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ORIGINAL ARTICLE

Self-resonant bifurcations of the Sitnikov family and the appearance of 3D isolas in the restricted three-body problem

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Abstract The association of the Sitnikov family with families of multiple three-dimensional periodic orbits is studied. In particular, the families consisting of three-dimensional periodic orbits bifurcating from self-resonant orbits of the Sitnikov family at double, triple and quadruple period of the bifurcation orbit are considered. The branch families close upon themselves and remain 3D up to their terminations having two common members with the Sitnikov family. By varying the mass parameter we also study the evolution of some of the computed families and find that they become isolas and disappear gradually in three-dimensions by shrinking to point size.

Keywords Periodic orbits · Sitnikov problem · Three-body problem · Three-dimensional isolas · Self-resonant orbits

1 Introduction

The Sitnikov problem is a special case of the three-body problem where the third body of negligible mass moves along the z-axis perpendicular to the plane of motion of the primaries and passes through the inner collinear equilibrium point (Pavanini 1907; Mac Millan 1913; Sitnikov 1960). The case where the primaries perform elliptic motion has been studied by several authors (Hagel 1992; Alfaro and Chiralt 1993; Dvorak 1993; Lara and Buendía 2001; Faruque 2003; Hagel and Lhotka 2005; Llibre and Ortega 2008; Kovács and Érdi 2009; Ruzza and Lhotka 2011) while the case where the primaries perform circular motion has been studied by Perdios and Markellos (1988), Belbruno et al. (1994) and Perdios (2007). Also, for the circular problem, Sidorenko (2011) has studied the alternation of stability and

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instability of the Sitnikov family at large values of the oscillation amplitude and has shown the occurrence of complex saddle instability.

Modifications of the circular Sitnikov problem where the equal primaries are oblate spheroids and/or radiation sources have been studied by Kalantonis et al. (2008) and Perdios et al. (2008). Recently, Douskos et al. (2012) have shown the existence of a new type of straight-line periodic oscillations, different from the well known Sitnikov motions, in the restricted three-body problem when the primaries are prolate spheroids. They found new types of families of three-dimensional periodic orbits bifurcating from the Sitnikov family and consisting of orbits located entirely above or below the orbital plane of the two primaries.

The Sitnikov family of rectilinear orbits may be considered as a generator of families of three-dimensional periodic orbits. Perdios (2007) found that all families of three-dimensional periodic orbits bifurcating from one-to-one critical orbits of the Sitnikov family terminate at planar periodic orbits. Thus the Sitnikov family represents an alternative way to look for families of three-dimensional periodic orbits instead of generating them from vertical self-resonant coplanar periodic orbits. He also made a first study on period doubling bifurcations from the Sitnikov family and found that the corresponding bifurcating families of three-dimensional periodic orbits do not end on vertical critical coplanar periodic orbits. This result shows the importance of this motion of straight line oscillations since families bifurcating from it can not have be generated from coplanar periodic orbits.

Our aim here is to complement the work of that paper by considering the self-resonance bifurcations of the Sitnikov family and the behaviour of the branch families not only in the Copenhagen case, i.e. for $\mu = 0.5$, but also for varying mass parameter of the restricted three-body problem. The classical way to determine three-dimensional periodic orbits, except these emanating from the equilibrium points, is to generate them from vertical critical or vertical self-resonant orbits of the plane (see Robin and Markellos 1980, 1983). In this work we look for three-dimensional orbits bifurcating from self-resonant orbits of the rectilinear Sitnikov motion, i.e. three-dimensional orbits which branch out of an one-dimensional orbit. In particular, we look for bifurcating families consisting of orbits of double, triple and quadruple the period of the basic family and try to establish the general pattern of their behaviour in the full range of the mass parameter for which they exist. Our results show that all the computed branch families for $\mu = 0.5$ do not terminate at coplanar vertical self-resonant periodic orbits since they form closed curves in three-dimensions. This means that the only direct method to determine them is their computation as bifurcations of the Sitnikov family. The evolution of these families with respect to the mass parameter shows that they become "isolas" and disappear gradually in three-dimensions by shrinking to point size (for isolas see Dellwo et al. 1981). This remarkable result of three-dimensional isolas shows the high importance of a so simple motion, i.e. the Sitnikov motion, since these three-dimensional isolas could not have been discovered otherwise.

