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ORIGINAL ARTICLE

The two-body problem of a pseudo-rigid body and a rigid sphere

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Abstract In this paper we consider the two-body problem of a spherical pseudo-rigid body and a rigid sphere. Due to the rotational and "re-labelling" symmetries, the system is shown to possess conservation of angular momentum and circulation. We follow a reduction procedure similar to that undertaken in the study of the two-body problem of a rigid body and a sphere so that the computed reduced non-canonical Hamiltonian takes a similar form. We then consider relative equilibria and show that the notions of locally central and planar equilibria coincide. Finally, we show that Riemann's theorem on pseudo-rigid bodies has an extension to this system for planar relative equilibria.

Keywords Full two-body problem · Symmetry reduction · Pseudo-rigid/affine bodies · Riemann theorem

1 Introduction

With the recent progress of observational techniques and the increased interest in binary asteroids, the full Newtonian two-body problem with non-spherical bodies has attracted significant attention (Wang et al. 1991; Maciejewski 1995; Scheeres 2006; Bellerose and Scheeres 2008). In fact studies have indicated that about 16% of the near-Earth asteroids consists of systems of relative orbiting asteroid pairs (Margot et al. 2002). The formulation of

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this problem has been posed and studied in many references (Wang et al. 1991; Maciejewski 1995; Scheeres 2006; Bellerose and Scheeres 2008; Vereshchagin et al. 2010). The main effect is the coupling between the orbital and rotational motion. Moreover, the introduction of local coordinates leads to singularities and very complicated equations of motion. There is therefore significant gains to be obtained by exploiting the symmetries under-pinning the system to reduce and simplify the equations of motion. The reference (Wang et al. 1991) was probably the first attempt in a systematic way to make use of the rotational symmetry of the Newtonian system of a sphere and a non-spherical rigid body to reduce the equations in a coordinate-free way. Since then this reduction procedure has been used and extended by many authors, see for example Maciejewski (1995); Scheeres (2006); Bellerose and Scheeres (2008); Vereshchagin et al. (2010).

More recently, the question whether the binary systems have random shapes or instead obey some general rules of rotating fluid bodies has been addressed (see e.g. Deschamps 2008). Most of the binary asteroids discovered so far are considered rubble pile asteroids which are cohesion-less gravitational aggregates. Therefore it is believed that fluid studies can give insight into the shapes of asteroids, although this assumption of fluidity is not realistic for actual bodies. To approach this problem, the models have to be extended to account for the deformation of the body. This can be attained, at least to a first approximation, by replacing the rigid bodies with pseudo-rigid bodies whose shapes are described through the action of a general orientation-preserving, invertible matrix. Formally, this replaces the configuration space of the rigid non-spherical body $SO(3)$ by $GL^+(3) = \{Q \in \mathbb{R}^{3\times 3} | \det Q > 0\}$. The problem of gravitationally interacting pseudo-rigid body and sphere was also considered in O'Reilly and Thoma (2003); Sharma (2009). The main drive of this paper is to develop these models further.

Without the gravitational interaction from another body, such pseudo-rigid bodies have received attention in many references (Dedekind 1861; Dirichlet 1861; Jacobi 1834; Riemann 1860; Chandrasekhar 1987; Lewis and Simó 1990; Roberts and Sousa Dias 1999; Slawianowski 1974). The interest was initiated by Newton in *Principia*, where he showed that a spinning axi-symmetric self-gravitating body of fluid that is rotating slowly about the symmetry axis will be oblate. Jacobi (Jacobi 1834) extended the work of Newton, but also work of Maclaurin, to show that a self-gravitating fluid can also take on ellipsoidal shapes. The solutions of Jacobi, Maclaurin and Newton were, however, still all rigid. In a frame rotating with the body the fluid is stationary. Dirichlet and Dedekind, (Dirichlet 1861 and Dedekind 1861), respectively, opened a new direction when they found a symmetry that applied to Jacobi's solution generated a new solution in which the body is stationary in shape but the fluid particles follow elliptical paths in planes orthogonal to a principal axis of the ellipsoid. Dirichlet's paper inspired Riemann to turn his attention to the problem. In Riemann (1860) he gave a classification of the solutions of Dirichlet's equations for which the ellipsoidal shape of the body remains constant. At the heart of this classification lies what is now known as Riemann's theorem: the angular velocity and circulation (i) lie in the same principal plane and (ii) if the angular velocity is parallel to a principal axis then the circulation vector must also lie along that same principal axis.

Since then, the work of Riemann et al. has been united and extended by e.g. Chandrasekhar (1987) and Roberts and Sousa Dias (1999). Pseudo rigid bodies have also been applied to elasticity, spinning gas clouds, atomic nuclei etc. (see Roberts and Sousa Dias 1999 for references therein).

In this article we aim to put the work of O'Reilly and Thoma (2003); Sharma (2009) into the language of geometric theory of Hamiltonian systems while aligning our approach and notation with the now standard reduction procedure for the two-body problem of a rigid body and a sphere, see e.g. Wang et al. (1991); Maciejewski (1995); Vereshchagin et al. (2010). We will assume that the reference shape is spherical. By the singular value decomposition any matrix $\mathbf{Q} \in GL^+(3)$ can be decomposed as a product:

$$\mathbf{Q} = \mathbf{R}\tilde{\mathbf{A}}\mathbf{S}^T, \tag{1}$$

where $\mathbf{R}, \mathbf{S} \in SO(3)$, and $\tilde{\mathbf{A}} \in \mathcal{D} = \{\text{diag}\,(d_1, d_2, d_3) | d_1 \geq d_2 \geq d_3 > 0\}$. It therefore follows that the shape of the pseudo-rigid body at any time is ellipsoidal with principal axis half-lengths equal to the singular values of $\mathbf{Q}$, the diagonal entries of $\tilde{\mathbf{A}}$. We will also make use of a "re-labelling" symmetry, which in the classical work gives rise to the conservation of circulation. The references (O'Reilly and Thoma 2003; Sharma 2009) do not discuss this symmetry even though their systems do possess such. Compared to O'Reilly and Thoma (2003); Sharma (2009), our equations may also account for the pseudo rigid body being incompressible without the use of multipliers.

The main results and novelties of the paper are:

(i) The development of non-canonical and canonical Hamiltonian systems for the reduced problem with non-truncated potential, and the identification of the two conserved quantities: angular momentum and circulation. Compared to O'Reilly and Thoma (2003); Sharma (2009) we provide the following improvements:

  (a) the related rigid body system is a natural subsystem;
  (b) the identification of a second conserved quantity, the circulation of the pseudo-rigid body;
  (c) our results all go through essentially unchanged if the pseudo-rigid body is assumed incompressible.

(ii) In Theorem 2 we show that the notions of locally central equilibria and planar equilibria coincide. The proof is only based upon the potential being symmetric, and since the problem of a rigid body and a sphere is a natural subsystem of our equations, this result therefore also extends to this case, where to the authors' knowledge it has never been proven.

(iii) In Theorem 4 we present an extension of Riemann's theorem to this present two body problem. To the authors' knowledge this, too, has not been addressed previously, in particular not in O'Reilly and Thoma (2003); Sharma (2009). Proposition 3 includes a very interesting result: All "S-type relative equilibria" (Chandrasekhar 1987; Roberts and Sousa Dias 1999), where the angular velocities are both directed along the same principal axis, are "in general" planar. The meaning of "in general" are made precise in Proposition 3.

Even though the gravitational problem was our motivation to look at the problem, in fact these results all apply to general two-body problems of the given form. Theorem 5 is the only exception which makes use of the explicit form of the gravitational potential to exclude some exceptions to the validity of Riemann's theorem in the more general version (Theorem 4). The hypotheses are

H1 The pseudo-rigid body is spherically symmetric;

H2 The actual shape $\mathcal{B}$ of the pseudo-rigid body is not spheroidal ($\mathbf{Q}$ has distinct singular values d_1, d_2 and d_3);

and for our investigation of relative equilibria we shall assume the following:

H3 The gradient of the potential U with respect to the relative position vector is nowhere zero.

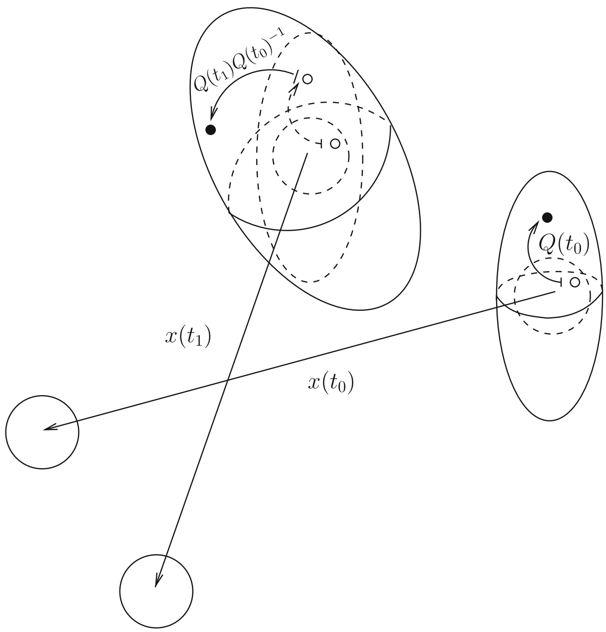
This last condition excludes relative equilibria where the relative position vector is fixed in inertial space. It holds true for the gravitational problem when the rigid sphere is outside the pseudo-rigid body. We will introduce the hypotheses chronologically in the manuscript. Whereas all of our results in our section on relative equilibria will rely on H1 and H2, some results do not require H3, for example the general version of Riemann's theorem (Theorem 4). The hypotheses used will therefore be stated explicitly in each of our results.

The article is structured as follows: In Sect. 2 we introduce the model and the relevant potential energies and then derive the unreduced equations of motion. In the following section, we show that the system possesses two symmetries and, upon making explicit use of the decomposition (1), we reduce the equations to the appropriate quotient space. In Sect. 4 we finally investigate the relative equilibria of the system.

2 The model

We consider a rigid sphere and a deformable body with masses m_1 and m_2 respectively. See Fig. 1. We model this system on the configuration space $\mathbb{R}^3 \times \mathbb{R}^3 \times GL^+(3)$. The former two spaces describes the centres of masses of the sphere and the pseudo-rigid body while the latter describes the deformation of the pseudo-rigid body $\mathcal{B}$ with respect to its centre of mass. Applying a $GL^+(3)$ matrix to the pseudo-rigid body preserves the centre of mass. We assume that the potential only depends on the the relative position and the configuration of the pseudo-rigid body so that the system possesses translational symmetry, and the centre of mass of the system moves with constant velocity. This is the case of the gravitational problem. We can reduce the system by introducing a centre of mass of the system and relative coordinates. Let $\mathbf{x}$ be the relative position of the two centres of masses and let $\mathbf{Q} \in GL^+(3)$. Then, upon proper scaling, see e.g. Maciejewski (1995), the kinetic energy of the system is:

Fig. 1 The two-body problem of a sphere and a pseudo-rigid body. Here $\mathbf{x}$ denotes the relative position. The matrix $\mathbf{Q} \in GL^+(3)$ describes the shapes of the pseudo-rigid body so that any point, say, $\mathbf{w}$ of the pseudo-rigid body in its reference shape is mapped to a new point $\mathbf{Qw}$ in the deformed pseudo-rigid body. We assume that the reference shape is spherical and it therefore follows that the shape of the pseudo-rigid body is ellipsoidal at all times



$$K(\dot{\mathbf{x}}, \dot{\mathbf{Q}}) = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \langle\langle \dot{\mathbf{Q}}, \dot{\mathbf{Q}}\mathbb{J}\rangle\rangle,$$

where $|\mathbf{z}| = \sqrt{\langle \mathbf{z}, \mathbf{z}\rangle}$ is the Euclidean distance,

$$\langle\langle \mathbf{V}, \mathbf{W}\rangle\rangle = \tfrac{1}{2}\mathrm{tr}(\mathbf{V}\mathbf{W}^T) \tag{2}$$

is the inner product[1] on the tangent spaces of $GL^+(3)$, and $\mathbb{J} \in \mathrm{diag}^+(3)$ is the moment coefficient of inertia of the reference shape, see e.g. Holm et al. (2009). It is without loss of generality to assume that spherical reference shape $\mathcal{B}_0$ satisfy $\mathbb{J} = \mathbf{I}$. Here $\mathbf{I}$ denotes the identity. Indeed, we can just replace $\mathbf{Q}$ by $\mathbf{Q}\mathbb{J}^{-1/2}$ to achieve this.

