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ORIGINAL ARTICLE

The rhomboidal symmetric four-body problem

Jörg Waldvogel

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Abstract We consider the planar symmetric four-body problem with two equal masses $m_1 = m_3 > 0$ at positions $(\pm x_1(t),\ 0)$ and two equal masses $m_2 = m_4 > 0$ at positions $(0,\ \pm x_2(t))$ at all times t, referred to as the *rhomboidal symmetric four-body problem*. Owing to the simplicity of the equations of motion this problem is well suited to study regularization of the binary collisions, periodic solutions, chaotic motion, as well as the four-body collision and escape manifolds. Furthermore, resonance phenomena between the two interacting rectilinear binaries play an important role.

Keywords Symmetric four-body problem · Levi-Civita regularization · Periodic motion · Resonance · Invariant tori · Collision manifold

1 Introduction

Classical Celestial Mechanics began with Isaac Newton's discovery of the two basic laws governing the motion of celestial bodies: the law of gravitation as the physical basis and the law of motion of a point mass under external forces as the mathematical basis. Up to the present day a large part of Celestial Mechanics is still based on these two fundamental principles.

A simple, but highly relevant, non-trivial case, the motion of two point masses in $\mathbb{R}^3$ under their mutual gravitational attraction, often referred to as spacial *Kepler motion,* can be described completely in closed form in terms of elementary functions. The motion is regular, predictable over long time intervals; the problem is *integrable.* The addition of one more point mass, i.e. the transition to the *three-body problem,* causes a quantum leap in complexity.

The set of possible solutions of the three-body problem is so large and complex that any attempt of a complete description is bound to fail. Typically, the distance between corresponding points on neighbouring orbits increases exponentially with time, which prevents

J. Waldvogel (✉)
Seminar for Applied Mathematics, Swiss Federal Institute of Technology ETH, 8092 Zurich, Switzerland
e-mail: waldvogel@sam.math.ethz.ch

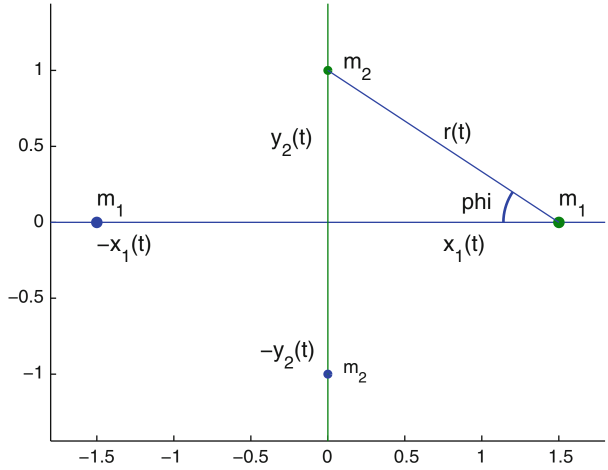


Fig. 1 The rhomboidal symmetric four-body problem, notation $x_2(t) := y_2(t)$

predictions over long time intervals. This type of motion is referred to as *chaotic*, the problem is *non integrable*.

The goal of this paper is to consider systems of point masses beyond the three-body problem, still showing chaoticity, yet being simpler. Sufficiently simple systems may bear the chance of permitting theoretical advances. A good candidate is the symmetric four-body problem introduced by Steves and Roy (1998), where two pairs of equal masses move in a fixed plane, occupying positions of central symmetry with respect to the origin (the "Caledonian" problem). This system has four degrees of freedom like the planar three-body problem, and the solutions of both problems have similar complexity.

Two particular cases with only two degrees of freedom are even simpler, but they still have chaotic solutions: the one-dimensional symmetric four-body problem and the rhomboidal symmetric four-body problem. The former problem has been studied by Sweatman (2002) and by Ouyang and Yan (2011); it will not be considered here. The latter problem, also simply referred to as the *rhomboidal four-body problem*, is similar in many aspects, but there are differences as well. It has been studied by Lacomba and Perez-Chavela (1993), is also mentioned by Shibayama (2011) in a study of periodic orbits. In Sects. 2 through 7 of this paper we give brief introductions to the principal aspects of the rhomboidal four-body problem.

