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ORIGINAL ARTICLE

On the third-body perturbations of high-altitude orbits

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Abstract The long-term effects of a distant third-body on a massless satellite that is orbiting an oblate body are studied for a high order expansion of the third-body disturbing function. This high order may be required, for instance, for Earth artificial satellites in the so-called MEO region. After filtering analytically the short-period angles via averaging, the evolution of the orbital elements is efficiently integrated numerically with very long step-sizes. The necessity of retaining higher orders in the expansion of the third-body disturbing function becomes apparent when recovering the short-periodic effects required in the computation of reliable osculating elements.

Keywords Third-body perturbation · Lie transforms · Averaging · High-altitude orbits

1 Introduction

Semi-analytical integration is the common approach used by aerospace engineers in the mathematical problem of investigating the long-term dynamics of uncontrolled man-made Earth satellites, like non-operational orbits and debris. Short-periodic angles are filtered analytically

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via averaging procedures and then, the averaged system is integrated numerically with very long step-sizes (Green 1979; Slutsky 1983; McClain 1977; Valk et al. 2009).

The secular contribution of zonal terms of the gravitational potential, as well as their influence in the pericenter dynamics, is essential in the description of the orbit evolution in the long-term (Brouwer 1959; Kozai 1962). Also the influence of some tesseral harmonics can be crucial when dealing with resonant motion, a case in which tesseral resonant terms must be left unaveraged (Gedeon 1969; Proulx et al. 1981; Collins 1981; Ely and Howell 1996). Besides, in the case of Earth-like bodies the second order zonal coefficient, J_2, clearly dominates over other coefficients in the expansion of the gravitational potential, and hence the averaging must be extended to consider terms of the order of J_2^2 even for a qualitative description of the dynamics (Coffey et al. 1986). Third-body perturbations may also show important effects in the long-term dynamics—as well as drag for those celestial bodies with a dense enough atmosphere—depending on the orbital characteristics (Kozai 1959; Giacaglia 1974). Besides, the effect of solar radiation pressure may become an important issue for satellites with high area-to-mass ratio (Anselmo and Pardini 2010).

When dealing with the long-term dynamics induced by gravitational perturbations of a distant body, the third-body disturbing function is expanded in the ratio of the distances from the central body as an infinite series in Legendre polynomials, that, commonly, is truncated up to the second degree. Useful as it can be in the description of the qualitative dynamics (Broucke 2003) this early truncation is not acceptable, in general, in the time history description of real orbits, where the third-body disturbing function must be truncated to a higher degree (Collins and Cefola 1979; Métris and Exertier 1995; Steichen 1998; Prado 2003; Cefola et al. 2003; Vashkov'yak 2005; Laskar and Boué 2010).

In many cases third-body perturbations are of second order when compared with the (first order) J_2 effect; that is the case, for instance, of satellites in low Earth orbits where the effect of lunisolar perturbations is small. Then, it is enough to retain the contribution of the second degree term of the expansion of the third-body disturbing function. However, third-body perturbations are much more important for orbits at higher altitudes, the so-called MEO region (medium Earth orbits) as well as high Earth orbits. In fact, lunisolar perturbations are comparable in strength to the J_2 disturbing effect at the altitudes of geostationary orbits. For this and other cases, the inclusion of higher order terms of the third-body perturbation is essential in the construction of semi-analytical theories. Truncations up to the fifth degree in the parallactic ratio are considered quite accurate (Métris and Exertier 1995; Steichen 1998); however, available recurrence relations allow the extension of the series expansion to any degree (Cefola and Broucke 1975; Collins and Cefola 1979; Laskar and Boué 2010).

We investigate the long-term dynamics of orbits about an axisymmetric body for such orbital configurations that the third-body perturbation cannot be considered to be second order when compared to J_2. More specifically, we focus on orbital configurations for which second order effects of J_2 can be of the same order of magnitude as perturbations due to the P_2 thru P_5 terms in the Legendre polynomials expansion of the third-body's disturbing function. Because this is the case for traditional Global Navigation Satellite Systems (GNSS) constellations we call this problem a GNSS-type problem. Notice that we do not study the real case of an Earth's GNSS satellite, where different perturbations may have important effects, as illustrated in Fig. 1 (see also Valk et al. 2009; Rossi 2008); specifically, we do not deal with perturbations of the Sun, which have a similar effect to those of the Moon, or with the case of tesseral resonances that is so common in this kind of orbit. We further simplify the model by assuming that the third-body is in circular orbit, thus neglecting periodic effects in the orbital parameters due to the eccentricity of the third-body's orbit. However, at least for Earth's orbits, it is known that the circular orbit approximation does not cause any noticeable

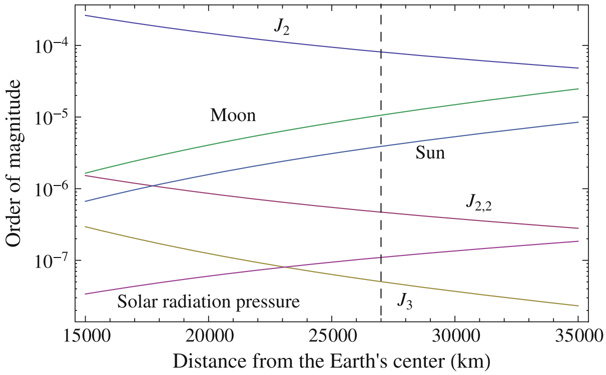


Fig. 1 Order of magnitude of the main perturbations acting on objects in the MEO region, relative to the Keplerian attraction (after Montenbruck and Gill 2001). The area-to-mass ratio used in the computation of the solar radiation pressure is $0.01\ \mathrm{m}^2/\mathrm{kg}$

degradation in the long-term GNSS orbit propagation (Chao and Gick 2004). Note, however, that in the very long-term, say in the order of hundreds of years, third body effects begin to present a diffusion process that cause the eccentricity to grow along lines of resonances with the third body, that can, in time, yield a hyperbolic orbit (Ely 2002). We limit our study here to scales of tens of years, rather than hundreds, in which the diffusion effects are still not apparent.

We approach the GNSS-type problem using perturbation theory based on Lie transforms (Hori 1966; Deprit 1969; Campbell and Jefferys 1970), and perform a higher order averaging to investigate the long-term evolution of GNSS-type orbits. We use the standard approach except for we include Kozai-like constants in each term of the generating function in order to get the mean elements of the theory as close as possible to the average value of the osculating elements (Kozai 1962; Deprit and Rom 1970; Métris 1991b; Steichen 1998). With our Hamiltonian arrangement, the coupling between J_2 terms and third-body perturbations occurs at the ninth order of the theory, what compels us to take up to the sixth degree in the expansion of the third-body disturbing function. This high order in the perturbation theory is required if one aims at computing accurate osculating elements when recovering the short-periodic effects that are excluded from the averaged solution. Differences between a single averaged model, in which the mean anomaly of the satellite is removed, and a double averaged model, where the mean anomaly of the third-body is also removed, are shown to be quite relevant for short-term propagations of the mean elements alone, but both averaging approaches are equivalent in the computation of osculating elements. Systematic application of the semi-analytical theory to different examples shows the importance of the initial argument of the node in the long-term evolution of the orbital eccentricity and inclination. This results from resonances in the angular motion of the node and periapsis of the satellite with the angular motion of the third body node (Ely and Howell 1997).