2.1 Potential

The potential of the system naturally splits into two parts $\tilde{U} = U + U_{\text{pres}}$. The first part U may for example include a term U_{grav} due to gravitational interaction between the sphere and the pseudo-rigid body. This potential is simply the Newtonian inter-particle gravitational interaction integrated up over the pseudo-rigid body $\mathcal{B}$:

$$U_{\text{grav}} = -\int_{\mathcal{B}} \frac{\mu\rho(\mathbf{z})}{|\mathbf{x}+\mathbf{z}|} d\mathbf{z}.$$

Here μ is the universal gravitational constant, $\rho = \rho(\mathbf{z})$ is the density and $\mathbf{z}$ is the position of particles in the body relative to its centre of mass. There are simplifications available for ellipsoids (see Bellerose and Scheeres 2008), but we do not need them here. The potential U may also include a term U_{self}, the potential due to self-gravitating forces on the pseudo-rigid body. The expression for U_{self} for a homogeneous ellipsoid is given by Dirichlet's formula:

$$U_{\text{self}} = \frac{1}{2}\int_{\mathcal{B}} \mu\rho(\mathbf{z})^2 \left(\int_0^\infty \Phi(u, \mathbf{z}) du\right) d\mathbf{z},$$

$$\Phi(u, \mathbf{z}) = \frac{d_1 d_2 d_3}{\sqrt{\left(d_1^2 + u\right)\left(d_2^2 + u\right)\left(d_3^2 + u\right)}}\left(\sum_{i=1}^{3} \frac{z_i^2}{d_i^2 + u} - 1\right),$$

where d_i are the half-lengths of the principal axis, see O'Reilly and Thoma (2003). Finally, U may include a term U_{elas} due to possible elastic forces on the body and its surface. Such potentials are considered and described in Lewis and Simó (1990).

The second term U_{pres}, given by

$$U_{\text{pres}} = -4\Xi\langle\langle \mathbf{Q}^{-T}\det\mathbf{Q}, \dot{\mathbf{Q}}\rangle\rangle, \tag{3}$$

in the total potential $\tilde{U}$ has to be included if the pseudo-rigid body models an incompressible ideal fluid with $\det\mathbf{Q} \equiv 1$. Equation (3) is based on methods developed in Maddocks and Pego (1995) for un-constrained Hamiltonians of ideal fluids. At this stage, Ξ is a Lagrange multiplier. We will, however, later see that it can be directly related to the fluid pressure, which vanishes on the surface $\partial\mathcal{B}_0 = S^2$. We will include this term in our calculations henceforth. If the pseudo-rigid body is not incompressible, then one can simply set $\Xi \equiv 0$ in the following to recover the governing equations. Notice that

$$\frac{\delta U_{\text{pres}}}{\delta\dot{\mathbf{Q}}} = -4\Xi\mathbf{Q}^{-T}\det\mathbf{Q}.$$

[1] The reason for including the factor of $\frac{1}{2}$ will become apparent later.

Compared to O'Reilly and Thoma (2003) we do not require the body to be homogeneous. Instead we restrict attention to the larger class of *spherically symmetric* pseudo-rigid bodies:

Definition 1 We call a pseudo-rigid body *spherically symmetric* if *in its reference spherical shape* the potential U is rotationally invariant:

$$U(\mathbf{gx}, \mathbf{gQh}) = U(\mathbf{x}, \mathbf{Q}) \quad \text{for every} \quad (\mathbf{g}, \mathbf{h}) \in SO(3) \times SO(3).$$

A pseudo-rigid body is only spherically symmetric if material parameters, such as density and elasticity, in *its reference spherical shape* only depend upon the distance from the centre of body. We add the hypothesis:

H1 The pseudo-rigid body is spherically symmetric.

Note that the gravitational problem considered in O'Reilly and Thoma (2003) and Sharma (2009) with $U = U_{\text{grav}} + U_{\text{self}}$ and $\rho = \rho(|\mathbf{z}|)$ satisfy this hypothesis. We will add a second H2 and third H3 hypothesis later on. From the kinetic energy we define the following Legendre transformations:[2]

$$\begin{aligned}\langle Fl_{\mathbf{x}}(\dot{\mathbf{x}}), \mathbf{v}\rangle + \langle\langle Fl_{\mathbf{Q}}(\dot{\mathbf{Q}}), \mathbf{V}\rangle\rangle &= dK(\dot{\mathbf{x}}, \dot{\mathbf{Q}})(\mathbf{v}, \mathbf{V}) - dU_{\text{pres}}(\dot{\mathbf{Q}})(\mathbf{V}) \\ &= \langle \dot{\mathbf{x}}, \mathbf{v}\rangle + \langle\langle 2\dot{\mathbf{Q}} + 4\Xi\mathbf{Q}^{-T}\det\mathbf{Q}, \mathbf{V}\rangle\rangle,\end{aligned}$$

for every $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{V} \in T_{\mathbf{Q}}GL^+(3)$, so that

$$\begin{aligned}\mathbf{y} &= \dot{\mathbf{x}}, \\ \mathbf{P} &= 2\dot{\mathbf{Q}} + 4\Xi\mathbf{Q}^{-T}\det\mathbf{Q},\end{aligned} \tag{4}$$

are the momenta canonically associated with $\mathbf{x}$ and $\mathbf{Q}$, respectively (Marsden and Ratiu 1994). The Hamiltonian is the function on the phase space $\mathcal{P} \equiv T^*\left(\mathbb{R}^3 \times GL^+(3)\right)$ defined by:

$$\begin{aligned}H(\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}) &= \langle \mathbf{y}, \dot{\mathbf{x}}\rangle + \langle\langle \mathbf{P}, \dot{\mathbf{Q}}\rangle\rangle - K(\dot{\mathbf{x}}, \dot{\mathbf{Q}}) + \tilde{U} \\ &= \frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle + \langle\langle \dot{\mathbf{Q}}, \dot{\mathbf{Q}}\rangle\rangle + U(\mathbf{x}, \mathbf{Q}),\end{aligned} \tag{5}$$

equipped with canonical symplectic structure associated with the Poisson bracket:

$$\{f, g\}(\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}) = \langle \partial_{\mathbf{x}} f, \partial_{\mathbf{y}} g\rangle - \langle \partial_{\mathbf{x}} g, \partial_{\mathbf{y}} f\rangle + \left\langle\left\langle \frac{\delta f}{\delta \mathbf{Q}}, \frac{\delta g}{\delta \mathbf{P}}\right\rangle\right\rangle - \left\langle\left\langle \frac{\delta g}{\delta \mathbf{Q}}, \frac{\delta f}{\delta \mathbf{P}}\right\rangle\right\rangle,$$

for $f, g \in C^\infty(\mathcal{P})$. Here $U = \tilde{U} - U_{\text{pres}}$ is the "true" potential. When the pseudo rigid body is incompressible, Hamilton's equations define the true equations of motion only when letting Ξ be the unique quantity satisfying:

$$\partial_\Xi H = 0. \tag{6}$$

The uniqueness follows from $\frac{\partial^2 H}{\partial \Xi^2} > 0$. Using (4) this gives

$$\operatorname{tr}(\mathbf{Q}^{-T}\dot{\mathbf{Q}})\det\mathbf{Q} = 0,$$

or in the more transparent form $\nabla_{\mathbf{q}} \cdot \dot{\mathbf{q}} = 0$ with $\mathbf{q} = \mathbf{QX}$. This is the incompressibility condition. Defining Ξ this way, is what allow us to ignore the dependency of Ξ on the other variables. Indeed if we denote the right hand side of (5) by $\tilde{H} = \tilde{H}(\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}, \Xi)$, then through (6) we have $\frac{\delta \tilde{H}}{\delta \mathbf{z}} = \frac{\delta H}{\delta \mathbf{z}}$ for $\mathbf{z} = \mathbf{x}, \mathbf{y}, \mathbf{Q}$ and $\mathbf{z} = \mathbf{P}$.

[2] We identify the dual $T^*_{\mathbf{Q}}GL^+(3)$ with $T_{\mathbf{Q}}GL^+(3)$ via the inner product (2).

The differential equations for $\mathbf{Q}$ and $\mathbf{P}$ are (4) and

$$\dot{\mathbf{P}} = -\frac{\delta U}{\delta \mathbf{Q}} - 4\Xi \mathbf{Q}^{-T}\dot{\mathbf{Q}}^T\mathbf{Q}^{-T}\det \mathbf{Q}.$$

Therefore

$$\begin{aligned}\ddot{\mathbf{Q}} &= \frac{1}{2}\partial_t(\mathbf{P} - 4\Xi\mathbf{Q}^{-T}\det\mathbf{Q})\\ &= \frac{1}{2}\Big(-\frac{\delta U}{\delta \mathbf{Q}} - 4\Xi\mathbf{Q}^{-T}\dot{\mathbf{Q}}^T\mathbf{Q}^{-T}\det\mathbf{Q} - 4\dot{\Xi}\mathbf{Q}^{-T}\det\mathbf{Q}\\ &\qquad + 4\Xi\mathbf{Q}^{-T}\dot{\mathbf{Q}}^T\mathbf{Q}^{-T}\det\mathbf{Q}\Big)\\ &= -\frac{1}{2}\frac{\delta U}{\delta \mathbf{Q}} - 2\dot{\Xi}\mathbf{Q}^{-T}. \end{aligned} \tag{7}$$

Here we have used that $\det \mathbf{Q} \equiv 1$. The factor $\frac{1}{2}$ is the expense we have to pay for introducing the factor $\frac{1}{2}$ in the definition of the trace inner product. In terms of the gradient $\left(\frac{\delta U}{\delta \mathbf{Q}}\right)_*$ of U with respect to the inner product $\langle\langle \cdot, \cdot\rangle\rangle_* = 2\langle\langle \cdot, \cdot\rangle\rangle$ Eq. (7) takes the canonical form:

$$\ddot{\mathbf{Q}} = -\left(\frac{\delta U}{\delta \mathbf{Q}}\right)_* - 2\dot{\Xi}\mathbf{Q}^{-T}.$$

By for example comparing with the equations in Chandrasekhar (1987) section 27, we realise that $\dot{\Xi}$ is the pressure at the centre of mass of the pseudo rigid body.

The system described by (5) is not very convenient to work with. First of all Hamilton's equations will include matrix equations. Furthermore, the symmetries of the system have not been used to reduce the dimension of the system. We will address these issues in the following.

