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ORIGINAL ARTICLE

Global geometry of non-planar 3-body motions

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Abstract The aim of this paper is to study the global geometry of non-planar 3-body motions in the realms of equivariant Differential Geometry and Geometric Mechanics. This work was intended as an attempt at bringing together these two areas, in which geometric methods play the major role, in the study of the 3-body problem. It is shown that the Euler equations of a three-body system with non-planar motion introduce non-holonomic constraints into the Lagrangian formulation of mechanics. Applying the method of undetermined Lagrange multipliers to study the dynamics of three-body motions reduced to the level of moduli space $\bar{M}$ subject to the non-holonomic constraints yields the generalized Euler-Lagrange equations of non-planar three-body motions in $\bar{M}$. As an application of the derived dynamical equations in the level of $\bar{M}$, we completely settle the question posed by A. Wintner in his book [The analytical foundations of Celestial Mechanics, Sections 394–396, 435 and 436. Princeton University Press (1941)] on classifying the constant inclination solutions of the three-body problem.

Keywords Three-body problem · Kinematic geometry · Non-holonomic mechanical systems · Generalized Euler-Lagrange equations · Constant inclination solutions

1 Introduction

Mechanics is quite obviously geometric, yet the traditional approach to the subject is based mainly on differential equations. This approach was recently augmented using modern geometric methods such as differential geometry and the theory of Lie groups and Lie algebras to reveal qualitative aspects of the theory; the newborn field is known as geometric mechanics. In particular, classical and celestial mechanics, as the oldest branches of science, have

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undergone a long evolution since 1687, the year of publication of Newton's *Principia mathematica*.

In his study of mechanics around 1840, Jacobi initiated a geometric approach by reformulating Lagrange's least action principle; but, due to the lack of necessary mathematical tools at that time such as topology, fiber bundles, Lie groups, etc., the approach was eventually abandoned by Jacobi himself who switched over to the new Hamiltonian approach and developed the theory jointly with Hamilton. However, his original ideas were not forgotten because modern topology and differential geometry grew out of Jacobi's geometric approach a few decades later when Riemann and Poincaré developed the ideas in a more general mathematical setting, which continued into the twentieth century. But Jacobi's original approach, as applied to the 3-body problem, has somehow been overlooked. In recent years, work of Hsiang and Straume (2007, 2008) has revitalized the Riemannian geometric approach to the 3-body problem—in the framework of modern equivariant differential geometry.

In this paper, our purpose is to bring together the frameworks of equivariant differential geometry and geometric mechanics in the study of the general 3-body problem. For non-planar 3-body motions, we derive the generalized Euler-Lagrange equations in the level of moduli space $\bar{M}$. As an application of the derived dynamical equations, a problem raised by Wintner (1941) on the classification of constant inclination solutions of the 3-body problem is answered.

According to Wintner's description of the 3-body problem in space, the equations of motions can be written as a three-degrees-of-freedom non-autonomous Hamiltonian system, where the variables are the three mutual distances, their conjugate momenta, and time which is taken as the inclination ψ of the plane of motion (i.e., the plane of the m-triangle formed by the three bodies) with respect to the invariable plane (i.e., the plane passing through the center of mass and perpendicular to the angular momentum vector). This program has been carried out by Wintner (1941). If the inclination $\psi = \psi(t)$ is not constant along a solution curve, then the time can be eliminated from the Hamiltonian, giving rise to an autonomous Hamiltonian system with three degrees of freedom by having the inclination as the time. This makes it important to classify all the solutions of the three-body problem for which the inclination is constant. An obvious case is that of planar motions, where $\psi = 0$. But $\psi(t) =$ constant is possible for some non-planar motions as well, namely two classes of isosceles three-body motions with $\psi = \pi/2$: those having the line of the angular momentum (which lies on the plane of motion with this inclination) and the other having the line of nodes (i.e., the line of intersection of the plane of motion and the invariable plane) as the symmetry axis of the isosceles triangles. In his study of constant inclination solutions, Cabral (1990) proved that those isosceles solutions are the only ones with $\psi = \pi/2$, while leaving the question of whether or not there exist solutions with constant inclination between 0 and $\pi/2$ as an interesting open problem.

We study the problem of existence of these constant inclination solutions in Sect. 4; in fact, we show that the planar solutions and the two types of non-planar isosceles ones are the only constant inclination solutions of the Newtonian three-body problem.

2 The setting

Let $\Re^3$ denote the Euclidean 3-space. We consider three bodies of masses m_i, $i = 1, 2, 3$, in $\Re^3$ and study their motions under Newtonian attraction.

2.1 The local and global characterization of 3-body trajectories

The classical 3-body problem in Celestial Mechanics studies the local and global geometry of trajectories of a 3-body system as a conservative system with potential energy $-U$, where

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \tag{1}$$

is the Newtonian potential function and $r_{ij} = \left|\mathbf{a}_i - \mathbf{a}_j\right|$ are the mutual distances. Here $m_i > 0$ are the masses, and $\mathbf{a}_i = (x_i, y_i, z_i)$ are the position vectors in Euclidean 3-space with respect to an inertial frame. Then the trajectories are locally characterized by Newton's equations

$$m_i \ddot{\mathbf{a}}_i = \frac{\partial U}{\partial \mathbf{a}_i} = \frac{m_i m_j}{r_{ij}^3}\left(\mathbf{a}_j - \mathbf{a}_i\right) + \frac{m_i m_k}{r_{ik}^3}\left(\mathbf{a}_k - \mathbf{a}_i\right), \quad \{i, j, k\} = \{1, 2, 3\}. \tag{2}$$

This equation is a 2-order ODE in Euclidean space $\Re^9$, and hence a trajectory is completely determined by the initial positions and velocities of the particles—in agreement with the deterministic laws of classical mechanics.