2 Model

We just deal with a simplified model in which a particle of negligible mass (the satellite) moves in the gravitational field of an oblate body under the gravitational pull of a distant third-body. Thus, the motion of the satellite is derived from the potential

$$\mathscr{V} = -\frac{\mu}{r} + \frac{\mu}{r}\frac{\alpha^2}{r^2}J_2\left(\frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2}\right) + V' \tag{1}$$

where μ is the central body's gravitational parameter, r is the distance from the origin, α is the equatorial radius of the oblate body, J_2 is the zonal coefficient of second degree, and z is the satellite's coordinate in the direction of the symmetry axis of the oblate body. V' is the third-body disturbing potential that, in the mass-point approximation, is

$$V' = -\frac{\mu'}{r'}\left(\frac{r'}{||\mathbf{r}-\mathbf{r}'||} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r'^2}\right) \tag{2}$$

where μ' is the third-body's gravitational parameter, $\mathbf{r}$ and $\mathbf{r}'$ are the radius vector of the satellite and of the third-body, respectively, of corresponding modulus r and r'. We must note that the model in Eq. (1) is made of a truncation of the geopotential at the second degree zonal term, whereas the third-body contribution is not truncated.

For a close satellite to the central body the ratio r/r' is small and the third-body potential in Eq. (2) can be expanded in power series of (r/r')

$$V' = -\beta\,\frac{n'^2\,a'^3}{r'}\sum_{j\geq 2}\left(\frac{r}{r'}\right)^j P_j(\cos\psi), \tag{3}$$

where ψ is the angle encompassed by $\mathbf{r}$ and $\mathbf{r}'$, $\beta = m'/(m'+m)$ is the third-body's reduced mass, a' is the semi-major axis of the third-body's orbit, n' is the third-body's mean motion, and P_j are Legendre polynomials. Note that a term $-\mu'/r'$ has been neglected from Eq. (3) because it has no effect on the equations of motion in the restricted problem approximation.

The state vector of the satellite and of the third-body are conveniently expressed by their orbital elements, noted $(a, e, i, \Omega, \omega, M)$ in the usual notation; when we refer to the third-body, the orbital elements are written in prime notation.

The potential Eq. (1) is conveniently expressed in orbital elements by recalling that the Cartesian and orbital frames are related by three angles of the Euler type. Thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_3(-\Omega)\,R_1(-i)\,R_3(-\theta)\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

where θ is the argument of latitude and R_1 and R_3 are the usual rotation matrices about the x and z axis, respectively.

Besides, we further simplify the model by assuming that the third-body is moving with circular Keplerian motion, and that the inertial origin is defined by the intersection of the equatorial plane of the central body and the orbital plane of the third body. Then, $r' = a'$, $M' = M'_0 + n'\,t$, $\Omega' = 0$, and

$$\begin{aligned} x' &= a'\cos(\omega' + M') \\ y' &= a'\,c'\sin(\omega' + M') \\ z' &= a'\,s'\sin(\omega' + M') \end{aligned} \tag{5}$$

where we use the abbreviations c and s for $\cos i$ and $\sin i$, respectively.

3 Perturbation theory

In order to filter the short-period effects in the potential Eq. (1) we proceed by canonical perturbation theory using Lie transforms (Hori 1966; Deprit 1969; Campbell and Jefferys 1970), which these days is customarily used as a standard procedure in artificial satellite theory (see, for instance, Palacián 2007; Lara et al. 2010). Thus, we set the Hamiltonian

$$\mathscr{H} = \sum_{m \geq 0} \frac{1}{m!} \mathscr{H}_m \tag{6}$$

where $\mathscr{H}_0 = -\frac{1}{2} n^2 a^2$ is the Keplerian term, and the first order term $\mathscr{H}_1 = \Lambda\, n'$ is introduced for convenience to avoid the explicit appearance of time in the Hamiltonian by moving to an "extended" phase space (see, for instance, Brouwer and Clemence 1961). Because we work in the restricted problem approximation, in which the motion of the third-body is not affected by the motion of the satellite, the real value and time evolution of Λ, the conjugate momentum to the third-body's mean anomaly, is irrelevant in our study.

Although we are not tackling the problem of lunisolar perturbations in this paper, we arrange our Hamiltonian model based on the physical parameters of artificial satellites in the high-MEO region, where traditional GNSS constellations reside. That is why we call this model a GNSS-type problem. For the GNSS-type problem, the (formal) small parameter in the perturbation approach is taken to be of the order of one tenth. In consequence, we set $\mathscr{H}_2 = \mathscr{H}_3 = 0$ and put the J_2 perturbation at the fourth order

$$\mathscr{H}_4 = \frac{\mu}{r} \frac{\alpha^2}{r^2} J_2 \left(\frac{3}{2} \frac{z^2}{r^2} - \frac{1}{2} \right).$$

The first term in the expansion of the third-body perturbation appears, then, at the fifth order

$$\mathscr{H}_5 = -\beta\, n'^2\, r^2\, P_2(\cos \psi),$$

and the terms $\mathscr{H}_6, \mathscr{H}_7, \ldots$, are given by the consecutive next degrees in the expansion of Eq. (3), respectively. We truncate the Hamiltonian at the ninth order in the small parameter, corresponding to the sixth degree in the expansion of the third-body disturbing function, for reasons that will be apparent below.

To express the Legendre polynomials in orbital elements we must compute

$$\cos \psi = \frac{x\, x' + y\, y' + z\, z'}{r\, a'}$$

using Eqs. (4) and (5). Besides,

$$r = \frac{a\, \eta^2}{1 + e \cos f},$$

where f is the true anomaly of the satellite. It deserves to mention that the orbital elements used are not canonical variables. Therefore, hereafter we assume that they are expressed as functions of Delaunay variables $\ell = M, g = \omega, h = \Omega, L = n\, a^2, G = L\, \eta$, and $H = G \cos i$, where $\eta = \sqrt{1 - e^2}$ is the usual eccentricity function. One additional remark is that Delaunay variables are singular for zero eccentricity and/or zero inclination, hence nonsingular variables like for instance Poincaré variables or semi-equinoctial elements are commonly used in semi-analytic integration. Nevertheless, as the generating function of the Lie transform can be applied to any function of Delaunay variables, the theory is naturally computed in Delaunay variables and conveniently reformulated in non-singular variables if required.

Once the Hamiltonian (6) is expressed in Delaunay variables, we apply perturbation theory by Lie transforms to compute the transformation from osculating to mean elements. More specifically, we base on Deprit's approach that is specifically devised for automatic computation by machine (Deprit 1969). The Lie transforms algorithm is used as a standard perturbation method these days; in addition to the mentioned research papers, an introduction to the topic can also be found in textbooks such as Meyer and Hall (1992), Boccaletti and Pucacco (1998). Because we put the term $\Lambda\, n'$ at the first order we are ignoring the problem of small divisors, and hence we are implicitly assuming that we are not dealing with resonances of the mean motion of the satellite with the mean motion of the third-body. Moreover, this means that we are also assuming that the third-body mean anomaly varies sufficiently slowly such that its variation may be ignored for the purpose of averaging over the mean anomaly of the satellite. Therefore, to get the double averaged Hamiltonian we can proceed in two steps, using two different canonical transformations. The first one will eliminate the mean anomaly of the satellite from the Hamiltonian, while a second canonical transformation will remove also remaining periodic terms that depend on the mean anomaly of the third-body. This splitting of the canonical transformation has the advantage that the generating function of the transformation at each step only requires solving quadratures—contrary to partial differential equations. Recall that the perturbation algorithm provides also the transformation equations of each canonical transformation in explicit form.