The paper is organized as follows: In Sect. 2, we recall the equations of motion of the circular Sitnikov problem and consider the transversal stability of the Sitnikov family and the occurring self-resonant Sitnikov orbits. In Sects. 3, 4 and 5, we study the bifurcating one-to-two, one-to-three and one-to-four families, respectively, as well as their variation with respect to the mass parameter. Finally, in Sect. 6, we present some concluding remarks.

2 Equations of motion and stability

We consider a barycentric, rotating and dimensionless coordinate system $Oxyz$, where the Ox axis always contains the two primaries m_1 and m_2 of masses $1 - \mu$ and μ, respectively

($\mu \leqslant 1/2$). The equations of motion of the third body of infinitesimal mass moving under the gravitational attraction of the two primaries are (Szebehely 1967):

$$\begin{aligned}
\ddot{x} - 2\dot{y} &= x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu-1) = \frac{\partial \Omega}{\partial x},\\
\ddot{y} + 2\dot{x} &= y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = \frac{\partial \Omega}{\partial y},\\
\ddot{z} &= z\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = \frac{\partial \Omega}{\partial z},
\end{aligned} \tag{1}$$

where Ω is the potential function, $\mu = m_2/(m_1+m_2)$ is the mass parameter and:

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}, \tag{2}$$

are the distances of the third body of negligible mass from the two primaries. System (1) admits the well-known integral:

$$C = (x^2 + y^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \tag{3}$$

where C is the Jacobi constant. The Sitnikov motion $z(t)$ of the restricted three-body problem can be obtained from System (1) by putting $\mu = 0.5$ and $x(t) = y(t) = 0$ and is described by:

$$\ddot{z} = -\frac{z}{(z^2 + \frac{1}{4})^{3/2}}, \tag{4}$$

while the equation of the Jacobi integral (3) becomes:

$$C = \frac{2}{(z^2 + \frac{1}{4})^{1/2}} - \dot{z}^2. \tag{5}$$

In order to study the stability of the Sitnikov family we consider small perturbations $x = \xi$ and $y = \eta$ of the zero horizontal components of the rectilinear motion obtaining the variational equations:

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx0}(z)\xi, \qquad \ddot{\eta} + 2\dot{\xi} = \Omega_{yy0}(z)\eta, \tag{6}$$

where $\Omega_{xx0}(z)$ and $\Omega_{yy0}(z)$ are the second partial derivatives of the potential function with respect to the variables x and y, respectively, computed at $\mu = 1/2$, while the corresponding mixed derivative is eliminated since $\Omega_{xy0}(z) = 0$. These can be written in matrix form:

$$\dot{\Xi} = A[z(t)]\Xi, \quad \text{with} \quad \Xi = (\xi, \eta, \dot{\xi}, \dot{\eta})^{\mathrm{T}}, \tag{7}$$

where the periodic coefficients of matrix $A[z(t)]$ as well as details for the determination of the stability of the rectilinear motion can be found in Perdios and Markellos (1988). The stability conditions are:

$$|a_1| \leqslant 2, \qquad |a_2| \leqslant 2, \tag{8}$$

where a_1, a_2 are the stability parameters (for the corresponding diagrams of the stability of the Sitnikov family the reader may also address the papers by Perdios and Markellos (1988) and Belbruno et al. (1994)). The cases $a_i = -2$ and $a_i = 2$, $i = 1, 2$, represent the critical orbits of the Sitnikov family at which families of three-dimensional periodic orbits of the same and double period (respectively) bifurcate. The first case has been extensively studied in Perdios (2007) whereas a first study of the second case has been done in the same paper (these were named in that paper as one-to-one and one-to-two critical orbits, respectively).