The basic kinematic quantities are

$$I = \sum m_i \left|\mathbf{a}_i\right|^2, \quad T = \frac{1}{2} \sum m_i \left|\dot{\mathbf{a}}_i\right|^2, \quad \mathbf{\Omega} = \sum m_i \left(\mathbf{a}_i \times \dot{\mathbf{a}}_i\right) \tag{3}$$

which are, respectively, the total (polar) moment of inertia, kinetic energy, and the angular momentum. Their interactions with the potential function U play a major role in the dynamics of the 3-body problem. In fact, it is fairly easy to deduce the classical conservation laws from the system (2), namely the invariance of the linear momentum $\sum m_i \dot{\mathbf{a}}_i$, that of the angular momentum vector $\mathbf{\Omega}$, and of the total energy

$$h = T - U. \tag{4}$$

On the other hand, the trajectories can be globally characterized using the basic action principles in mechanics due to Lagrange and Hamilton. These principles are quite different but somehow dual to each other. In either case, trajectories can be determined as solutions of a suitable boundary value problem—namely that for a given pair of points P, Q, what the trajectories

$$\gamma(t), \ t_0 \leq t \leq t_1$$

with $\gamma(t_0) = P$ and $\gamma(t_1) = Q$ are. The solutions are extrema of an action integral $J(\gamma)$ of any of the following two types:

$$\textit{Lagrange}: \quad J_1(\gamma) = \int_\gamma T\,dt, \quad \text{fixed energy } h \tag{5}$$

$$\textit{Hamilton}: \quad J_2(\gamma) = \int_\gamma (T + U)\,dt, \quad \text{fixed time interval } [t_0, t_1]. \tag{6}$$

Now, as usual, one can utilize the invariance of linear momentum by choosing the origin of the inertial frame at the center of mass. This reduces the study of 3-body trajectories to that of the associated time parametrized curves

$$\gamma(t) = \left(\mathbf{a}_1(t), \mathbf{a}_2(t), \mathbf{a}_3(t)\right)$$

in the 6-dimensional Euclidean configuration space

$$M = \left\{(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)\,;\ \sum m_i \mathbf{a}_i = 0\right\} \cong \Re^6. \tag{7}$$

The geometric reduction method that will be used in this project dates back to Jacobi (1840), who geometrized classical mechanics by reformulating Lagrange's least action principle. It is in fact worth noting that in the action integral $J_1(\gamma)$ (as in (5)), time is allowed to vary, i.e., the limit of integration is not fixed. This awkwardness led Jacobi to suggest that the time differential be eliminated from $J_1(\gamma)$. He introduced the *kinematics metric* on M

$$ds^2 = 2T\,dt^2 = \sum m_i \left(dx_i^2 + dy_i^2 + dz_i^2\right) \tag{8}$$

which represents the kinetic energy. Then, for a fixed energy level h, he considered the modified *dynamical* metric

$$ds_h^2 = (U + h)\,ds^2 \tag{9}$$

and observed that

$$\sqrt{2}J_1(\gamma) = \sqrt{2}\int_\gamma T\,dt = \int_\gamma \sqrt{U+h}\,ds = \int_\gamma ds_h$$

is the arc-length of the virtual motion γ in M with the Riemannian metric (9). Consequently, trajectories of Newton's equations at a fixed energy level h are precisely the geodesics in M with respect to the metric ds_h^2 (for further information on this geometric approach, see Hsiang and Straume 2007, 2008).

2.2 $SO(3)$-symmetry and reduction to the congruence moduli and shape space level

In order to contribute to Hsiang-Straume's setting (see e.g., Hsiang and Straume 2007), we shall assume the following setup. An *oriented m-triangle* is a pair $(\mathbf{X}, \mathbf{n})$, where $\mathbf{X} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ represents the position of the 3-body system and $\mathbf{n}$ is a unit normal vector (i.e., $\mathbf{a}_i \cdot \mathbf{n} = 0$ for all i). An m-triangle is called *degenerate* if the three masses are aligned. A non-degenerate m-triangle is said to be *positively* (resp. *negatively*) *oriented* if $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n})$ is a right-handed (resp. left-handed) frame at every instant of time. The squared norm of $\mathbf{X}$ with respect to the kinematic metric is the moment of inertia $|\mathbf{X}|^2 = I = \rho^2$, so ρ is the natural size function for m-triangles. The rotation group $SO(3)$ acts naturally on m-triangles, and the $SO(3)$-orbit of an m-triangle is its congruence class in the usual geometric sense.

It is convenient to replace the Euclidean space M in (7) by the space of all oriented m-triangles having the obvious induced action of $SO(3)$ and an invariant kinematic Riemannian structure. Then, there is the map projection

$$\pi : M \to \bar{M} = M/SO(3) \cong \Re^3 \tag{10}$$

which identifies the orbit space $\bar{M}$, called the *congruence moduli space*, with the usual 3-space. This fits in such a fashion that the equatorial plane $z = 0$ represents congruence classes of degenerate triangles, and the semi-space $z > 0$ (resp. $z < 0$) represents positively (resp. negatively) oriented m-triangles. It turns out that $\bar{M}$ is homeomorphic to $\Re^3$, as indicated in (10), and they are also diffeomorphic away from the origin ($\rho = 0$). Naturally, the subset $M^* = (\rho = 1)$ or *unit sphere* of $\bar{M}$ represents similarity classes of m-triangles and is, therefore, called the *shape space*. Namely, a point in M^* represents a homothety class of a triangle (of size $\rho > 0$), and it is an important fact that the shape space is actually the 2-sphere

$$M^* \cong S^2 : x^2 + y^2 + z^2 = \rho^4 = 1. \tag{11}$$

However, with the induced metric $d\sigma^2 = d\bar{s}^2|_{M^*}$, the shape space is actually a round sphere of radius $1/2$

$$\left(M^*, d\sigma^2\right) = S^2\left(1/2\right).$$

Therefore, as a Riemannian cone over M^*, the kinematic metric on the moduli space $\bar{M}$ can be expressed as

$$d\bar{s}^2 = d\rho^2 + \rho^2 d\sigma^2 = d\rho^2 + \rho^2 \frac{\left(d\varphi^2 + \sin^2\varphi d\theta^2\right)}{4}, \tag{12}$$

where (ρ, φ, θ) are the spherical coordinates on $\bar{M} \cong \Re^3$, and (φ, θ) is any choice of the spherical polar coordinates on $M^* \cong S^2$.