3.1 Averaging the mean anomaly

We first perform a *Delaunay normalization* (Deprit 1982)

$$T^\star : (\ell, g, h, L, G, H) \longrightarrow (\ell^\star, g^\star, h^\star, L^\star, G^\star, H^\star)$$

from original to star variables that averages the Hamiltonian over the mean anomaly. With our Hamiltonian arrangement, second order terms in J_2 appear at the eighth order of the Hamiltonian, and the coupling of the J_2 perturbation with the third body makes apparent at the ninth order, at which we stop the perturbation algorithm. Note that reaching the ninth order in the Lie transforms theory implies that the original Hamiltonian (6) must include up to the sixth degree in the expansion in Legendre polynomials of the third-body disturbing function:

$$\mathscr{H}_9 = -9!\,\beta\, n'^2 r^2 \left(\frac{r}{a'}\right)^4 P_6(\cos\psi).$$

We note that the Hamiltonian term $\mathscr{H}_4$ is naturally expressed as a Poisson series in the true anomaly of the satellite

$$\sum_{j,k} Q_{j,k} \cos(jf + kg)$$

whose coefficients $Q_{j,k}$ depend only on momenta, and that consists of 11 terms. On the contrary, terms related to the third-body disturbing function are naturally expressed as Poisson series in the eccentric anomaly u of the satellite

$$\sum_{j,k,l,m} Q_{j,k,l,m} \cos(ju + kg + lh + m\lambda)$$

that are made of 113, 392, 1013, 2178 and 4141 summands, for terms H_5 to H_9 respectively. This duality in the anomalies will not cause problems in the perturbation approach where the necessary integrals are solved by using the known differential relations $r\,\mathrm{d}u = a\,\mathrm{d}M$, between the eccentric and mean anomaly, and $r^2\,\mathrm{d}f = a^2\eta\,\mathrm{d}M$, between the true and mean anomalies.

The simultaneous appearance of the true and eccentric anomalies first takes place, with our Hamiltonian arrangement, at the eighth order of the perturbation approach. At this order, second order terms in J_2 depending on sine and cosine functions of f, as well as the equation of the center, must be handled jointly with terms of the fifth degree in the expansion of the

Table 1 Integrals $I_k = \langle (f - \ell) \sin kf \rangle$, $k = 1, \ldots, 5$, cf. recurrences in Métris (1991a)

$I_1 = \frac{\eta}{e}\left(\frac{3}{2} - \eta - \frac{1}{2}\eta^2\right)$
$I_2 = -\frac{\eta}{e^2}\left(3 - 2\eta - 3\eta^2 + 2\eta^3 + 4\eta^2 \log\frac{2\eta}{1+\eta}\right)$
$I_3 = \frac{\eta}{e^3}\left(\frac{9}{2} - 3\eta - 4\eta^2 + 7\eta^3 - \frac{9}{2}\eta^4 + 16\eta^2 \log\frac{2\eta}{1+\eta}\right)$
$I_4 = -\frac{\eta}{e^4}\left[6 - 4\eta + 2\eta^2 + 16\eta^3 - 28\eta^4 + 8\eta^5 + 8\eta^2(5+\eta^2)\log\frac{2\eta}{1+\eta}\right]$
$I_5 = \frac{\eta}{e^5}\left[\frac{15}{2} - 5\eta + \frac{131}{6}\eta^2 + 30\eta^3 - \frac{183}{2}\eta^4 + \frac{149}{3}\eta^5 - \frac{25}{2}\eta^6 + 16\eta^2(5+3\eta^2)\log\frac{2\eta}{1+\eta}\right]$

third-body disturbing function, which depend of periodic functions of u. Nevertheless, at this stage there are not coupled terms still, and the computation of the eighth order averaged Hamiltonian and disturbing function can be carried out in closed form—the closed form integration of terms related to the center equation is achieved with relations provided by Kelly (1989), Métris (1991a). Thus, cf. Métris (1991a), $\langle (f - \ell) \cos kf \rangle = 0$ for k integer, and the other nontrivial integrals that appear in the computation of the eighth order averaged Hamiltonian are given in Table 1.

The singularities of the closed form expressions in Table 1 for the zero-eccentricity case are just virtual, and these integrals are efficiently evaluated for the case of small eccentricities by a simple series expansion in powers of e (Métris 1991a). Notwithstanding, the series expansion is never required in our case because the combination of terms affected of negative powers of the eccentricity as derived form the Lie triangle construction (Deprit 1969) simplifies after a simple rearrangement. Specifically, we find out that

$$2e\,(6+e^2)\,I_1 + 4(2+3e^2)\,I_2 + 3e\,(4+e^2)\,I_3 + 6e^2\,I_4 + e^3\,I_5 = \frac{4}{3}\frac{1-\eta}{1+\eta}\,(1+2\eta)\,\eta^3. \tag{7}$$

On the other side, the closed form expression of the eighth order generating function, as well as the transformation equations derived from it, depends on special functions related to the integration of the equation of the center (Osácar and Palacián 1994). At this stage, we decided to expand the eighth order generating function in power series of the eccentricity. For consistency with terms in the averaged Hamiltonian that result from the closed form averaging of $\mathscr{H}_9$, we truncate the series up to the eighth order. Then, differentiations in the Poisson brackets operator needed in the computation of the ninth order averaged Hamiltonian, reduce this order by two to e^6, cf. Deprit and Rom (1970), which is precisely the maximum power of the eccentricity that appears in the closed form expression of $\langle \mathscr{H}_9 \rangle_\ell$. This truncation, of course, limits the application of the theory to periods of time where there is no significant eccentricity growth in the orbits being evaluated. This is exactly the case of the GNSS-type problem, at least in the case of time scales of decades as studied in this paper.

After averaging, the new Hamiltonian is notably simplified

$$T^\star \mathscr{H} \equiv \mathscr{K} = \sum_{m\ge 0} \frac{1}{m!} K_m, \tag{8}$$

where $K_i = \mathscr{H}_i$, $i = 0, \ldots 3$, and with our Hamiltonian arrangement we check that

$$K_j = \langle \mathscr{H}_j \rangle_\ell, \quad j = 4, \ldots, 7,$$
$$K_8 = \langle \mathscr{H}_8 \rangle_\ell$$

$$+8!\left(-\frac{n^2a^2}{2}\right)\frac{\alpha^4}{a^4}\,J_2^2\,\frac{3}{64\eta^7}\left\{-5(1-2c^2-7c^4)+4(1-3c^2)^2\eta+(5-18c^2+5c^4)\,\eta^2\right.$$
$$\left.+2\,\frac{1-\eta}{1+\eta}\,s^2\left[5(1-7c^2)\,(1+2\eta)+(1-15c^2)\,\eta^2\right]\cos 2g\right\}$$

where both Delaunay variables and functions of them are now assumed to be expressed in the star notation, although we alleviate the notation suppressing the stars whenever there is no risk of confusion. Terms in addition to $\langle\mathscr{H}_8\rangle_\ell$ that appear in K_8 are due to second order effects of J_2. The coupling of J_2 (term $\mathscr{H}_4$) with the third-body terms $\mathscr{H}_5$ occurs at the ninth order. Thus, K_9 is made of corresponding terms in addition to $\langle\mathscr{H}_9\rangle_\ell$. The expression is quite involved and is not presented.[1] Terms K_5 to K_8 are made of 23, 56, 113 and 200 summands, respectively. The term K_9 has been computed up to the sixth order in the eccentricity and is made of 319 terms.