However, the stability diagram of the Sitnikov family indicates the existence of higher order resonances, i.e. orbits of the rectilinear motion exist at which families of three-dimensional periodic orbits of multiple the period of the corresponding Sitnikov orbit bifurcate. These are called Self-Resonant (SR) orbits and correspond to a stability parameter a_i, $i = 1, 2$, taking a value:

$$a_i = -2\ \cos\left(2\pi\frac{n}{m}\right), \tag{9}$$

where m is the multiplicity of the bifurcating family (one-to-m bifurcation). Our aim here is to complement the work of Perdios (2007), in the sense that we shall also consider the cases of one-to-three and one-to-four SR orbits of the Sitnikov family, i.e. the orbits at which families of three-dimensional periodic orbits of triple and quadruple the period of the corresponding orbit bifurcate, and study the behaviour of these bifurcating families in the Copenhagen case as well as in the case for which $\mu \neq 0.5$. The first eight one-to-three and one-to-four SR Sitnikov orbits are given in Tables 1 and 2.

Table 1 SR orbits of the Sitnikov family from which families of 3D periodic orbits of triple period bifurcate (case $a_1 = 1$)

	$\dot{z}_0$	$T/4$	C	a_2
D1	1.81998986	5.71407748	0.68763692	−37.58465695
D2	1.83040542	6.20547653	0.64961598	−14.18456858
D3	1.84720586	7.17706330	0.58783052	−32.70297297
D4	1.85714933	7.88829391	0.55099635	−9.12902560
D5	1.86655065	8.68377563	0.51598869	−27.54518412
D6	1.87498420	9.52587162	0.48443424	−6.72688763
D7	1.88092907	10.20971121	0.46210583	−23.32295086
D8	1.88800746	11.14327674	0.43542784	−5.26964054

Table 2 SR orbits of the Sitnikov family from which families of 3D periodic orbits of quadruple period bifurcate (case $a_1 = 0$)

	$\dot{z}_0$	$T/4$	C	a_2
E1	1.76799488	4.04487173	0.87419411	−49.58721419
E2	1.78929529	4.60612902	0.79842235	−16.49498743
E3	1.81422117	5.47100867	0.70860155	−36.23594758
E4	1.83280731	6.32963882	0.64081737	−9.09029928
E5	1.84386800	6.96329152	0.60015081	−25.98242544
E6	1.85836905	7.98412837	0.54646448	−6.22634827
E7	1.86431287	8.48198367	0.52433754	−19.16456732
E8	1.87576172	9.61070155	0.48151797	−4.63672333

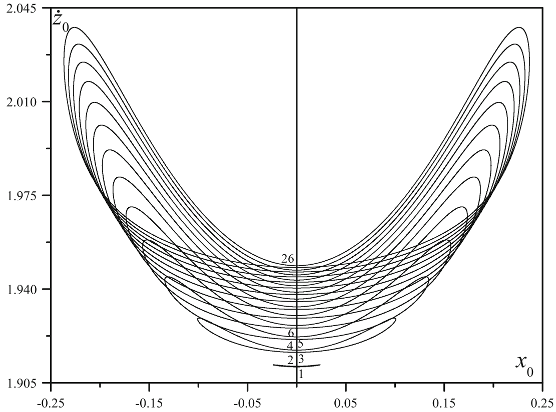

Fig. 1 The manifold of families of 3D periodic orbits, in the Copenhagen case ($\mu = 0.5$), bifurcating from the one-to-two critical orbits of the Sitnikov family

3 One-to-two families

We start our computations by giving the manifold of the families bifurcating from the one-to-two critical orbits for which $z(T/4) \leqslant 10$ (26 such orbits). Note that, the solution $z(t)$ has been chosen such that $z(0) = 0$ and $\dot{z}(0) = \dot{z}_0 > 0$, so that $z(t)$ reaches its maximum at $T/4$. These critical orbits can be found in Perdios (2007), named C1, C2, ..., C26 in that paper where the first six bifurcating families were computed. The corresponding manifold is shown in Fig. 1 here. As it can be seen from this figure, the statement of Perdios (2007) that all these families emanate from the rectilinear motion in pairs, namely the family emanating from the first one-to-two critical orbit returns to the second one-to-two critical orbit and so on, is valid (at least for the first 26 bifurcations) creating thus a "tree" of closed families. The stability of all the computed branch families was determined and it was found that many of them contain stable parts. Note that, all the 3D periodic orbits of the computed families have been determined using the symmetry of the Ox-axis, i.e. the initial state vector of a 3D periodic orbit is of the form $(x_0, 0, 0, 0, \dot{y}_0, \dot{z}_0)$. For simplicity, however, in all figures of the present paper we present the projections of the initial state vector in the $(x_0, \dot{z}_0)$ plane. Note also that, in Fig. 1 and all figures presenting such projections of the initial state in the case of $\mu = 0.5$, the family of rectilinear Sitnikov motions is represented by the vertical $\dot{z}_0$-axis at $x_0 = 0$.