3 Symmetry and reduction

In this section we shall make use of symmetries to reduce the system. We shall throughout the rest of the paper make use of the hat-map which is defined by

$$\widehat{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \in so(3),$$

for every $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$. This map defines an isomorphism between the Lie-algebra

$$so(3) = \{\text{space of skew-symmetric matrices in } \mathbb{R}^{3\times 3} \text{ equipped with the commutator algebra } [\cdot,\cdot]\},$$

and $(\mathbb{R}^3, \cdot \times \cdot)$ but also between $(so(3), \langle\langle\cdot,\cdot\rangle\rangle)$ and $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ as inner-product spaces. The latter property is the reason for the factor of $\frac{1}{2}$ introduced in the definition of the trace inner product. Finally, it also has the following properties (Roberts and Sousa Dias 1999; Maciejewski 1995):

$$\widehat{\mathbf{z}} = -\widehat{\mathbf{z}}^T,\ \ \widehat{\mathbf{z}}\widehat{\mathbf{w}} = \mathbf{w}\mathbf{z}^T - \mathbf{z}^T\mathbf{w},\ \ \widehat{\mathbf{z}\times\mathbf{w}} = \widehat{\mathbf{z}}\widehat{\mathbf{w}} - \widehat{\mathbf{w}}\widehat{\mathbf{z}},\ \ \mathbf{g}(\mathbf{z}\times\mathbf{w}) = (\mathbf{g}\mathbf{z})\times(\mathbf{g}\mathbf{w}),$$

and

$$\widehat{\mathbf{z}}\mathbf{w} = \mathbf{z}\times\mathbf{w}, \tag{8}$$

$$\widehat{\mathbf{g}\mathbf{z}} = \mathbf{g}\widehat{\mathbf{z}}\mathbf{g}^T, \tag{9}$$

for any vectors $\mathbf{z},\ \mathbf{w} \in \mathbb{R}^3$ and $\mathbf{g} \in SO(3)$. We now define two actions of $SO(3)$ on $\mathcal{P}$ by:

$$l_{\mathbf{g}} : \mathcal{P} \ni (\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}) \mapsto (\mathbf{gx}, \mathbf{gy}, \mathbf{gQ}, \mathbf{gP}) \in \mathcal{P},$$
$$r_{\mathbf{g}} \ \ : \mathcal{P} \ni (\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}) \mapsto (\mathbf{x}, \mathbf{y}, \mathbf{Qg}, \mathbf{Pg}) \in \mathcal{P},$$

for $\mathbf{g} \in SO(3)$. We then have the following:

Proposition 1 *The Hamiltonian system* (5)*, subject to the hypothesis H1, is invariant under* $l_{\mathbf{g}}$ *and* $r_{\mathbf{g}}$*, i.e.* $H(r_{\mathbf{h}} \circ l_{\mathbf{g}}(\mathbf{z})) = H(\mathbf{z}),\ \forall\, \mathbf{z} \in \mathcal{P}$ *and* $\forall\, (\mathbf{g}, \mathbf{h}) \in SO(3)^2$.

Proof If $\mathbf{Q} \mapsto \mathbf{gQh}$ and $\mathbf{P} \mapsto \mathbf{gPh}$ then $\dot{\mathbf{Q}} \mapsto \mathbf{g}\dot{\mathbf{Q}}\mathbf{h}$. Therefore

$$H(r_{\mathbf{h}} \circ l_{\mathbf{g}}(\mathbf{z})) = \frac{1}{2}\langle \mathbf{gy}, \mathbf{gy}\rangle + \langle\langle \mathbf{g}\dot{\mathbf{Q}}\mathbf{h}, \mathbf{g}\dot{\mathbf{Q}}\mathbf{h}\rangle\rangle + U(\mathbf{gx}, \mathbf{gQh}).$$

For the invariance of the first two terms notice

$$\begin{aligned}
\langle \mathbf{gy}, \mathbf{gy}\rangle &= (\mathbf{gy})^T \mathbf{gy} = \mathbf{y}^T \mathbf{g}^T \mathbf{gy} = \langle \mathbf{y}, \mathbf{y}\rangle,\\
\langle\langle \mathbf{g}\dot{\mathbf{Q}}\mathbf{h}, \mathbf{g}\dot{\mathbf{Q}}\mathbf{h}\rangle\rangle &= \frac{1}{2}\mathrm{tr}\left(\mathbf{g}\dot{\mathbf{Q}}\mathbf{h}\left(\mathbf{g}\dot{\mathbf{Q}}\mathbf{h}\right)^T\right)\\
&= \frac{1}{2}\mathrm{tr}\left(\mathbf{g}\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\mathbf{g}^T\right) = \frac{1}{2}\mathrm{tr}\left(\mathbf{g}^T\mathbf{g}\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right)\\
&= \langle\langle \dot{\mathbf{Q}}, \dot{\mathbf{Q}}\rangle\rangle.
\end{aligned}$$

Here we have in the second last equality used the conjugation symmetry of the trace inner product:

$$\mathrm{tr}(\mathbf{CD}^T) = \mathrm{tr}(\mathbf{D}^T\mathbf{C}),$$

for all matrices $\mathbf{C},\ \mathbf{D} \in \mathbb{R}^{3\times 3}$. We therefore obtain

$$H(r_{\mathbf{h}} \circ l_{\mathbf{g}}(\mathbf{z})) = \frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle + \langle\langle \dot{\mathbf{Q}}, \dot{\mathbf{Q}}\rangle\rangle + U(\mathbf{gx}, \mathbf{gQh}).$$

The invariance of the potential

$$U(\mathbf{gx}, \mathbf{gQh}) = U(\mathbf{x}, \mathbf{Q}),$$

follows immediately from the hypothesis H1 and so

$$H(r_{\mathbf{h}} \circ l_{\mathbf{g}}(\mathbf{z})) = H(\mathbf{z}),$$

completing the proof.

The first part regarding $l_{\mathbf{g}}$ is precisely what is exploited in the work in the two-body problem of a rigid body and a sphere. See for example Wang et al. (1991); Maciejewski (1995); Scheeres (2006); Bellerose and Scheeres (2008); Vereshchagin et al. (2010). By Noether's theorem (Holm et al. 2009) the symmetries $l_{\mathbf{g}}$ and $r_{\mathbf{g}}$ generate conserved quantities $\widehat{\mathbf{J}}_l$ and $\widehat{\mathbf{J}}_r$, respectively. Since the symmetries are due to the left and right actions of $SO(3)$ the conserved quantities are maps from $\mathcal{P}$ to the dual $so(3)^*$ defined by:

$$\langle\langle \widehat{\mathbf{J}}_l(\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathbf{P}), \widehat{\mathbf{\Sigma}}\rangle\rangle = \langle \mathbf{y}, \mathbf{\Sigma} \times \mathbf{x}\rangle + \langle\langle \mathbf{P}, \widehat{\mathbf{\Sigma}}\mathbf{Q}\rangle\rangle,$$
$$\langle\langle \widehat{\mathbf{J}}_r(\mathbf{Q}, \mathbf{P}), \widehat{\mathbf{\Sigma}}\rangle\rangle = \langle\langle \mathbf{P}, \mathbf{Q}\widehat{\mathbf{\Sigma}}\rangle\rangle,$$

for every $\widehat{\mathbf{\Sigma}} \in so(3)$ (see, for example, Holm et al. 2009, Chapter 8). Therefore we have:

$$\langle\langle \widehat{\mathbf{J}}_l(\mathbf{Q}, \mathbf{P}), \widehat{\mathbf{\Sigma}}\rangle\rangle = \langle \mathbf{x} \times \mathbf{y}, \mathbf{\Sigma}\rangle + \langle\langle \mathbf{PQ}^T, \widehat{\mathbf{\Sigma}}\rangle\rangle = \langle\langle \widehat{\mathbf{x} \times \mathbf{y}} + \mathbf{PQ}^T, \widehat{\mathbf{\Sigma}}\rangle\rangle,$$
$$\langle\langle \widehat{\mathbf{J}}_r(\mathbf{Q}, \mathbf{P}), \widehat{\mathbf{\Sigma}}\rangle\rangle = \langle\langle \mathbf{Q}^T\mathbf{P}, \widehat{\mathbf{\Sigma}}\rangle\rangle,$$

so that upon identifying $so(3)^*$ with $so(3)$ through the inner product and taking the skew-symmetric part to ensure that $\widehat{\mathbf{J}}_l,\ \widehat{\mathbf{J}}_r \in so(3)$:

$$\widehat{\mathbf{J}}_l(\mathbf{x},\mathbf{y},\mathbf{Q},\mathbf{P}) = \widehat{\mathbf{x}\times\mathbf{y}} + \frac{1}{2}\left(\mathbf{P}\mathbf{Q}^T - \mathbf{Q}\mathbf{P}^T\right), \tag{10}$$

$$\widehat{\mathbf{J}}_r(\mathbf{Q},\mathbf{P}) = \frac{1}{2}\left(\mathbf{Q}^T\mathbf{P} - \mathbf{P}^T\mathbf{Q}\right). \tag{11}$$

Here $\widehat{\mathbf{J}}_l$ and $\widehat{\mathbf{J}}_r$ are the total angular momentum and the circulation, respectively. See also Roberts and Sousa Dias (1999). The momentum maps are left and right equivariant to the action of $SO(3)$ in the following sense:

$$\widehat{\mathbf{J}}_l(\mathbf{gx},\mathbf{gy},\mathbf{gQ},\mathbf{gP}) = \mathbf{g}\widehat{\mathbf{J}}_l(\mathbf{x},\mathbf{y},\mathbf{Q},\mathbf{P})\mathbf{g}^T = \widehat{\mathbf{g}\mathbf{J}_l}, \tag{12}$$

$$\widehat{\mathbf{J}}_r(\mathbf{Qh},\mathbf{Ph}) = \mathbf{h}^T\widehat{\mathbf{J}}_r(\mathbf{Q},\mathbf{P})\mathbf{h} = \widehat{\mathbf{h}^T\mathbf{J}_r}, \tag{13}$$

for every $(\mathbf{g},\mathbf{h}) \in SO(3)^2$. Here, we have used (9) in the last equality. Furthermore, the momentum maps are right and left invariant in the following sense:

$$\widehat{\mathbf{J}}_l(\mathbf{x},\mathbf{y},\mathbf{Qh},\mathbf{Ph}) = \widehat{\mathbf{J}}_l(\mathbf{x},\mathbf{y},\mathbf{Q},\mathbf{P}),$$
$$\widehat{\mathbf{J}}_r(\mathbf{gQ},\mathbf{gP}) = \widehat{\mathbf{J}}_r(\mathbf{Q},\mathbf{P}).$$

We shall make use of these facts in the following lemma which will be useful later on when proving the extension of Riemann's theorem. At a relative equilibrium the trajectory is an orbit of a one-parameter subgroup of the symmetry group $SO(3)^2$:

$$\mathbf{x}(t) = \exp(\widehat{\boldsymbol{\Omega}}t)\mathbf{x}_0, \quad \mathbf{Q}(t) = \exp(\widehat{\boldsymbol{\Omega}}t)\tilde{\mathbf{A}}_0\exp(-\widehat{\boldsymbol{\Lambda}}t),$$

where $\widehat{\boldsymbol{\Omega}}$ and $\widehat{\boldsymbol{\Lambda}}$ are skew-symmetric matrices and $\tilde{\mathbf{A}}_0$ is a constant diagonal matrix. $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ are the angular velocities associated with the rigid body and orbital rotation and the internal rotation of particles of the body, respectively. We have the following fundamental property:

Lemma 1 *At relative equilibria:*

$$\mathbf{J}_l \times \boldsymbol{\Omega} = \mathbf{0}, \quad \mathbf{J}_r \times \boldsymbol{\Lambda} = \mathbf{0}.$$

Proof The proof is straightforward (see also Roberts and Sousa Dias 1999). It relies on the conservation of the momenta $\mathbf{J}_l$ and $\mathbf{J}_r$ and their equivariance with respect to the left and right action, see (12) and (13). Indeed, the conservation of $\mathbf{J}_l$ and $\mathbf{J}_r$ imply that:

$$\mathbf{J}_l(\mathbf{x}(t),\mathbf{y}(t),\mathbf{Q}(t),\mathbf{P}(t)) = \mathbf{J}_l(\mathbf{x}(0),\mathbf{y}(0),\mathbf{Q}(0),\mathbf{P}(0)),$$
$$\mathbf{J}_r(\mathbf{Q}(t),\mathbf{P}(t)) = \mathbf{J}_r(\mathbf{Q}(0),\mathbf{P}(0)). \tag{14}$$

