd})}(0) = \pm e$. When $e \in [\frac{3}{11}, 1)$, we have $\frac{e}{1-e} \geq \frac{3}{8}$, and by Lemma 2.1,

$$\sup_{\substack{k\geq 3\\ k \text{ odd}}} \left(\frac{e}{k!(1-e)}\right)^{\frac{1}{k-1}} = \sqrt{\frac{e}{6(1-e)}}.$$

In this case, Smale's α-test translates into $\frac{M\sqrt{e}}{\sqrt{6}(1-e)^{3/2}} < \alpha_0$, which corresponds to the region R_2. For the remaining case, $e \in [0, \frac{3}{11}]$, we have that $\frac{e}{1-e} \leq \frac{3}{8}$, so

$$\left(\frac{e}{k!(1-e)}\right)^{\frac{1}{k-1}} \leq \left(\frac{1}{16}\right)^{\frac{1}{k-1}} \leq \frac{1}{4} \quad \forall k \geq 3.$$

This means that Smale's condition is implied by $\frac{M}{4(1-e)} < \alpha_0$, which corresponds to the region R_1. □

Theorem 2.3 *$\tilde{E} = \pi$ is an approximate zero of $f_{e,M}(E)$ in the region $R_3 \cup R_4$, where*

$$R_3 = \left\{\pi - 4\alpha_0(1+e) < M \leq \pi,\ 0 \leq e \leq \frac{3}{5}\right\},$$
$$R_4 = \left\{\pi - \frac{\sqrt{6}\alpha_0(1+e)^{\frac{3}{2}}}{\sqrt{e}} < M \leq \pi,\ \frac{3}{5} \leq e < 1\right\}.$$

Proof Since $f(\pi) = \pi - M$, $f'(\pi) = 1+e$, $f^{(\text{even})}(\pi) = 0$ and $f^{(\text{odd})}(\pi) = \pm e$, Smale's α-test is equivalent to

$$\frac{\pi - M}{1+e} \sup_{\substack{k\geq 3\\ k \text{ odd}}} \left(\frac{e}{k!(1+e)}\right)^{\frac{1}{k-1}} < \alpha_0.$$

For any $e \in [0, \frac{3}{5}]$, we have $\frac{e}{1+e} \leq \frac{3}{8}$. This gives the following estimate for the supremum:

$$\left(\frac{e}{k!(1+e)}\right)^{\frac{1}{k-1}} \leq \left(\frac{\frac{3}{8}}{k!}\right)^{\frac{1}{k-1}} \leq \left(\frac{1}{16}\right)^{\frac{1}{k-1}} \leq \frac{1}{4}, \quad \forall k \geq 3.$$

This means that Smale's condition is implied by $\frac{\pi - M}{4(1+e)} < \alpha_0$, which corresponds exactly to the region R_3. For the other case, where $e \in [\frac{3}{5}, 1)$, the supremum is $\sqrt{\frac{e}{6(1-e)}}$ by Lemma 2.1, so the α-condition is reduced to

$$\frac{(\pi - M)\sqrt{e}}{\sqrt{6}(1+e)^{\frac{3}{2}}} < \alpha_0,$$

which corresponds to the region R_4. □

Theorem 2.4 *$\tilde{E} = M$ is an approximate zero of $f_{e,M}(E)$ in the region*

$$\left\{0 \leq e \leq \frac{1}{2}\right\} \cup \left\{\frac{2\pi}{3} \leq M \leq \pi\right\} \cup R_2,$$

where R_2 is defined as in Theorem 2.2.

Proof Consider first the strip $M \geq \frac{2\pi}{3}$.

$$\beta(f_{e,M}, M) = \left|\frac{e\sin(M)}{1 - e\cos(M)}\right| \leq \left|\frac{\sin(M)}{1 - \cos(M)}\right| = \cot\left(\frac{M}{2}\right) \leq \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}.$$

By Lemma 2.1, we have that for any even integer $k \geq 2$,

$$\left|\frac{e\sin(M)}{k!(1 - e\cos(M))}\right|^{\frac{1}{k-1}} \leq \left|\frac{\frac{1}{\sqrt{3}}}{k!}\right|^{\frac{1}{k-1}} \leq \frac{1}{2\sqrt{3}},$$

and for any odd integer $k \geq 3$,

$$\left|\frac{e\cos(M)}{k!(1 - e\cos(M))}\right|^{\frac{1}{k-1}} \leq \left|\frac{\frac{1}{2}}{k!}\right|^{\frac{1}{k-1}} \leq \frac{1}{2\sqrt{3}}.$$

The last two inequalities together imply $\gamma(f_{e,M}, M) \leq \frac{1}{2\sqrt{3}}$ and $\alpha(f_{e,M}, M) \leq \frac{1}{6} < \alpha_0$. This proves that the starter $\tilde{E} = M$ satisfies the α-test in the strip $M \geq \frac{2\pi}{3}$.

In the region $\left\{\frac{\pi}{2} \leq M \leq \frac{2\pi}{3}, 0 \leq e \leq \frac{1}{2}\right\}$, we have that $\sin(M) \in [\frac{\sqrt{3}}{2}, 1]$ and $\cos(M) \in [-\frac{1}{2}, 0]$, so

$$\beta(f_{e,M}, M) = \left|\frac{f(M)}{f'(M)}\right| = \frac{e\sin(M)}{1 - e\cos(M)} \leq \frac{1}{2}.$$