Consider two equal point masses $m_1 = m_3 > 0$ at positions $(\pm x_1(t),\ 0)$ in a fixed plane (described by Cartesian coordinates x, y), and two equal masses $m_2 = m_4 > 0$ at positions $(0,\ \pm y_2(t))$, see Fig. 1. Since the resulting forces acting on m_1 and on m_3 have a vanishing y-component and are of equal magnitude, m_1 and m_3 can move on the x-axis. Equally, the motion of m_2 and m_4 is restricted to the y-axis. Therefore, the rhomboidal nature of the constellation is preserved, provided the initial velocities satisfy the same symmetries. The two binaries $(m_1,\ m_3)$ and $(m_2,\ m_4)$ move rectilinearly on the x-axis or on the y-axis, respectively.

The rhomboidal symmetric four-body problem is well suited for studying regularization of the binary collisions, periodic solutions, homothetic solutions and central configurations, as well as the four-body collision and escape manifolds. We will also study the role of resonance phenomena between the two interacting rectilinear binaries for generating periodic orbits.

2 Equations of motion and regularization

In the following we will use the notation $x_2(t) := y_2(t)$ for simplicity. Direct application of the Newtonian laws yields the equations of motion

$$\ddot{x}_j + \frac{m_j}{4\,x_j^2} + \frac{2\,m_{3-j}\,x_j}{r^3} = 0\,, \quad j = 1, 2\,, \tag{1}$$

where $r := \sqrt{x_1^2 + x_2^2}$ is the distance between m_1 and m_2, and dots denote differentiation with respect to time t. For conveniently regularizing the frequent collisions in the two binaries we will use the Hamiltonian formalism in the following. With the kinetic and potential energies

$$T = m_1\,\dot{x}_1{}^2 + m_2\,\dot{x}_2{}^2, \quad U = -\frac{m_1^2}{2\,x_1} - \frac{m_2^2}{2\,x_2} - \frac{4\,m_1\,m_2}{r} \tag{2}$$

and the momenta $p_j := m_j\,\dot{x}_j,\ j = 1, 2$, the Hamiltonian $H := \frac{1}{2}(T + U)$ corresponding to Eq. (1) becomes

$$H = \frac{p_1^2}{2\,m_1} + \frac{p_2^2}{2\,m_2} - \frac{m_1^2}{4\,x_1} - \frac{m_2^2}{4\,x_2} - \frac{2\,m_1\,m_2}{\sqrt{x_1^2 + x_2^2}}\,, \tag{3}$$

and the Hamiltonian (canonical) equations of motion and the energy integral read

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \quad j = 1, 2\,, \quad H(t) = H_0 = \text{const.} \tag{4}$$

Now, regularization of the binary collisions by the classical technique of Levi-Civita (1920) will be discussed. It is not necessary to consider the more general regularization concept of Easton (1971). A technique for regularizing the n-body problem by doubling the dimension of the phase space was suggested by Heggie (1974). This technique was applied by Sivasankaran et al. (2010) to regularize the symmetric four-body problem. A potential difficulty are simultaneous binary collisions possible in this system. ElBialy (1996) proved that simultaneous binary collisions in n-body systems are always regularizable.

Owing to its simplicity the rhomboidal symmetric four-body problem allows for a much simpler regularization procedure, without doubling the dimension of the phase space and without the necessity to consider simultaneous binary collisions. For this purpose simultaneous collisions of all four bodies will be excluded, i.e. $r > 0$ is assumed. Then the system behaves as two weakly coupled rectilinear binaries. Following the ideas of Sundman (1907) and of Birkhoff (1915), we first introduce the new independent variable τ (the fictitious time), according to the differential relation given in Eq. (5) below. The canonical form of the equations of motion is preserved if K is adopted as the new Hamiltonian, where H_0 is the (constant) total energy on the orbit considered:

$$dt = x_1\,x_2\,d\tau, \quad K = x_1\,x_2\,(H - H_0). \tag{5}$$

Next, new coordinates ξ_j and new momenta $\pi_j,\ j = 1, 2$ are introduced by

$$x_j = \xi_j^2\,, \quad p_j = \frac{\pi_j}{2\,\xi_j}\,, \quad j = 1, 2. \tag{6}$$