We now consider what happens to these families when we vary the value of the mass parameter. Let us begin with the first family bifurcating from the first one-to-two critical orbit which returns on the second one-to-two critical orbit. For a slightly different value of the mass parameter ($\mu = 0.4995$) we can see in the left frame of Fig. 2 that the characteristic curve of this family remains a closed curve, but has now become an isola since it does not bifurcate from the family of rectilinear Sitnikov motions as the Sitnikov family does not exist for $\mu \neq 0.5$. Note that, the Sitnikov family as the family of rectilinear orbits existing only for $\mu = 0.5$ can be thought of as a particular case of the family L_1^3 of 3D periodic orbits emanating from the inner collinear equilibrium point L_1 which exists for all values of the mass parameter. For $\mu = 0.4995$ the family L_1^3 coexists with the families plotted in Fig. 2 (left frame) and is shown in a separate frame (right frame) due to difference in scale in both x_0 and $\dot{z}_0$. By continuing to vary the mass parameter to lower values we see, in the same figure, that the family remains an isola for all values of the mass parameter for which it exists and disappears in three-dimensions by shrinking into point size.

The evolution w.r.t. the mass parameter of the second family of three-dimensional periodic orbits bifurcating from the third one-to-two critical orbit and returning on the fourth

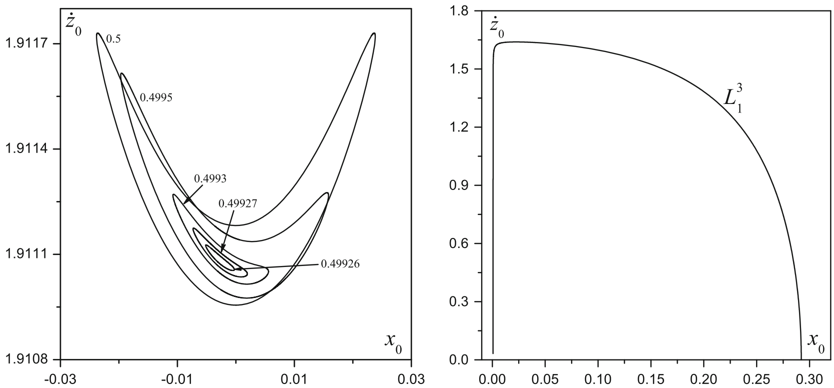

Fig. 2 *Left frame* Evolution w.r.t. the mass parameter of the family of 3D periodic orbits bifurcating from the first and second one-to-two critical orbits of the Sitnikov family. *Right frame* The Lyapunov family L_1^3 for $\mu = 0.4995$

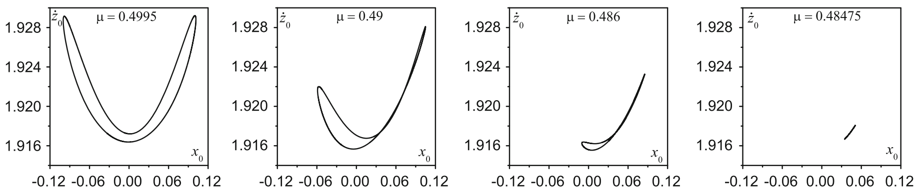

Fig. 3 Evolution w.r.t. the mass parameter of the family of 3D periodic orbits bifurcating from the third and fourth one-to-two critical orbits of the Sitnikov family

one-to-two critical orbit is shown in Fig. 3. As we see in this figure the behaviour of its characteristic curve is the same as previously but this family exists for a wider range of the mass parameter. Following the third family emanating from the fifth one-to-two critical orbit (it returns on the sixth one-to-two critical orbit) as we vary the mass parameter we see, in Fig. 4, that its characteristic curve remains an isola down to a certain value of the mass parameter and then splits into two separate isolas which shrink into point size separately.