On the other hand, using Lemma 2.1 gives us

$$\begin{aligned}\sup_{\substack{k\geq 2\\ k \text{ even}}} \left|\frac{f^{(k)}(M)}{k!f'(M)}\right|^{\frac{1}{k-1}} &\leq \sup_{\substack{k\geq 2\\ k \text{ even}}} \left|\frac{1}{2k!}\right|^{\frac{1}{k-1}} = \max\left\{\frac{1}{4}, \sup_{\substack{k\geq 4\\ k \text{ even}}} \left|\frac{1}{2k!}\right|^{\frac{1}{k-1}}\right\}\\ &= \max\left\{\frac{1}{4}, \frac{1}{\sqrt[3]{48}}\right\} = \frac{1}{\sqrt[3]{48}} \approx 0.2752,\end{aligned}$$

$$\begin{aligned}\sup_{\substack{k\geq 3\\ k \text{ odd}}} \left|\frac{f^{(k)}(M)}{k!f'(M)}\right|^{\frac{1}{k-1}} &\leq \sup_{\substack{k\geq 3\\ k \text{ odd}}} \left|\frac{1}{4k!}\right|^{\frac{1}{k-1}} = \max\left\{\frac{1}{\sqrt{24}}, \sup_{\substack{k\geq 4\\ k \text{ even}}} \left|\frac{1}{4k!}\right|^{\frac{1}{k-1}}\right\}\\ &= \max\left\{\frac{1}{\sqrt{24}}, \frac{1}{\sqrt[4]{480}}\right\} = \frac{1}{\sqrt[4]{480}} \approx 0.2136.\end{aligned}$$

Therefore, $\gamma(f_{e,M}, M) \leq \frac{1}{\sqrt[3]{48}}$ and the α-test holds because $\frac{1}{2}\frac{1}{\sqrt[3]{48}} < \alpha_0$.

In the region $\left\{0 \leq M \leq \frac{\pi}{2}, 0 \leq e \leq \frac{1}{2}\right\}$,

$$\frac{e\sin(M)}{1 - e\cos(M)} \leq \frac{\frac{1}{2}\sin(M)}{1 - \frac{1}{2}\cos(M)} \leq \frac{1}{\sqrt{3}} \tag{1}$$

and using Lemma 2.1 we obtain that

$$\sup_{\substack{k\geq 2\\ k \text{ even}}} \left|\frac{f^{(k)}(M)}{k!f'(M)}\right|^{\frac{1}{k-1}} \leq \max\left\{g_2, g_4, \sup_{\substack{k\geq 6\\ k \text{ even}}}\left|\frac{\frac{1}{2}\sin(M)}{k!(1-\frac{1}{2}\cos(M))}\right|^{\frac{1}{k-1}}\right\}$$

$$\leq \max\left\{g_2, g_4, \sup_{\substack{k\geq 6\\ k \text{ even}}}\left|\frac{1}{k!}\right|^{\frac{1}{k-1}}\right\} = \max\left\{g_2, g_4, \sqrt[5]{\frac{1}{6!}}\right\},$$

where $g_k = \left(\frac{\frac{1}{2}\sin(M)}{k!(1-\frac{1}{2}\cos(M))}\right)^{\frac{1}{k-1}}$ for $k = 2, 4$. Similarly,

$$\sup_{\substack{k\geq 3\\ k \text{ odd}}} \left|\frac{f^{(k)}(M)}{k!f'(M)}\right|^{\frac{1}{k-1}} \leq \max\left\{g_3, g_5, \sup_{\substack{k\geq 7\\ k \text{ odd}}}\left|\frac{\frac{1}{2}\cos(M)}{k!(1-\frac{1}{2}\cos(M))}\right|^{\frac{1}{k-1}}\right\}$$

$$\leq \max\left\{g_3, g_5, \sup_{\substack{k\geq 7\\ k \text{ odd}}}\left|\frac{1}{k!}\right|^{\frac{1}{k-1}}\right\} = \max\left\{g_3, g_5, \sqrt[6]{\frac{1}{7!}}\right\},$$

where $g_k = \left(\frac{\frac{1}{2}\cos(M)}{k!(1-\frac{1}{2}\cos(M))}\right)^{\frac{1}{k-1}}$ for $k = 3, 5$. Therefore,

$$\gamma(f_{e,M}, M) \leq \max\left\{g_2, g_3, g_4, g_5, \sqrt[5]{\frac{1}{6!}}, \sqrt[6]{\frac{1}{7!}}\right\} = \max\left\{g_2, g_3, g_4, g_5, \sqrt[5]{\frac{1}{6!}}\right\}.$$

As an immediate consequence of the second inequality in (1), we get $g_2 < g_4$, $\frac{\frac{1}{2}\sin(M)}{1-\frac{1}{2}\cos(M)} g_4 < \alpha_0$ and $\frac{\frac{1}{2}\sin(M)}{1-\frac{1}{2}\cos(M)}\sqrt[5]{\frac{1}{6!}} < \alpha_0$. It remains to see that $\frac{\frac{1}{2}\sin(M)}{1-\frac{1}{2}\cos(M)} g_k \leq \alpha_0$ for $k = 3, 5$, which is equivalent to proving

$$\frac{\sin^3(M)\cos(M)}{\left(1-\frac{1}{2}\cos(M)\right)^3} < 48\alpha_0^2 \approx 1.41, \text{ and } \frac{\sin^4(M)\cos(M)}{\left(1-\frac{1}{2}\cos(M)\right)^5} < 3840\alpha_0^4 \approx 3.33.$$

In both cases, the left-hand side function has a maximum and the inequalities are true at it.

Finally, note that $f_{e,M}(M) = -e\sin M \leq 0$ and $f_{e,M}$ is increasing, so $0 \leq M \leq E$, where E represents the exact solution of Kepler's equation. In particular, M is always closer to E than 0, hence for any point in R_2, the starter $\tilde{E} = M$ gives an approximate solution. □

Theorem 2.5 *$\tilde{E} = \frac{M}{1-e}$ is an approximate zero of $f_{e,M}(E)$ in the region $R_5 \cup R_6$, where*

$$R_5 = \left\{0 \leq M < \min\left\{\sqrt[4]{12\alpha_0}\frac{(1-e)^{\frac{3}{2}}}{e^{\frac{1}{2}}}, \sqrt[3]{24\alpha_0}\frac{(1-e)^{\frac{4}{3}}}{e^{\frac{1}{3}}}\right\}, 0 \leq e \leq \frac{3}{11}\right\},$$

$$R_6 = \left\{0 \leq M < \sqrt[4]{12\alpha_0}\frac{(1-e)^{\frac{3}{2}}}{e^{\frac{1}{2}}}, \frac{3}{11} \leq e < 1\right\}.$$

This region contains the region of Theorem 2.2.