A motion of m-triangles is a parametrized curve $t \to \gamma(t)$ in the 6-dimensional configuration space M. Such a curve may, for example, be a solution of Newton's equations and hence represents a solution (or trajectory) of the 3-body problem. The above reduction technique can replace $\gamma(t)$ by either its moduli curve or shape curve

$$\bar{\gamma}(t) = (\rho(t), \varphi(t), \theta(t)), \ \gamma^*(t) = (\varphi(t), \theta(t)) \tag{13}$$

respectively in the 3-dimensional space $\bar{M} = \Re^3$ and its 2-sphere $M^* = S^2$. The kinetic energy $\bar{T}$ of $\bar{\gamma}(t)$ is encoded by the above metric (12), namely

$$d\bar{s}^2 = 2\bar{T}dt^2, \ \bar{T} = T - T^\omega, \tag{14}$$

where T^ω is the purely rotational kinetic energy of the motion $\gamma(t)$, which can be determined explicitly from $\gamma(t)$ and the angular momentum vector $\mathbf{\Omega}$. The reconstruction of the motion $\gamma(t)$ from the knowledge of the curve $\bar{\gamma}(t)$, with a given constant angular momentum vector $\mathbf{\Omega}$, is a purely mechanical lifting procedure in (10), which yields a unique curve $\gamma(t)$ up to congruence.

2.3 Kinematics in the configuration space and the Euler equations of motion

The size of an m-triangle $\mathbf{X}$, which is naturally measured by its Euclidean length in the configuration space M, can also be given by the following mass-dependent inner product

$$\mathbf{X} \cdot \mathbf{Y} = m_1\mathbf{a}_1 \cdot \mathbf{b}_1 + m_2\mathbf{a}_2 \cdot \mathbf{b}_2 + m_3\mathbf{a}_3 \cdot \mathbf{b}_3, \tag{15}$$

where $\mathbf{X} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $\mathbf{Y} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$; this construction is due to Jacobi. For convenience, we define

$$\begin{aligned} \omega \times \mathbf{X} &= (\omega \times \mathbf{a}_1, \omega \times \mathbf{a}_2, \omega \times \mathbf{a}_3), \quad \omega \in \Re^3, \\ \mathbf{X} \times \mathbf{Y} &= \sum m_i \mathbf{a}_i \times \mathbf{b}_i, \end{aligned} \tag{16}$$

for which we have the scalar triple product (also called the mixed product) identity

$$\omega \times \mathbf{X} \cdot \mathbf{Y} = \omega \cdot \mathbf{X} \times \mathbf{Y}, \quad \omega \in \Re^3. \tag{17}$$

The infinitesimal generators of the $SO(3)$-action on M are the rotational (or Killing) vector fields

$$\mathbf{X} \to \omega \times \mathbf{X}, \quad \omega \in \Re^3 \simeq so(3)$$

of fixed angular velocity ω. These vectors are tangent to the $SO(3)$-orbits. Thus, at each $\mathbf{X}$ the tangent space $T_{\mathbf{X}}M \simeq M$ has an orthogonal decomposition into *vertical* and *horizontal* vectors, where the vertical ones are the above Killing vectors $\omega \times \mathbf{X}$ and the horizontal vectors $\mathbf{Y}$ are characterized by $\mathbf{X} \times \mathbf{Y} = 0$, due to (17).

For any virtual motion $\mathbf{X}(t)$ in M, the velocity vector at every instant of time t has the above-mentioned type of splitting, namely

$$\dot{\mathbf{X}} = \frac{d}{dt}\mathbf{X} = \dot{\mathbf{X}}^{\omega} + \dot{\mathbf{X}}^{h} = (\omega \times \mathbf{X}) + \dot{\mathbf{X}}^{h}, \tag{18}$$

where $\omega = \omega(t)$ is commonly referred to as the (instantaneous) *angular velocity* of the motion. Correspondingly, kinetic energy splits as the sum

$$T = \frac{1}{2}\left|\omega \times \mathbf{X}\right|^2 + \frac{1}{2}\left|\dot{\mathbf{X}}^{h}\right|^2 = T^{\omega} + T^{h} \tag{19}$$

of purely rotational and horizontal kinetic energy, respectively. The motion is called *horizontal* if the velocity is always horizontal.

Using (17) we can also deduce the following relationship between the angular momentum and angular velocity of a virtual motion, namely

$$\boldsymbol{\Omega} = \mathbf{X} \times \dot{\mathbf{X}} = \mathbf{X} \times \dot{\mathbf{X}}^{\omega} = \mathbf{X} \times (\omega \times \mathbf{X}). \tag{20}$$

Indeed, to each m-triangle $\mathbf{X}$, there correspond the *inertia operator*

$$\mathbf{I}_{\mathbf{X}} : \Re^3 \to \Re^3, \quad \omega \mapsto \mathbf{X} \times (\omega \times \mathbf{X}) \tag{21}$$

relating the two vectors ω and $\boldsymbol{\Omega}$, and the *inertia tensor*

$$B_{\mathbf{X}}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \times \mathbf{X}) \cdot (\mathbf{v} \times \mathbf{X}) = \sum m_i\, (\mathbf{u} \times \mathbf{a}_i) \cdot (\mathbf{v} \times \mathbf{a}_i)\,, \quad \mathbf{u}, \mathbf{v} \in \Re^3 \tag{22}$$

which is a bilinear symmetric form on Euclidean 3-space satisfying

$$B_{\mathbf{X}}(\mathbf{u}, \mathbf{v}) = \mathbf{I}_{\mathbf{X}}(\mathbf{u}) \cdot \mathbf{v}.$$