The transformation of the momenta must be such that the π_j are canonical conjugates of the ξ_j. This may be done via the generating function $W(p, \xi)$, see e.g., Siegel and Moser (1971):

$$\pi_j = \frac{\partial W}{\partial \xi_j}, \quad j = 1, 2, \quad \text{with} \quad W(p, \xi) = p_1 x_1 + p_2 x_2. \tag{7}$$

With the transformation (6) being canonical, it suffices to express the Hamiltonian K of Eq. (5) in terms of ξ_j and π_j in order to obtain the regularized Hamiltonian

$$K = \frac{1}{8}\left(\frac{\pi_1^2 \xi_2^2}{m_1} + \frac{\pi_2^2 \xi_1^2}{m_2}\right) - \frac{1}{4}\left(m_1^2 \xi_2^2 + m_2^2 \xi_1^2\right) - \frac{2 m_1 m_2 \xi_1^2 \xi_2^2}{\sqrt{\xi_1^4 + \xi_2^4}} - H_0 \xi_1^2 \xi_2^2 \tag{8}$$

with $K(\tau) = 0$ on the orbit. The regularized equations of motion become

$$\xi_j{}' = \frac{\partial K}{\partial \pi_j}, \quad \pi_j{}' = -\frac{\partial K}{\partial \xi_j}, \quad j = 1,\ 2, \quad (\,)' = \frac{d}{d\tau}(\,) \tag{9}$$

or, explicitly for $j = 1,\ 2$ with $k := 3 - j$,

$$\begin{aligned}
\xi_j{}' &= \frac{\pi_j \xi_k^2}{4 m_j} \\
\pi_j{}' &= \xi_j \left(-\frac{\pi_k^2}{4 m_k} + \frac{m_k^2}{2} + 4 m_1 m_2 \left(\frac{\xi_k^4}{\xi_1^4 + \xi_2^4}\right)^{3/2} + 2 H_0 \xi_k^2\right) \\
t' &= \xi_1^2 \xi_2^2 .
\end{aligned} \tag{10}$$

This system of differential equations is well suited for theoretical considerations and for numerical experiments. In each of the following sections we present a few results on separate topics related to the rhomboidal problem. Each topic may lead to further investigations.

3 Escape

In this section we present a typical orbit in terms of physical coordinates and in terms of regularized variables. As it happens often, after a couple of collisions in each binary a close quadruple encounter will redistribute the energy, leaving behind a tight binary and an escaping hyperbolic binary.

A convenient way of defining initial conditions for numerical experiments is to begin in a binary collision (only possible in regularized variables). To this end, it is necessary to establish the initial terms of the Taylor series of a generic solution of Eqs. (10) in a collision. With no loss of generality consider collisions at $\tau = 0$ with $\xi_1(0) = 0$. For $\xi_2(0) \neq 0$, the condition $K = 0$ yields

$$\pi_1(0) = \pm\sqrt{2 m_1^3}. \tag{11}$$

The three parameters of the motion are $A := \xi_2(0) \neq 0$, $B := \pi_2(0)$ and the total energy H_0. Then the initial terms of the Taylor series are found to be

$$
\begin{aligned}
\xi_1(\tau) &= \pi_1(0)\,\frac{A^2}{4m_1}\,\tau + O(\tau^3)\\
\xi_2(\tau) &= A + O(\tau^3)\\
\pi_1(\tau) &= \pi_1(0) + O(\tau^2)\\
\pi_2(\tau) &= B + O(\tau^3).
\end{aligned}
\tag{12}
$$

Up to the signs of $\xi_1(\tau)$ and $\pi_1(\tau)$ the series are uniquely determined by m_1, m_2, A, B, H_0.