By setting $\mathbf{g} = \exp(\widehat{\boldsymbol{\Omega}}t)$ and $\mathbf{h} = \exp(-\widehat{\boldsymbol{\Lambda}}t)$ in (12) and (13) we have:

$$\mathbf{J}_l(\mathbf{x}(0),\mathbf{y}(0),\mathbf{Q}(0),\mathbf{P}(0)) = \exp(\widehat{\boldsymbol{\Omega}}t)\mathbf{J}_l(\mathbf{x}(0),\mathbf{y}(0),\mathbf{Q}(0),\mathbf{P}(0)),$$
$$\mathbf{J}_r(\mathbf{Q}(0),\mathbf{P}(0)) = \exp(\widehat{\boldsymbol{\Lambda}}t)\mathbf{J}_r(\mathbf{Q}(0),\mathbf{P}(0)). \tag{15}$$

Differentiating with respect to t at $t = 0$ gives:

$$\widehat{\boldsymbol{\Omega}}\mathbf{J}_l = \mathbf{0}, \quad \widehat{\boldsymbol{\Lambda}}\mathbf{J}_r = \mathbf{0},$$

or simply by (8):

$$\boldsymbol{\Omega} \times \mathbf{J}_l = \mathbf{0}, \quad \boldsymbol{\Lambda} \times \mathbf{J}_r = \mathbf{0}.$$

3.1 Decomposition

In our reduction procedure we will make use of the singular value decomposition $\mathbf{Q} = \mathbf{R}\tilde{\mathbf{A}}\mathbf{S}^T$ (1). This decomposition is not unique. First of all, if the singular values are not distinct, then we may re-arrange the corresponding identical diagonal components of $\tilde{\mathbf{A}}$ without changing $\mathbf{Q}$. We will exclude this scenario by including a second hypothesis:

H2 The actual shape $\mathcal{B}$ of the pseudo-rigid body is not spheroidal ($\mathbf{Q}$ has distinct singular values d_1,d_2 and d_3).

We therefore replace $\{\tilde{\mathbf{A}}\} = \mathcal{D}$ by $\mathcal{D}^\circ = \{\mathrm{diag}\,(d_1, d_2, d_3)|d_1 > d_2 > d_3 > 0\}$ in (1). Now $\tilde{\mathbf{A}}$ is unique. However, the decomposition is still not unique. Let $\mathbf{Q} = \mathbf{R}\tilde{\mathbf{A}}\mathbf{S}^T = \breve{\mathbf{R}}\tilde{\mathbf{A}}\breve{\mathbf{S}}^T$. Then

$$\tilde{\mathbf{A}} = \mathbf{R}^T\breve{\mathbf{R}}\tilde{\mathbf{A}}(\mathbf{S}^T\breve{\mathbf{S}})^T$$

so

$$\breve{\mathbf{R}} = \mathbf{R}\mathbf{P}_i, \quad \breve{\mathbf{S}} = \mathbf{S}\mathbf{P}_i, \tag{16}$$

for some $i = 0, \ldots, 3$ where

$$\mathbf{P}_0 = \mathbf{I},\ \mathbf{P}_1 = \mathrm{diag}\,(1, -1, -1),\ \mathbf{P}_2 = \mathrm{diag}\,(-1, 1, -1),\ \mathbf{P}_3 = \mathrm{diag}\,(-1, -1, 1); \tag{17}$$

the elements of $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The mapping $(\mathbf{R}, \tilde{\mathbf{A}}, \mathbf{S}) \in SO(3) \times \mathcal{D}^\circ \times SO(3) \mapsto \mathbf{R}\tilde{\mathbf{A}}\mathbf{S}^T \in GL^+(3)$ is therefore four-to-one. By considering the determinant of the Jacobian of the mapping, one can show that it is a submersion. See also Fasso and Lewis (2001).[3] The decomposition therefore lifts the system to a (branched) cover of the original configuration space and $GL^+(3)$ is the quotient of $SO(3)\times\mathcal{D}^\circ\times SO(3)$, by the action $(\mathbf{R}, \mathbf{S}) \mapsto (\mathbf{R}\mathbf{P}_i, \mathbf{S}\mathbf{P}_i)$ of the dihedral symmetry group D_2. This action lifts to an action on the cotangent bundle. The actions on both the *covering* configuration space and its cotangent bundle are free, and $T^*GL^+(3)$ can therefore be identified with the quotient of $T^*(SO(3) \times \mathcal{D}^\circ \times SO(3))$ by this free action. We will denote the covering phase space by $\mathcal{P}_{\mathrm{cov}} = T^*(\mathbb{R}^3 \times SO(3) \times \mathcal{D}^\circ \times SO(3))$ so that $\mathcal{P} = \mathcal{P}_{\mathrm{cov}}/D_2$ with the action defined above.

The reference (Holm et al. 2009) also gives the following easy interpretation:

- $\mathbf{S}^T$ rotates the coordinates in the reference frame.
- $\tilde{\mathbf{A}}$ stretches the body along the instantaneous principal axis of $\mathbf{S}^T(\mathcal{B}_0)$.
- $\mathbf{R}$ rotates the deformed body.

See also Fig. 2. Upon replacing $GL^+(3)$ as configuration space by the product $SO(3) \times \mathrm{diag}^+(3) \times SO(3)$, we obtain a new expression for the kinetic energy:

$$K(\dot{\mathbf{x}}, (\mathbf{R}\tilde{\mathbf{A}}\mathbf{S})^{\cdot}) = \frac{1}{2}\langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle + \langle\langle\dot{\mathbf{R}}, \dot{\mathbf{R}}\tilde{\mathbf{A}}^2\rangle\rangle + \langle\langle\dot{\tilde{\mathbf{A}}}, \dot{\tilde{\mathbf{A}}}\rangle\rangle$$
$$+2\langle\langle\mathbf{R}\tilde{\mathbf{A}}\dot{\mathbf{S}}^T, \dot{\mathbf{R}}\tilde{\mathbf{A}}\mathbf{S}^T\rangle\rangle + \langle\langle\dot{\mathbf{S}}, \dot{\mathbf{S}}\tilde{\mathbf{A}}^2\rangle\rangle,$$

and, through straightforward calculations, momenta conjugate to $\mathbf{R}$, $\tilde{\mathbf{A}}$ and $\mathbf{S}$:

$$\mathbf{R}\widehat{\mathbf{M}} = \dot{\mathbf{R}}\tilde{\mathbf{A}}^2 - \mathbf{R}\tilde{\mathbf{A}}^2\left(\mathbf{R}^T\dot{\mathbf{R}}\right)^T + 2\mathbf{R}\tilde{\mathbf{A}}\dot{\mathbf{S}}^T\mathbf{S}\tilde{\mathbf{A}}, \tag{18}$$

$$\tilde{\mathbf{B}} = 2\dot{\tilde{\mathbf{A}}} + 4\Xi\tilde{\mathbf{A}}^{-T}\det\tilde{\mathbf{A}}, \tag{19}$$

$$\mathbf{S}\widehat{\mathbf{N}} = \dot{\mathbf{S}}\tilde{\mathbf{A}}^2 - \mathbf{S}\tilde{\mathbf{A}}^2\left(\mathbf{S}^T\dot{\mathbf{S}}\right)^T + 2\mathbf{S}\tilde{\mathbf{A}}\dot{\mathbf{R}}^T\mathbf{R}\tilde{\mathbf{A}}, \tag{20}$$

[3] In Fasso and Lewis (2001) the authors have, however, mistaken the group D_2 with $\mathbb{Z}_4$. Furthermore, they have not written (16) correctly.

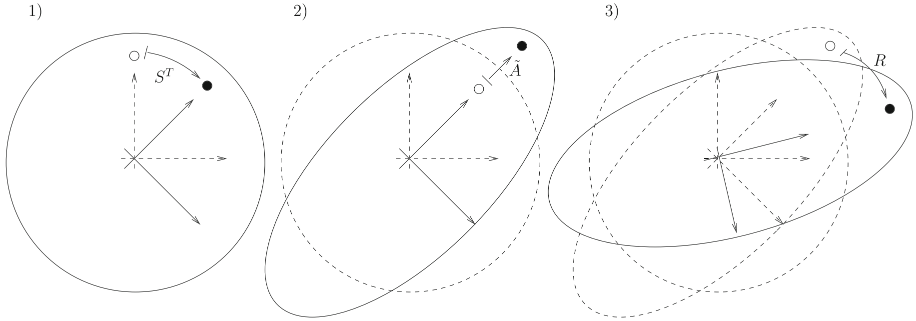


Fig. 2 The action of $\mathbf{Q}$ can through singular value decomposition be decomposed into the following steps: (1) a rotation $\mathbf{S}^T$ of the reference sphere; (2) a deformation $\tilde{\mathbf{A}}$ along the instantaneous principal axis; (3) a rotation $\mathbf{R}$ of the ellipsoid

respectively. Here we have again taken skew-symmetric parts to ensure that $\widehat{\mathbf{M}}$ and $\widehat{\mathbf{N}}$ belong to $so(3)$. For $\tilde{\mathbf{B}}$ we have also made use of (3). The Hamiltonian then takes the following form:

$$H(\mathbf{z}) = \langle \mathbf{y}, \dot{\mathbf{x}} \rangle + \langle\langle \mathbf{R}\widehat{\mathbf{M}}, \dot{\mathbf{R}} \rangle\rangle + \langle\langle \dot{\tilde{\mathbf{A}}}, \dot{\tilde{\mathbf{A}}} \rangle\rangle + \langle\langle \mathbf{S}\widehat{\mathbf{N}}, \dot{\mathbf{S}} \rangle\rangle - K(\dot{\mathbf{x}}, (\mathbf{R}\tilde{\mathbf{A}}S)^{\cdot}) \\ + U(\mathbf{x}, \mathbf{R}\tilde{\mathbf{A}}\mathbf{S}^T) \tag{21}$$

equipped with the canonical symplectic structure associated with the Poisson bracket:

$$\{f, g\}(\mathbf{z}) = \langle \partial_{\mathbf{x}} f, \partial_{\mathbf{y}} g \rangle - \langle \partial_{\mathbf{x}} g, \partial_{\mathbf{y}} f \rangle + \left\langle\left\langle \frac{\delta f}{\delta \mathbf{R}}, \frac{\delta g}{\delta \mathbf{R}\widehat{\mathbf{M}}} \right\rangle\right\rangle - \left\langle\left\langle \frac{\delta g}{\delta \mathbf{R}}, \frac{\delta f}{\delta \mathbf{R}\widehat{\mathbf{M}}} \right\rangle\right\rangle \\ + \left\langle\left\langle \frac{\delta f}{\delta \tilde{\mathbf{A}}}, \frac{\delta g}{\delta \tilde{\mathbf{B}}} \right\rangle\right\rangle - \left\langle\left\langle \frac{\delta g}{\delta \tilde{\mathbf{A}}}, \frac{\delta f}{\delta \tilde{\mathbf{B}}} \right\rangle\right\rangle + \left\langle\left\langle \frac{\delta f}{\delta \mathbf{S}}, \frac{\delta g}{\delta \mathbf{S}\widehat{\mathbf{N}}} \right\rangle\right\rangle - \left\langle\left\langle \frac{\delta g}{\delta \mathbf{S}}, \frac{\delta f}{\delta \mathbf{S}\widehat{\mathbf{N}}} \right\rangle\right\rangle,$$

for $f, g \in C^\infty(\mathcal{P}_{\text{cov}})$, where $\mathbf{z} = (\mathbf{x}, \mathbf{y}, \mathbf{R}, \mathbf{R}\widehat{\mathbf{M}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{S}, \mathbf{S}\widehat{\mathbf{N}})$. Here $\dot{\tilde{\mathbf{A}}}$ is given by (19). The action $r_{\mathbf{h}} \circ l_{\mathbf{g}}$ is through the decomposition mapped to the action

$$\Phi_{\mathbf{gh}}(\mathbf{z}) = (\mathbf{gx}, \mathbf{gy}, \mathbf{gR}, \mathbf{gR}\widehat{\mathbf{M}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{h}^T\mathbf{S}, \mathbf{h}^T\mathbf{S}\widehat{\mathbf{N}}), \quad (\mathbf{g}, \mathbf{h}) \in SO(3)^2, \tag{22}$$

which leaves the Hamiltonian (21) invariant. The Hamiltonian therefore descends to a Hamiltonian function h on the quotient space $\mathcal{P}_{\text{cov}}/SO(3)^2$. We may define a model for $\mathcal{P}_{\text{cov}}/SO(3)^2$ by taking $(\mathbf{g}, \mathbf{h}) = (\mathbf{R}^T, \mathbf{S})$ in (22) so that

$$\mathbf{z} = (\mathbf{R}^T\mathbf{x}, \mathbf{R}^T\mathbf{y}, \mathbf{I}, \widehat{\mathbf{M}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{I}, \widehat{\mathbf{N}}) \in \mathcal{P}_{\text{cov}}.$$

