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ORIGINAL ARTICLE

Global bifurcation of planar and spatial periodic solutions in the restricted n-body problem

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Abstract The paper deals with the study of a satellite attracted by n primary bodies, which form a relative equilibrium. We use orthogonal degree to prove global bifurcation of planar and spatial periodic solutions from the equilibria of the satellite. In particular, we analyze the restricted three body problem and the problem of a satellite attracted by the Maxwell's ring relative equilibrium.

Keywords Global bifurcation · Orthogonal degree · Restricted n-body problem · Ring configuration · Restricted three-body problem

1 Introduction

The restricted n-body problem is the study of the movement of a satellite attracted by n primary bodies which are moving, at a constant angular speed, around an axis. Since the mass of the satellite is small, one assumes that the satellite does not perturb the trajectories of the primaries. We shall suppose that these trajectories form a relative equilibrium and, as such, are in a plane, let us say the (x, y)-plane. In this paper, the primaries are assumed to be point masses or, equivalently, homogeneous spheres.

The purpose of this paper is to prove the existence of a global bifurcation of periodic solutions for the satellite, starting from the relative equilibria of the satellite. These solutions will form a continuum in the plane of the primaries and we shall also prove that there are other global branches of solutions out of that plane. The proof is based on the use of a topological degree for maps that commute with some symmetries and are orthogonal to the infinitesimal generators for these symmetries. We give results for a general situation and applications to

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some special cases such as the restricted three body problem and the Maxwell's Saturn ring, that is when there are n primaries, of the same mass, forming a regular polygon, and a central larger mass, as a classical model for Saturn and one ring around it. However, for the general result the primaries may have different masses and may be located at any relative equilibrium.

The study of relative equilibria for the restricted n-body setting is a classical problem and there is a vast literature for it. For instance, in the case of the restricted three-body problem, the local bifurcation of planar periodic orbits from the Lagrange points is well known (see Marchal (1990); Meyer and Hall (1991)). There is a huge number of numerical explorations for this restricted three-body problem, under a variety of hypotheses, such as the bifurcation near L_4, where the mass of the primary is the bifurcation parameter above Routh's number, Bardin (2002); Sicardy (2010), with a period doubling cascade. Sandor et al. (2000) has a study of the phase space for solutions near L_4 and Érdi et al. (2009) treats the elliptic case where one has four periods for solutions close to L_4. The stability of the orbits close to L_4 is studied in Efthymiopoulos (2005) and the connection from E_3 to L_4 is explored in Pinotsis (2009). A very complete numerical study, Doedel et al. (2007), using AUTO, shows the many different types of periodic orbits and the connections between the Lagrange points and also the secondary bifurcations along the curves in the x, y, μ space, where μ is the mass of one of the primaries. From a very applied point of view, one may cite Kasdin et al. (2005) and Gómez et al. (2000).

In the case of the Maxwell ring, besides the theoretical results of Siegel and Moser (1971) and Meyer (1999), one has also many numerical results, such as Pinotsis (2005), where the author studies numerically some families of solutions around the central body and around the ring for a low number of peripherals, with a theoretical approximation for the case of a satellite far from the ring. A theoretical study, with averaging techniques is given in Llibre and Stoica (2011) for orbits far from the set of primaries (comets) and close to one of the primaries (Hill solutions). Similarly Mavraganis and Kalvouridis (2007) proposes a regularization for collision orbits. Closer to the spirit of the present paper, we mention some of the more recent papers in the bibliography, in particular Arribas and Elipe (2004); Bang and Elmabsout (2004); Kalvouridis (2008) and Barrio et al. (2008), where a numerical classification of the different types of orbits is done.

The paper which is closer to ours, in the sense that is based on topological arguments similar to ours and giving global results for the possible connections between the relative equilibria is Maciejewski and Rybicki (2004), for the restricted three-body problem.

A final introductory comment, about topological methods, in particular in bifurcation problems, may be useful: the degree arguments, coupled with group representation ideas, give global information, i.e., an indication of where the bifurcation branches could go. Also, since the results are valid for problems which are deformation of the original problem, the method does not require high order computations and they may be applied in some degenerate cases (for instance it is not necessary that the bifurcation parameter crosses a critical value with non-zero speed; it is enough that it crosses eventually). However, knowledge of some generic property, like a Morse condition, implies an easy application of the argument. This may be not the case for problems with more parameters (see however Ize (1995)). An immediate drawback of this approach is that topological methods do not provide a detailed information on the local behavior of the bifurcating branch, such as stability or the existence of other type of solutions, like KAM tori. Other methods, such as normal forms or special coordinates, should be used for these purposes, but they only provide local information near the critical point. In a similar way, our degree arguments give only partial results on resonances and other tools should be used. Topological methods provide an interesting complement of information.

2 Setting the problem

Newton's equations describing the movement of a satellite, in rotating coordinates and with angular speed equal to 1, are

$$\ddot{x} + 2\bar{J}\dot{x} = \nabla V(x) \text{ with}$$
$$V(x) := \frac{\|\bar{I}x\|^2}{2} + \sum_{j=1}^{n} m_j \phi_\alpha(\|x - (a_j, 0)\|),$$

where $x \in \mathbb{R}^3$ is the position of the satellite and $(a_j, 0)$ is the position of a primary body with mass m_j. The function ϕ_α represents the attraction between the bodies, where we suppose that $\phi'_\alpha = -1/x^\alpha$, and we include the gravitational potential for $\alpha = 2$. The matrices $\bar{I}$ and $\bar{J}$ are defined by

$$\bar{I} = \mathrm{diag}(I, 0) \quad \text{and} \quad \bar{J} = \mathrm{diag}(J, 0),$$

where J and I are the symplectic and identity 2×2 matrices.