For generating Fig. 2a, b we arbitrarily chose $m_1 = 2$, $m_2 = 1$, $x_1(0) = 3$, $x_2(0) = 2$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$; in Fig. 2a we plotted x_1 and x_2 (bold face) versus physical time t. The repeated collisions appear as cusps. Near $t = 4$ all 4 masses are near the origin, which causes a change in the amplitudes of the oscillations. The fifth collision in the m_1-binary, near $t = 19.5$, is a near-quadruple encounter which dramatically tightens the m_1-binary and sends the m_2-binary into an escape.

In Fig. 2b the regularized coordinates ξ_j (solid) and the regularized momenta π_j (dashed) are plotted versus fictitious time τ for the same orbit (bold face for $j = 2$). It is seen that regularization stretches the independent variable at every collision and flips every other oscillation of Fig. 2a at the τ-axis, such that all functions become smooth. The instantaneous near-quadruple encounter near $t = 19.5$ is stretched to an interval approximately $17.5 < \tau < 35$, an extremely useful property of our regularization, allowing to closely observe the history of near-quadruple encounters.

The graphics of Fig. 2b nicely corroborate the Taylor series of Eq. (12). Consider, e.g., the collision of the m_1-binary near $\tau = 36$. Since ξ_1 is increasing, π_1 has a positive extreme, and the graphs of ξ_2 and π_2 have terrasse points.

4 Periodic solutions and resonance

For finding periodic solutions we use initial conditions in a collision, see Eq. (12), e.g.

$$
\xi_1(0) = 0\,, \quad \pi_1(0) = -\sqrt{2\,m_1^3}\,, \quad \pi_2(0) = 0 \tag{13}
$$

and treat the fourth initial value, $u := \xi_2(0)$, as an unknown quantity. Equation (8) automatically implies $K = 0$; it does not allow to determine the total energy H_0, however. Therefore, H_0 may be chosen as a quantity characteristic of the periodic orbit.

In order to find a periodic orbit corresponding to given values m_1, m_2, H_0, choose an approximate value of u and find q such that $\xi_2(q) = 0$. The quantity q is a tentative value of the quarter period; computationally it can be generated directly by numerical integrators with *event capability*. Then the condition for periodicity is

$$
\boxed{\pi_1(q) = 0}\,. \tag{14}
$$

Since q depends on u, this is a nonlinear equation for the unknown u, to be solved numerically by an iterative algorithm. The *secant method* is very well suited since it doesn't need derivatives with respect to u.

In Fig. 3a, b we show the results of the example $m_1 = m_2 = 1$, $H_0 = -0.9$. The algorithm described above yields $\xi_2(0) = -1.34776\,71645$, $q = 3.25359\,53267$.

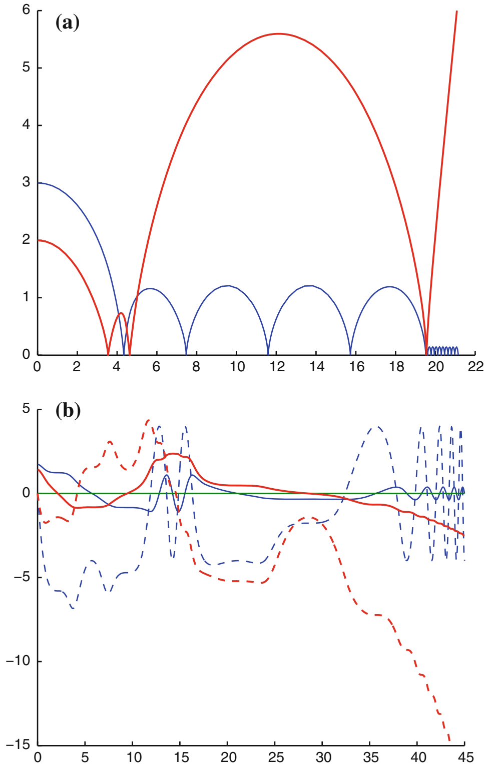


Fig. 2 **a** Escape solution in physical coordinates x_j versus time t; $m_1 = 2$ (*thin*, *blue*), $m_2 = 1$ (*bold*, *red*), $x_1(0) = 3,\ x_2(0) = 2$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$. **b** Solution of Fig. 2a in regularized variables ξ_j, π_j versus fictitious time τ; *thin lines* $m_1 = 2$, *bold face* $m_2 = 1$, *dashed* momenta