4 One-to-three families

In Fig. 5 we present the manifold of families of three-dimensional periodic orbits bifurcating from the first eight one-to-three SR orbits of the rectilinear motion for $\mu = 0.5$. As we see

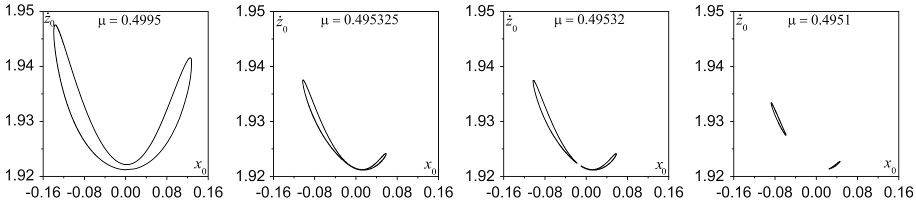

Fig. 4 Evolution w.r.t. the mass parameter of the family of 3D periodic orbits bifurcating from the fifth and sixth one-to-two critical orbits of the Sitnikov family

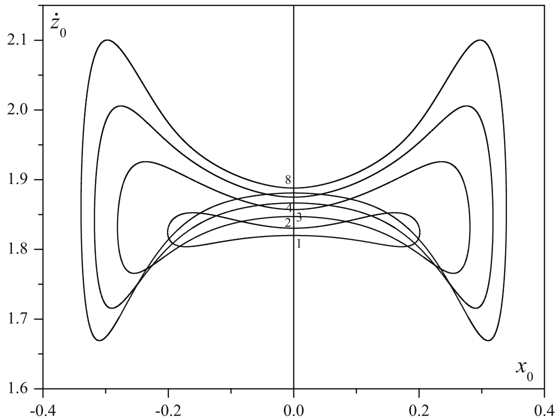

Fig. 5 The manifold of families of 3D periodic orbits, in the Copenhagen case ($\mu = 0.5$), bifurcating from the one-to-three SR orbits of the Sitnikov family

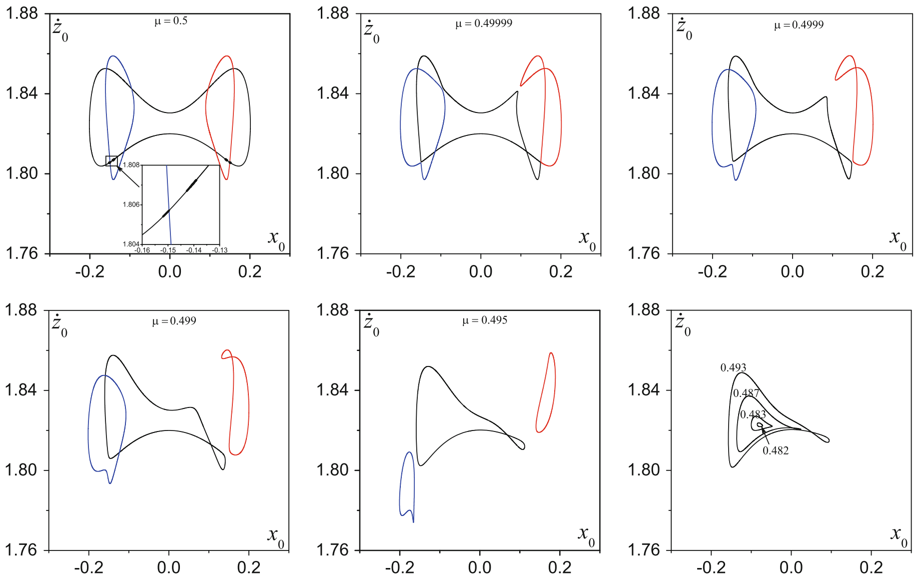

Fig. 6 Evolution w.r.t. the mass parameter of the family of 3D periodic orbits bifurcating from the first and second one-to-three SR orbits of the Sitnikov family. For $\mu = 0.5$ the stable parts of this family are shown *bold* in the *top left frame* (inset)

in this figure, the mechanism of the bifurcating families is the same as in the case of the families bifurcating from the one-to-two critical orbits, i.e. they emanate from the rectilinear motion in pairs (the family emanating from the first one-to-three SR orbit returns to the second one-to-three SR orbit and so on) forming thus closed curves in three dimensions. This can be seen from the family plots presented in Fig. 5.