Proof In this case we have

$$|f(\tilde{E})| = e\left|\frac{M}{1-e} - \sin\left(\frac{M}{1-e}\right)\right| \leq \frac{eM^3}{6(1-e)^3},$$

$|f'(\tilde{E})| \geq 1-e$ and $|f^{(k)}(\tilde{E})| \leq e$ for all $k \geq 2$. Besides,

$$\gamma\left(f_{e,M}, \frac{M}{1-e}\right) \leq \max\left\{\frac{e\frac{M}{1-e}}{2(1-e)}, \sup_{k\geq 3}\left|\frac{e}{k!(1-e)}\right|^{\frac{1}{k-1}}\right\}.$$

In particular, Smale's α-test is satisfied if

$$\frac{M^4e^2}{12(1-e)^6} < \alpha_0 \quad \text{and} \quad \frac{eM^3}{6(1-e)^4}\sup_{k\geq 3}\left|\frac{e}{k!(1-e)}\right|^{\frac{1}{k-1}} < \alpha_0.$$

The first condition is equivalent to $M < \frac{\sqrt[4]{12\alpha_0}(1-e)^{\frac{3}{2}}}{e^{\frac{1}{2}}}$, which is true in both R_5 and R_6. The second inequality needs to be discussed depending on the value of e.

When $e \in [\frac{3}{11}, 1)$, we have by Lemma 2.1 that

$$\sup_{k\geq 3}\left|\frac{e}{k!(1-e)}\right|^{\frac{1}{k-1}} = \sqrt{\frac{e}{6(1-e)}},$$

so the second inequality becomes $M < \sqrt{6}\sqrt[3]{\alpha_0}\frac{(1-e)^{3/2}}{e^{1/2}}$, which is automatically true in R_6 since $\sqrt{6}\sqrt[3]{\alpha_0} > \sqrt[4]{12\alpha_0}$.

In the other case, i.e. when $e \in [0, \frac{3}{11}]$, we have $\frac{e}{1-e} \leq \frac{3}{8}$. In particular, we can estimate the supremum from above as follows:

$$\sup_{k\geq 3}\left|\frac{e}{k!(1-e)}\right|^{\frac{1}{k-1}} \leq \sup_{k\geq 3}\left|\frac{3}{8k!}\right|^{\frac{1}{k-1}} = \frac{1}{4},$$

where we have used Lemma 2.1. Therefore, in the case $e \in [0, \frac{3}{11}]$, the α-test is satisfied when

$$M < \frac{\sqrt[4]{12\alpha_0}(1-e)^{\frac{3}{2}}}{e^{\frac{1}{2}}} \quad \text{and} \quad M < \sqrt[3]{24\alpha_0}\frac{(1-e)^{\frac{4}{3}}}{e^{\frac{1}{3}}},$$

which is the definition of the region R_5.

Finally, the inclusion $R_2 \subseteq R_6$ follows immediately from $\sqrt{6}\alpha_0 < \sqrt[4]{12\alpha_0}$ and $R_1 \subseteq R_5$ from the fact that $4\alpha_0(1-e) < \sqrt[4]{12\alpha_0}\frac{(1-e)^{\frac{3}{2}}}{e^{\frac{1}{2}}}$ and $4\alpha_0(1-e) < \sqrt[3]{24\alpha_0}\frac{(1-e)^{\frac{4}{3}}}{e^{\frac{1}{3}}}$ for all $e \in [0, \frac{3}{11}]$. □

Theorem 2.6 *The exact solution of the cubic equation* $\tilde{E}(1-e) + e\frac{\tilde{E}^3}{6} - M = 0$ *is an approximate zero of* $f_{e,M}(E)$ *in the entire region* $[0, 1) \times [0, \pi]$.

Proof First, note that the derivative of the left-hand side of the equation is $(1-e)+e\tilde{E}^2/2 > 0$, so the expression is increasing. This means that the cubic has only one real root. Moreover, the values of the cubic at 0 and π are $-M \leq 0$ and $\pi(1-e) + e\frac{\pi^3}{6} - M \geq \pi - M \geq 0$ respectively, so the real root $\tilde{E}$ must be in $[0, \pi]$. In particular, we have that $\tilde{E} < \sqrt{42}$, so

$$|f(\tilde{E})| = |\tilde{E} - e\sin(\tilde{E}) - M| = \left|\tilde{E}(1-e) + e\left(\frac{\tilde{E}^3}{3!} - \frac{\tilde{E}^5}{5!} + \cdots\right) - M\right| \leq e\frac{\tilde{E}^5}{120}.$$

Let us now consider two different cases depending on the value of $\tilde{E}$.

If $\tilde{E} \leq \frac{\pi}{2}$, we have that $f'(\tilde{E}) \geq 1 - \cos(\tilde{E}) = 2\sin^2(\frac{\tilde{E}}{2}) \geq \frac{4}{\pi^2}\tilde{E}^2$ and

$$\gamma(f_{e,M}, \tilde{E}) \leq \sup_{k\geq 2} \left| \frac{1}{k!(1-\cos(\tilde{E})} \right|^{\frac{1}{k-1}} \leq \sup_{k\geq 2} \left| \frac{\pi^2}{4k!\tilde{E}^2} \right|^{\frac{1}{k-1}} = \frac{\pi^2}{8\tilde{E}^2}$$

by Lemma 2.1. Therefore, the α-test follows if we prove

$$\frac{e\frac{\tilde{E}^5}{120}}{\frac{4}{\pi^2}\tilde{E}^2} \frac{\pi^2}{8\tilde{E}^2} < \frac{\pi^4 \tilde{E}}{3840} < \alpha_0 \Leftrightarrow \tilde{E} < \frac{3840\alpha_0}{\pi^4} \approx 6.76,$$

which is always true in this region.