Here we assume that the primary bodies form a relative equilibrium. Because of the homogeneity of the potential, we may rescale the space so that the angular velocity is 1. As all relative equilibria are planar for the n-body problem, thus the positions of the primary bodies $a_j \in \mathbb{R}^2$ must satisfy the relation

$$a_i = \sum_{j=1\ (j \neq i)}^{n} m_j \frac{a_i - a_j}{\|a_i - a_j\|^{\alpha+1}}.$$

The equilibria of the satellite are just the critical points of the potential V. From the potential we can prove that all equilibria are planar. Now, we wish to find the Hessian of the potential at a planar equilibrium.

Proposition 1 *Let d_j be the distance between $x_0 = (x, y, 0)$ and the primary body $(a_j, 0) = (x_j, y_j, 0)$. The Hessian matrix of the potential is*

$$D^2V(x_0) = \left(I + \sum_{j=1}^{n} m_j A_j, -\sum_{j=1}^{n} m_j / d_j^{\alpha+1} \right),$$

where the matrices A_j are defined by

$$A_j = \frac{(\alpha+1)}{d_j^{\alpha+3}} \begin{pmatrix} (x - x_j)^2 & (x - x_j)(y - y_j) \\ (x - x_j)(y - y_j) & (y - y_j)^2 \end{pmatrix} - \frac{I}{d_j^{\alpha+1}}. \tag{1}$$

Proof Since the function $\phi_\alpha(d_j)$ has Hessian

$$D^2\phi_\alpha(d_j) = \frac{\alpha+1}{d_j^{\alpha+3}} \begin{pmatrix} (x - x_j)^2 & (x - x_j)(y - y_j) & 0 \\ (x - x_j)(y - y_j) & (y - y_j)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{I}{d_j^{\alpha+1}},$$

hence $D^2\phi_\alpha(d_j) = \mathrm{diag}(A_j, -1/d_j^{\alpha+1})$. From this fact we get the Hessian of V. □

Now we want to estimate the number of equilibria provided that the potential is a Morse function, which is more than a reasonable condition. This is a generic condition, which is met in our applications, but which could not hold in some cases. As a matter of fact, we only need that the critical points should be isolated. Because all equilibria are in the plane, we may restrict the potential to planar points.

Proposition 2 *Let us assume that the potential of the satellite is in the plane with* $\alpha \in [1, \infty)$. *Then the potential does not have maximum points. In addition, if the potential is a Morse function, then*

$$\#\textit{saddle points} = n - 1 + \#\textit{ minimum points.}$$

Moreover, since the potential has a global minimum, there are at least n saddle points.

Proof The potential in the plane has Hessian $D^2V(x_0) = I + \sum_{j=1}^{n} m_j A_j$, and the trace of $D^2V(x_0)$ is

$$T = 2 + (\alpha - 1) \sum_{j=1}^{n} \frac{m_j}{d_j^{\alpha+1}}. \tag{2}$$

Consequently, the potential does not have maximum points as the trace is positive for $\alpha \in [1, \infty)$. Moreover, we know that $V(x)$ is positive and that $V(x) \to \infty$ as $x \to \{\infty, a_1, \ldots, a_n\}$, then V has at least a global minimum in Ω. Since the gradient of $V(x)$ is dominated by the identity, for large $\|x\|$, the critical points are bounded.

Let us define the set Ω as a ball of radius ρ, minus small balls of radii ρ^{-1} with centers at a_j. Since the gradient ∇V points outward in $\partial\Omega$ provided ρ is big enough, then by the Poincaré–Hopf theorem the degree of $\nabla V(x)$ is equal to $1 - n$. Furthermore, since $V(x)$ is a Morse function, that is the critical points are non-degenerate, then this degree is the sum of the local indices. Each of these indices is the sign of the determinant of the Hessian matrix, that is 1 for a minimum and -1 for a saddle point. Then

$$1 - n = \deg_{\Omega} \nabla V = \#\text{minimum points} - \#\text{saddle points}$$

□

3 Bifurcation theorem

In order to explain our results, we may give a short description of the steps to prove the bifurcation theorem.

We wish to remark that we follow the ideas from the book (Ize and Vignoli 2003), where more general bifurcation theorems are proven. In addition, in the thesis (García-Azpeitia 2010) one may find a systematic application to different Hamiltonian systems and situations.

3.1 The bifurcation operator

Our aim is to find bifurcation of periodic solutions from the equilibria of the satellite. First, we make the change of variables from t to t/ν. Hence, the $2\pi/\nu$-periodic solutions of the differential equations are the 2π-periodic solutions of

$$-\nu^2\ddot{x} - 2\nu\bar{J}\dot{x} + \nabla V(x) = 0.$$

Let $H_{2\pi}^2(\mathbb{R}^n)$ be the Sobolev space of 2π-periodic functions, with the corresponding regularity. We define the collision points as the set $\Psi = \{a_1, \ldots, a_n\}$ and the collision-free paths as the set

$$H_{2\pi}^2(\mathbb{R}^3 \backslash \Psi) = \{x \in H_{2\pi}^2(\mathbb{R}^3) : x(t) \neq a_j\}.$$

Recall that functions in this space are continuous. Hence, we define the bifurcation operator $f : H^2_{2\pi}(\mathbb{R}^3\backslash\Psi) \times \mathbb{R}^+ \to L^2_{2\pi}$ as

$$f(x, \nu) = -\nu^2\ddot{x} - 2\nu\bar{J}\dot{x} + \nabla V(x).$$

In view of the definitions, the collision-free 2π-periodic solutions are zeros of the bifurcation operator $f(x, \nu)$. Furthermore, the operator f is well defined and continuous.