In the orthogonal splitting (18) of the velocity of a virtual motion $\mathbf{X}(t)$, the horizontal component can further split into two summands

$$\dot{\mathbf{X}}^{h} = \dot{\mathbf{X}}^{\rho} + \dot{\mathbf{X}}^{\sigma} = \frac{\dot{\rho}}{\rho}\mathbf{X} + \dot{\mathbf{X}}^{\sigma} \tag{23}$$

representing the change in size and shape, respectively. Correspondingly, the total kinetic energy splits as

$$T = T^{\omega} + T^{h} = T^{\omega} + \left(T^{\rho} + T^{\sigma}\right) = \frac{1}{2}\left|\omega \times \mathbf{X}\right|^2 + \left(\frac{1}{2}\dot{\rho}^2 + T^{\sigma}\right). \tag{24}$$

Let $(\mathbf{X}, \mathbf{n})$ be a non-degenerate oriented m-triangle. It is possible to choose eigenvectors of the inertia tensor (22) that constitute a positive orthonormal frame

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}) \in SO(3),$$

where $\mathbf{u}_1, \mathbf{u}_2$ lie in the plane of motion spanned by the position vectors of the bodies and $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{n}$.

By definition, $B_{\mathbf{X}}(\mathbf{n}, \mathbf{n}) = I$ (which explains the name of $B_{\mathbf{X}}$) and

$$B_{\mathbf{X}}(\mathbf{u}_1, \mathbf{u}_1) = \lambda_1, \quad B_{\mathbf{X}}(\mathbf{u}_2, \mathbf{u}_2) = \lambda_2, \quad B_{\mathbf{X}}(\mathbf{u}_1, \mathbf{u}_2) = 0, \tag{25}$$

where the two eigenvalues λ_i can be expressed (Hsiang and Straume 2007) as

$$\{\lambda_1, \lambda_2\} = \frac{1}{2}\left(I \pm \sqrt{I^2 - 16m_1m_2m_3\Delta^2}\right) = \frac{I}{2}(1 \pm \sin\varphi), \tag{26}$$

using spherical polar coordinates (φ, θ) on the 2-sphere M^*, in which Δ is the area of the non-degenerate oriented m-triangle $(\mathbf{X}, \mathbf{n})$. The eigenvalue in the normal direction $\pm\mathbf{n}$ is the largest one

$$\lambda_3 = \lambda_1 + \lambda_2 = I = \rho^2.$$

To a continuous motion of oriented m-triangles, there corresponds a moving orthonormal eigenframe

$$F(t) = \{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{n}(t)\}. \tag{27}$$

In particular, $t \mapsto F(t)$ is a parametrized curve in $SO(3)$.

For non-planar 3-body motions, the normal vector $\mathbf{n} = \mathbf{n}(t)$ of the m-triangles formed by the three bodies moving along $\gamma(t) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ describes a curve on the 2-sphere S^2. Therefore, the whole 3-body motion is essentially encoded into two spherical curves $\gamma^*(t)$ and $\mathbf{n}(t)$ together with the size function $\rho(t)$.

Let F be the chosen moving intrinsic orthonormal frame along $\gamma(t)$, where $\mathbf{u}_1, \mathbf{u}_2$ point in the principal inertia directions in the plane of the m-triangle, and the associated principal moments of inertia that depend solely on the moduli curve $\bar{\gamma}(t)$ be

$$\lambda_1 = \frac{I}{2}(1 + \sin\varphi), \quad \lambda_2 = \frac{I}{2}(1 - \sin\varphi), \quad \lambda_3 = I = \lambda_1 + \lambda_2.$$

Then the coordinates of the conserved angular momentum vector $\mathbf{\Omega}$ relative to the moving frame F, namely

$$g_1 = \mathbf{\Omega}\cdot\mathbf{u}_1, \quad g_2 = \mathbf{\Omega}\cdot\mathbf{u}_2, \quad g_3 = \mathbf{\Omega}\cdot\mathbf{n}, \tag{28}$$

satisfy the following system of ODEs

$$\begin{aligned} \dot{g}_1 &= g_2\left[\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right)g_3 + \frac{1}{2}\dot{\theta}\cos\varphi\right] \\ \dot{g}_2 &= g_1\left[\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right)g_3 - \frac{1}{2}\dot{\theta}\cos\varphi\right] \\ \dot{g}_3 &= g_1g_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right), \end{aligned} \tag{29}$$

which will be referred to as the *kinematic Euler equations* of motion (Hsiang and Straume 2008). This is a first order system on the 2-sphere of radius $|\mathbf{\Omega}|$ and is in fact a natural generalization of the classical Euler equations for a rigid body (see e.g., Arnold 1978).

We now turn to the angular momentum

$$\mathbf{\Omega} = \mathbf{X} \times \dot{\mathbf{X}}^{\omega} = \mathbf{X} \times (\omega \times \mathbf{X})$$

which is usually assumed to be a fixed vector along the z-axis. The expansion

$$\mathbf{\Omega} = g_1\mathbf{u}_1 + g_2\mathbf{u}_2 + g_3\mathbf{n} \tag{30}$$

defines its (time-dependent) coordinate vector (g_1, g_2, g_3) relative to the moving frame $F(t)$. The inner product of $\mathbf{\Omega}$ with a vector $\mathbf{v}$ may be written as

$$\mathbf{\Omega}\cdot\mathbf{v} = (\mathbf{X}\times\omega)\times\mathbf{X}\cdot\mathbf{v} = (\mathbf{X}\times\omega)\cdot(\mathbf{X}\times\mathbf{v}) = B_{\mathbf{X}}(\omega, \mathbf{v}).$$

Hence, by letting $\mathbf{v} = \mathbf{v}_i$ be any of the vectors from F,

$$g_i = \Omega \cdot \mathbf{v}_i = B_{\mathbf{X}}(\omega, \mathbf{v}_i) = \lambda_i \omega \cdot \mathbf{v}_i, \quad i = 1, 2, 3,$$

and the angular velocity reads

$$\boldsymbol{\omega} = \frac{g_1}{\lambda_1}\mathbf{u}_1 + \frac{g_2}{\lambda_2}\mathbf{u}_2 + \frac{g_3}{I}\mathbf{n}. \tag{31}$$

It follows from (29)–(31) that

$$\boldsymbol{\Omega} \cdot \omega \times \mathbf{n} = \dot{g}_3, \tag{32}$$

which may provide an elegant and suggestive viewpoint in the study of non-planar 3-body motions in general.