If $\tilde{E} > \frac{\pi}{2}$, then $\gamma(f_{e,M}, \tilde{E}) = \max\{g_2, g_3, g_4, g_5\}$, where

$$g_2 = \frac{1}{2(1-e\cos(\tilde{E}))}, \quad g_3 = \sqrt{\frac{|\cos(\tilde{E})|}{6(1-e\cos(\tilde{E}))}},$$

$$g_4 = \sup_{\substack{k\geq 4 \\ k \text{ even}}} \left| \frac{1}{k!(1-e\cos(\tilde{E}))} \right|^{\frac{1}{k-1}} = \sqrt[3]{\frac{1}{24(1-e\cos(\tilde{E}))}},$$

$$g_5 = \sup_{\substack{k\geq 5 \\ k \text{ odd}}} \left| \frac{1}{k!(1-e\cos(\tilde{E}))} \right|^{\frac{1}{k-1}} = \sqrt[4]{\frac{1}{120(1-e\cos(\tilde{E}))}} \leq g_4.$$

Therefore, the α-test is satisfied if $\frac{e\tilde{E}^5}{120(1-e\cos(\tilde{E}))} g_i < \alpha_0$ for $i = 2, 3, 4$.

Since g_2, g_3 and g_4, are increasing in M, it is enough to prove the inequalities when $M = \pi$. Moreover, $\tilde{E}(e, \pi)$ is decreasing, so $\tilde{E}(e, \pi) \in [\sqrt[3]{6\pi}, \pi]$ and $1 - e\cos(\tilde{E}(e, \pi)) \geq 1 - e\cos(\sqrt[3]{6\pi})$.

We also have that $\pi = e\frac{\tilde{E}^3(e,\pi)}{6} + (1-e)\tilde{E}(e, \pi) \geq e\frac{\tilde{E}^3(e,\pi)}{6} + (1-e)\sqrt[3]{6\pi}$, hence

$$\tilde{E}(e, \pi) \leq \sqrt[3]{\frac{6\left(\pi - (1-e)\sqrt[3]{6\pi}\right)}{e}}. \tag{2}$$

Let us now study the three different cases.

When $i = 2$, it is enough to prove that

$$\frac{e\tilde{E}^5}{120(1-e\cos(\tilde{E}))} g_2 < \frac{e\tilde{E}(e,\pi)^5}{240(1-e\cos(\sqrt[3]{6\pi}))^2} < \alpha_0,$$

which is true using that $\tilde{E} \leq \pi$ in $e \in [0, 0.17]$, $\tilde{E}(e, \pi) \leq 2.92$ in $e \in [0.17, 0.3]$, $\tilde{E}(e, \pi) \leq 2.84$ in $e \in [0.3, 0.4]$ and Eq. (2) in $e \in [0.4, 1]$.

When $i = 3$, it suffices to show that

$$\frac{e\tilde{E}^5}{120(1-e\cos(\tilde{E}))} g_3 < \frac{e\tilde{E}^5(e,\pi)\sqrt{|cos(\tilde{E}(e,\pi))|}}{120\sqrt{6}(1-e\cos(\sqrt[3]{6\pi}))^{\frac{3}{2}}} < \alpha_0,$$

which is true using that

- $\tilde{E} \leq \pi$ and $\sqrt{|cos(\tilde{E}(e,\pi)|} \leq 1$ in $e \in [0, 0.2]$,
- Eq. (2) and $\sqrt{|cos(\tilde{E}(e,\pi)|} \leq 1$ in $e \in [0.2, 0.7]$,

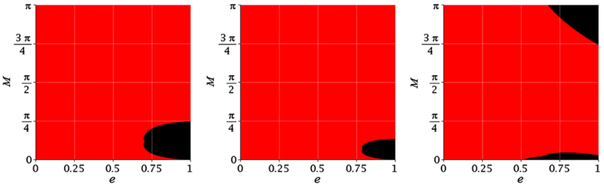

Fig. 4 The regions of S_2, S_3 and S_4

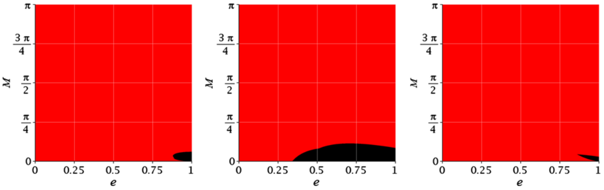

Fig. 5 The regions of S_5, S_6 and S_7

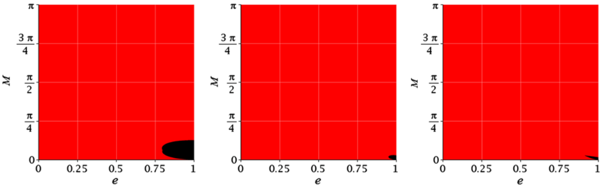

Fig. 6 The regions of S_8, S_9 and S_{CEMR}

- Eq. (2) and $\sqrt{|cos(\tilde{E}(e,\pi)|} < 0.91$ in $e \in [0.7, 1]$.

Lastly, the case $i = 4$ follows by using $\tilde{E} \le \pi$ in $e \in [0, 0.2]$ and Eq. (2) in $e \in [0.2, 1]$. ☐

3 Numerical comparison of classical starters via α-theory

We tested numerically the α-condition on a fine grid (dividing each axis in 1000 points) for the starters $S_2, \ldots, S_9$, defined in Odell and Gooding (1986), and the improved S_7 starter obtained in (Prop. 1 of Calvo et al. 2013), which we denote S_{CEMR}. Note in Figs. 4, 5 and 6 that none of the starters produce approximate zeros near the corner $(1, 0)$.

4 A simple new starter that covers the entire region

We devote this section to proving Theorem 1.2. We study each branch separately.

Theorem 4.1 *$\tilde{E} = \frac{2\pi}{3}$ is an approximate zero of $f_{e,M}(E)$ in the region*

$$\left\{\frac{\pi}{4} \le M \le \frac{2\pi}{3}, \frac{1}{2} \le e < 1\right\}.$$

Proof First of all, we have that

$$\beta\left(f_{e,M}, \frac{2\pi}{3}\right) \le \frac{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}e - \frac{\pi}{4}}{1 + \frac{e}{2}} = \frac{\frac{5\pi}{12} - \frac{\sqrt{3}}{2}e}{1 + \frac{e}{2}}.$$

On the other hand,

$$\gamma\left(f_{e,M}, \frac{2\pi}{3}\right) = \max\left\{\sup_{\substack{k\geq 2\\ k \text{ even}}}\left|\frac{e\frac{\sqrt{3}}{2}}{k!(1+\frac{e}{2})}\right|^{\frac{1}{k-1}}, \sup_{\substack{k\geq 3\\ k \text{ odd}}}\left|\frac{\frac{e}{2}}{k!(1+\frac{e}{2})}\right|^{\frac{1}{k-1}}\right\}.$$