Now, we define the actions of the group $\mathbb{Z}_2 \times S^1$ on $H^2_{2\pi}(\mathbb{R}^3\backslash\Psi)$ as

$$\rho(\kappa)x = Rx(t) \quad \text{and} \quad \rho(\varphi)x = x(t+\varphi),$$

where $R = \mathrm{diag}(1, 1, -1)$ is just the reflection which fixes the plane.

Since $V(x)$ is invariant with respect to the reflection, the gradient ∇V is a $\mathbb{Z}_2$-equivariant map. Moreover, since the equation is autonomous and R commutes with the matrix $\bar{J}$, then

$$f(\rho(\kappa, \varphi)x) = \rho(\kappa, \varphi)f(x).$$

Therefore $f(x)$ is a $\mathbb{Z}_2 \times S^1$-equivariant map.

Now, the generator of the group S^1 on the space $H^2_{2\pi}$ is $D_\varphi(\rho(\varphi)x)_{\varphi=0} = \dot{x}$. As the operator $f(x)$ satisfies the equality

$$\langle f(x), \dot{x}\rangle_{L^2_{2\pi}} = (-\nu^2 |\dot{x}|^2/2 + V(x))|_0^{2\pi} = 0,$$

then the operator $f(x)$ is orthogonal to the generator $\dot{x}$ in $L^2_{2\pi}$. Given this condition we say that the operator $f(x)$ is a $\mathbb{Z}_2 \times S^1$-orthogonal map. The orthogonality corresponds to the conservation of energy.

Finally, since all the equilibria are planar, the isotropy subgroup of an equilibrium x_0 is $\mathbb{Z}_2 \times S^1$. This means that all equilibria are fixed by the action of $\mathbb{Z}_2 \times S^1$.

3.2 The Lyapunov-Schmidt reduction

We want to use the orthogonal degree in order to prove bifurcation, but since this degree is defined only in finite dimensions, we need to reduce the bifurcation operator to finite dimensions. To achieve this, let us set the Fourier series of the bifurcation operator as

$$f(x) = \sum_{l\in\mathbb{Z}} \left(l^2\nu^2 x_l - 2il\nu\bar{J}x_l + g_l\right) e^{ilt},$$

where x_l and g_l are the Fourier modes of x and $\nabla V(x)$, respectively. Since $l^2\nu^2 I - 2il\nu\bar{J}$ is invertible for all l's except a finite number, we can make a Lyapunov-Schmidt reduction to a finite space. In fact, we perform a global reduction, using the global implicit function theorem, with the right bounds taking care of the collision points Ψ.

In that way, we get that the zeros of the bifurcation operator are the zeros of the bifurcation function

$$f(x_1, x_2(x_1, \nu), \nu) = \sum_{|l|\le p} (l^2\nu^2 x_l - 2il\nu\bar{J}x_l + g_l)e^{ilt},$$

where x_1 corresponds to the $2p+1$ modes and x_2 to the complement.

Consequently, the linearized bifurcation function at an equilibrium x_0 is

$$f'(x_0, \nu)x_1 = \sum_{|l|\le p} \left(l^2\nu^2 I - 2il\nu\bar{J} + D^2V(x_0)\right) x_l e^{ilt}.$$

In fact, $\nabla V(x) = D^2V(x_0)(x - x_0) + \cdots$, close to x_0 and the Fourier components of $x - x_0$ are x_l for $l \neq 0$ and we rename the stationary mode as x_0.

So the linearized bifurcation equation has blocks $M(l\nu)$ for $l \in \{0, \ldots, p\}$, where the block $M(\lambda)$ is

$$M(\lambda) = \lambda^2 I - 2i\lambda \bar{J} + D^2V(x_0).$$

3.3 Irreducible representations

In the following part, we analyze the symmetries of the group $\mathbb{Z}_2 \times S^1$. Since the action of $(\kappa, \varphi) \in \mathbb{Z}_2 \times S^1$ on Fourier modes $e^{ilt}x_l$ is

$$\rho(\kappa, \varphi)(e^{ilt}x_l) = Re^{il\varphi}(e^{ilt}x_l),$$

then the action on the block $M(l\nu)$ is given by $\rho(\kappa, \varphi)x_l = Re^{il\varphi}x_l$.

Now, as the action of $\mathbb{Z}_2$ on $\mathbb{C}^3$ is $\rho(\kappa) = \mathrm{diag}(1, 1, -1)$, the space $\mathbb{C}^3$ has two irreducible representations: $V_0 = \mathbb{C}^2 \times \{0\}$ and $V_1 = \{0\} \times \mathbb{C}$. That is, the group $\mathbb{Z}_2$ acts on V_0 as $\rho(\kappa) = 1$ and on V_1 as $\rho(\kappa) = -1$. Hence, by Schur's lemma we know that the matrix $M(\lambda)$ must satisfy

$$M(\lambda) = \mathrm{diag}(M_0(\lambda), M_1(\lambda)).$$

Actually, from the explicit Hessian $D^2V(x_0)$ we have

$$M_1(\lambda) = \lambda^2 - \sum_{j=1}^{n} m_j/d_j^{\alpha+1} \text{ and} \tag{3}$$

$$M_0(\lambda) = \lambda^2 I - 2iJ\lambda + \left(I + \sum_{j=1}^{n} m_j A_j\right).$$