In particular, the rotational kinetic energy can be expressed as

$$T^{\omega} = \frac{1}{2}\left|\dot{\mathbf{X}}^{\omega}\right|^2 = \frac{1}{2}B_{\mathbf{X}}(\omega, \omega) = \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{I}\right). \tag{33}$$

2.4 Geometry of shape curves with a shape potential function

Since shape curves play a major role in this paper, it is convenient to collect some basic formulas concerning their differential geometry and those concerning their shape potential function.

For a given oriented shape curve γ^*, let τ^* (resp. ν^*) be the unit tangent vector (resp. unit normal vector) in the positive direction such that (τ^*, ν^*) is a positively oriented frame of the 2-sphere M^*. We consider a (regular) time, parametrized curve $\gamma^*(t) = (\varphi(t), \theta(t))$ and set $s = s(t) \geq 0$ to be the arc-length along the curve. Then,

$$\tau^* = \frac{d\gamma^*}{ds} = \frac{1}{v}\left(\dot{\varphi}\frac{\partial}{\partial\varphi} + \dot{\theta}\frac{\partial}{\partial\theta}\right), \quad \nu^* = \frac{1}{v}\left(-\dot{\theta}\sin\varphi\frac{\partial}{\partial\varphi} + \dot{\varphi}\frac{1}{\sin\varphi}\frac{\partial}{\partial\theta}\right), \tag{34}$$

where the velocity vector field of the curve is

$$\frac{d\gamma^*}{dt} = v\tau^*, \quad v = \sqrt{\dot{\varphi}^2 + (\sin^2\varphi)\dot{\theta}^2}. \tag{35}$$

For an arc-length parametrized curve $\mathbf{x}(s) = (x(s), y(s), z(s))$, the geodesic curvature function K^* can be calculated by

$$K^*(s) = \left(\mathbf{x}(s) \times \mathbf{x}'(s)\right) \cdot \mathbf{x}''(s),$$

where $\mathbf{x}'(s) = \frac{d\mathbf{x}}{ds}$, etc. Applying the above formula for the shape curve $\gamma^*(s) = \mathbf{x}(s)$ with $x = \sin\varphi\cos\theta,\ \ y = \sin\varphi\sin\theta$, and $z = \cos\varphi$ yields

$$\begin{aligned} K^* &= (\cos\varphi)\theta'(1 + \varphi'^2) + \sin\varphi(\varphi'\theta'' - \theta'\varphi'') \\ &= \frac{1}{v^3}\left\{(\cos\varphi)\dot{\theta}(v^2 + \dot{\varphi}^2) + \sin\varphi(\dot{\varphi}\ddot{\theta} - \dot{\theta}\ddot{\varphi})\right\}. \end{aligned} \tag{36}$$

2.5 The Newtonian (shape) potential function

Let U^* be the restriction of the Newtonian potential function U in (1) to the unit sphere M^*. Hsiang and Straume (2007) showed that U can be defined at the moduli space level $\bar{M}$ in terms of spherical coordinates $\left(\rho^2, \varphi, \theta\right)$, and consequently

$$U = \frac{1}{\rho} U^*(\varphi, \theta)$$

which is due to the homogeneity of the potential function U. Therefore, the shape potential function reads

$$U^* = \sum_{i=1}^{3} \frac{\hat{m}_i^{3/2} \left(m_i^*\right)^{-1/2}}{\sqrt{1 - \sin\varphi \cos(\theta - \theta_i)}}, \tag{37}$$

where $\hat{m}_i = m_j m_k$, $m_i^* = \frac{1}{2}(1 - m_i)$, and $(\theta_1, \theta_2, \theta_3) = (0, \beta_3, -\beta_2)$, which are given by $\beta_i = \arccos\left(\frac{\hat{m}_i - m_i}{\hat{m}_i + m_i}\right)$, for $\{i, j, k\} = \{1, 2, 3\}$.

Recall that the induced metric of the shape space $M^* = S^2(1)$ from the Euclidean 3-space is that of the round sphere of radius one (cf. (12)): $d\varphi^2 + \sin^2\varphi d\theta^2$.

Thus, the gradient field of $U^* = U^*(\varphi, \theta)$ on the shape space $\left(M^*, d\varphi^2 + \sin^2\varphi d\theta^2\right)$ is the following vector field

$$\nabla U^* = U_\varphi^* \frac{\partial}{\partial\varphi} + \frac{U_\theta^*}{\sin^2\varphi} \frac{\partial}{\partial\theta}. \tag{38}$$

It follows that the tangential and normal derivatives of U^* along the shape curve γ^* are, respectively

$$U_\tau^* = \frac{\partial U^*}{\partial \boldsymbol{\tau}^*} = \nabla U^* \cdot \tau^* = \frac{1}{v}\left(\dot{\varphi} U_\varphi^* + \dot{\theta} U_\theta^*\right), \tag{39}$$

$$U_\nu^* = \frac{\partial U^*}{\partial \nu^*} = \nabla U^* \cdot \nu^* = \frac{1}{v}\left(-\dot{\theta} \sin\varphi U_\varphi^* + \dot{\varphi} \frac{1}{\sin\varphi} U_\theta^*\right). \tag{40}$$