Since $\frac{\frac{e}{2}}{1+\frac{e}{2}} \in [\frac{1}{3}, \frac{1}{5}]$, we can apply Lemma 2.1 for $n = 4$ and $n = 5$:

$$\sup_{\substack{k\geq 2\\ k \text{ even}}}\left|\frac{e\frac{\sqrt{3}}{2}}{k!(1+\frac{e}{2})}\right|^{\frac{1}{k-1}} = \max\left\{\frac{e\frac{\sqrt{3}}{2}}{2!(1+\frac{e}{2})}, \left(\frac{e\frac{\sqrt{3}}{2}}{4!(1+\frac{e}{2})}\right)^{\frac{1}{3}}\right\},$$

$$\sup_{\substack{k\geq 3\\ k \text{ odd}}}\left|\frac{\frac{e}{2}}{k!(1+\frac{e}{2})}\right|^{\frac{1}{k-1}} = \max\left\{\left(\frac{\frac{e}{2}}{3!(1+\frac{e}{2})}\right)^{\frac{1}{2}}, \left(\frac{\frac{e}{2}}{5!(1+\frac{e}{2})}\right)^{\frac{1}{4}}\right\}.$$

Comparing the four functions, we obtain

$$\gamma\left(f_{e,M}, \frac{2\pi}{3}\right) = \left(\frac{e\frac{\sqrt{3}}{2}}{4!(1+\frac{e}{2})}\right)^{\frac{1}{3}}.$$

Therefore, the α-test is satisfied if

$$\frac{\frac{5\pi}{12} - \frac{\sqrt{3}}{2}e}{1+\frac{e}{2}}\left(\frac{e\frac{\sqrt{3}}{2}}{4!(1+\frac{e}{2})}\right)^{\frac{1}{3}} < \alpha_0.$$

Taking derivatives, it can be shown that the left-hand side of the inequality is a decreasing function of e. Also, its value at $e = \frac{1}{2}$ is approximately 0.1706, which is less than α_0. □

Theorem 4.2 *$\tilde{E} = \frac{\pi}{2}$ is an approximate zero of $f_{e,M}(E)$ in the region*

$$\left\{\frac{\pi}{7} \leq M \leq \frac{\pi}{4}, \frac{1}{2} \leq e < 1\right\}.$$

Proof We have that $f_{e,M}(\frac{\pi}{2}) = \frac{\pi}{2} - e - M \leq \frac{\pi}{2} - e - \frac{\pi}{7} = \frac{5\pi}{14} - e$ and $f'_{e,M}(\frac{\pi}{2}) = 1$. Moreover, $f^{(\text{odd})}(\frac{\pi}{2}) = 0$, hence

$$\gamma\left(f_{e,M}, \frac{\pi}{2}\right) = \sup_{\substack{k\geq 2\\ k \text{ even}}}\left|\frac{e}{k!}\right|^{\frac{1}{k-1}} = \max\left\{\frac{e}{2}, \sup_{\substack{k\geq 4\\ k \text{ even}}}\left|\frac{e}{k!}\right|^{\frac{1}{k-1}}\right\} = \max\left\{\frac{e}{2}, \sqrt[3]{\frac{e}{24}}\right\}$$

by Lemma 2.1. The α-test is satisfied because

$$\left(\frac{5\pi}{14} - e\right)\frac{e}{2} \leq \left(\frac{5\pi}{14} - \frac{5\pi}{28}\right)\frac{\frac{5\pi}{28}}{2} \approx 0.1573 < \alpha_0,$$

$$\left(\frac{5\pi}{14} - e\right)\sqrt[3]{\frac{e}{24}} \leq \left(\frac{5\pi}{14} - \frac{1}{2}\right)\sqrt[3]{\frac{\frac{1}{2}}{24}} \approx 0.1711 < \alpha_0,$$

which ends the proof. □

Theorem 4.3 $\tilde{E} = \frac{\sqrt[3]{6Me^2}}{e} - \frac{2(1-e)}{\sqrt[3]{6Me^2}}$ *is an approximate zero of* $f_{e,M}(E)$ *in the region* R_7*, where*

$$R_7 = \left\{ \frac{8(1-e)^{3/2}}{27\sqrt{6}\alpha_0 e^{1/2}} < M \leq \frac{\pi}{7}, \frac{3}{11} \leq e < 1 \right\}.$$

Proof The first condition we have to impose is that $\tilde{E} \geq 0$, which is equivalent to $M \geq \frac{\sqrt{2}(1-e)^{\frac{3}{2}}}{3\alpha_0 e^{\frac{1}{2}}}$ and true in R_7. We also show that $\tilde{E} \leq \frac{\pi}{2}$ in $[0, \frac{\pi}{7}] \times [0, 1)$, which includes R_7.

Indeed, $\tilde{E} \leq \frac{\pi}{2}$ is equivalent to

$$h(e, M) = \frac{\sqrt[3]{36} e^{\frac{1}{3}} M^{\frac{2}{3}}}{2(1-e)} - \frac{\pi \sqrt[3]{6} e^{\frac{2}{3}} M^{\frac{1}{3}}}{4(1-e)} \leq 1. \tag{3}$$

For a fixed e, the function h has a minimum at $M = \frac{\pi^3}{384} e$ and no other critical points. Therefore, the inequality (3) holds if and only if $h(e, 0) \leq 1$ and $h\left(e, \frac{\pi}{7}\right) \leq 1$. The first one is trivial since $h(e, 0) = 0$ and the second one is equivalent to

$$2\sqrt[3]{36} \left(\frac{\pi}{7}\right)^{\frac{2}{3}} e^{\frac{1}{3}} - \pi \sqrt[3]{6} \left(\frac{\pi}{7}\right)^{\frac{1}{3}} e^{\frac{2}{3}} - 4(1-e) < 0.$$

The substitution $e = x^3$ transforms the inequality above into

$$2\sqrt[3]{36} \left(\frac{\pi}{7}\right)^{\frac{2}{3}} x - \pi \sqrt[3]{6} \left(\frac{\pi}{7}\right)^{\frac{1}{3}} x^2 - 4(1-x^3) < 0,$$

which is verified for all $x \in [0, 1]$ since the expression in x is increasing and the inequality is true at $x = 1$.