Consequently, the action of the group $\mathbb{Z}_2 \times S^1$ on the block $M_0(\nu)$ is $(\kappa, \varphi)x = e^{i\varphi}x$. Therefore the element $(\kappa, 0)$ leaves fixed the points for $M_0(\nu)$, so the isotropy subgroup for $M_0(\nu)$ is the one generated by $(\kappa, 0)$,

$$\mathbb{Z}_2 = \langle(\kappa, 0)\rangle.$$

For $M_1(\nu)$ the action of the group $\mathbb{Z}_2 \times S^1$ is $(\kappa, \varphi)x = -e^{i\varphi}x$. It follows that (κ, π) leaves fixed the points for $M_1(\nu)$, thus the isotropy subgroup for $M_1(\nu)$ is generated by (κ, π),

$$\tilde{\mathbb{Z}}_2 = \langle(\kappa, \pi)\rangle.$$

3.4 The orthogonal degree

The orthogonal degree is defined for orthogonal maps that are non-zero on the boundary of some open bounded invariant set. The degree is made of integers, one for each orbit type, and it has all the properties of the usual Brouwer degree. Hence, if one of the integers is non-zero, then the map has a zero corresponding to the orbit type of that integer. In addition, the degree is invariant under orthogonal deformations that are non-zero on the boundary. The degree has other properties such as sum, products and suspensions, for instance, the degree of two pieces of the set is the sum of the degrees.

Now, if one has an isolated orbit, then its linearization at one point of the orbit x_0 has a block diagonal structure, due to Schur's lemma, where the isotropy subgroup of x_0 acts as $\mathbb{Z}_n$

or as S^1. Therefore, the orthogonal index of the orbit is given by the signs of the determinants of the submatrices where the action is as $\mathbb{Z}_n$, for $n = 1$ and $n = 2$, and the Morse indices of the submatrices where the action is as S^1. In particular, for problems with a parameter, if the orthogonal index changes at some value of the parameter, one will have bifurcation of solutions with the corresponding orbit type. Here, the parameter is the frequency ν.

Any Fourier mode will give rise to an orbit type (modes which are multiples of it), hence one has an element of the orthogonal degree for each mode. Furthermore, if $x(t)$ is a periodic solution, with frequency ν, then $y(t) = x(nt)$ is a $2\pi/n$-periodic solution, with frequency ν/n. Hence, any branch arising from the fundamental mode will be reproduced in the harmonic branch. If one wishes to study period-doubling, then one has to consider the branch corresponding to π-periodic solutions.

The complete study of the orthogonal degree theory is given in Ize and Vignoli (2003).

Theorem 1 *Supposing that the matrix $M(0) = D^2V(x_0)$ is invertible, we define*

$$\eta_k(\lambda) = \sigma(n_k(\lambda - \rho) - n_k(\lambda + \rho)), \tag{4}$$

where $\sigma = \mathrm{sgn}(\det M_0(0))$ and $n_k(\lambda)$ is the Morse index of $M_k(\lambda)$ for $k \in \{0, 1\}$.

In general, if x_0 is an isolated critical point, then σ is the index of $\nabla V(x)$ at x_0.

If $\eta_k(\nu_k)$ is nonzero, then the equilibrium has a global bifurcation of periodic solutions starting from the period $2\pi/\nu_k$ with isotropy group G_k.

Proof Since $M_1(0)$ is a negative number, the sign of the determinant of $M(0)$ is the opposite of σ. Furthermore, there will be a change of the Morse number only at values of λ where $M_1(\lambda)$ is 0 or where the self-adjoint matrix $M_0(\lambda)$ has one of its two eigenvalues equal to 0 (the other is not 0, given that the trace is positive). Finally, since $\lambda = l\nu$, what happens for the fundamental mode ($l = 1$) is reproduced for higher modes and frequencies which are quotients of the fundamental frequency by the mode l. Here we take the fundamental mode. One is then in the position of applying Proposition 3.1, p. 255 of Ize and Vignoli (2003), after one sees the change of orthogonal index. Finally, if x_0 is an isolated critical point, then one may perform an orthogonal deformation of the map to $(\nabla V(x), M(l\nu)x_l)$, for $l \in \{1, \ldots, p\}$, near (x_0, ν_k), with a jump at ν_k given by the above formula. □

We say that the bifurcation is *non-admissible* when either: i) the global branch goes to infinity in norm or period or ii) the branch ends in a collision path. In any other case we say that the bifurcation is *admissible*. By global bifurcation we mean either that the bifurcation is non-admissible or, if the bifurcation is admissible, that the bifurcation branch returns to other bifurcation points and that the sum of the jumps of the indices at the bifurcation points, $\eta_k(\nu_k)$, is zero.

4 Spectral analysis

Now, we wish to find the bifurcation points of an equilibrium. In order to do so, we need to analyze the spectrum of the blocks $M_0(\lambda)$ and $M_1(\lambda)$. But let us first find the symmetries of the solutions that bifurcate from these blocks.

For $M_0(\lambda)$ we get solutions with isotropy subgroup $\mathbb{Z}_2$. As $\kappa \in \mathbb{Z}_2$ has action $\rho(\kappa)x_0(t) = Rx_0(t)$, this means that the solutions with symmetry $\mathbb{Z}_2$ satisfies $x_0(t) = Rx_0(t)$, i.e. $z(t) = 0$. Therefore, solutions with symmetry $\mathbb{Z}_2$ are just planar solutions.