Owing to the equality of the masses, $m_1 = m_2$, this periodic solution displays remarkable symmetries: $\xi_1(\tau)$ and $\xi_2(\tau)$ behave like the symmetrically distorted graphs of the trigonometric functions $-\sin(\pi/2 \cdot \tau/q)$ and $-\cos(\pi/2 \cdot \tau/q)$. The stability of similar orbits with simultaneous binary collisions has been investigated by Bakker et al. (2010, 2011).

Even more surprising is the robustness of this periodic solution against varying the initial conditions ("stickiness"). This periodic solution has neighbouring solutions that do not run into a quadruple close encounter with subsequent escape for hundreds of periods. A crude explanation of this phenomenon is the fact that the two coupled binaries of the periodic solution are in a 1:1 resonance. In this way they are locked away from a close quadruple encounter, which would eventually result in an escape (see the example of Sect. 3). A more rigorous explanation by means of Poincaré sections will be given in the next section.

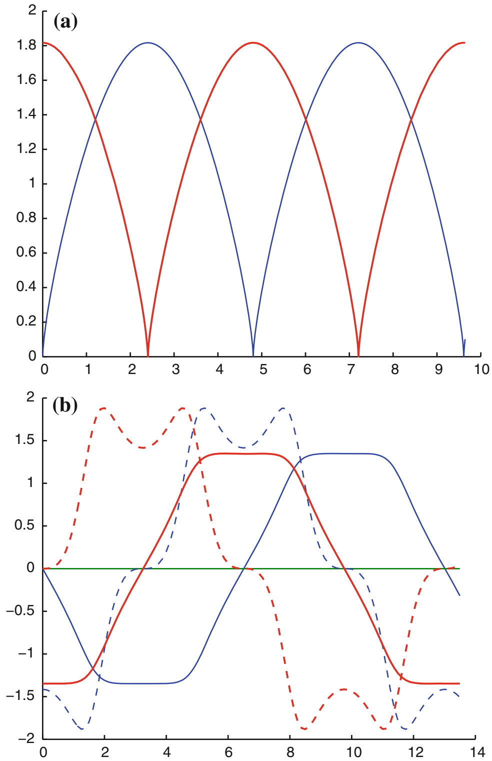
5 Poincaré sections and quasiperiodic solutions

Instead of the entire orbit $\big(\xi_j(\tau),\ \pi_j(\tau)\big)$ we now only consider its intersection points with the *surface of section*

$$\xi_1 = 0 \quad \text{with} \quad \xi_1{}' > 0\,, \quad \pi_1 = -\sqrt{2\,m_1^3}. \tag{15}$$

We consider a fixed value H_0 of the energy, and we plot the sequence of intersection points in the $(\xi_2,\ \pi_2)$-plane for various initial points.

Fig. 3 **a** Periodic solution in physical coordinates x_j versus time t; (*thin, blue*) $m_1 = 1$, (*bold, red*) $m_2 = 1$, $H_0 = -0.9$. **b** Periodic solution of Fig. 3a in regularized variables ξ_j, π_j versus τ; *thin lines* $m_1 = 1$, *bold face* $m_2 = 1$, *dashed* momenta, $H_0 = -0.9$



In Fig. 4a the point at the center (marked by an asterisk) corresponds to the periodic solution of Sect. 4 with energy $H_0 = -0.9$. The ovals around it visualize quasiperiodic solutions (see below). The 6 asterisks between the 5th and the 6th oval mark a periodic solution of longer period (near $\xi_2 = -1.49906$, $\pi_2 = 0$), closing only after all 6 points have been visited. Around every asterisk there are "islands" of quasiperiodic orbits similar to the islands around the center of Fig. 4a. In the outermost set of intersection points (corresponding to $\xi_2 = -1.57$, $\pi_2 = 0$) the onset of *chaos* is documented by the irregular arrangement of some of the points.