The equations of motion are obtained from Hamilton equations

$$\frac{\mathrm{d}(\ell,g,h)}{\mathrm{d}t}=\frac{\partial K}{\partial(L,G,H)},\qquad \frac{\mathrm{d}(L,G,H)}{\mathrm{d}t}=-\frac{\partial K}{\partial(\ell,g,h)}. \tag{9}$$

After averaging the mean anomaly $\mathrm{d}L/\mathrm{d}t=0$ and, therefore, L and $a=L^2/\mu$ are constant. Then, the time evolution of the mean anomaly decouples from the two degrees of freedom, time-dependent system

$$\frac{\mathrm{d}(g,h)}{\mathrm{d}t}=\frac{\partial K}{\partial(G,H)},\qquad \frac{\mathrm{d}(G,H)}{\mathrm{d}t}=-\frac{\partial K}{\partial(g,h)}.$$

The numerical integration of this system can be done with longer stepsizes than the original one because of the filtering of short periodic effects via averaging. At each step of the numerical integration the osculating elements can be recovered analytically using the transformation equations computed also by the Lie transforms method.

We note that, in order to keep the star variables as close as possible to the average value of corresponding osculating ones $\xi^\star\approx\langle\xi\rangle_\ell$, $\xi\in(\ell,g,h,L,G,H)$, we introduced Kozai's term

$$k=-3n\,J_2\,\alpha^2\,\frac{1+2\eta}{\eta^3}\,\frac{1-\eta}{1+\eta}\,s^2\sin 2g$$

in the solution of the fourth order generating function as an arbitrary constant (Kozai 1962, last equation of section II) so to ensure that the corresponding transformation equations are free from hidden long-period terms. Analogously, because of the closed form averaging of terms related to the third-body disturbing function, adequate arbitrary constants have been introduced in the fifth and higher orders of the generating function in order to guarantee their average to zero. In spite of that, the equivalence between mean elements and the average of corresponding osculating elements is no longer possible for higher orders than the seventh, within our Hamiltonian arrangement, due to non-trivial terms derived from the Lie triangle. "Centered elements" could be obtained if desired by means of a new non-canonical transformation (Métris 1991b; Métris and Exertier 1995), but we do not follow this interesting approach because we are better interested in carrying out a new Lie transform that removes the time from the system.

[1] Readers interested in having the literal expressions of this section should contact the first author of this paper.

3.2 Time removal

The third-body's mean anomaly is straightforwardly removed performing a new Lie transform

$$T^*:(\ell^\star,g^\star,h^\star,L^\star,G^\star,H^\star)\longrightarrow(\ell^*,g^*,h^*,L^*,G^*,H^*),$$

from star to asterisk variables. After the double averaging we get

$$T^*\mathscr{K}\equiv\mathscr{S}=\sum_{m\geq 0}\frac{1}{m!}S_m \tag{10}$$

where, doing without the asterisk notation without risk of confusion, $S_i=\mathscr{H}_i$, $i=0,\ldots,3$,

$$\frac{S_4}{S_0}=-4!\,\frac{\alpha^2}{a^2}\,J_2\,\frac{1-3c^2}{2\eta^3} \tag{11}$$

$$\begin{aligned}\frac{S_5}{S_0}=5!\,\frac{n'^2}{n^2}\,\frac{\beta}{16}\Big\{&(1-3c'^2)\left[(1-3c^2)\,(2+3e^2)-15e^2s^2\cos 2g\right]\\&+3(2+3e^2)\,(4c'\,s'\,c\,s\cos h+s'^2s^2\cos 2h)\\&\frac{15}{2}\,e^2\left[s'^2(1-c)^2\cos(2g-2h)+s'^2(1+c)^2\cos(2g+2h)\right.\\&\left.+4c'\,s'\,(1-c)\,s\cos(2g-h)-c'\,s'\,(1+c)\,s\cos(2g+h)\right]\Big\}\end{aligned} \tag{12}$$

$$S_6=0, \tag{13}$$

$$\frac{S_7}{S_0}=7!\,\frac{n'^2}{n^2}\,\beta\,\frac{a^2}{a'^2}\sum_{k=0}^{4}D_k\sum_{j=-2}^{2}X_j\,Y_{2j,k}\cos(2jg+kh) \tag{14}$$

$$\begin{aligned}\frac{S_8}{S_0}=8!\,\frac{\alpha^4}{a^4}\,J_2^2\,\frac{3}{64\eta^7}\Big\{&-5(1-2c^2-7c^4)\\&+4(1-3c^2)^2\eta+(5-18c^2+5c^4)\,\eta^2\\&+2\,\frac{1-\eta}{1+\eta}\,s^2\left[5(1-7c^2)\,(1+2\eta)+(1-15c^2)\,\eta^2\right]\cos 2g\Big\}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{S_9}{S_0}=9!\frac{n'^2}{n^2}\,\beta\Bigg[&\frac{a^4}{a'^4}\sum_{k=0}^{6}Q_k\sum_{j=-3}^{3}Z_j\,q_{2j,k}\cos(2jg+kh)\\&-\frac{\alpha^2}{a^2}\,J_2\sum_{k=0}^{2}T_k\sum_{j=-2}^{2}\left(\sum_{m=|j|}^{3}p_{2j,k,m}e^{2m}\right)\cos(2jg+kh)\Bigg]\end{aligned} \tag{16}$$

where we split S_9 into two parts to separate effects coming from the simple averaging of higher order terms of the third-body disturbing function, from other terms produced by the coupling of lower orders of the Hamiltonian. Constant coefficients D_k, Q_k and T_k, as well as inclination polynomials $Y_{2j,k}$, $q_{2j,k}$ and $p_{2j,k,m}$, and eccentricity polynomials X_j and T_j, corresponding to S_7 and S_9 are given in the appendix.

Recall that, up to the eighth order, the contribution of the third-body to the double averaged Hamiltonian is just the double average of the original term. Hence, $S_5\equiv\langle\mathscr{H}_5\rangle_{\ell,\lambda}$, $S_6\equiv\langle\mathscr{H}_6\rangle_{\ell,\lambda}$, $S_7\equiv\langle\mathscr{H}_7\rangle_{\ell,\lambda}$, and S_8 just adds second order terms on J_2 to $\langle\mathscr{H}_8\rangle_{\ell,\lambda}$. Besides, $\langle\mathscr{H}_6\rangle_{\ell,\lambda}=\langle\mathscr{H}_8\rangle_{\ell,\lambda}=0$ because they only involve odd degrees (3 and 5, respectively) of the third-body disturbing function, which are known to double average to zero (Collins and Cefola 1979). Finally, as mentioned before, terms $Q_k\, Z_j\, q_{k,2j}\cos(2jg+kh)$ in Eq. (16) are the result of the double averaging in closed form $\langle\mathscr{H}_9\rangle_{\ell,\lambda}$, while those factored by J_2 show the coupling between oblateness and third-body perturbations up to the sixth power in the eccentricity.

After the double averaging the system is still of two degrees of freedom, but now it is time independent. The numerical integration is now much more efficient, and the osculating elements are recovered analytically using the transformation equations from asterisk variables to star variables, first, and to star variables to original, osculating ones, then.