For $M_1(\lambda)$ we get solutions with isotropy subgroup $\tilde{\mathbb{Z}}_2$. As $(\kappa, \pi) \in \tilde{\mathbb{Z}}_2$ has action $\rho((\kappa, \pi)x_0(t) = Rx_0(t+\pi)$, then the solutions with symmetry $\tilde{\mathbb{Z}}_2$ satisfy $x_0(t) = Rx_0(t+\pi)$, i.e.

$$x(t) = x(t+\pi), \; y(t) = y(t+\pi) \quad \text{and} \quad z(t) = -z(t+\pi). \tag{5}$$

Since the projection of this solution on the (x, y)-plane is a π-periodic curve, that solution follows twice this planar curve, one time with the spatial coordinate $z(t)$ and a second time with $-z(t)$. Consequently, there is at least one t_0 where $z(t_0) = z(t_0 + \pi) = 0$. For instance, if only one t_0 exists, then the solution looks like a spatial eight near the equilibrium. For this reason, we will call eight-solutions the solutions with isotropy subgroup $\tilde{\mathbb{Z}}_2$.

Remark 1 Actually, the solutions of the satellite are defined in rotating coordinates, so that the periodic solutions are in general quasiperiodic in fixed coordinates.

4.1 Planar solutions

Let T and D be the trace and determinant of the matrix $M_0(0)$. We point out that the block $M_0(0)$ is just the Hessian of the planar potential at the equilibrium point. In addition, in the first section we have proven that the trace T is always positive. Now, we want to show that the bifurcation depends essentially on the sign of D.

Proposition 3 *Let us define* $\nu_\pm$ *as*

$$\nu_\pm = \left(2 - T/2 \pm \sqrt{(2 - T/2)^2 - D}\right)^{1/2}.$$

(a) *If* $D < 0$, *then* x_0 *has a global bifurcation of periodic planar solutions from* $2\pi/\nu_+$ *with*

$$\eta_0(\nu_+) = -1.$$

(b) *If* $D > 0$, $(2 - T/2)^2 > D$ *and* $T < 4$, *then* x_0 *has a global bifurcation of periodic planar solutions from* $2\pi/\nu_+$ *and* $2\pi/\nu_-$ *with*

$$\eta_0(\nu_+) = 1 \quad \text{and} \quad \eta_0(\nu_-) = -1.$$

Proof Since $M_0(0)$ is selfadjoint, there is an orthonormal matrix $P \in SO(2)$ such that $M_0(0) = P^T \Lambda P$, where Λ is the eigenvalue matrix $\mathrm{diag}(\lambda_1, \lambda_2)$. Since $M_0(\nu) = \nu^2 I - 2iJ\nu + M_0(0)$ and J commutes with P, then

$$PM_0(\nu)P^T = \mathrm{diag}(\nu^2 + \lambda_1, \nu^2 + \lambda_2) - 2\nu(iJ).$$

In view of $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$, the determinant of $M_0(\nu)$ is

$$\det M_0(\nu) = \nu^4 - 2(2 - T/2)\nu^2 + D.$$

It follows that the determinant has the factorization

$$\det M_0(\nu) = (\nu^2 - \nu_+^2)(\nu^2 - \nu_-^2).$$

Consequently, the Morse index of $M_0(\nu)$ can change only at $\pm\sqrt{\nu_\pm}$.

For (a), only ν_+ is positive, and $\sigma = \mathrm{sgn}(D) = -1$. The Morse index of $M_0(0)$ is $n_0(0) = 1$ due to $D < 0$, and $n_0(\infty) = 0$ due to the fact that $M_0(\nu)$ has only positive eigenvalues for ν big enough. Therefore $\eta_0(\nu_+) = \sigma(1 - 0) = -1$.

For (b), both numbers $\nu_\pm$ are positive, and $\sigma = \mathrm{sgn}(D) = 1$. Moreover, we see that the determinant of $M_0(\nu)$ is negative between ν_- and ν_+, thus $n_0(\nu) = 1$ for $\nu \in (\nu_-, \nu_+)$. As the Morse index at infinity is $n_0(\infty) = 0$, we conclude that $\eta_0(\nu_+) = 1 - 0$. Now, the Morse index of $M_0(0)$ is $n_0(0) = 2$ if $T < 0$ and $n_0(0) = 0$ if $T > 0$. It follows that $\eta_0(\nu_-) = 2-1$ if $T < 0$ and $\eta_0(\nu_-) = 0 - 1$ if $T > 0$.

Note that this proof is independent of the form of the potential. For the case of the specific potential of this paper, Eq. (2) implies that $T > 0$. □

Remark 2 In the case (b), the two local bifurcations can locally collide when the resonance condition $\nu_+ = m\nu_-$ holds. Moreover, it is easy to prove that the resonance condition is equivalent to

$$(4 - T)D^{-1/2} \in \{m + m^{-1} : m \in \mathbb{N}\}.$$

Remark 3 In all other cases different from (a), (b) and $(2 - T/2)^2 = D$, there is no bifurcation, if D is not 0, since then the matrix $M_0(\lambda)$ is always invertible. In addition, in the case $(2 - T/2)^2 = D > 0$, both points $\nu_\pm$ coincide and $\eta_0(\nu_+) = 0$, then we cannot assure or discard the existence of bifurcation, but probably of a different kind, as found in Bardin (2002) and Sicardy (2010). Finally, if $D = 0$, then $\nu_- = 0$ and $\nu_+ = (4 - T)^{1/2}$, if $T < 4$, i.e. $V(x)$ is not a Morse function at x_0. In this last case, one may have a bifurcation of relative equilibria if the masses of the primaries are chosen as parameter and one has a change in σ, when one of the masses crosses the critical value, or one could have a secondary bifurcation of periodic solutions if the unfolding has the right properties, see Ize (1995). However, in the applications of the present paper, the potential is a Morse function.