For $\mu = 0.5$ the stability of the families bifurcating from the one-to-three SR orbits was computed and it was found that only the first branch family contains stable parts. Specifically, it contains four stable parts, two of which are illustrated in the top left frame of Fig. 6 (inset).

The evolution w.r.t. the mass parameter of the first family which bifurcates at the first one-to-three SR orbit of the Sitnikov family and returns at the second one is shown in Fig. 6. In the first frame of this figure we see that for $\mu = 0.5$ there exist two additional families of three-dimensional periodic orbits bifurcating from the first family (shown with blue and red

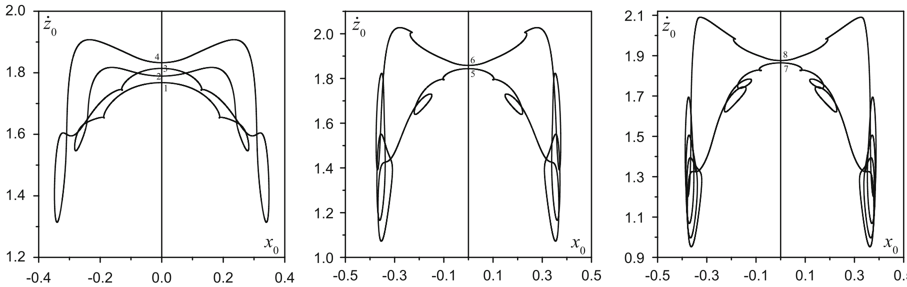

Fig. 7 The manifold of families of 3D periodic orbits, in the Copenhagen case ($\mu = 0.5$), bifurcating from the one-to-four SR orbits of the Sitnikov family

colors). For a slightly lower value of the mass parameter ($\mu = 0.49999$) we see in the second frame, that the first family which bifurcate from the rectilinear motion and the other two families still exist but now some segments of their characteristic curves have come together creating different formations. For lower values of the mass parameter these formations constituting three isolas can be seen more clearly (frames 3, 4 and 5 of this figure). In the last frame of this figure we follow the evolution of the first family w.r.t. the mass parameter and observe that this constantly shrinks and ends at a point, representing the birth of this isola, for $\mu \cong 0.481$.

5 One-to-four families

In Fig. 7 we show the characteristic curves of the families of three-dimensional periodic orbits bifurcating from the first eight one-to-four SR orbits of the Sitnikov family. All these families which emanate from the Sitnikov family for $\mu = 0.5$ follow the same pattern as in the previous cases. Their characteristic curves remain closed, but they are more complicated since they create many loops before they close upon themselves. For $\mu = 0.5$ the stability of these branch families was computed and it was found that none of them contains stable parts.

In Fig. 8 we show the evolution w.r.t. the mass parameter of the first family emanating from the first one-to-four SR orbit and terminating on the second one-to-four SR orbit. In the first frame of this figure we observe that for $\mu = 0.5$ there are four more families of three-dimensional periodic orbits bifurcating from the first family (shown with blue, magenta, red and green colors). For a slightly different value of the mass parameter ($\mu = 0.4999$) we see, in the second frame, a remarkable change in the formation of the characteristic curves of these families. Branches of all these families have joined with each other in such a way that now there are two less families of three-dimensional periodic orbits. Their characteristic curves remain closed forming isolas and by varying the mass parameter further we observe the same situation (frames 3 and 4). In the last frame we see that for $\mu = 0.425$ the additional families shrink faster than the first family and for $\mu = 0.374$ they cease to exist while the first family continues to shrink in size.