Substituting the expression for $\tilde{E}$ and using the Taylor expansion of $\sin(\tilde{E})$, we obtain

$$|f(\tilde{E})| = |\tilde{E} - e\sin(\tilde{E}) - M| = \left| \tilde{E}(1-e) + e\left(\frac{\tilde{E}^3}{3!} - \frac{\tilde{E}^5}{5!} + \cdots \right) - M \right|$$
$$\leq \left| \tilde{E}(1-e) + e\frac{\tilde{E}^3}{6} - M \right| + \left| \frac{\tilde{E}^5}{120} \right| = \frac{2(1-e)^3}{9eM} + \left| \frac{\tilde{E}^5}{120} \right|,$$

where we have bounded the alternating series using Leibniz's criterion (possible because $\tilde{E} < \sqrt{42}$).

Since $\tilde{E} \leq \frac{\pi}{2}$, we have both $f'(\tilde{E}) \geq 1-e$ and $f'(\tilde{E}) \geq 1-\cos(\tilde{E}) = 2\sin^2(\frac{\tilde{E}}{2}) \geq \frac{4}{\pi^2}\tilde{E}^2$. Therefore, the α-test follows if we prove the stronger conditions

$$\frac{2(1-e)^2}{9eM} \gamma(f_{e,M}, \tilde{E}) < \frac{3\alpha_0}{4} \text{ and } \left| \frac{\tilde{E}^3 \pi^2}{480} \right| \gamma(f_{e,M}, \tilde{E}) < \frac{\alpha_0}{4}. \tag{4}$$

The second one holds because

$$\gamma(f_{e,M}, \tilde{E}) \leq \sup_{k\geq 2} \left| \frac{1}{k!(1-\cos(\tilde{E})} \right|^{\frac{1}{k-1}} \leq \sup_{k\geq 2} \left| \frac{\pi^2}{4k!\tilde{E}^2} \right|^{\frac{1}{k-1}} = \frac{\pi^2}{8\tilde{E}^2},$$

by Lemma 2.1, and

$$\left| \frac{\tilde{E}^3 \pi^2}{480} \right| \frac{\pi^2}{8\tilde{E}^2} = \frac{\pi^4 \tilde{E}}{3840} < \frac{\alpha_0}{4} \Leftrightarrow \tilde{E} < \frac{960\alpha_0}{\pi^4} \approx 1.69,$$

which is true since $\tilde{E} \leq \frac{\pi}{2}$ in R_7.

For the first inequality in (4), we need

$$\gamma(f_{e,M}, \tilde{E}) \leq \max\left\{\frac{e\sin(\tilde{E})}{2!(1-e)}, \sup_{k\geq 3}\left|\frac{e}{k!(1-e)}\right|^{\frac{1}{k-1}}\right\}$$
$$\leq \max\left\{\frac{e\tilde{E}}{2!(1-e)}, \left|\frac{e}{3!(1-e)}\right|^{\frac{1}{2}}\right\},$$

true by Lemma 2.1 when $e \geq \frac{3}{11}$. Therefore,

$$\frac{2(1-e)^2}{9eM}\left|\frac{e}{3!(1-e)}\right|^{\frac{1}{2}} < \frac{3\alpha_0}{4} \Leftrightarrow M > \frac{8(1-e)^{\frac{3}{2}}}{27\sqrt{6}\alpha e^{\frac{1}{2}}},$$

which is one of the conditions of the region R_7.

It only remains to show that

$$\frac{2(1-e)^2}{9eM}\frac{e\tilde{E}}{2!(1-e)} = \frac{\tilde{E}(1-e)}{9M} < \frac{3\alpha_0}{4},$$

which is equivalent to

$$M - \frac{4}{27\alpha_0}(1-e)\tilde{E} > 0 \text{ or } \underbrace{\frac{\sqrt[3]{6}e^{2/3}}{(1-e)^2}M^{4/3} - \frac{4\sqrt[3]{36}e^{1/3}}{27\alpha_0(1-e)}M^{2/3}}_{g(e,M)} > -\frac{8}{27\alpha_0}.$$

This is true for every $e \in [0, 1)$ and $M \in [0, \pi]$ because, if we fix e, the function g has a minimum at $M = \sqrt{\frac{48}{27^3\alpha_0^3}}$ and

$$g\left(e, \sqrt{\frac{48}{27^3\alpha_0^3}}\right) = -\frac{24}{27^2\alpha_0^2} > -\frac{8}{27\alpha_0}.$$

□

Proof of Theorem 1.2 It follows immediately from Theorems 2.4, 2.5, 4.1, 4.2 and 4.3, and the inequality $\sqrt[4]{12\alpha_0} > \frac{8}{27\sqrt{6\alpha_0}}$ that implies that the "otherwise" region is included in the one from Theorem 4.3. □

5 Approximate solutions near $e = 1$ and $M = 0$

In this section we will prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3 Given $\varepsilon > 0$, let us take a natural number N such that $N > \frac{\pi+2}{2\alpha_0\varepsilon^2}$. Given two integers $i \in \{0, \ldots, N-1\}$ and $j \in \{0, \ldots, N\}$, we define the constants $E_{ij}^{\text{low}} = \frac{\pi j}{N}$ and $E_{ij}^{\text{up}} = \pi$, which satisfy

$$E_{ij}^{\text{low}} - \frac{i}{N}\sin(E_{ij}^{\text{low}}) - \frac{\pi j}{N} = -\frac{i}{N}\sin\left(\frac{\pi j}{N}\right) \leq 0,$$
$$E_{ij}^{\text{up}} - \frac{i}{N}\sin(E_{ij}^{\text{up}}) - \frac{\pi j}{N} = \pi - \frac{\pi j}{N} \geq 0,$$

respectively. By the bisection method, we can thus find E_{ij} such that

$$\frac{\pi j}{N} = E_{ij}^{\text{low}} \leq E_{ij} \leq E_{ij}^{\text{up}} = \pi \quad \text{and} \quad \left| E_{ij} - \frac{i}{N}\sin(E_{ij}) - \frac{\pi j}{N} \right| < \frac{1}{N}.$$

Given $(e, M) \in ([0,1) \times [0,\pi]) \setminus ([1-\varepsilon, 1] \times [0, \arccos(1-\varepsilon)])$, we now define $\tilde{E}(e, M) = E_{ij}$, where $i = \lfloor Ne \rfloor \in \{0, \dots, N-1\}$ and $j = \lceil \frac{MN}{\pi} \rceil \in \{0, \dots, N\}$. Therefore, $\tilde{E}$ is a piecewise constant function and it only remains to show that it satisfies the α-test.