3 Dynamics in the moduli space and intrinsic geometry of the shape curve

Recall that for non-planar 3-body motions, the rotational kinetic energy can be expressed by (33) as

$$T^\omega = \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{\lambda_3}\right)$$

where

$$\lambda_1 = \frac{\rho^2}{2}(1 + \sin\varphi), \quad \lambda_2 = \frac{\rho^2}{2}(1 - \sin\varphi), \quad \lambda_3 = \lambda_1 + \lambda_2 = \rho^2$$

and the coordinates of the angular momentum vector $\mathbf{\Omega}$ in (30) read

$$g_1^2 + g_2^2 + g_3^2 = |\mathbf{\Omega}|^2.$$

Setting

$$\bar{T} := T - T^\omega, \quad \bar{U} := U - T^\omega, \tag{41}$$

we can define the Lagrangian of our dynamical mechanical system on the level of moduli space $\bar{M}$, which reads $\bar{L} = \bar{T} + \bar{U}$.

From (12), (14) and (24), it follows that the Lagrangian $\bar{L}$ on $\bar{M}$ is given by

$$\bar{L} = \left(T^{\rho} + T^{\sigma}\right) + \left(U - T^{\omega}\right) = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}\left(\dot{\varphi}^2 + \sin^2\varphi\dot{\theta}^2\right) + \frac{U^*}{\rho} - \frac{1}{2}\left\{g_1^2\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) + g_2^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) + \frac{|\mathbf{\Omega}|^2}{\lambda_3}\right\}. \tag{42}$$

In the study of the general three-body problem, the inclination angle of the plane of motion $\psi = \psi(t)$ with respect to the plane passing through the center of mass and perpendicular to the angular momentum vector $\mathbf{\Omega}$ is allowed to vary in $[0, \pi/2]$. Thus g_i's are not constant.

Therefore, the configurations of our dynamical system can be specified by the *generalized coordinates* $q = (q_1, q_2, q_3, q_4, q_5) = (\rho, \varphi, \theta, g_1, g_2)$ in which $g_1,\ g_2$ are referred to as the *kinematical* coordinates and are subject to the following constraints (cf. (29)):

$$\begin{aligned} C_\alpha &= \dot{g_1} + g_2 g_3\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) - \frac{g_2\cos\varphi}{2}\dot{\theta} = 0, \\ C_\beta &= \dot{g_2} + g_1 g_3\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_1}\right) + \frac{g_1\cos\varphi}{2}\dot{\theta} = 0. \end{aligned} \tag{43}$$

Since the constraints C_α and C_β are functions of both the generalized coordinates q and the generalized velocities $\dot{q}$, they are *non-holonomic*. Comparing C_α and C_β with the standard form of linearly non-holonomic constraints (see e.g., Whittaker 1937), we can rewrite them as

$$C_\alpha = \sum_{i=1}^{5}\left(a_{\alpha i}\dot{q}_i\right) + b_\alpha = 0$$

$$C_\beta = \sum_{i=1}^{5}\left(a_{\beta i}\dot{q}_i\right) + b_\beta = 0,$$

where

$$\begin{aligned} &a_{\alpha 1} = 0 = a_{\alpha 2}, \quad a_{\alpha 3} = -\frac{g_2\cos\varphi}{2}, \quad a_{\alpha 4} = 1, \quad a_{\alpha 5} = 0 \\ &a_{\beta 1} = 0 = a_{\beta 2}, \quad a_{\beta 3} = \frac{g_1\cos\varphi}{2}, \quad a_{\beta 4} = 0, \quad a_{\beta 5} = 1. \end{aligned}$$

As the constraints are non-holonomic, the equations expressing them are non-integrable and so it is not possible to express the kinematical coordinates (i.e., $g_1,\ g_2$) in terms of the others. Therefore, in order to derive the dynamical equations on the level of $\bar{M}$, it is convenient to apply the method of *undetermined Lagrange multipliers*.

It is evident that since $C_\alpha = 0 = C_\beta$,

$$\lambda_\alpha C_\alpha + \lambda_\beta C_\beta = 0$$

where $\lambda_\alpha,\ \lambda_\beta$ are Lagrange multipliers.

The dynamical equations resulting from the extension of Hamilton's principle to our non-holonomic system (see e.g., Whittaker 1937) read

$$\frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{q}_i}\right) - \frac{\partial\bar{L}}{\partial q_i} = \lambda_\alpha a_{\alpha i} + \lambda_\beta a_{\beta i}\ ; \quad i = 1, \ldots, 5 \tag{44}$$

It follows immediately that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\rho}}\right)-\frac{\partial \bar{L}}{\partial \rho}=0=\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\varphi}}\right)-\frac{\partial \bar{L}}{\partial \varphi}\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\theta}}\right)-\frac{\partial \bar{L}}{\partial \theta}=-\lambda_\alpha\frac{g_2\cos\varphi}{2}+\lambda_\beta\frac{g_1\cos\varphi}{2}\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{g}_1}\right)-\frac{\partial \bar{L}}{\partial g_1}=g_1\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_3}\right)=\lambda_\alpha\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{g}_2}\right)-\frac{\partial \bar{L}}{\partial g_2}=g_2\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)=\lambda_\beta.
\end{aligned}\tag{45}$$

By the last two equations in (45), the Lagrange multipliers read

$$\lambda_\alpha=\frac{g_1}{\rho^2}\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right),\quad \lambda_\beta=\frac{g_2}{\rho^2}\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right).\tag{46}$$