6 Concluding remarks

Self-resonant bifurcations of the Sitnikov family were considered. All the computed bifurcating families of three-dimensional periodic orbits close upon themselves having two common

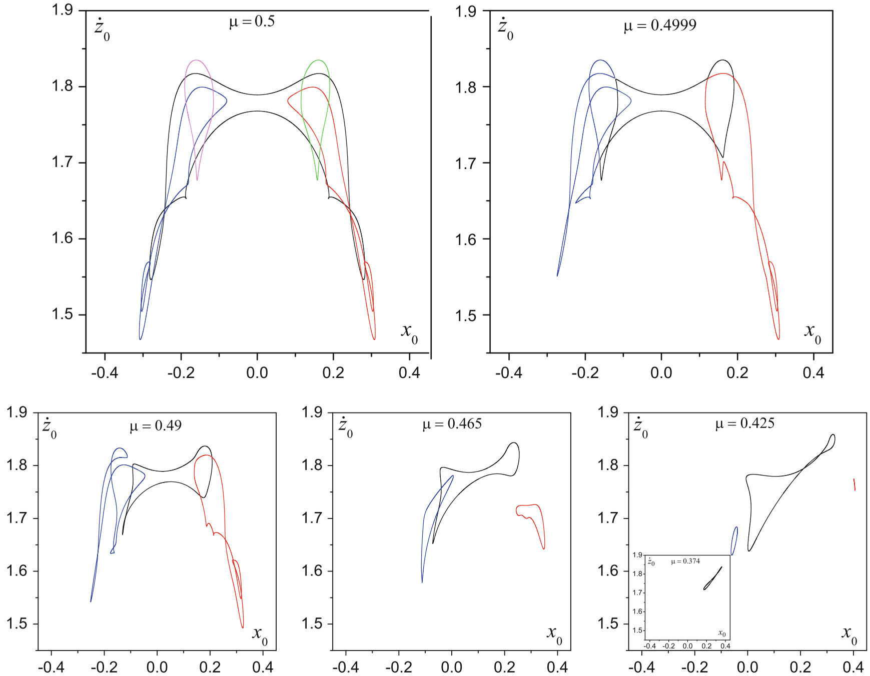

Fig. 8 Evolution w.r.t. the mass parameter of the family of 3D periodic orbits bifurcating from the first and second one-to-four SR orbits of the Sitnikov family

members with the Sitnikov family. The evolution of these branch families with respect to the mass parameter shows that they become three-dimensional isolas and each one of them disappears gradually by shrinking to a point representing the birth of the respective isola. The significance of the Sitnikov family is enhanced by these results since such rectilinear oscillations may generate infinite branch families of three-dimensional periodic orbits, for equal and not equal masses, leading to the discovery of 3D isolas and their birth points, which could not have been found by classical methods.

An interesting result is that the bifurcating families from self-resonant orbits of the Sitnikov family bifurcate with other families of three-dimensional periodic orbits and their characteristic curves combine with each other in their evolution with respect to the mass parameter creating new formations of family characteristics.

In the case of equal masses, i.e. $\mu = 0.5$, we found several branch families that have stable parts. More precisely, the one-to-two families bifurcating from the pairs of Sitnikov orbits C3–C4, C5–C6, C7–C8, C9–C10, C11–C12, C13–C14 and C15–C16, contain stable parts. For the one-to-three families only the bifurcating family from the pair of Sitnikov orbits D1–D2 has stable parts while for the one-to-four bifurcating families none of the computed families were found to have stable parts. Note that, the stable parts of these families are far from the corresponding self-resonant rectilinear orbits.

Regarding the symmetries of three-dimensional periodic orbits which are members of the bifurcating families from self-resonant orbits of the Sitnikov family we point out that they fall into two types: (I) double symmetry w.r.t. both the Ox-axis and the Oy-axis for families that consist of 3D periodic orbits where their period is even multiple of the self-resonant orbit's period and (II) double symmetry w.r.t. both the Ox-axis and the Oyz-plane for families that consist of 3D periodic orbits where their period is odd multiple of the self-resonant orbit's period. To the authors' knowledge doubly symmetric periodic orbits of the type (I) have not been found before in the three-dimensional restricted three-body-problem.

A remarkable result, which remains unexplained, is that the families which bifurcate from self-resonant orbits of the Sitnikov family with $a_i = -2\cos(2\pi\ n/m)$, $i = 1, 2$, and $m \geqslant 2$, consist entirely of three-dimensional periodic orbits, in contrast to the families bifurcating from the one-to-one critical orbits of the Sitnikov family ($m = 1$) which have been found to terminate with planar periodic orbits in previous works.