Indeed, we have that

$$\begin{aligned} |f(\tilde{E})| &= |E_{ij} - e\sin(E_{ij}) - M| \\ &= \left| \left(E_{ij} - \frac{i}{N}\sin(E_{ij}) - \frac{\pi j}{N} \right) - \left(e - \frac{i}{N} \right)\sin(E_{ij}) - \left(M - \frac{\pi j}{N} \right) \right| \\ &< \frac{1}{N} + \left| e - \frac{i}{N} \right| + \left| M - \frac{\pi j}{N} \right| \leq \frac{\pi + 2}{N}. \end{aligned}$$

On the other hand, $|f'(\tilde{E})| = 1 - e\cos(\tilde{E}) \geq \varepsilon$ because

$$|f'(\tilde{E})| \geq \begin{cases} 1 - e \geq \varepsilon & \text{if } e \in [0, 1-\varepsilon], \\ 1 - \cos(E_{ij}) \geq 1 - \cos(M) \geq \varepsilon & \text{if } \tilde{E} \in [0, \frac{\pi}{2}],\, M \geq \arccos(1-\varepsilon), \\ 1 \geq \varepsilon & \text{if } \tilde{E} \in [\frac{\pi}{2}, \pi], \end{cases}$$

where we have used that $E_{ij} \geq E_{ij}^{\text{low}} = \frac{\pi j}{N} = \frac{\pi \lceil \frac{MN}{\pi} \rceil}{N} \geq M$.

Since $|f^{(k)}(\tilde{E})| \leq 1$, we obtain using Lemma 2.1 and the hypothesis over N that

$$\alpha(f_{e,M}, \tilde{E}) \leq \frac{\pi + 2}{N\varepsilon} \sup_{k \geq 2} \left| \frac{1}{k!\varepsilon} \right|^{\frac{1}{k-1}} \leq \frac{\pi + 2}{2N\varepsilon^2} < \alpha_0,$$

which ends the proof. □

Proof of Theorem 1.4 We proceed by contradiction, i.e. we assume that $\tilde{E}(e, M)$ is an approximate zero of $f_{e,M}$ for all $e \in [0, 1)$ and $M \in [0, \pi]$. Since the branches of $\tilde{E}$ are given by polynomial inequalities, there is an open set $U \subseteq \mathbb{R}^2$ and $\varepsilon > 0$ such that $\overline{U} \supset \{1\} \times [0, \varepsilon]$ and $\tilde{E}$ is a rational function on $U \cap ([0, 1) \times [0, \pi])$. We also assume that $U \subseteq [\frac{1}{2}, 1) \times [0, 0.0001]$.

By definition of approximate zero, we have that

$$|f(\tilde{E})| \underbrace{\max \left\{ \frac{e|\sin(\tilde{E})|}{2(1 - e\cos\tilde{E})^2}, \sqrt{\frac{e|\cos(\tilde{E})|}{6(1 - e\cos\tilde{E})^3}}, \sqrt[3]{\frac{e|\sin(\tilde{E})|}{24(1 - e\cos\tilde{E})^4}} \right\}}_{B} < \alpha_0.$$

It can be readily verified that $B \geq 0.14433$ for all $e \in [\frac{1}{2}, 1)$ and any $\tilde{E} \in \mathbb{R}$, so $|f(\tilde{E})| < \frac{\alpha_0}{0.14433} \leq 1.1888$ in U. By the triangle inequality, this implies that $|\tilde{E}| < 1.1888 + e + M < 2.1889$ in U. Repeating the argument, but using that $|\tilde{E}| < 2.1889$, it can be shown that $B \geq 0.176$, so $|\tilde{E}| < \frac{\alpha_0}{0.176} + e + M \leq 1.975$ in U. Doing this one more time, gives $B \geq 0.2368$ and the estimate $|\tilde{E}| < 1.725$ in U.

Since $\tilde{E}$ is bounded in U, it can be extended analytically to $\{1\} \times (0, \delta)$ for some $0 < \delta < \varepsilon \leq 0.0001$. To show this, recall that $\tilde{E}(e, M) = \frac{p(e,M)}{q(e,M)}$ for some polynomials p and q with no common factors. Now, if $q(1, M)$ were zero (as a polynomial), then q would be divisible by $e-1$ and p would not, so $\tilde{E}$ would not be bounded, in contradiction with our previous result. This proves that $q(1, M) \not\equiv 0$, so we can take $\delta > 0$ small enough to ensure that $q(1, M)$ has no roots in $(0, \delta)$, hence $\tilde{E}(1, M)$ is well defined.

Denote $\tilde{E}_1(M) = \tilde{E}(1, M)$ for $M \in (0, \delta)$. Using that $B \geq \frac{e|\sin(\tilde{E})|}{2(1-e\cos\tilde{E})^2}$, we get

$$|\tilde{E} - e\sin\tilde{E} - M| \leq \frac{2\alpha_0(1-e\cos\tilde{E})^2}{e|\sin(\tilde{E})|}.$$

Taking limit as $e \to 1^-$, we obtain

$$|\tilde{E}_1 - \sin\tilde{E}_1 - M| \leq \frac{2\alpha_0(1-\cos\tilde{E}_1)^2}{|\sin(\tilde{E}_1)|} = \frac{4\alpha_0|\sin(\frac{\tilde{E}_1}{2})|^3}{|\cos(\frac{\tilde{E}_1}{2})|} \leq \frac{\alpha_0|\tilde{E}_1|^3}{2|\cos(\frac{\tilde{E}_1}{2})|} < 0.133|\tilde{E}_1|^3$$

for all $M \in (0, \delta)$. By the power series expansion of $\sin(\tilde{E}_1)$,

$$\left|\frac{\tilde{E}_1^3}{3!} - \frac{\tilde{E}_1^5}{5!} + \cdots - M\right| < 0.133|\tilde{E}_1^3|.$$