4 Numerical tests

Because the theory includes the transformation equations from double-averaged (asterisk) to single-averaged elements (star) and from single-averaged (star) to osculating elements, at any step of the numerical integration we can recover analytically the short-periodic effects removed in the averaging. Therefore, the theory can be used either to compute ephemerides semi-analytically within a good precision, or to investigate the long-term evolution of the orbital elements by means of very fast numerical integrations. Below we provide a sample test in which we compare the numerical integration of the original model, without truncation of the third-body potential Eq. (2), with results obtained with the single- and double-averaged models.

For our tests, we base on the physical parameters of the Earth-Moon system, although we are aware that the Keplerian approximation does not fit to the motion of the Moon. Thus, we take $\mu=398600.4415\,\mathrm{km}^3/\mathrm{s}^2$, $\alpha=6378.1363\,\mathrm{km}$, $J_2=0.0010826$, $a'=384\,400\,\mathrm{km}$, $i'=23.5\,\mathrm{deg}$, $\beta=1/28.8245$, and $n'=2\pi/27.32\,\mathrm{rad/day}$.

4.1 Short-term propagation

For the first test case we choose a GNSS-type orbit with initial orbital elements $a=28\,560\,\mathrm{km}$, $e=0.02$, $i=56\,\mathrm{deg}$, and $\omega=\Omega=M=\lambda=0$ and perform a short-term propagation of two months. Note that these initial conditions must be acted by the corresponding Lie transforms before propagated either in the single or double-averaged models. Thus, for the single-averaged model we need to apply the corrections $\Delta a=-1.9$ km, $\Delta e=-5.7\times10^{-5}$, $\Delta i=-3.96$ arc seconds, and $\Delta\omega=\Delta\Omega=\Delta M=0$, to the original orbital elements at epoch. For the double-averaged case we find the same correction to the semimajor axis because it became constant after removing the mean anomaly, and $\Delta e=-6.42\times10^{-5}$, $\Delta i=-14.04$ arc seconds, and, again, $\Delta\omega=\Delta\Omega=\Delta M=0$.

Figure 2 illustrates a short-term propagation comparing results of the direct numerical integration of the original problem with the numerical integration of the Hamilton equations of the single and double-averaged Hamiltonians. As noticed in the top plot of Fig. 2, the semimajor axis suffers periodic short and medium period oscillations induced by the frequencies of both the third-body and the satellite; besides, the single and double-averaged models both provide the constant value to which the osculating semimajor axes averages. Similar behavior is found for the other orbital elements, the main difference between the single- and double-averaged models being that the single-averaged model reveals the two weeks-period oscillations induced by the third-body, which are flattened by the double averaged-model. The most conspicuous case are those of the inclination and argument of the node, third from

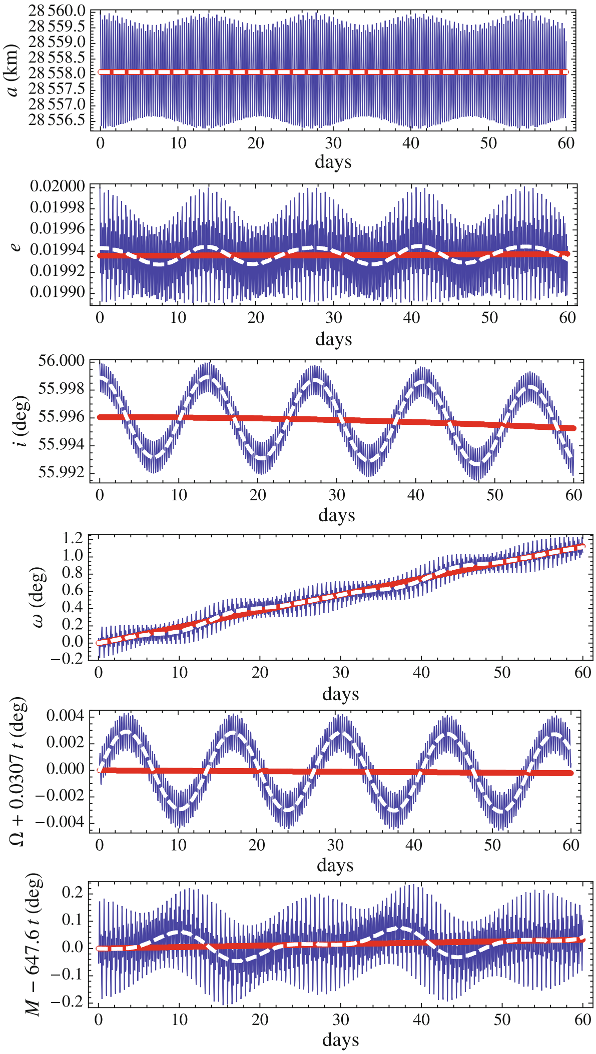


Fig. 2 Sample short-term propagation: numerical integration of the original problem (*blue lines*) versus the single averaged problem (*white dashed*) and the double averaged one (*red dots*)

the top and second to bottom plots, respectively in Fig. 2, because the amplitude of the oscillations induced by the third-body clearly surpasses the amplitude of those oscillations related to the mean motion of the satellite. Note in the second to bottom plot of Fig. 2 (respectively bottom) that a linear drift with mean rate of 0.0307 deg/day (respectively 646.7 deg/day) has been added (respectively subtracted) to the argument of the node curves (respectively mean anomaly). These rates correspond to the main part of the mean motion of the node and of the mean anomaly, respectively, and have been added simply to highlight the differences between averaged and non-averaged results.

Differences between the real evolution of the orbital elements and that provided by any of the averaged models are checked to remain below 2 km for the semimajor axes, 0.65×10^{-4} for the eccentricity, and 0.22°. for both the argument of the pericenter and mean anomaly. For the argument of the node and inclination, the differences are only ∼15 arc seconds for both models, but notably smaller in the case of the single-averaged model. These differences are, of course, unacceptably large in an ephemeris computation. However this situation changes

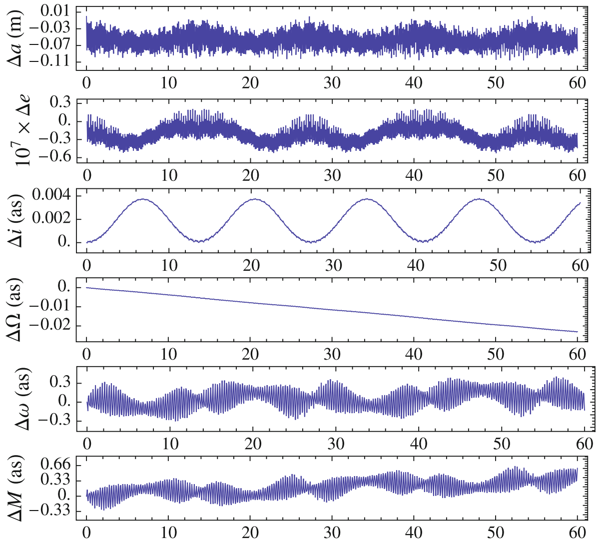


Fig. 3 Errors of the semi-analytical theory after two months propagation. Abscissas are days

considerably, as expected, when recovering the short-periodic effects. As presented in Fig. 3, the errors of the numerical integration of the singly averaged Hamiltonian, followed by the restoration of the short-periodic terms, are only a few centimeters in the semimajor axis, almost negligible for the eccentricity, and fall below the arc second (as) for all angles. These errors would roughly reflect in an error of about 100 m in the position of the satellite after two months.

4.2 Long-term propagation

For longer propagations the errors of the semi-analytical theory degrade, but still remain with very good values. Thus, as appreciated in Fig. 4, short-periodic errors reduce to small values, disclosing long-period errors that are consequence of the Hamiltonian truncation. Nevertheless, these errors still remain small after 5 years, a time interval in which the position of the satellite can be determined within 2 km.