Substituting the Lagrange multipliers into (45) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\rho}}\right)-\frac{\partial \bar{L}}{\partial \rho}=0\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\varphi}}\right)-\frac{\partial \bar{L}}{\partial \varphi}=0\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\theta}}\right)-\frac{\partial \bar{L}}{\partial \theta}=2\left(\frac{g_1g_2}{\rho^2}\right)\tan\varphi
\end{aligned}\tag{47}$$

and then, by straightforward calculations using (42), we obtain the *generalized Euler-Lagrange equations* of non-planar 3-body motions on the level of the moduli space $\bar{M}$:

$$\begin{aligned}
\ddot{\rho}-\frac{\rho}{4}\left(\dot{\varphi}^2+\sin^2\varphi\dot{\theta}^2\right)+\frac{U^*}{\rho^2}-\frac{1}{\rho^3}\left\{g_1^2\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)+g_2^2\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+|\Omega|^2\right\}&=0\\
\ddot{\varphi}+2\left(\frac{\dot{\rho}}{\rho}\right)\dot{\varphi}-\frac{1}{2}\dot{\theta}^2\sin(2\varphi)-\frac{4}{\rho^3}U^*_\varphi-\frac{4\cos\varphi}{\rho^4}\left(\frac{g_1^2}{(1+\sin\varphi)^2}-\frac{g_2^2}{(1-\sin\varphi)^2}\right)&=0\\
\ddot{\theta}+2\left(\frac{\dot{\rho}}{\rho}+\dot{\varphi}\cot\varphi\right)\dot{\theta}-\left(\frac{4}{\rho^3\sin^2\varphi}\right)U^*_\theta-\frac{16g_1g_2}{\rho^4\sin(2\varphi)}&=0.
\end{aligned}$$

3.1 Geodesic curvature function of the shape curve

Substituting $\ddot{\varphi}$ and $\ddot{\theta}$ from the last two equations of the above system of generalized Euler-Lagrange equations of motion into (36) gives

$$K^*=\frac{1}{\rho^3}\left(\frac{4U^*_\nu}{\upsilon^2}\right)+\frac{1}{\rho^4}\left(\frac{2G}{\upsilon^3}\right),\tag{48}$$

where $\upsilon=|\frac{d\gamma^*}{dt}|=\sqrt{\dot{\varphi}^2+(\sin^2\varphi)\dot{\theta}^2}$ is the speed of shape curve, U^*_ν is the normal derivative of the shape potential U^* along γ^* as in (40), and

$$G = \left(\frac{4g_1g_2}{\cos\varphi}\right)\dot{\varphi} - \dot{\theta}\sin(2\varphi)\left(\frac{g_1^2}{(1+\sin\varphi)^2} - \frac{g_2^2}{(1-\sin\varphi)^2}\right), \tag{49}$$

the latter of which will be referred to as the $G\text{-}function$.

4 Constant inclination 3-body motions

As an application of the system of generalized Euler-Lagrange equations, in this section, we shall prove a conjecture of Wintner (1941) on classifying the constant inclination solutions of the three-body problem.

By a constant inclination solution we mean a three-body motion whose plane of motion makes a constant inclination angle $\psi \in \left[0, \frac{\pi}{2}\right]$ with respect to a fixed reference plane passing through the center of mass. Planar 3-body motions that include vanishing angular momentum motions are those with zero inclination. Cabral (1990) proved that the only constant inclination solutions with $\psi = \frac{\pi}{2}$ are the two types of non-planar isosceles solutions: one with the line of the angular momentum (which lies on the plane of motion with this inclination) as the symmetry axis of the isosceles triangles and the other with the line of intersection of the plane of motion and the reference plane as the symmetry axis.

In this section we will be concerned only with existence of solutions with constant inclination $0 < \psi < \frac{\pi}{2}$. For non-planar motions, the fixed reference plane is called the *invariable plane*, which is perpendicular to the (non-zero) angular momentum vector $\mathbf{\Omega}$. Hence, the constant inclination ψ is the angle between $\mathbf{\Omega}$ and the unit normal vector $\mathbf{n}$ of the plane of motion, the latter being an eigenvector of the moving eigenframe $F(t) = \{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{n}(t)\}$ defined in Sect. 2.

Theorem *There exists no solution of the three-body problem with constant inclination* $0 < \psi < \frac{\pi}{2}$.

Proof of Theorem On the contrary, suppose that there is a solution $\gamma(t)$ of the 3-body problem with constant inclination $\psi \in (0, \pi/2)$.

Recall that the coordinates (g_1, g_2, g_3) of $\mathbf{\Omega}$ with respect to the moving eigenframe $F(t)$ satisfy the kinematic Euler equations (29).

Since the inclination angle ψ is constant,

$$g_3 = \mathbf{\Omega}\cdot\mathbf{n} = Constant,$$

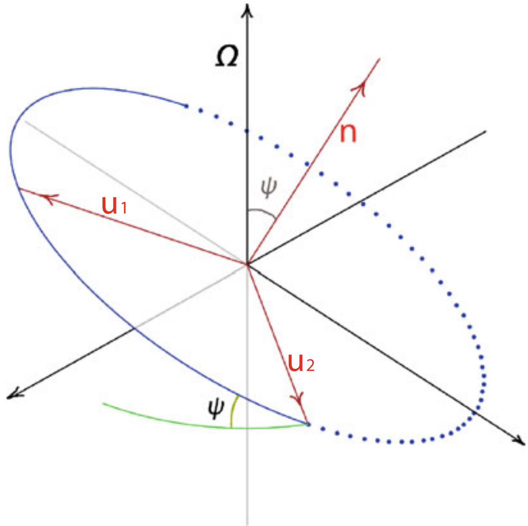
and hence

$$0 = \dot{g}_3 = g_1g_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right).$$

It follows that $g_1g_2 = 0$ (note that $\lambda_1 = \lambda_2$ just amounts to the fact that the solution curve γ is a shape invariant one: m-triangles always have the constant shape of an equilateral triangle, i.e., the associated shape curve γ^* only consists of the North pole ($\varphi = 0$) on the shape sphere; it follows that the motion is planar, and so $\psi = 0$).