Figure 4b concentrates on the remaining 10 asterisks of Fig. 4a. They correspond to two periodic orbits generated by the initial points $\xi_2 = -1.55325$, $\pi_2 = 0$ and $\xi_2 = -0.96016$, $\pi_2 = 0$, respectively, each closing after visiting 5 of the 10 marked points. Some of the islands around these points are visible, provided that a sufficient number of contributing points has been generated. Between the islands there exist *hyperbolic equilibrium points* whose stable and unstable manifolds generate *chaotic zones*.

The ovals of Fig. 4a have been selected by prescribing initial points by means of rounded values of its coordinates. As a consequence, the points of an oval generally form a continuous curve that separates its outer region from its inner region. The corresponding orbit is quasiperiodic, referred to as a *torus*. The set of tori is dense in the set of real numbers (a Cantor set); between the tori there are periodic orbits with islands and chaotic regions, at a microscopic scale. This structure explains the robustness of the periodic orbit of Sect. 4: an orbit beginning between two tori stays there for ever. In summary: The neighbourhood of the periodic orbit of Sect. 4 resembles the well known picture of chaotic dynamics extensively

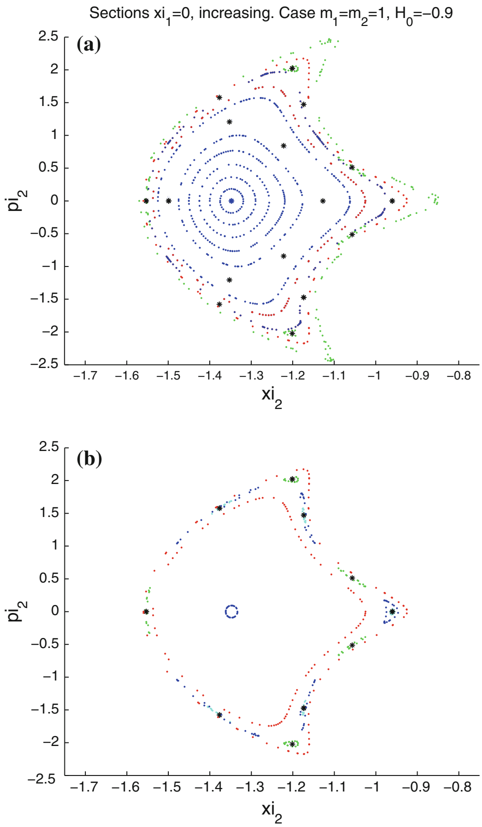


Fig. 4 **a** Sections $\xi_1 = 0$, increasing, plotted in the ξ_2, π_2-plane. Case $m_1 = m_2 = 1,\ H_0 = -0.9$. **b** Details of the outermost region of Fig. 4a. Case $m_1 = m_2 = 1,\ H_0 = -0.9$

discussed by many authors (see, e.g. Froeschlé and Lega 2006, for an overview and many more references).

6 Homothetic solutions and central configurations

A homothetic solution of an n-body problem is defined as a solution with the property that every mass point moves on a rectilinear Keplerian orbit. The constellation of the mass points remains similar to itself; such constellations are referred to as *central configurations*. In the case of the rhomboidal four-body problem the equations of motion (1) need to be solved by

$$x_j(t) = c_j\, f(t)\,,\ \ j = 1, 2,\ \ c_1 = c\, \cos(\varphi),\ \ c_2 = c\, \sin(\varphi) \tag{16}$$

with an appropriate constant $c > 0$ and the angle $\varphi \in (0,\ \pi/2)$ being defined in Fig. 1. The function $f(t)$ describes a rectilinear Kepler motion,

$$\ddot{f}(t) + \frac{m}{f(t)^2} = 0\,, \quad m > 0.$$

This yields the two conditions

$$\frac{m_1}{\cos^3(\varphi)} + 8\, m_2 = \frac{m_2}{\sin^3(\varphi)} + 8\, m_1 = 4\, c^3 m, \tag{17}$$