The direct propagation of the mean elements of the double-averaged model is very fast, generally hundreds of times faster that the propagation of the single-averaged model. Hence, the double-averaged is useful in the exploration of the long-term behavior for different initial conditions. As presented in Fig. 5, these long term propagations show the important influence in the eccentricity and inclination evolution of the initial argument of the node of the satellite—or, more precisely, the difference between the arguments of the node of the third-body and the satellite's orbit. The initial argument of the periapsis has also an important effect in the long-term evolution of the eccentricity, as shown in Fig. 6, but a negligible effect in the inclination.

In order to check that simulations using the mean elements propagation present the true qualitative and quantitative nature of the propagated orbit, all the cases presented here have been compared with a propagation of the original osculating equations of motion derived from Eqs. (1) and (2). We found a perfect qualitative agreement between the propagation of secular terms and the non-averaged motion, with errors that generally remain below one thousandth for the eccentricity, and below one arc minute for the inclination. An example is

Fig. 4 Sample long-term propagation: errors between the semi-analytical integration of the double averaged model and the numerical integration of the original problem. Abscissas are years

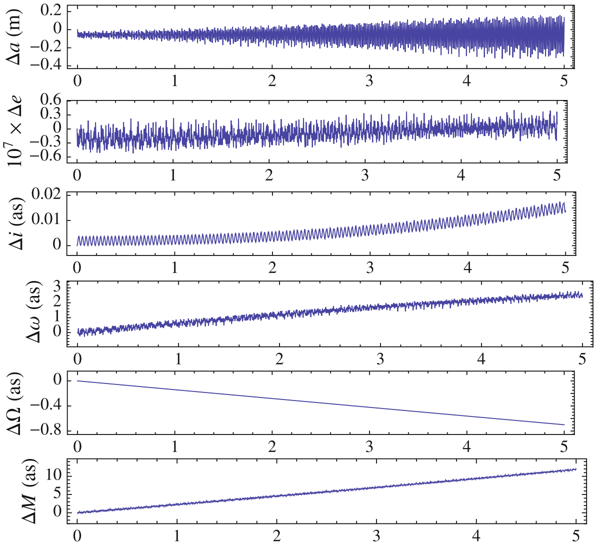


Fig. 5 Sample long-term propagations for $\omega = 0$ and: $\Omega = 0$, *full line*, 90, *dashed*, 180, *dotted*, and 270°, *dash-dotted*. *Top plot*: eccentricity variation. *Bottom plot*: inclination variations

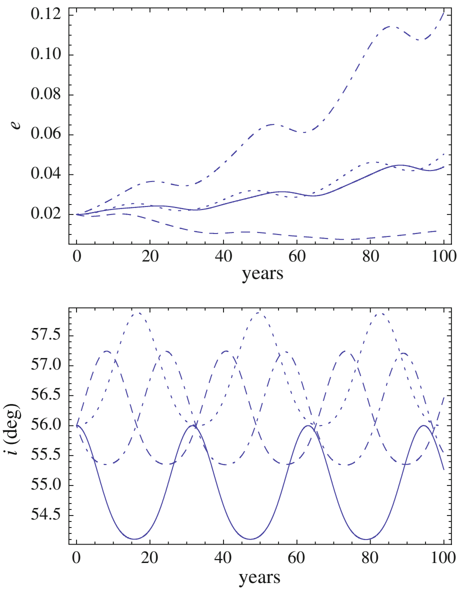
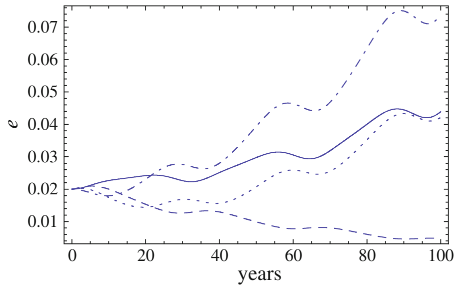


Fig. 6 Sample long-term propagations for $\Omega = 0$ and: $\omega = 0$, *full line*, 30, *dashed*, 60, *dotted*, and 90°, *dash-dotted*. Eccentricity variation



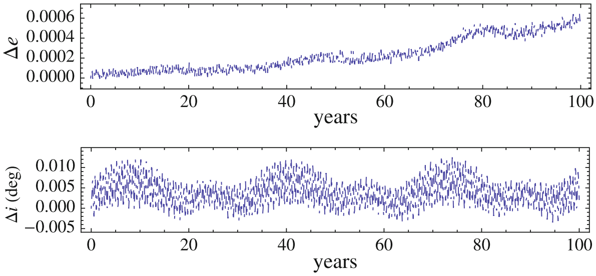


Fig. 7 Errors in the long-term propagations of the secular elements for $\omega = 0$ and $\Omega = 270°$. *Top*: eccentricity errors; *bottom*: inclination errors

provided in Fig. 7 for the orbit in Fig. 5 with the initial node at 270°, which is the case with the highest increase in the eccentricity value.

Besides, for the mean elements propagation we checked that the errors obtained with the eighth order theory are roughly the same as those obtained when using lower order truncations. Hence, truncating the double averaged Hamiltonian up to the fifth order will be enough in the investigation of the long-term behavior of GNSS-type orbits, thus obtaining a further increase in the speed of the numerical integration.

5 Conclusions

The case of an artificial satellite problem in which the third-body perturbation is comparable, but smaller, than the J_2 perturbation is efficiently integrated in a semi-analytical way. This is the case for traditional GNSS constellations (GPS, GLONASS, and Galileo). But, in addition to the perturbations studied here, the problem of GNSS orbits must also to take into account the non-Keplerian motion of the Moon, possible resonances with tesseral harmonics and the contribution of higher order zonal harmonics.

Despite we have not dealt with the long-term evolution of GNSS orbits, some of the conclusions may apply. Specifically, for both the satellite's inclination and argument of the node the amplitude of periodic terms related to the mean anomaly of the third-body clearly exceeds that of terms related to the mean anomaly of the satellite. This fact makes that the mean inclination and argument of the node derived from the double averaged model may notably depart from the average value of corresponding osculating elements during short-term propagations. This difference is of course recovered by applying the terms resulting from the long period generator. Also, the initial difference between the arguments of the node of the third-body and that of the satellite has a radical effect in the long-term behavior of the orbital eccentricity. This initial node has also important repercussions in the orbital inclination, whose mean value can vary by several degrees. The initial argument of the periapsis is also crucial in the orbit evolution in the long-term, but in this case the effect is only relevant for the secular evolution of the eccentricity.

This orbital behavior is efficiently investigated by the numerical integration of secular Hamilton equations that are obtained after a Lie transforms averaging procedure that removes both the satellite's and the third-body's mean anomalies. We also checked that all the relevant features, are apparent from a simplified Hamiltonian that only considers the main term in the Legendre polynomials expansion of the third-body disturbing function. Nevertheless, results provided by this early truncation are not accurate enough for the computation of precise ephemeris, where a higher order truncation of the third-body disturbing function is required, at least up to the sixth degree, because corresponding terms are of the same order of coupling terms with the J_2 perturbation.

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Appendix: Tables with coefficients of the double averaged Hamiltonian

For brevity, the notation $\gamma \equiv 1 \pm c$ is used in following tables (Tables 2, 3, 4, 5, 6).