Remark 4 Actually, the satellite equation on the plane is a Hamiltonian system with two degrees of freedom. We can relate the linear stability of the system with the bifurcation analysis. Indeed, it can be proven that the equilibrium x_0 is linearly stable on the plane if and only if condition (b) is satisfied. Note that one could argue about the usefulness of a bifurcation result for the satellite if the arrangement of the primaries is unstable. This is a quite valid argument from the practical point of view, taking into account the reality of this model for a problem of mechanics. However, the mathematical result is independent of the stability of the primaries and furthermore, as proved in García-Azpeitia (2010) and in an article in preparation, the primaries may loose their stability and generate stable periodic solutions of the whole system. In that case, it is much simpler to prove the bifurcation of periodic solutions for the satellite, assuming, as a first approximation, that the primaries are at their position of relative equilibrium. Hence, the mathematical study of the bifurcation is also justified in this framework.

In the case of the Maxwell ring, it is well known that the system of the primaries is unstable if n is between 3 and 6 and the stability is treated, for $n > 6$ and large central mass, in Roberts (2000); Vanderbei and Kolemen (2007) and others. A complete mathematical study of the stability is given in García-Azpeitia and Ize (2011). Thus, if one insists, on physical grounds, that the stability of the relative equilibrium configuration must be insured in other to have a study of the bifurcation, one has to restrict to the case $n > 6$ and large central mass, or assume that the primaries are fixed in the rotating frame.

Remark 5 Because there is only one bifurcation value for the frequency in case (a), the global branch cannot return to the same equilibrium point, so the bifurcation branch is non-admissible or it is connected to the bifurcation point of another equilibrium. In fact, if the potential is a Morse function, then one should get a connection to the small period branch of a minimum (that is with a jump of 1). This implies that, in this case, there are at least $n - 1$ non admissible branches starting from saddle points, (see our previous proposition). In Maciejewski and Rybicki (2004), one finds other possibilities for branches starting from a minimum, and ν_+, for the restricted three-body problem.

4.2 Spatial solutions

As before, let $\nu_{\pm}$ be the points where $M_0(\lambda)$ is not invertible.

Proposition 4 *Let us define* ν_1 *as the positive root of*

$$\nu_1^2 = \sum_{j=1}^{n} m_j / d_j^{\alpha+1}.$$

Then every equilibrium x_0 *has a global bifurcation of periodic eight solutions with*

$$\eta_1(\nu_1) = \sigma.$$

In addition, the local bifurcation branch from $2\pi/\nu_1$ *is truly spatial,* $z(t) \neq 0$*, provided the nonresonant condition* $\nu_1 \neq \nu_{\pm}/2l$ *holds.*

Proof It is clear that $M_1(\nu)$ is zero only for $\pm\nu_1$. Since $M_1(\infty)$ is positive and $M_1(0)$ is negative, the Morse indices at infinity and zero are $n_1(\infty) = 0$ and $n_1(0) = 1$. Therefore $\eta_1(\nu_1) = \sigma(1 - 0)$. Thus, one has the global bifurcation of periodic eight solutions.

It remains only to prove that the solutions are truly spatial. In order to achieve this, we need to prove the nonexistence of solutions of the kind

$$x(t) = x(t + \pi),\ y(t) = y(t + \pi) \quad \text{and} \quad z(t) = 0 \tag{6}$$

near (x_0, ν_1). In fact, the solutions (6) are in the fixed point space of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $\kappa \in \mathbb{Z}_2$ and $\pi \in S^1$.

Now, the restriction of the derivative of the bifurcation equation to the fixed point space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ has blocks $M_0(2l\nu_1)$. Since the matrix $M_0(\nu)$ is invertible except for the points $\nu_{\pm}$, and we suppose $\nu_{\pm} \neq 2l\nu_1$, the blocks $M_0(2l\nu_0)$ are invertible. Consequently, the derivative of the bifurcation equation in the fixed point space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is invertible. Therefore, we get the nonexistence of planar solutions (6) near (x_0, ν_1) from the implicit function theorem. □

Remark 6 Although the nonresonant condition $\nu_1 \neq \nu_{\pm}/2l$ is sufficient to assure that the bifurcation from $2\pi/\nu_1$ is really spatial, it is not a necessary condition. If one considers the full three-dimensional problem, without any special symmetry (except periodicity), then, if one has the resonance $\nu_{\pm} = 2l\nu_1$, the jump of orthogonal index has two components $\eta_1(\nu_1)$ for the fundamental mode and $\eta_0(\nu_{\pm})$ for the $2l$-mode. Since this jump is different from the one caused by the rescaling of the jump for the solution in the fixed-point subspace of $\mathbb{Z}_2$, which has only the second component for the $2l$-mode, one obtains a new branch of periodic solutions. In the case of the restricted three-body problem, this is the branch given in Maciejewski and Rybicki (2004).

5 Applications

5.1 A Morse potential

We have proven that the potential for the satellite problem has at least n saddle points and a global minimum, provided it is a Morse function. Consequently, we get the following result:

Theorem 2 *Each one of the saddle points has a global bifurcation of planar periodic solutions and a global bifurcation of periodic eight solutions.*

Each of the minimum points satisfy one of the following options: (a) it has two global bifurcations of planar periodic solutions and one bifurcation of periodic eight solutions, or (b) it has one bifurcation of spatial periodic eight solutions.