There is no loss of generality in assuming $0 = g_2 = \mathbf{\Omega}\cdot\mathbf{u_2}$. Hence, $\mathbf{u_2}\perp\mathbf{\Omega}$, and consequently $\mathbf{u_2}$ lies along the line of intersection of the plane of motion and the invariable plane, which is known as the *line of nodes* (see e.g., Saari 1984). It follows that $\mathbf{\Omega} \in Span_{\Re}\{\mathbf{n}, \mathbf{u_1}\}$ (see Fig. 1), which implies

Fig. 1



$$g_1 = |\mathbf{\Omega}| \cos\left(\frac{\pi}{2} - \psi\right) = |\mathbf{\Omega}| \sin\psi > 0,$$
$$g_3 = |\mathbf{\Omega}| \cos\psi > 0.$$

By the kinematic Euler equations (29),

$$0 = \dot{g}_2 = g_1 \left[\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3} \right) g_3 - \frac{1}{2} \dot{\theta} \cos\varphi \right],$$

which yields the following *first integral*

$$\dot{\theta} = 2 \left(\frac{\cos\varphi}{(1 + \sin\varphi)^2} \right) \left(\frac{|\mathbf{\Omega}| \cos\psi}{\rho^2} \right). \tag{50}$$

Geometrically speaking, the first integral (50) says that to the solution curve $\gamma(t)$ with constant inclination $0 < \psi < \pi/2$ there corresponds on the 2-sphere M^* a shape curve $\gamma^*(\varphi(t), \theta(t))$ for which $\dot{\theta}$ is positive whenever γ^* is on the upper-hemisphere and is negative on the lower-hemisphere; therefore $\dot{\theta} = 0$ whenever γ^* meets the equator, and hence the shape curve does so orthogonally.

It is worth pointing out that changing the orientation of m-triangles formed by the three bodies moving along a given solution curve γ amounts to replacing φ by $(\pi - \varphi)$ and fixing θ along the associated shape curve $\gamma^*(\varphi, \theta)$ on the 2-sphere M^*, i.e., changing the orientation of m-triangles of a three-body motion γ_φ to which the shape curve γ^*_φ is associated yields another solution curve $\gamma_{(\pi-\varphi)}$ whose associated shape curve $\gamma^*_{(\pi-\varphi)} \subset M^*$ is the reflectional image of γ^*_φ in the equatorial plane. Note that, throughout the proof, all the geometric quantities of the above-mentioned curves are labeled with indexes φ or $(\pi - \varphi)$ appropriately.

Our original system of Euler-Lagrange equations on the level of $\bar{M}$ coupled with the first integral in (50) gives us an over-determined system of ODEs by which the moduli curve $\bar{\gamma}(t)$ associated to our solution curve must also be given since $\bar{\gamma}(t)$ is a solution of the original system of ODEs (47) on $\bar{M}$. Therefore, we may study the behavior of any geometric quanties of $\bar{\gamma}$ as a solution of the original system of ODEs (47) regardless of the first integral (50).

By (36), (40) and assuming that γ^* is the associated shape curve of $\bar{\gamma}$ as a solution of the original system (47), it is immediate that

$$K^*_{(\pi-\varphi)} = -K^*_{\varphi}, \tag{51}$$

$$\left(U^*_{\nu}\right)_{(\pi-\varphi)} = -\left(U^*_{\nu}\right)_{\varphi}. \tag{52}$$

On the other hand, for the constant inclination solution γ with $\psi \in (0, \pi/2)$ and its associated moduli curve $\bar{\gamma}$ as the solution of the above-mentioned over-determined system of ODEs, we can rewrite K^* in (48) as

$$K^* - \frac{1}{\rho^3}\left(\frac{4U^*_{\nu}}{\upsilon^2}\right) = \frac{1}{\rho^4}\left(\frac{2G}{\upsilon^3}\right) \tag{53}$$

in which the G-function (49) now reads

$$G = -4\frac{g_1^2 g_3}{\rho^2}\left(\frac{\sin\varphi\cos^2\varphi}{(1+\sin\varphi)^4}\right) \leq 0, \tag{54}$$

which follows by substituting the first integral (50) into (49). Hence, it is clear that

$$G_{(\pi-\varphi)} = G_{\varphi}. \tag{55}$$

From (51), (52) and (55), it follows that changing the orientation of m-triangles formed by the three bodies moving along the solution curve $\gamma(t)$ changes only the sign of the left-hand side of (53) while keeping the other side invariant. Therefore, G in (54) must be identically 0.

There are only two ways this quantity can be zero: either

(A) $\sin\varphi\cos\varphi \equiv 0$, or
(B) $g_1 g_3 = |\mathbf{\Omega}|^2 \sin\psi\cos\psi = 0$.

First, if $(\sin\varphi\cos\varphi) \equiv 0$, then either $\varphi \equiv 0$ or $\varphi \equiv \frac{\pi}{2}$. Suppose that $\varphi \equiv 0$. This asserts that $\lambda_1 = \lambda_2$ and the solution curve $\gamma(t)$ is shape invariant, and so the inclination $\psi = 0 \notin (0, \pi/2)$. If $\varphi \equiv \frac{\pi}{2}$ then from (50) it follows that $\dot{\theta} \equiv 0$, and hence θ is constant. Thus, the associated shape curve γ^* consists of only a single eclipse point (i.e., a point on the equator of the 2-sphere M^*) given by the constant φ and θ, which implies that the solution curve is shape invariant and so $\psi = 0 \notin (0, \pi/2)$.

Next suppose that $(|\mathbf{\Omega}|^2 \sin\psi\cos\psi) = 0$, which implies that either $\psi = 0$ or $\psi = \frac{\pi}{2}$; contrary again to the assumption that $\psi \in \left(0, \frac{\pi}{2}\right)$.

This finishes the proof of the theorem.

Corollary *The planar solutions ($\psi = 0$) and the two types of non-planar isosceles solutions (with $\psi = \pi/2$) are the only constant inclination solutions of the Newtonian three-body problem.*

This answers the question raised by Wintner (1941) on classifying the constant inclination solutions of the three-body problem.

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