Table 2 Constant coefficients D_j, Q_j, and T_j

j	D	Q	T
0	$3 - 30c'^2 + 35c'^4$	$5 - 105c'^2 + 315c'^4 - 231c'^6$	$1 - 3c'^2$
1	$s'\,c'\,(3 - 7c'^2)$	$s'\,c'\,(5 - 30c'^2 + 33c'^4)$	$s'\,c'$
2	$1 - 8c'^2 + 7c'^4$	$s'^2\,(1 - 18c'^2 + 33c'^4)$	s'^2
3	$s'^3\,c'$	$s'^3\,c'\,(3 - 11c'^2)$	
4	s'^4	$s'^4(1 - 11c'^2)$	
5		$s'^5\,c'$	
6		s'^6	

Table 3 Eccentricity polynomials X_j and Z_j

j	X	Z
0	$8 + 40e^2 + 15\,e^4$	$1 + \frac{21}{2}\,e^2 + \frac{105}{8}\,e^4 + \frac{35}{16}\,e^6$
± 1	$e^2\,(2 + e^2)$	$e^2 + \frac{5}{3}\,e^4 + \frac{5}{16}\,e^6$
± 2	e^4	$e^4\,(10 + 3e^2)$
± 3		e^6

Table 4 Inclination polynomials $Y_{2j,k}$ in Eq. (14)

k	$j = 0$	$j = \pm 1$	$j = \pm 2$
0	$\frac{9}{16384}\,(3 - 30c^2 + 35c^4)$	$-\frac{315}{8292}\,(1 - 8c^2 + 7c^4)$	$+\frac{6615}{32768}\,s^4$
1	$\frac{45}{2048}\,s\,c\,(3 - 7c^2)$	$\mp\frac{315}{2048}\,s\,\gamma\,(1 \pm 7c - 14c^2)$	$\pm\frac{6615}{4096}\,s^3\,\gamma$
2	$\frac{45}{4096}\,(1 - 8c^2 + 7c^4)$	$-\frac{315}{2048}\,\gamma^2(1 \mp 7c + 7c^2)$	$-\frac{6615}{8192}\,s^2\,\gamma^2$
3	$\frac{315}{2048}\,s^3\,c$	$\mp\frac{2205}{2048}\,s\,\gamma^2(1 \mp 2c)$	$\mp\frac{6615}{4096}\,s\,\gamma^3$
4	$\frac{315}{16384}\,s^4$	$+\frac{2205}{8192}\,s^2\,\gamma^2$	$+\frac{6615}{32768}\,\gamma^4$

Table 5 Inclination polynomials $q_{2j,k}$ in Eq. (16)

j	$k = 0$	$k = 1$
0	$+\frac{25}{32768}\,(5 - 105c^2 + 315c^4 - 231c^6)$	$+\frac{525}{8192}\,s\,c\,(5 - 30c^2 + 33c^4)$
± 1	$-\frac{4725}{131072}\,s^2\,(1 - 18c^2 + 33c^4)$	$\mp\frac{4725}{32768}\,s\,\gamma\,(1 \pm 18c - 36c^2 \mp 66c^3 + 99c^4)$
± 2	$+\frac{10395}{1048576}\,s^4\,(1 - 11c^2)$	$\pm\frac{10395}{262144}\,s^3\,\gamma\,(2 \pm 11c - 33c^2)$
± 3	$-\frac{495495}{2097152}\,s^6$	$\mp\frac{1486485}{524288}\,s^5\,\gamma$

Table 5 continued

j	$k=2$	$k=3$	
0	$+\frac{2625}{65536}\,s^2\,(1-18c^2+33c^4)$	$+\frac{2625}{16384}\,s^3\,c\,(3-11c^2)$	
±1	$-\frac{4725}{262144}\,\gamma^2(17\mp108c-90c^2\pm660c^3-495c^4)$	$\mp\frac{14175}{65536}\,s\,\gamma^2(3\pm5c-55c^2\pm55c^3)$	
±2	$+\frac{51975}{2097152}\,s^2\,\gamma^2\,(1\mp22c+33c^2)$	$\pm\frac{51975}{524288}\,s\,\gamma^3\,(2\mp11c+11c^2)$	
±3	$+\frac{7432425}{4194304}\,s^4\,\gamma^2$	$\pm\frac{2477475}{1048576}\,s^3\,\gamma^3$	
j	$k=4$	$k=5$	$k=6$
0	$+\frac{1575}{32768}\,s^4\,(1-11c^2)$	$+\frac{17325}{16384}\,s^5\,c$	$\frac{5775}{65536}\,s^6$
±1	$-\frac{14175}{131072}\,s^2\,\gamma^2\,(1\mp22c+33c^2)$	$\mp\frac{155925}{65536}\,s^3\,\gamma^2\,(1\mp3c)$	$\frac{155925}{262144}\,s^4\,\gamma^2$
±2	$-\frac{10395}{1048576}\,\gamma^4\,(13\mp44c+33c^2)$	$\mp\frac{114345}{524288}\,s\,\gamma^4\,(2\mp3c)$	$\frac{114345}{2097152}\,s^2\,\gamma^4$
±3	$-\frac{1486485}{2097152}\,s^2\,\gamma^4$	$\mp\frac{1486485}{1048576}\,s\,\gamma^5$	$\frac{495495}{4194304}\,\gamma^6$

Table 6 Inclination polynomials $p_{2j,k,m}$ in Eq. (16)

j	m	$k=0$	$k=1$	$k=2$
0	0	$\frac{9}{64}-\frac{51}{32}c^2+\frac{141}{64}c^4$	$s\,c\left(\frac{51}{16}-\frac{141}{16}c^2\right)$	$s^2\left(\frac{39}{64}-\frac{141}{64}c^2\right)$
	1	$\frac{27}{64}-3c^2+\frac{273}{64}c^4$	$s\,c\left(6-\frac{273}{16}c^2\right)$	$s^2\left(\frac{81}{64}-\frac{273}{64}c^2\right)$
	2	$\frac{141}{256}-\frac{117}{32}c^2+\frac{1359}{256}c^4$	$s\,c\left(\frac{117}{16}-\frac{1359}{64}c^2\right)$	$s^2\left(\frac{423}{256}-\frac{1359}{256}c^2\right)$
	3	$-\frac{8795}{6144}+\frac{4721}{1024}c^2-\frac{10689}{2048}c^4$	$s\,c\left(-\frac{4721}{512}+\frac{10689}{512}c^2\right)$	$s^2\left(-\frac{1247}{2048}+\frac{10689}{2048}c^2\right)$
±1	1	$s^2\left(\frac{3}{8}-\frac{45}{16}c^2\right)$	$\pm s\gamma\left(\frac{93}{64}\pm\frac{225}{64}c-\frac{45}{4}c^2\right)$	$-\gamma^2\left(\frac{21}{64}\pm\frac{225}{128}c-\frac{45}{16}c^2\right)$
	2	$s^2\left(\frac{69}{128}-\frac{525}{128}c^2\right)$	$\pm s\gamma\left(\frac{273}{128}\pm\frac{645}{128}c-\frac{525}{32}c^2\right)$	$-\gamma^2\left(\frac{33}{64}\pm\frac{645}{256}c-\frac{525}{128}c^2\right)$
	3	$s^2\left(-\frac{6743}{4096}+\frac{7569}{4096}\right)$	$\mp s\gamma\left(\frac{1701}{256}\mp\frac{6513}{1024}c^2-\frac{7569}{1024}c^2\right)$	$\gamma^2\left(\frac{6987}{4096}\mp\frac{6513}{2048}c-\frac{7569}{4096}c^2\right)$
±2	2	$-\frac{15}{2048}s^4$	$\mp\frac{15}{512}s^3\gamma$	$\frac{15}{2048}s^2\gamma^2$
	3	$-\frac{177}{40960}s^4$	$\mp\frac{177}{10240}s^3\gamma$	$\frac{177}{40960}s^2\gamma^2$