Let $\boldsymbol{\lambda} = \mathbf{R}^T\mathbf{x}$, $\boldsymbol{\mu} = \mathbf{R}^T\mathbf{y}$ and

$$\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\Omega}}, \quad \dot{\mathbf{S}} = \mathbf{S}\widehat{\boldsymbol{\Lambda}}. \tag{23}$$

In particular, (18) and (20) then simplify to:

$$\widehat{\mathbf{M}} = \widehat{\mathbb{I}_d \boldsymbol{\Omega}} - \widehat{\mathbb{I}_c \boldsymbol{\Lambda}}, \tag{24}$$

$$\widehat{\mathbf{N}} = \widehat{\mathbb{I}_d \boldsymbol{\Lambda}} - \widehat{\mathbb{I}_c \boldsymbol{\Omega}}, \tag{25}$$

with inverses

$$\widehat{\boldsymbol{\Omega}} = \widehat{\mathbb{I}^d \mathbf{M}} + \widehat{\mathbb{I}^c \mathbf{N}}, \tag{26}$$

$$\widehat{\boldsymbol{\Lambda}} = \widehat{\mathbb{I}^d \mathbf{N}} + \widehat{\mathbb{I}^c \mathbf{M}}, \tag{27}$$

where

$$\mathbb{I}_d = \mathrm{tr}\tilde{\mathbf{A}}^2\mathbf{I} - \tilde{\mathbf{A}}^2 = \mathrm{diag}\left(d_2^2 + d_3^2, d_1^2 + d_3^2, d_1^2 + d_2^2\right), \tag{28}$$

$$\mathbb{I}_c = \mathrm{diag}\left(2d_2d_3, 2d_1d_3, 2d_1d_2\right), \tag{29}$$

$$\mathbb{I}^d = \left(\mathbb{I}_d^2 - \mathbb{I}_c^2\right)^{-1}\mathbb{I}_d, \tag{30}$$

$$\mathbb{I}^c = \left(\mathbb{I}_d^2 - \mathbb{I}_c^2\right)^{-1}\mathbb{I}_c, \tag{31}$$

and $\tilde{\mathbf{A}} = \mathrm{diag}(d_1, d_2, d_3)$.

Remark 1 (Singularities): Notice from (30) and (31) that $\mathbb{I}^c$ and $\mathbb{I}^d$ do not exist if $d_i = d_j$, for some $i, j \in \{1, 2, 3\}$ with $i \neq j$, i.e. if the body is spheroidal. This is a consequence of our decomposition. The spheroidal shapes are, however, exceptional. There is therefore little loss of generality by restricting attention to ellipsoidal shapes and hypothesis H2, as in Riemann's classical theorem.

Upon identifying ($\tilde{\mathbf{A}} = \mathrm{diag}\,(d_1, d_2, d_3), \tilde{\mathbf{B}} = 2\mathrm{diag}\,(b_1, b_2, b_3)$) with ($\mathbf{A} = (d_1, d_2, d_3)$, $\mathbf{B} = (b_1, b_2, b_3)) \in T^*\mathbb{R}^3_+$ a few calculations give

$$h(\mathbf{w}) \equiv H(\tilde{\mathbf{z}}) = \frac{1}{2}\langle\dot{\mathbf{A}}, \dot{\mathbf{A}}\rangle + \frac{1}{2}\langle\boldsymbol{\mu}, \boldsymbol{\mu}\rangle + \frac{1}{2}\langle\mathbf{M}, \mathbb{I}^d\mathbf{M}\rangle + \frac{1}{2}\langle\mathbf{N}, \mathbb{I}^d\mathbf{N}\rangle + \langle\mathbf{N}, \mathbb{I}^c\mathbf{M}\rangle$$
$$+ u(\boldsymbol{\lambda}, \mathbf{A}), \quad \mathbf{w} = (\mathbf{M}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{N}, \mathbf{A}, \mathbf{B}), \tag{32}$$

where $u(\boldsymbol{\lambda}, \mathbf{A}) = U(\boldsymbol{\lambda}, \tilde{\mathbf{A}})$. Here $\dot{A} = (\dot{d}_1, \dot{d}_2, \dot{d}_3)$ is given by

$$\mathbf{B} = \dot{\mathbf{A}} + 2\Xi d_1 d_2 d_3 \begin{pmatrix} d_1^{-1} \\ d_2^{-1} \\ d_3^{-1} \end{pmatrix}.$$

The Poisson structure also descends to a non-canonical Poisson structure on the reduced space. In Wang et al. (1991) it is shown how one obtains the reduced brackets for the system of a rigid body and a sphere. One can perform similar calculations to obtain the part of the reduced bracket related to the left invariance. Similarly, it can be shown that the reduced bracket related to the right invariance is the standard reduced rigid body bracket (see, for example, Holm et al. 2009). We therefore have:

Theorem 1 *The reduced system on $\mathcal{P}_{\mathrm{cov}}/SO(3)^2$ is described by the Hamiltonian* (32) *equipped with the Poisson structure matrix:*

$$\boldsymbol{\Lambda} = \begin{pmatrix} \widehat{\mathbf{M}} & \widehat{\boldsymbol{\lambda}} & \hat{\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\lambda}} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\mu}} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{N}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \end{pmatrix}.$$

Since $\partial_{\mathbf{M}} h = \mathbb{I}^d\mathbf{M} + \mathbb{I}^c\mathbf{N} = \boldsymbol{\Omega}$ and $\partial_{\mathbf{N}} h = \mathbb{I}^d\mathbf{N} + \mathbb{I}^c\mathbf{M} = \boldsymbol{\Lambda}$, Hamilton's equations read:

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\Omega} + \boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u,$$
$$\dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \times \boldsymbol{\Omega} + \boldsymbol{\mu},$$
$$\dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \boldsymbol{\Omega} - \partial_{\boldsymbol{\lambda}} u,$$
$$\dot{\mathbf{N}} = \mathbf{N} \times \boldsymbol{\Lambda},$$

$$\dot{\mathbf{A}} = \mathbf{B} - 2\Xi \boldsymbol{A}^{-1},$$
$$\dot{\mathbf{B}} = -\partial_{\mathbf{A}} h. \tag{33}$$

Here $\boldsymbol{A}^{-1} \equiv \begin{pmatrix} d_1^{-1} \\ d_2^{-1} \\ d_3^{-1} \end{pmatrix}$. *The quantity* Ξ *is determined through* (6) $\partial_\Xi h = 0$*:*

$$\langle \boldsymbol{A}^{-1}, \dot{\boldsymbol{A}} \rangle = 0 \Rightarrow \Xi = \frac{1}{2} \frac{\langle \boldsymbol{A}^{-1}, \boldsymbol{B} \rangle}{\langle \boldsymbol{A}^{-1}, \boldsymbol{A}^{-1} \rangle}$$

The theorem gives the reduction of the lifted system. The following computations show that the lifted angular momentum map $(\mathbf{RL}, \mathbf{SN})$ is invariant with respect to the D_2-action:

$$\begin{aligned}
\mathbf{R}\widehat{\mathbf{M}}\mathbf{R}^T &\mapsto \mathbf{R}\mathbf{P}_i \left(\mathbf{P}_i \mathbf{R}^T \dot{\mathbf{R}} \mathbf{P}_i \tilde{\mathbf{A}}^2 - \tilde{\mathbf{A}}^2 \mathbf{P}_i \left(\mathbf{R}^T \dot{\mathbf{R}} \right)^T \mathbf{P}_i + 2\tilde{\mathbf{A}} \mathbf{P}_i \dot{\mathbf{S}}^T \mathbf{S} \mathbf{P}_i \tilde{\mathbf{A}} \right) \mathbf{P}_i \mathbf{R}^T \\
&= \mathbf{R}\widehat{\mathbf{M}}\mathbf{R}^T, \\
\mathbf{S}\widehat{\mathbf{N}}\mathbf{S}^T &\mapsto \mathbf{S}\mathbf{P}_i \left(\mathbf{P}_i \mathbf{S}^T \dot{\mathbf{S}} \mathbf{P}_i \tilde{\mathbf{A}}^2 - \tilde{\mathbf{A}}^2 \mathbf{P}_i \left(\mathbf{S}^T \dot{\mathbf{S}} \right)^T \mathbf{P}_i + 2\mathbf{S}\tilde{\mathbf{A}} \mathbf{P}_i \dot{\mathbf{R}}^T \mathbf{R} \mathbf{P}_i \tilde{\mathbf{A}} \right) \mathbf{P}_i \mathbf{S}^T \\
&= \mathbf{S}\widehat{\mathbf{N}}\mathbf{S}^T.
\end{aligned}$$

Here we have used that $\mathbf{P}_i \tilde{\mathbf{A}} \mathbf{P}_i = \tilde{\mathbf{A}}$. Therefore its fibres $(\mathbf{RL}, \mathbf{SN}) = (\mathbf{j}_l, \mathbf{j}_r) \in \mathbb{R}^6$ are invariant under the D_2-action and the reduced spaces of the original system are just the quotients of the reduced spaces of the lifted system by the action of D_2.

Remark 2 There reduced system has the following symmetry:

$$(\mathbf{M}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{N}) \mapsto (\mathbf{P}_i \mathbf{M}, \mathbf{P}_i \boldsymbol{\lambda}, \mathbf{P}_i \boldsymbol{\mu}, \mathbf{P}_i \mathbf{N}),$$

where $\mathbf{P}_i$ is an element of D_2 (17). The pseudo-rigid body is invariant under this action and the symmetry-related solutions obtained hereby therefore coincide with those obtained in the original system through $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{P}_i \mathbf{x}, \mathbf{P}_i \mathbf{y})$

We can replace the equation for $\mathbf{A}$ and $\mathbf{B}$ by a second order equation for $\mathbf{A}$. This way it becomes clearer how the pressure enters the equations. We have

$$\ddot{\mathbf{A}} = \dot{\mathbf{B}} - 2\dot{\Xi}\mathbf{A}^{-1} + 2\Xi \tilde{\mathbf{A}}^{-2} \dot{\mathbf{A}},$$

and so by using

$$\partial_{\mathbf{A}} \left(\frac{1}{2} \langle \dot{\mathbf{A}}, \dot{\mathbf{A}} \rangle \right) = 2\Xi \tilde{\mathbf{A}}^{-2} \dot{\mathbf{A}},$$

we finally get

$$\ddot{\mathbf{A}} = -\partial_{\mathbf{A}} (\psi + u) - 2\dot{\Xi}\mathbf{A}^{-1}, \tag{34}$$

where we have introduced $\psi = \frac{1}{2} \langle \mathbf{M}, \mathbb{I}^d \mathbf{M} \rangle + \frac{1}{2} \langle \mathbf{N}, \mathbb{I}^d \mathbf{N} \rangle + \langle \mathbf{N}, \mathbb{I}^c \mathbf{M} \rangle$.