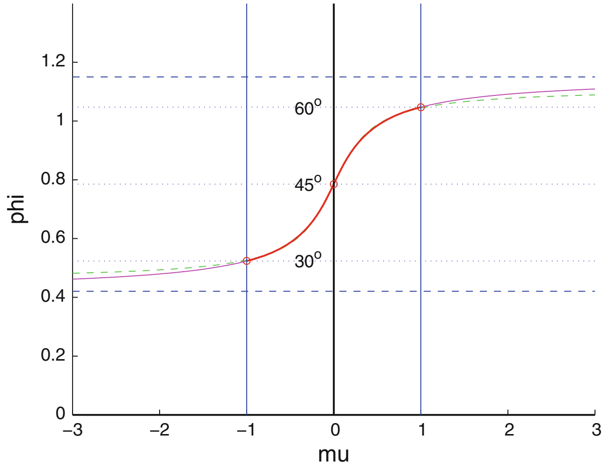


Fig. 5 Rhomboidal central configurations, plotted as φ (defined in Fig. 1) vs. μ from Eq. (19)

resulting in the following condition for the geometry of the symmetric *rhomboidal central configurations* of four pairwise equal masses:

$$\frac{m_1}{\cos^3(\varphi)} - \frac{m_2}{\sin^3(\varphi)} = 8\,(m_1 - m_2)\,, \quad 0 < \varphi < \pi/2. \tag{18}$$

For discussing computational aspects and results we introduce the mass parameter

$$\mu := \frac{m_1 - m_2}{m_1 + m_2} \in (-1,\ 1). \tag{19}$$

For arbitrary (real) μ Eq. (17) has a unique real solution φ with

$$\left|\varphi - \frac{\pi}{4}\right| < 0.36474\,52742\,36650.$$

Equation (17) reduces to a polynomial equation of degree 12 for $\tan(\frac{\varphi}{2})$ (see Waldvogel 2001).

Alternatively, Eq. (17) may be solved numerically by the *Newton-Raphson* iteration, e.g. by using the initial approximation (dashed curve in Fig. 5)

$$\varphi_0 = \frac{\pi}{4} + \frac{1}{f}\arctan\left(f\,b\,\mu\right) \quad \text{with} \quad f = 1.528545\,\pi,\ b = \frac{2\sqrt{2}-1}{3}.$$

In the interval $-1 < \mu < 1$ the absolute error of φ_0 is less than 0.003255, and 3 iterations yield an accuracy of 15 digits. The graph of φ versus μ, together with its asymptotes as $\mu \to \pm\infty$, and the graph of φ_0 (dashed lines) are shown in Fig. 5. In general we have $\varphi(\mu) + \varphi(-\mu) = \pi/2$; particular values are $\varphi(-1) = \pi/6,\ \varphi(0) = \pi/4,\ \varphi(1) = \pi/3,\ \varphi(\pm\infty) = \pi/4 \pm 0.364745,\ \varphi'(0) = (2\sqrt{2}-1)/3$.

7 The quadruple-collision manifold

In this final section we will apply the technique introduced by McGehee (1974) for adequately describing the events during a close encounter of all four bodies. The general idea is to introduce normalized coordinates, momenta, and fictitious time $\tilde{\xi}_j,\ \tilde{\pi}_j,\ \tilde{\tau}$, adapted to the current size and rate of change of a four-body system in a close quadruple encounter or in a quadruple collision.

A convenient length is the radius of inertia ρ, defined by means of the moment of inertia I, (see McGehee 1974, and Waldvogel 1982):

$$\rho^2 = I = 2\left(m_1 x_1^2 + m_2 x_2^2\right). \tag{20}$$

Note that Eqs. (1) through (6) imply

$$\dot{I} = 2\left(\pi_1 \xi_1 + \pi_2 \xi_2\right), \quad \ddot{I} = 4T + 2U = 8H_0 - 2U = 4H_0 + 2T. \tag{21}$$

Therefore we use the scaling transformations

$$x_j = \rho\, \tilde{x}_j, \quad p_j = \rho^{-1/2}\, \tilde{p}_j, \quad \xi_j = \rho^{1/2}\, \tilde{\xi}_j, \quad \pi_j = \tilde{\pi}_j, \quad d\tau = \rho^{-1/2}\, d\tilde{\tau} \tag{22}$$

in order to normalize the equations of motion.