For the planar bifurcation, each saddle point has a bifurcation with index $\eta_0 = -1$ and each minimum point has two bifurcations, if any, one with $\eta_0 = 1$ and another with $\eta_0 = -1$. Because an admissible bifurcation branch has sum of indices η_0 equal to zero, the sum over all admissible branches is 0. If s_a denotes the number of saddle points which belong to an admissible branch, m_{-a} the number of minima with jump of -1 which are on an admissible branch and m_{+a} those with jump 1, one has that $s_a + m_{-a} = m_{+a}$. Let s_i, m_{-i}, m_{+i} be the numbers of points which are on non-admissble branches and let s be the total number of saddle points, m the number of minima (including m_0 those which are not on any branch), then one gets that $m = m_0 + m_{-a} + m_{-i} = m_0 + m_{+a} + m_{+i}$ and, since $s = n - 1 + m$, one has $s_i + m_{-i} - m_{+i} = n - 1 + m$, that is the number $s_i + m_{-i}$ of points with jump -1 belonging to non-admissible branches is at least $n - 1 + m$. Thus, the number of points on non-admissible branches is at least the number of saddle points.

Now, since every minimum has a spatial bifurcation with $\eta_1 = 1$ and every saddle point has a spatial bifurcation with $\eta_1 = -1$, then a bifurcation branch of eight solutions is non-admissible or the total number of saddle and minimum points that it connects is the same and the number of saddle points which are on non-admissible branches of eight solutions is at least $n - 1$.

5.2 The restricted three-body problem

In the restricted three-body problem, the primary bodies are at $a_1 = (1 - \mu, 0)$ and $a_2 = (-\mu, 0)$ with masses $m_1 = \mu$ and $m_2 = 1 - \mu$. Hence, the potential of the satellite is

$$V(x) = \frac{1}{2}\left\|\bar{I}x\right\|^2 + \sum_{j=1}^{2} m_j \phi_\alpha\left(\|x - (a_j, 0)\|\right).$$

This problem is well known on the plane, see for instance Meyer and Hall (1991). There are only five equilibrium points called Lagrangians. Two of these equilibrium points form an equilateral triangle with the primary bodies a_1 and a_2, and they are minima of the planar potential. Three of the equilibrium points are collinear with the primaries, also called Eulerian points, and they are saddle points of the potential. All of these relative equilibria are non-degenerate, that is $V(x)$ is a Morse function.

Also, it is well known that the minimum points have two bifurcation frequencies $\nu_\pm$ for $\mu < \mu_1$, where $\mu_1 = (1 - (\alpha + 1)^{-1}\sqrt{\alpha(30 - \alpha) - 33)/12})/2$, when α is in the interval$(15 - 8\sqrt{3}, 3)$, is the critical Routh ratio and without any restriction on μ if α belongs to the interval $(1, 15 - 8\sqrt{3})$. This comes from the fact that the trace $T = \alpha + 1$ and the determinant $D = 3(\alpha + 1)^2\mu(1 - \mu)/4$, with the conditions $T < 4$ and $(2 - T/2)^2 > D$. In that case,

$$\nu_\pm^2 = \left(3 - \alpha \pm \sqrt{(3 - \alpha)^2 - 3(\alpha + 1)^2\mu(1 - \mu)}\right)/2.$$

Note that ν_+/ν_- tends to infinity when μ tends to 0, thus there is an infinite number of resonance values for μ, when μ goes to 0.

For the saddle points we have only the bifurcation point ν_+, where

$$\nu_+^2 = 1 - (\alpha - 1)\nu_1^2/2 + \left((\alpha + 1)^2\nu_1^4/4 - 2(\alpha - 1)\nu_1^2\right)^{1/2}$$

with $\nu_1^2 = \sum_{j=1}^{2} m_j/d_j^{\alpha+1} > 1$, since $D^2V(x_0) = \mathrm{diag}(1 + \alpha\nu_1^2, 1 - \nu_1^2)$.

Consequently, we get the classical global bifurcation of planar periodic solutions, with at least three equilibria on non-admissible branches, see Maciejewski and Rybicki (2004) for the case $\alpha = 2$.

Now, we wish to find bifurcation of spatial periodic eight-solutions.

Theorem 3 *In the restricted three-body problem each one of the five equilibria has a global bifurcation of spatial periodic eight-solutions.*

Proof We only need to prove the nonresonant condition $\nu_1 > \nu_\pm/2l$ at equilibrium points. For the triangular Lagrangian points we have that $\nu_1 = 1$ and $\nu_\pm \in (0, \sqrt{(3-\alpha)})$ for $\mu \in (0, 1)$, therefore $\nu_1 > \nu_\pm/2l$.

For the collinear Lagrangian points, since ν_+ is given in terms of ν_1, we need to prove that $4l^2\nu_1^2 \neq \nu_+^2$, or equivalently $(\alpha+1)^2\nu_1^4/4 - 2(\alpha-1)\nu_1^2 \neq ((4l^2 + (\alpha-1)/2)\nu_1^2 - 1)^2$. The last inequality is also equivalent to $a\nu_1^4 - 2b\nu_1^2 + 1 \neq 0$, where $a = \left(4l^2 + (\alpha-1)/2\right)^2 - (\alpha+1)^2/4$ and $b = 4l^2 - (\alpha-1)/2$. But since $b^2 - a = (\alpha+1)^2/4 - 8(\alpha-1)l^2 < 0$ is satisfied for all $l \geq 1$, if $\alpha \in (15 - 8\sqrt{3}, 3)$, then the quadratic equation $a\nu_1^4 - 2b\nu_1^2 + 1 = 0$ does not have solutions and $4l^2\nu_1^2 \neq \nu_+^2$. On the other hand, if $\alpha \in (1, 15 - 8\sqrt{3})$ and $l = 1$, then the quadratic equation has its largest root less than 1, which contradicts the fact that at the saddle point $\nu_1 > 1$. Thus, there is no resonance and the branch is truly spatial and at least one branch is non-admissible. □

5.3 The Maxwell's Saturn ring

In this section, we analyze the satellite problem when the primaries form a polygonal relative equilibrium. Hereafter, we identify the real and complex planes.