Since $\det \tilde{\mathbf{A}} = d_1 d_2 d_3 = 1$ we can eliminate one component $d_i = (d_j d_k)^{-1}$ with (i, j, k) a permutation of $(1, 2, 3)$ and solve for the pressure at the centre of mass:

$$\dot{\Xi} = \frac{1}{2d_i} \left(\ddot{d}_i + \partial_{d_i} (\psi + u) \right).$$

This can then be inserted into the Eqs. (34) for d_j and d_k.

Besides decreasing the necessary degrees of freedom, the introduced reduction also allows us to decouple the dependency of the effective rotations $\mathbf{R}$ and $\mathbf{S}$ (23). In particular, in the rotational frame described by $\mathbf{R}$ the mutual attraction between the bodies is independent of the attitude of the pseudo-rigid body. Instead, the rotations affect the orbital motion, and vice versa, via the dependency of the angular momenta $\mathbf{M}$ and $\mathbf{N}$.

Now, $\mathbf{L} = \mathbf{M} + \boldsymbol{\lambda} \times \boldsymbol{\mu}$ and $\mathbf{N}$ are the body angular momentum and body circulation, respectively, so that $\mathbf{J}_l = \mathbf{RL}$ and $\mathbf{J}_r = \mathbf{SN}$ (see (10) and (11)). By virtue of the reduction we have:

Proposition 2 *The functions* $C = C(|\mathbf{N}|, |\mathbf{L}|)$, *are Casimir functions of the system. In particular,* $|\mathbf{N}|$ *and* $|\mathbf{L}|$ *are conserved.*

These conserved quantities can be introduced as new coordinates leading to further reduction in the number of equations. One particular nice way of doing this is obtained by setting

$$\mathbf{L} = \begin{pmatrix} \sqrt{L^2 - L_3^2}\cos\theta \\ \sqrt{L^2 - L_3^2}\sin\theta \\ L_3 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \sqrt{N^2 - N_3^2}\cos\phi \\ \sqrt{N^2 - N_3^2}\sin\phi \\ N_3 \end{pmatrix}$$

where $L = |\mathbf{L}|$ and $N = |\mathbf{N}|$ are the conserved quantities. The new equations in these coordinates are still Hamiltonian. Indeed, the new Hamiltonian function is just (32) written in these coordinates:

$$\begin{aligned} h_{L,N}(\mathbf{u}) &= \frac{1}{2}\langle \dot{\mathbf{A}}, \dot{\mathbf{A}}\rangle + \frac{1}{2}\langle \boldsymbol{\mu}, \boldsymbol{\mu}\rangle + \frac{1}{2}\langle \mathbf{L} - \boldsymbol{\lambda} \times \boldsymbol{\mu}, \mathbb{I}^d\,(\mathbf{L} - \boldsymbol{\lambda} \times \boldsymbol{\mu})\rangle \\ &\quad + \frac{1}{2}\langle \mathbf{N}, \mathbb{I}^d \mathbf{N}\rangle + \langle \mathbf{N}, \mathbb{I}^c\,(\mathbf{L} - \boldsymbol{\lambda} \times \boldsymbol{\mu})\rangle + u(\boldsymbol{\lambda}, \mathbf{A}). \\ \mathbf{u} &= (\theta, L_3, \boldsymbol{\lambda}, \boldsymbol{\mu}, \phi, N_3, \mathbf{A}, \mathbf{B}), \end{aligned}$$

depending on the constants L and N as *parameters*, while the Poisson structure descends to a canonical symplectic form:

$$\omega = d\theta \wedge dL_3 + d\boldsymbol{\lambda} \wedge \boldsymbol{\mu} + d\phi \wedge N_3 + d\mathbf{A} \wedge d\mathbf{B}.$$

In other words, the new Hamiltonian equations take the following form:

$$\begin{aligned} \dot{\theta} &= \partial_{L_3} h_{L,N}, \\ \dot{L}_3 &= -\partial_\theta h_{L,N}, \\ \dot{\boldsymbol{\lambda}} &= \partial_{\boldsymbol{\mu}} h_{L,N}, \\ \dot{\boldsymbol{\mu}} &= -\partial_{\boldsymbol{\lambda}} h_{L,N}, \\ \dot{\phi} &= \partial_{N_3} h_{L,N}, \\ \dot{N}_3 &= -\partial_\phi h_{L,N}, \\ \dot{\mathbf{A}} &= \partial_{\mathbf{B}} h_{L,N}, \\ \dot{\mathbf{B}} &= -\partial_{\mathbf{A}} h_{L,N}. \end{aligned}$$

The equations $L = |\mathbf{L}|$ and $N = |\mathbf{N}|$ therefore define the symplectic leaves (Marsden and Ratiu 1994) of the reduced Poisson structure. Their topologies are $S_L^2 \times \mathbb{R}^6 \times S_N^2 \times \mathbb{R}^6$. Here S_c^2 denotes $S_c^2 = \{\mathbf{z} \in \mathbb{R}^3 | |\mathbf{z}| = c\}$.

4 Relative equilibria

By the reduction, the relative equilibria of the system are solutions of the following system:

$$\mathbf{0} = \mathbf{M} \times \boldsymbol{\Omega} + \boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u, \tag{35a}$$
$$\mathbf{0} = \boldsymbol{\lambda} \times \boldsymbol{\Omega} + \boldsymbol{\mu}, \tag{35b}$$
$$\mathbf{0} = \boldsymbol{\mu} \times \boldsymbol{\Omega} - \partial_{\boldsymbol{\lambda}} u, \tag{35c}$$
$$\mathbf{0} = \mathbf{N} \times \boldsymbol{\Lambda}, \tag{35d}$$
$$\mathbf{0} = \dot{\mathbf{A}}, \tag{35e}$$
$$\mathbf{0} = \partial_{\mathbf{A}} h, \tag{35f}$$

For further simplifications it is advantageous to eliminate $\boldsymbol{\mu}$ from (35b) so that (35c) and the total angular momentum read:

$$\mathbf{0} = \boldsymbol{\lambda}|\boldsymbol{\Omega}|^2 - \boldsymbol{\Omega}\langle\boldsymbol{\lambda}, \boldsymbol{\Omega}\rangle - \partial_{\boldsymbol{\lambda}} u. \tag{36}$$

and

$$\mathbf{L} = \mathbf{M} + \boldsymbol{\Omega}|\boldsymbol{\lambda}|^2 - \boldsymbol{\lambda}\langle\boldsymbol{\lambda}, \boldsymbol{\Omega}\rangle, \tag{37}$$

respectively. At relative equilibria (12) and (13) give:

$$\mathbf{L} = \mathbf{J}_l(\mathbf{x}_0, \mathbf{y}_0, \tilde{\mathbf{A}}_0, \mathbf{P}_0), \quad \mathbf{N} = \mathbf{J}_r(\tilde{\mathbf{A}}_0, \mathbf{P}_0).$$

Lemma 1 may therefore be restated as:

Corollary 1 *At relative equilibria:*

$$\mathbf{L} \times \boldsymbol{\Omega} = \mathbf{0}, \quad \mathbf{N} \times \boldsymbol{\Lambda} = \mathbf{0}.$$

Proof Here we show that this result can also be deduced directly from the reduced equations. Indeed, the latter condition coincides with Eq. (35d). For the former condition, take the right outer product of $\mathbf{L}$ expressed by (37) with $\boldsymbol{\Omega}$, so that

$$\mathbf{M} \times \boldsymbol{\Omega} - \boldsymbol{\lambda} \times \boldsymbol{\Omega}\langle\boldsymbol{\lambda}, \boldsymbol{\Omega}\rangle = \mathbf{0}.$$

The first item of this equation, using Eq. (35a), is equal to $-\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u$. In turn $\partial_{\boldsymbol{\lambda}} u$ can be eliminated from (35c). After these substitutions, the first item is equal to the negative of the second one, and hence $\mathbf{L} \times \boldsymbol{\Omega} = 0$. The corollary is therefore completed.

We will now add the final hypothesis of the paper:

H3 The reduced potential u satisfy $\partial_{\boldsymbol{\lambda}} u \neq 0$.

Cf. (36) this condition is equivalent to $\boldsymbol{\lambda} \not\parallel \boldsymbol{\Omega}$. The hypothesis therefore excludes relative equilibria where the relative position vector $\mathbf{x}$ is fixed in inertial space. For the gravitational problem, this means that the rigid sphere is external to the pseudo-rigid body.

As for the rigid body case the relative equilibria can be divided into two types: *locally central* and *non-locally central* (see also Scheeres 2006). We also define planar equilibria in the following definition:

Definition 2 1. A relative equilibrium is said to be locally central if the mutual attraction and relative position vectors are parallel, i.e. $\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u = \mathbf{0}$.
2. A relative equilibrium is said to be planar if the total angular momentum vector $\mathbf{L}$ is perpendicular to the relative position vector $\boldsymbol{\lambda}$, i.e. $\langle\mathbf{L}, \boldsymbol{\lambda}\rangle = 0$.

However, the following theorem implies that the two notions of locally central and planar equilibria actually coincide:

Theorem 2 *Assume that the pseudo-rigid body satisfies the hypotheses H1, H2 and H3. Then a relative equilibrium of the system is planar if and only if it is locally central:*

$$\langle \mathbf{L}, \boldsymbol{\lambda} \rangle = 0 \quad \Leftrightarrow \quad \boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u = 0.$$

Proof First notice that by using (37) that $\langle \mathbf{L}, \boldsymbol{\lambda} \rangle = 0$ is equivalent to $\langle \mathbf{M}, \boldsymbol{\lambda} \rangle = 0$. Let $\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u = 0$. Then from Eq. (35a) it follows that $\mathbf{M} \times \boldsymbol{\Omega} = 0$. Taking the left outer product of the last equation with $\boldsymbol{\lambda}$, we obtain:

$$\boldsymbol{\Omega} \langle \mathbf{M}, \boldsymbol{\lambda} \rangle = \mathbf{M} \langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle. \tag{38}$$

Then, taking the left outer product of Eq. (35c) with $\boldsymbol{\lambda}$, and applying the assumption $\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u = \mathbf{0}$, we obtain $\boldsymbol{\lambda} \times \boldsymbol{\Omega} \langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle = \mathbf{0}$. Thus, there are two feasible cases. If $\boldsymbol{\lambda} \times \boldsymbol{\Omega} = \mathbf{0}$ then from Eq. (35b) it follows that $\boldsymbol{\mu}$ vanishes and from Eq. (35c) it then follows that the gradient $\partial_{\boldsymbol{\lambda}} u$ vanishes. This contradicts hypothesis H3. Thus, $\langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle$ vanishes, and then, since $\boldsymbol{\Omega}$ does not vanish, from Eq. (38) it follows that $\langle \mathbf{M}, \boldsymbol{\lambda} \rangle$ vanishes. Thus, the sufficient condition has been proved.

Let $\langle \mathbf{L}, \boldsymbol{\lambda} \rangle = 0$. By Corollary 1 it follows that $\langle \boldsymbol{\Omega}, \boldsymbol{\lambda} \rangle = 0$. Therefore, by eliminating M in (37) and inserting this into (35a), we have:

$$\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u = \mathbf{0}.$$

Thus, the necessary condition has been proved.

We note that the two-body problem of a ellipsoidal rigid body and a sphere is a natural subsystem:

$$\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\Omega} + \boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u, \\
\dot{\boldsymbol{\lambda}} &= \boldsymbol{\lambda} \times \boldsymbol{\Omega} + \boldsymbol{\mu}, \\
\dot{\boldsymbol{\mu}} &= \boldsymbol{\mu} \times \boldsymbol{\Omega} - \partial_{\boldsymbol{\lambda}} u,
\end{aligned}$$

with $\mathbf{A}$ fixed, and hence the theorem also holds true for this case. In particular, the proof did not make use of any properties of the potential other than it being symmetric.