References

Anselmo, L., Pardini, C.: Long-term dynamical evolution of high area-to-mass ratio debris released into high earth orbits. Acta Astronaut. **67**, 204–216 (2010)

Boccaletti, D., Pucacco, G.: Theory of Orbits. 2: Perturbative and Geometrical Methods. pp. 10 Springer, Berlin (1998)

Broucke, R.A.: Long-term third-body effects via double averaging. J. Guidance Control Dyn. **26**(1), 27–32 (2003)

Brouwer, D.: Solution of the problem of aritifical satellite theory without drag. Astron. J. **64**(1274), 378–396 (1959)

Brouwer, D., Clemence, G.M.: Methods of Celestial Mechanics. pp. 296ff Academic Press, New York (1961)

Campbell, J.A., Jefferys, W.H.: Equivalence of the perturbation theories of Hori and Deprit. Celest. Mech. **2**(4), 467–473 (1970)

Cefola, P., Broucke, R.: On the Formulation of the Gravitational Potential in Terms of Equinoctial Variables. Paper AIAA 75-9, American Institute of Aeronautics and Astronautics (1975)
Cefola, P.J., Yurasov, V.S., Folcik, Z.J., Phelps, E.B., Proulx, R.J., Nazarenko, A.I.: Comparison of the DSST and the USM Semi-Analytical Orbit Propagators. Paper AAS 03-236, American Astronautical Society (2003)
Chao, C.C., Gick, R.A.: Long-term evolution of navigation satellite orbits: GPS/GLONASS/Galileo. Adv. Space Res. **34**, 1221–1226 (2004)
Coffey, S.L., Deprit, A., Miller, B.R.: The critical inclination in artificial satellite theory. Celest. Mech. **39**(4), 365–406 (1986)
Collins, S.K.: Long Term Prediction of High Altitude Orbits. PhD Thesis, MIT (1981)
Collins, S.K., Cefola, P.J.: Double Averaged Third Body Model for Prediction of Super-Synchronous Orbits over Long Time Spans, p. 63. Paper AAS 79-135, American Astronautical Society (1979)
Deprit, A.: Canonical transformations depending on a small parameter. Celest. Mech. **1**(1), 12–30 (1969)
Deprit, A.: Delaunay normalisations. Celest. Mech. **26**(1), 9–21 (1982)
Deprit, A., Rom, A.: The main problem of artificial satellite theory for small and moderate eccentricities. Celest. Mech. **2**, 166–206 (1970)
Ely, T.A.: Eccentricity impact on east-west stationkeeping for global positioning system class orbits. J. Guidance Control Dyn. **25**(2), 352–357 (2002)
Ely, T.A., Howell, K.C.: Long term evolution of artificial satellite orbits due to resonant tesseral harmonics. J. Astronaut. Sci. **44**, 167–190 (1996)
Ely, T.A., Howell, K.C.: Dynamics of artificial satellite orbits with tesseral resonances including the effects of luni-solar perturbations. Int. J. Dyn. Stab. Syst. **12**(4), 243–269 (1997)
Gedeon, G.S.: Tesseral resonance effects on satellite orbits. Celest. Mech. **1**(2), 167–189 (1969)
Giacaglia, G.: Lunar perturbations of artificial satellites of the earth. Celest. Mech. **9**, 239–267 (1974)
Green, A.J.: Orbit Determination and Prediction Processes for Low Altitude Satellites. PhD Thesis, MIT (1979)
Hori, G.-i.: Theory of general perturbation with unspecified canonical variable. Publ. Astron. Soc. Jpn. **18**(4), 287–296 (1966)
Kelly, T.S.: A note on first-order normalizations of perturbed Keplerian systems. Celest. Mech. Dyn. Astron. **46**, 19–25 (1989)
Kozai, Y.: On the Effects of the Sun and the Moon upon the Motion of a Close Earth Satellite, p. 10, SAO Special Report #22 Smithsonian Astrophysical Observatory, Cambridge, MA (1959)
Kozai, Y.: Second-order solution of artificial satellite theory without air drag. Astron. J. **67**(7), 446–461 (1962)
Lara, M., Palacián, J.F., Russell, R.P: Mission design through averaging of perturbed Keplerian systems: the paradigm of an enceladus orbiter. Celest. Mech. Dyn. Astron. **108**(1), 1–22 (2010)
Laskar, J., Boué, G.: Explicit expansion of the three-body disturbing function for arbitrary eccentricities and inclinations. Astron. Astrophys. **522**(A60), 11 (2010)
McClain, W.D.: A Recursively Formulated First-Order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging, Volume 1: The Generalized Method of Averaging Applied to the Artificial Satellite Problem, Computer Sciences Corporation CSC/TR-77/6010 (1977)
Métris, G.: Mean values of particular functions in the elliptic motion. Celest. Mech. Dyn. Astron. **52**, 79–84 (1991a)
Métris, G.: Théorie du mouvement des satellites artificiels—Développement des équations du mouvement moyen—Application à l'étude des longues périodes, p. 188, PhD Thesis, Observatoire de Paris (1991b)
Métris, G., Exertier, P.: Semi-analytical theory of the mean orbital motion. Astron. Astrophys. **294**, 278–286 (1995)
Meyer, K.R., Hall, G.R.: Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Vol. 90. Springer, New York (1992)
Montenbruck, O., Gill, E.: Satellite Orbits Models Methods and Applications. Springer, New york (2001)
Osácar, C., Palacián, J.F.: Decomposition of functions for elliptic orbits. Celest. Mech. Dyn. Astron. **60**, 207–223 (1994)
Palacián, J.F.: Dynamics of a satellite orbiting a planet with an inhomogeneous gravitational field. Celest. Mech. Dyn. Astron. **98**, 219–249 (2007)
Prado, A.F.B.A.: Third-body perturbation in orbits around natural satellites. J. Guidance Control Dyn. **26**(1), 33–40 (2003)
Proulx, R., McClain, W.C., Early, L., Cefola, P.: A Theory for the Short Periodic Motion due to the Tesseral Harmonic Gravity Field, p. 20, Paper AAS 81-180, American Astronautical Society, (1981)
Rossi, A.: Resonant dynamics of Medium earth orbits: space debris issues. Celest. Mech. Dyn. Astron. **100**(4), 267–286 (2008)

Slutsky, M.: First Order Short Periodic Motion of and Artificial Satellite Due to Third-Body Perturbations: Numerical Evaluation, p. 22, Paper 83-393, American Astronautical Society (1983)

Steichen, D.: An averaging method to study the motion of lunar artificial satellites. Celest. Mech. Dyn. Astron. **68**, 205–224; (part I) and 225–247 (part II) (1998)

Valk, S., Lemaître, A., Deleflie, F.: Semi-analytical theory of mean orbital motion for geosynchronous space debris under gravitational influence. Adv. Space Res. **43**, 1070–1082 (2009)

Vashkov'yak, M.A.: A numerical-analytical method for studying the orbital evolution of distant planetary satellites. Astron. Lett. **31**(1), 64–72 (2005)