References

Alfaro, J.M., Chiralt, C.: Invariant rotational curves in Sitnikov's problem. Celest. Mech. Dyn. Astron. **55**, 351–367 (1993)

Belbruno, E., Llibre, J., Ollé, M.: On the families of periodic orbits which bifurcate from the circular Sitnikov motions. Celest. Mech. Dyn. Astron. **60**, 99–129 (1994)

Dellwo, D., Keller, H.B., Matkowsky, B.J., Reiss, E.L.: On the birth of isolas. SIAM J. Appl. Math. **42**, 956–963 (1981)

Douskos, C., Kalantonis, V., Markellos, P., Perdios, E.: On Sitnikov-like motions generating new kinds of 3D periodic orbits in the R3BP with prolate primaries. Astrophys. Space Sci. **337**, 99–106 (2012)

Dvorak, R.: Numerical results to the Sitnikov-problem. Celest. Mech. Dyn. Astron. **56**, 71–80 (1993)

Faruque, S.B.: Solution of the Sitnikov problem. Celest. Mech. Dyn. Astron. **87**, 353–369 (2003)

Hagel, J.: A new analytic approach to the Sitnikov problem. Celest. Mech. Dyn. Astron. **53**, 267–292 (1992)

Hagel, J., Lhotka, C.: A high order perturbation analysis of the Sitnikov problem. Celest. Mech. Dyn. Astron. **93**, 201–228 (2005)

Kalantonis, V.S., Perdios, E.A., Perdiou, A.E.: The Sitnikov family and the associated families of 3D periodic orbits in the photogravitational RTBP with oblateness. Astrophys. Space Sci. **315**, 323–334 (2008)

Kovács, T., Érdi, B.: Transient chaos in the Sitnikov problem. Celest. Mech. Dyn. Astron. **105**, 289–304 (2009)

Lara, L.J., Buendía, A.E.: Symmetries and bifurcations in the Sitnikov problem. Celest. Mech. Dyn. Astron. **79**, 97–117 (2001)

Llibre, J., Ortega, R.: On the families of periodic orbits of the Sitnikov problem. SIAM J. Appl. Dyn. Syst. **7**, 561–576 (2008)

Mac Millan, W.D.: An integrable case in the restricted problem of three bodies. Astron. J. **27**, 11–13 (1913)

Pavanini, G.: Sopra una nuova categoria di solutioni periodiche nel problema dei tre corpi. Ann. Math. Serie III, Tomo XIII, 179–202 (1907)

Perdios, E.A.: The manifold of families of 3D periodic orbits associated to Sitnikov motions in the restricted three-body problem. Celest. Mech. Dyn. Astr. **99**, 85–104 (2007)

Perdios, E.A., Kalantonis, V.S., Douskos, C.N.: Straight-line oscillations generating three-dimensional motions in the photogravitational restricted three-body problem. Astrophys. Space Sci. **314**, 199–208 (2008)

Perdios, E.A., Markellos, V.V.: Stability and bifurcations of Sitnikov motion. Celest. Mech. **42**, 187–200 (1988)

Robin, I.A., Markellos, V.V.: Numerical determination of three-dimensional periodic orbits generated from vertical self-resonant satellite orbits. Celest. Mech. **21**, 395–434 (1980)

Robin, I.A., Markellos, V.V: The mechanism of branching of three-dimensional periodic orbits from the plane. In: Markellos, V.V., Kozai, Y. Dynamical Trapping and Evolution in the Solar System, pp. 213–224. D.Reidel, Dordrecht (1983)

Ruzza, S., Lhotka, C.: High order normal form construction near the elliptic orbit of the Sitnikov problem. Celest. Mech. Dyn. Astron. **111**, 449–464 (2011)

Sidorenko, V.V.: On the circular Sitnikov problem: the alternation of stability and instability in the family of vertical motions. Celest. Mech. Dyn. Astron. **109**, 367–384 (2011)

Sitnikov, K.: The existence of oscillatory motions in the three-body problem. Dokl. Akad. Nauk. **133**, 303–306 (1960)

Szebehely, V.: Theory of Orbits. Academic Press, Orlando (1967)