Riemann's theorem, in the particular form of our interest, describes geometrical properties of the angular velocity and the circulation of a pseudo-rigid body, without the gravitional interaction with another body, in a relative equilibrium:

Theorem 3 (Roberts and Sousa Dias 1999) *Consider a spherically symmetric pseudo-rigid body in a relative equilibrium:*

$$\boldsymbol{Q}(t) = \exp(\widehat{\boldsymbol{\Theta}} t) \tilde{\boldsymbol{A}}_0 \exp(-\widehat{\boldsymbol{\Sigma}} t),$$

with $\tilde{\boldsymbol{A}}_0 = diag\,(d_1, d_2, d_3)$ *a constant diagonal matrix. If* $\boldsymbol{\Theta}$ *and* $\boldsymbol{\Sigma}$ *are both non-zero and the equilibrium is ellipsoidal, i.e.* $d_1 > d_2 > d_3$*, then* $\boldsymbol{\Theta}$ *and* $\boldsymbol{\Sigma}$ *must satisfy one of the following conditions:*

(i) $\boldsymbol{\Theta}$ *and* $\boldsymbol{\Sigma}$ *lie in the same principal plane of the ellipsoid;*
(ii) *if one of the vectors is aligned with a principal axis, then the other vector is aligned along the same axis.*

In the following, we show that Riemann's theorem extends to the relative equilibria of the two-body systems of this paper whenever $\boldsymbol{\lambda}$ is aligned with one of the principal axes of the ellipsoid. First, however, we note that from Corollary 1 it follows that $\mathbf{L}$ and $\boldsymbol{\Omega}$, $\mathbf{N}$ and $\boldsymbol{\Lambda}$ are parallel pairwise. Here $\boldsymbol{\Omega} = \mathbf{0}$ would imply that $\boldsymbol{\mu} = \mathbf{0}$ and $\partial_{\boldsymbol{\lambda}} u = \mathbf{0}$, and would therefore contradict H3. If $\boldsymbol{\Lambda} \neq \mathbf{0}$ we therefore introduce k_Ω and k_Λ so that $\mathbf{L} = k_\Omega \boldsymbol{\Omega}$ and $\mathbf{N} = k_\Lambda \boldsymbol{\Lambda}$.

Theorem 4 *Assume that the two-body system is in a relative equilibrium satisfying the hypotheses H1 and H2 and where* $\boldsymbol{\lambda}$ *is aligned with the* l*'th principal axis of the pseudo-rigid body. Denote by the integers* m *and* n *the two remaining principal axes so that* (l, m, n) *is a cyclic permutation of* $(1, 2, 3)$. *Then:*

1° $\boldsymbol{\Lambda} \neq \mathbf{0}$ *and either of the following conditions hold true:*

(a)

$$k_\Lambda = d_n^2 + d_m^2 + 3\left(d_l^2 - d_m^2\right)\left(d_l^2 - d_n^2\right)\lambda_l^{-2};$$

(b) *or Riemann's theorem hold true:*

(i) *the angular velocity of the pseudo-rigid body vector* $\boldsymbol{\Omega}$ *and the internal rotation velocity vector* $\boldsymbol{\Lambda}$ *lie in the same principal plane of the body;*

(ii) *if one of the vectors is aligned with a principal axis, then the other vector is aligned along the same axis.*

2° $\boldsymbol{\Lambda} = \mathbf{0}$ *and* $\boldsymbol{\Omega}$ *is in a principal plane of the body.*

Proof For 1° assume that $\boldsymbol{\Lambda} \neq \mathbf{0}$. Then from (37), (24) and (25) it follows that

$$\begin{pmatrix} \mathbb{I}_d - \left(k_\Omega - |\boldsymbol{\lambda}|^2\right)\mathbf{I} - \boldsymbol{\lambda}\boldsymbol{\lambda}^T & -\mathbb{I}_c \\ -\mathbb{I}_c & \mathbb{I}_d - k_\Lambda \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Omega} \\ \boldsymbol{\Lambda} \end{pmatrix} = \mathbf{0}. \tag{39}$$

Since $\boldsymbol{\lambda}$ is assumed to be aligned with one principal axis, it follows that $\boldsymbol{\lambda}\boldsymbol{\lambda}^T = \operatorname{diag}\left(\lambda_1^2, \lambda_2^2, \lambda_3^2\right)$ with only one diagonal element non-zero. Therefore there exists a non-zero solution for (Ω_i, Λ_i) if and only if the determinant of the linear system

$$\begin{aligned} \left((\mathbb{I}_d)_{ii} - k_\Omega + |\boldsymbol{\lambda}|^2 - \lambda_i^2\right)\Omega_i - (\mathbb{I}_c)_{ii}\,\Lambda_i &= 0, \\ \left((\mathbb{I}_d)_{ii} - k_\Lambda\right)\Lambda_i - (\mathbb{I}_c)_{ii}\,\Omega_i &= 0 \end{aligned}$$

vanishes. We compute:

$$\left(k_\Omega - |\boldsymbol{\lambda}|^2 + \lambda_i^2\right)k_\Lambda - (\mathbb{I}_d)_{ii}\left(k_\Omega + k_\Lambda - |\boldsymbol{\lambda}|^2 + \lambda_i^2\right) + (\mathbb{I}_d)_{ii}^2 - (\mathbb{I}_c)_{ii}^2 = 0. \tag{40}$$

From the definitions of $\mathbb{I}_d$ (28) and $\mathbb{I}_c$ (29) we then obtain the following lemma:

Lemma 2 *If* (Ω_i, Λ_i) *and* $\left(\Omega_j, \Lambda_j\right)$ *are non-zero solutions of* (39) *with* $i \neq j$*, then either* $\lambda_i = \lambda_j = 0$ *and* $d_i = d_j$ *or the following two equations hold true*

$$\begin{aligned} k_\Omega + k_\Lambda &= d_i^2 + d_j^2 - 2d_k^2 + |\boldsymbol{\lambda}|^2 \\ &\quad + \lambda_i^2\left(d_j^2 + d_k^2 - k_\Lambda\right)\left(d_i^2 - d_j^2\right)^{-1} + \lambda_j^2\left(d_i^2 + d_k^2 - k_\Lambda\right)\left(d_j^2 - d_i^2\right)^{-1}, \end{aligned} \tag{41}$$

$$\begin{aligned} k_\Omega k_\Lambda &= d_i^2 d_j^2 + \left(d_i^2 + d_j^2\right)d_k^2 - 3d_k^4 + k_\Lambda|\boldsymbol{\lambda}|^2 \\ &\quad + \lambda_i^2\left(d_i^2 + d_k^2\right)\left(d_j^2 + d_k^2 - k_\Lambda\right)\left(d_i^2 - d_j^2\right)^{-1} \\ &\quad + \lambda_j^2\left(d_j^2 + d_k^2\right)\left(d_i^2 + d_k^2 - k_\Lambda\right)\left(d_j^2 - d_i^2\right), \end{aligned}$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$.

Proof We solve (40) for $k_\Omega + k_\Lambda$ and $k_\Omega k_\Lambda$ for $i = i$ and $i = j$.

If Riemann's theorem hold true then one of the pairs (Ω_i, Λ_i) vanishes. Now assume otherwise. Let us assume that $\lambda_l \neq 0$ and $\lambda_m = 0 = \lambda_n$ with (l, m, n) a cyclic permutation of $(1, 2, 3)$. Since the pseudo-rigid body is assumed to be ellipsoidal, it follows from Lemma 2, and in particular (41), that:

$$\begin{aligned} &d_l^2 + d_m^2 - 2d_n^2 + \lambda_l^2 + \lambda_l^2 \left(d_j^2 + d_k^2 - k_\Lambda\right) \left(d_l^2 - d_m^2\right)^{-1} \\ &= d_n^2 + d_l^2 - 2d_m^2 + \lambda_l^2 + \lambda_l^2 \left(d_n^2 + d_m^2 - k_\Lambda\right) \left(d_l^2 - d_n^2\right)^{-1} \\ &= d_m^2 + d_n^2 - 2d_l^2 + \lambda_l^2. \end{aligned}$$

After some straightforward manipulations this can be shown to be equivalent to

$$k_\Lambda = d_m^2 + d_n^2 + 3\left(d_l^2 - d_m^2\right)\left(d_l^2 - d_n^2\right)\lambda_l^{-2}.$$

In the following we will show the last part of Riemann's theorem. Since $\mathbb{I}_c$ is invertible it follows from (39) that

$$\boldsymbol{\Omega} = \mathbb{I}_c^{-1} \left(\mathbb{I}_d - k_\Lambda \mathbf{I}\right) \boldsymbol{\Lambda}. \tag{42}$$

Assume then that $\Lambda_i = \Lambda_j = 0, i \neq j$. Then by (42) it follows that either $\Omega_i = \Omega_j = 0$ or $d_i = d_k$. The latter contradicts H2 and the body being ellipsoidal. We can repeat the same arguments for $\Omega_i = \Omega_j = 0$. The last part of Riemann's theorem has therefore been shown.

For 2° let $\boldsymbol{\Lambda} = \mathbf{0}$. Then the first rows of (39) give

$$\left(\mathbb{I}_d - \left(k_\Omega - |\boldsymbol{\lambda}|^2\right)\mathbf{I} - \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right) \boldsymbol{\Omega} = \mathbf{0}. \tag{43}$$

For contradiction assume that all components of $\boldsymbol{\Omega}$ are non-zero. This implies that the entries of the diagonal matrix appearing in (43) all vanish. Therefore

$$d_l^2 = \frac{1}{2}k_\Omega - \lambda_l^2, \quad d_m^2,\, d_n^2 = \frac{1}{2}k_\Omega.$$

But $d_m \neq d_n$ according to H2. The proof is completed.

The proof of the theorem did only rely on the hypotheses H1 and H2, Corollary 1 and the properties of the conserved quantities and the angular velocities $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$. Therefore the result applies to general two-body problems of the given form, for example in molecular dynamics. By making further use of H3 we can show the following:

Proposition 3 *Assume the hypotheses of* Theorem 4 *and suppose furthermore that H3 holds. Then the second property of Riemann's theorem, see* 1° *(b) (ii) in* Theorem 4, *can only hold true in a locally central equilibrium.*

Remark 3 This proposition can be interpreted as follows: The "S-type relative equilibria" (Chandrasekhar 1987; Roberts and Sousa Dias 1999), where the angular velocities are both directed along the same principal axis, are "in general" planar for the gravitational two-body problem. The exception being given by 1° (a):

$$\frac{N}{\Lambda} = d_n^2 + d_m^2 + 3\left(d_l^2 - d_m^2\right)\left(d_l^2 - d_n^2\right)\lambda_l^{-2},$$

(l, m, n) being a cyclic permutation of $(1, 2, 3)$.

Proof Assume otherwise so that $\Omega_i = 0 = \Omega_j$, $i \neq j$ and therefore $\Lambda_i = 0 = \Lambda_j$. Then from (39) we have:

$$\lambda_i \langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle = 0 = \lambda_j \langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle,$$

so that $\langle \boldsymbol{\lambda}, \boldsymbol{\Omega} \rangle = 0$ or $\lambda_i = 0 = \lambda_j$. The former implies through (36) that the equilibrium is planar. By assumption the latter must therefore hold true. But then $\boldsymbol{\lambda} \parallel \boldsymbol{\Omega}$ and through (35b) and (35c) it follows that $\boldsymbol{\mu} = \mathbf{0}$ and $\partial_{\boldsymbol{\lambda}} u = \mathbf{0}$, respectively. This contradicts H3 and that the system is in relative equilibrium. This completes the proof.