By the triangle inequality,

$$\begin{aligned}
\left|\frac{\tilde{E}_1^3}{6} - M\right| &\leq 0.133|\tilde{E}_1^3| + \left|\frac{\tilde{E}_1^5}{5!} - \frac{\tilde{E}_1^7}{7!} + \ldots\right| \\
&\leq |\tilde{E}_1^3|\left(0.133 + \frac{\tilde{E}_1^2}{120}\left(1 + \frac{\tilde{E}_1^2}{6\cdot 7} + \frac{\tilde{E}_1^4}{6\cdot 7\cdot 8\cdot 9} + \ldots\right)\right) \\
&\leq |\tilde{E}_1^3|\left(0.133 + \frac{1.725^2}{120}\left(1 + \frac{1.725^2}{6^2} + \frac{1.725^4}{6^4} + \ldots\right)\right) \\
&\leq 0.161|\tilde{E}_1^3|,
\end{aligned}$$

for all $M \in (0, \delta)$. This implies that $(\frac{1}{6} - 0.161)|\tilde{E}_1^3| \leq M$, or equivalently,

$$|\tilde{E}_1| \leq \sqrt[3]{\frac{|M|}{\frac{1}{6} - 0.161}} \xrightarrow[M\to 0^+]{} 0.$$

This shows that $\tilde{E}_1$ has a removable singularity at $M = 0$, so it can be extended analytically to $[0, \delta)$ with $\tilde{E}_1(0) = 0$. Moreover, $\tilde{E}_1(M) = Mr(M)$ for some analytic function $r(M)$ in $[0, \delta)$, since the power series of $\tilde{E}_1$ cannot have a non-zero constant term.

Finally, by definition of approximate zero,

$$\begin{aligned}
\alpha_0 &> \frac{|f(\tilde{E})|}{1-e\cos(\tilde{E})}\max\left\{\frac{e|\sin(\tilde{E})|}{2(1-e\cos\tilde{E})}, \sqrt{\frac{e|\cos(\tilde{E})|}{6(1-e\cos\tilde{E})}}\right\} \\
&\geq \frac{|f(\tilde{E})|}{1-e\cos(\tilde{E})}\max\left\{\frac{e|\sin(\tilde{E})|}{\sqrt{6(1-e\cos(\tilde{E}))}}, \frac{e|\cos(\tilde{E})|}{\sqrt{6(1-e\cos(\tilde{E}))}}\right\} \\
&= \frac{e|f(\tilde{E})|}{\sqrt{6}(1-e\cos(\tilde{E}))^{\frac{3}{2}}}\max\{|\sin\tilde{E}|, |\cos\tilde{E}|\} \geq \frac{e|f(\tilde{E})|}{\sqrt{12}(1-e\cos(\tilde{E}))^{\frac{3}{2}}},
\end{aligned}$$

and taking limit as $e \to 1^-$,

$$\begin{aligned}|\tilde{E}_1 - \sin \tilde{E}_1 - M| &\leq \sqrt{12}\alpha_0(1-\cos(\tilde{E}_1))^{\frac{3}{2}} \\ &= \sqrt{96}\alpha_0 \left|\sin^3\left(\frac{\tilde{E}_1}{2}\right)\right| \leq \frac{\sqrt{96}\alpha_0|\tilde{E}_1^3|}{8} = \sqrt{\frac{3}{2}}\alpha_0|\tilde{E}_1|^3.\end{aligned}$$

Dividing by M, using that $\tilde{E}_1(M) = Mr(M)$ and taking limits as $M \to 0^+$,

$$\left|r(M) - \frac{\sin(Mr(M))}{M} - 1\right| \leq \sqrt{\frac{3}{2}}\alpha_0 M^2 |r(M)|^3,$$

which gives us the contradiction $1 \leq 0$. □

Remark 5.1 Note that in the proof of Theorem 1.4 we use the rationality of the function only to show that it can be analytically extended to a small segment $\{1\} \times [0, \varepsilon]$ for some $\varepsilon > 0$. If we start with an analytic function defined on $[0, 1] \times [0, \pi]$, this step is not necessary and the same contradiction is obtained.

This shows that the classical starters $S_1, \ldots, S_8$, as well as S_{CEMR}, are not approximate zeros in the entire domain, as Figs. 3, 4, 5 and 6 illustrate. The same argument can be used to show that no starter which is a linear combination of the former ones (for instance, $M + ke$ with $0 \leq k \leq 1$) gives an approximate zero in the entire region.

6 Conclusions

This paper proposes to quantify the efficiency of a starter for Kepler's equation $E - e\sin(E) = M$ by using Smale's α-theory. We certify analytically regions where certain classical starters give quadratic convergence under Newton's method. We also study numerically how efficient the classical starters are.

Our main contribution is the construction of a simple starter which converges quadratically in the entire domain. This starter is given by a piecewise-defined function that uses constant and rational functions everywhere except for a small part of the domain near the corner $e = 1$, $M = 0$, where a single cubic root is needed.

In the final section, we provide a technical analysis of the difficulties of solving Kepler's equation. More precisely, we show that it is possible to find approximate solutions given by piecewise-constant functions everywhere except for an arbitrarily small region near the corner $(1, 0)$. Additionally, we prove that no starter satisfying our efficiency criterion in the entire region can be obtained if only rational (or analytic) functions are used, thus showing that the cubic root in our proposed starter is unavoidable.

Acknowledgments The authors would like to thank Prof. Antonio Elipe for his valuable help. Second author is partially supported by the MINECO grant MTM2011-22621 and the FQM-327 group (Junta de Andalucía, Spain). Third author is partially supported by the MINECO grant MTM2010-21740-C02-02. Both are also partially supported by the Grupo consolidado E15 "Geometría" (Gobierno de Aragón, Spain) and the "Centro Universitario de la Defensa de Zaragoza" grant ID2013-15.

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