From Eqs. (20), (21_1) and the transformations (5_1), (22) we obtain

$$\frac{d\rho}{d\tilde{\tau}} = \rho\, \tilde{\xi}_1^2\, \tilde{\xi}_2^2\, \left(\pi_1 \tilde{\xi}_1 + \pi_2 \tilde{\xi}_2\right), \tag{23}$$

a differential equation for ρ allowing $\rho(\tilde{\tau}) \equiv 0$ as a solution. The remaining four equations for equivalently describing the motion follow from Eq. (10):

$$\begin{aligned}
\frac{d\tilde{\xi}_j}{d\tilde{\tau}} &= \tilde{\xi}_k^2 \left(\frac{\pi_j}{4m_j} - \frac{\tilde{\xi}_j^3}{2}\left(\pi_1 \tilde{\xi}_1 + \pi_2 \tilde{\xi}_2\right)\right), \quad k := 3-j, \; j = 1, 2 \\
\frac{d\pi_j}{d\tilde{\tau}} &= \tilde{\xi}_j \left(-\frac{\pi_k^2}{4m_k} + \frac{m_k^2}{2} + 4m_1 m_2 \left(\frac{\tilde{\xi}_k^4}{\tilde{\xi}_1^4 + \tilde{\xi}_2^4}\right)^{3/2} + 2\rho H_0 \tilde{\xi}_k^2\right) \\
\frac{dt}{d\tilde{\tau}} &= \rho^{3/2}\, \tilde{\xi}_1^2\, \tilde{\xi}_2^2.
\end{aligned} \tag{24}$$

Now the collision manifold $\mathcal{M}$ is defined as the limiting solution of the system (23), (24) characterized by $\rho(\tilde{\tau}) \equiv 0$. Equation (23) is satisfied, and (24_3) implies that time t does not advance. Therefore $\mathcal{M}$, i.e. the solution of (24_1), (24_2) with $\rho = 0$, describes the very instant of collision as seen in an infinitely slowed down and infinitely blown-up slow-motion picture.

As a consequence of (20) and (8), the collision manifold has the two integrals of motion

$$\begin{aligned}
& m_1 \tilde{\xi}_1{}^4 + m_2 \tilde{\xi}_2{}^4 = \frac{1}{2} \\
& \frac{1}{8}\left(\frac{\pi_1^2 \tilde{\xi}_2{}^2}{m_1} + \frac{\pi_2^2 \tilde{\xi}_1{}^2}{m_2}\right) - \frac{1}{4}\left(m_1^2 \tilde{\xi}_2{}^2 + m_2^2 \tilde{\xi}_1{}^2\right) - \frac{2 m_1 m_2 \tilde{\xi}_1{}^2 \tilde{\xi}_2{}^2}{\sqrt{\tilde{\xi}_1{}^4 + \tilde{\xi}_2{}^4}} = 0.
\end{aligned} \tag{25}$$

The global flow on the total-collision manifold has been studied by Delgado Fernandez and Perez-Chavela (1991).

8 Conclusions

A particular case of the "Caledonian" symmetric four-body problem has been investigated: two pairs of equal masses are moving symmetrically in the plane on two fixed perpendicular axes. This dynamical system, referred to as the rhomboidal symmetric four-body problem, is governed by a simple Hamiltonian of two degrees of freedom. It is a good practising ground for studying phenomena such as binary collisions, escape, resonance, periodic orbits, Poincaré sections, chaos, central configurations, quadruple collision. Of particular interest are nearly periodic orbits in the vicinity of a 1:1 resonance between the two binaries; they can be stable for very long time ("stickiness"). An introduction to each of these topics has been presented in the paper.

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