4.1 The gravitational two-body problem

We now state the only result which relies on the particular form of the potential for the gravitational two-body problem. In this case we will show that the exceptions to the validity of Riemann's theorem in Theorem 4 cannot hold true in a locally central relative equilibrium. This follows from Theorem 2 and the following lemma:

Lemma 3 *In a locally central equilibrium of the gravitational two-body problem, the rigid sphere is located along a principal axis of the body.*

Proof This follows directly from the fact that the pseudo-rigid body is ellipsoidal. See also Scheeres (2006).

Indeed, we have the following:

Theorem 5 *Consider the gravitational two-body problem and assume that the system satisfies the hypotheses H1, H2 and H3 and is in a locally central equilibrium, then Riemann's theorem hold true:*

(i) *the angular velocity of the pseudo-rigid body vector $\boldsymbol{\Omega}$ and the internal rotation velocity vector $\boldsymbol{\Lambda}$ lie in the same principal plane of the body;*
(ii) *if one of the vectors is aligned with a principal axis, then the other vector is aligned along the same axis.*

Proof By the assumptions of the theorem and Lemma 3 it follows that: (a) $\boldsymbol{\lambda}$ is aligned with a principal axis, say i, and (b) $\langle \boldsymbol{\Omega}, \boldsymbol{\lambda} \rangle = 0$ so that $\boldsymbol{\Omega}$ is contained in a principal plane with $\Omega_i = 0$. Therefore by (42) it follows that either k_Λ is such that $\left(\mathbb{I}_c^{-1}\left(\mathbb{I}_d - k_\Lambda \mathbf{I}\right)\right)_{ii} = 0$ or $\Lambda_i = 0$. If the latter holds then we are done. Assume therefore the former. By inserting (42) into (39) we obtain the following equation:

$$\mathbf{0} = \left(\left(\mathbb{I}_d - \left(k_\Omega + |\boldsymbol{\lambda}|^2\right)\mathbf{I} - \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\mathbb{I}_c^{-1}\left(\mathbb{I}_d - k_\Lambda \mathbf{I}\right) - \mathbb{I}_c\right)\boldsymbol{\Lambda}.$$

From this equation it follows that if $\left(\mathbb{I}_c^{-1}\left(\mathbb{I}_d - k_\Lambda \mathbf{I}\right)\right)_{ii} = 0$ then $\Lambda_i = 0$. The first part of Riemann's theorem has been completed. The last part is proved by repeating the arguments in Theorem 4.

It is now natural to ask what happens when the equilibrium is non-planar. We do not expect a generalisation of Riemann's theorem beyond Theorem 4 since in a general non-planar equilibrium the term $\boldsymbol{\lambda}\boldsymbol{\lambda}^T$ is not diagonal. In the following we shall instead investigate the non-planar equilibria with the particular aim of diminishing the necessary equations while gaining further insight into the underlying geometry. Although, we pretty much follow the method proposed by Scheeres in Bellerose and Scheeres (2008), we find our approach clearer and simpler as unlike Bellerose and Scheeres (2008) we present explicit formulas for obtaining *all* the variables describing the relative equilibrium once $\boldsymbol{\lambda}$ is found.

4.2 Non-locally central relative equilibrium

Let $\boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u \neq \mathbf{0}$ and assume first that $\Lambda \neq 0$. Then if we take the inner product of Eq. (35a) with $\boldsymbol{\Omega}$, we obtain that the vectors $\boldsymbol{\Omega}$, $\boldsymbol{\lambda}$ and $\partial_{\boldsymbol{\lambda}} u$ all lie in the same plane. Furthermore, by taking the inner product of Eq. (35c) with $\boldsymbol{\Omega}$, we obtain that $\langle \boldsymbol{\Omega}, \partial_{\boldsymbol{\lambda}} u \rangle = 0$. Hence, the vectors $\boldsymbol{\Omega}$ and $\partial_{\boldsymbol{\lambda}} u$ are perpendicular. The vectors $\partial_{\boldsymbol{\lambda}} u$, $\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}$ and $\partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right)$ therefore form an orthogonal basis in $\mathbb{R}^3$. Let us denote this basis by $\mathcal{F}_{\boldsymbol{\lambda}}$. In this basis the vector $\boldsymbol{\Omega}$ has only one non-zero component as it is parallel to $\partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right)$. This allows us to write $\boldsymbol{\Omega}$ in the following way:

$$\boldsymbol{\Omega} = \pm |\boldsymbol{\Omega}| \frac{\partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right)}{\left|\partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right)\right|}. \tag{44}$$

Note that since the system is symmetric with respect to the reflection $(\boldsymbol{\Omega}, \boldsymbol{\Lambda}, \boldsymbol{\mu}) \mapsto -(\boldsymbol{\Omega}, \boldsymbol{\Lambda}, \boldsymbol{\mu})$ the choice of a sign in (44) is not important. The magnitude of $\boldsymbol{\Omega}$ can be found by taking the inner product of Eq. (36) with $\partial_{\boldsymbol{\lambda}} u$:

$$|\boldsymbol{\Omega}|^2 = \frac{|\partial_{\boldsymbol{\lambda}} u|^2}{\langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}} u \rangle}, \tag{45}$$

where $\langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}} u \rangle$ does not vanish as otherwise $\partial_{\boldsymbol{\lambda}} u$ would vanish. Moreover, it is strictly positive. Finally, after some simplifications, the vector $\boldsymbol{\Omega}$ can be rewritten in terms of $\boldsymbol{\lambda}$ and the potential in the following way:

$$\boldsymbol{\Omega} = \pm \frac{\partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right)}{\sqrt{\langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}} u \rangle \left(\boldsymbol{\lambda}^2 |\partial_{\boldsymbol{\lambda}} u|^2 - \langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}} u \rangle^2\right)}}. \tag{46}$$

From (42) we have

$$(\mathbb{I}_d - k_\Lambda \mathbf{I}) \boldsymbol{\Lambda} = \mathbb{I}_c \boldsymbol{\Omega}, \tag{47}$$

where k_Λ is a parameter. The matrix $(\mathbb{I}_d - k_\Lambda \mathbf{I})$ is diagonal and may only have one zero non-diagonal component, as otherwise it would imply that the pseudo-rigid body was spheroidal. We therefore consider two different scenarios. In the first scenario the matrix is invertible so that:

$$\boldsymbol{\Lambda} = (\mathbb{I}_d - k_\Lambda \mathbf{I})^{-1} \mathbb{I}_c \boldsymbol{\Omega}. \tag{48}$$

Next, let the matrix have a zero component so that $\Omega_i = 0$. From this k_Λ can be determined along with the other components of $\boldsymbol{\Lambda}$ by inverting corresponding diagonal elements of the matrix. We then leave the remaining component Λ_i, rather than k_Λ as above, as a parameter. We have:

$$\Lambda_i = \Lambda_{i0}, \quad \Lambda_j = \frac{2 d_i d_k}{d_i^2 - d_j^2} \Omega_j, \quad \Lambda_k = \frac{2 d_i d_j}{d_i^2 - d_k^2} \Omega_k, \tag{49}$$

where (i, j, k) are in a cyclic permutation.

After the introduced eliminations of $\boldsymbol{\mu}$, $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ Eqs. (35a) and (35f) form a closed subsystem, which should be solved for the position $\boldsymbol{\lambda}$ and $\mathbf{A}$. These equations are vectorial. It means that there are six scalar equations. However, the amount of equations determining the relative equilibrium position can be diminished. Each of the equations can be treated as a vector which should vanish. Any vector can be resolved in an unique way along an orthogonal basis, and the condition that a vector vanishes is equivalent to the condition that all its component in an orthogonal basis vanish. We shall use the basis $\mathcal{F}_{\boldsymbol{\lambda}}$ for this purpose. In this basis Eq. (35a) has a zero component along the vector $\partial_{\boldsymbol{\lambda}} u \times (\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda})$. This allows us to reduce the Eq. (35a) to two scalar equations, and together with Eqs. (35f) they give us the minimum number of equations for finding the relative equilibria position:

$$\langle \mathbf{M}^*, \partial_{\boldsymbol{\lambda}} u\rangle + \langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}} u\rangle \left(\boldsymbol{\lambda}^2 |\partial_{\boldsymbol{\lambda}} u|^2 - \langle \boldsymbol{\lambda}, \partial_{\boldsymbol{\lambda}}\rangle^2\right) = 0, \tag{50a}$$

$$\langle \mathbf{M}^*, \boldsymbol{\lambda} \times \partial_{\boldsymbol{\lambda}} u\rangle = 0 \tag{50b}$$

$$\partial_{\mathbf{A}} h = \mathbf{0}. \tag{50c}$$

Here

$$\mathbf{M}^* = \pm \left[\mathbb{I}_d - \mathbb{I}_c (\mathbb{I}_d - k_\Lambda \mathbf{I})^{-1} \mathbb{I}_c\right] \partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right), \tag{51}$$

is a re-normalised angular momentum of the pseudo-rigid body valid only when $(\mathbb{I}_c - k_\Lambda \mathbf{I})$ is invertible. Otherwise we replace this vector by:

$$\mathbf{M}^* = \pm \mathbb{I}_d \partial_{\boldsymbol{\lambda}} u \times \left(\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda}\right) - \mathbb{I}_c \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is given by (49).

We now consider the case when $\boldsymbol{\Lambda}$ vanishes. Then $\mathbf{M} = \mathbb{I}_d \boldsymbol{\Omega}$ and condition (35d) is identically satisfied. We therefore have the same three equations as above but with a different $\mathbf{M}^*$. In this case it is given by $\mathbf{M}^* = \pm \mathbb{I}_d \partial_{\boldsymbol{\lambda}} u \times (\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda})$. Let us collect the results:

Proposition 4 *Consider a non-locally central relative equilibrium satisfying the hypothesis H1, H2 and H3. Then either*

1. $\boldsymbol{\Lambda} \neq 0$*:* $\boldsymbol{\lambda}$ *and* $\mathbf{A}$ *are given by* (50a), (50b) and (50c). *Once this system has been solved,* $\boldsymbol{\Omega}$ *is given by* (46). *If the matrix* $(\mathbb{I}_d - \frac{N}{\Lambda}\boldsymbol{I})$ *is invertible then* $\boldsymbol{\Lambda}$ *is given by* (48). *Otherwise it is given by* (49)*;*
2. $\boldsymbol{\Lambda} = 0$*:* $\boldsymbol{\lambda}$ *and* $\mathbf{A}$ *are given by* (50a), (50b) and (50c) *with* $\boldsymbol{M}^* = \pm \mathbb{I}_d \partial_{\boldsymbol{\lambda}} u \times (\partial_{\boldsymbol{\lambda}} u \times \boldsymbol{\lambda})$. *Once this system has been solved,* $\boldsymbol{\Omega}$ *is given by* (46).

Compared to the equations found by Scheeres (2006), we have obtained five equations as opposed to two equations due to the extra degrees of freedom in our system. If the pseudo-rigid body is incompressible, the five equations reduce to four equations through the constraint $d_1 d_2 d_3 = 1$. The angular momentum vector also has a more complicated form. Again the rigid body case can be considered as a subsystem considering $\boldsymbol{\Lambda} = 0$ and d_1, d_2 and d_3 as constants.

5 Conclusion

In this article the gravitational two-body problem of a pseudo-rigid body and a rigid sphere was considered. For the case of a spherical symmetric pseudo-rigid body we reduced the system by its symmetries through an extension of the reduction procedure of the two-body problem of a rigid body and a sphere. This way the corresponding rigid body problem became a natural subsystem. We then showed that the pseudo-rigid body problem possesses similar properties and structure to the corresponding rigid body problem. In particular, we showed that the notions of locally central and planar relative equilibria coincided. We also showed that Riemann's theorem of pseudo rigid bodies had a natural extension for planar relative equilibria.

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