The polygon consists of one body of mass μ at $a_0 = 0$, and n bodies of mass 1 at each vertex of a regular polygon, for instance $a_j = ae^{ij\zeta}$ for $j \in \{1, \ldots, n\}$, where $\zeta = 2\pi/n$. It is easy to prove that the positions a_j form a relative equilibrium provided that $a^{\alpha+1} = s + \mu$, where s is defined by

$$s = \frac{1}{2^\alpha} \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)}.$$

Moreover, we can make the change of variable $x = au$ in such a way that the equation is $\ddot{u} + 2\bar{J}\dot{u} = \nabla V(u)$ with the potential

$$V(u) = \frac{1}{2}\left\|\bar{I}u\right\|^2 + \sum_{j=1}^{n} \frac{1}{s+\mu}\phi_\alpha\left(\|u - (e^{ij\zeta}, 0)\|\right) + \frac{\mu}{s+\mu}\phi_\alpha(\|u\|).$$

Now we point out that the case $n = 2$ with $\mu = 0$ is just a particular case of the restricted three-body problem, hence we shall analyze only the cases $n = 2$ with $\mu > 0$ and $n \geq 3$ with $\mu \geq 0$.

Existence of equilibria

Remember that all equilibrium points of the satellite are in the plane. So, we assume, for this purpose, that the satellite is in the plane, i.e. the potential is

$$V(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{n} \frac{1}{s+\mu}\phi_\alpha\left(\|u - e^{ij\zeta}\|\right) + \frac{\mu}{s+\mu}\phi_\alpha(\|u\|)$$

with $u \in \mathbb{R}^2$.

Proposition 5 *For $\mu = 0$, the origin $u_0 = 0$ is a critical point. In addition, we have for $n \geq 3$ that $D^2V(0) = \lambda I$ with $\lambda > 0$.*

Proof That the origin is a critical point follows from the fact that

$$\nabla_u V(0) = \frac{1}{s}\sum_{j=1}^{n} e^{ij\zeta} = 0.$$

Now, since $D^2V(0)$ has real eigenvalues and $D^2V(0)$ is D_n-equivariant, by Schur's lemma we have $D^2V(0) = \lambda I$ for $n \geq 3$. That $\lambda > 0$ is due to the fact that the trace $T = 2\lambda$ is always positive. □

Now for $u \neq 0$, we may simplify the analysis if we change to polar coordinates. For these coordinates the potential is

$$V(r,\varphi) = r^2/2 + \frac{\mu}{s+\mu}\phi_\alpha(\|r\|) + \sum_{j=1}^{n}\frac{1}{s+\mu}\phi_\alpha\left(\|r - e^{i(j\zeta-\varphi)}\|\right).$$

Let us observe that the potential V is D_n-invariant for the action $\rho(\zeta)u = e^{i\zeta}u$ and $\rho(\kappa)u = \bar{u}$, thus, critical points will be D_n-orbits of points. It follows that the potential $V(r,\varphi)$ is even and $2\pi/n$-periodic in φ, hence, we may restrict our analysis to points with $\varphi \in [0, \pi/n]$.

Now, we will show that the potential has three orbits of critical points. To achieve this goal, we need first to prove the following lemma.

Lemma 1 *For $n \geq 3$, the derivative V_r at $e^{i\pi/n}$ is negative,*

$$V_r(1, \pi/n) < 0.$$

Proof The derivative of $V(r,\varphi)$ is

$$V_r(r,\varphi) = r - \frac{\mu}{s+\mu}\frac{1}{r^\alpha} - \frac{1}{s+\mu}\sum_{j=1}^{n}\frac{r - \cos(j\zeta - \varphi)}{\|r - e^{i(j\zeta-\varphi)}\|^{\alpha+1}}. \tag{7}$$

Therefore, at $e^{i\pi/n}$, we have

$$V_r(1,\pi/n) = \frac{s}{s+\mu} - \frac{1}{s+\mu}\left(\sum_{j=1}^{n}\frac{1}{2^\alpha}\frac{1}{\sin^{\alpha-1}(j-1/2)\zeta/2}\right) = \frac{s-\sigma}{s+\mu},$$

where σ is the sum between parentheses.

So it remains to prove that $s < \sigma$. In order to do so, we need some inequalities. Since $n \geq 3$, we have the first inequality

$$2^\alpha s = \sum_{j=1}^{n-1}\frac{1}{\sin^{\alpha-1}(j\zeta/2)} \leq 2\sum_{j\in[1,n/2]\cap\mathbb{N}}\frac{1}{\sin^{\alpha-1}(j\zeta/2)},$$

where equality holds for n odd. Similarly, we have the second inequality

$$2^\alpha\sigma = \sum_{j=1}^{n}\frac{1}{\sin^{\alpha-1}(j-1/2)\zeta/2} \geq 2\sum_{j\in[1,n/2]\cap\mathbb{N}}\frac{1}{\sin^{\alpha-1}(j-1/2)\zeta/2},$$

where equality holds for n even. Finally, since $\sin(j-1/2)\zeta/2 < \sin j\zeta/2$ for $j \in [1, n/2]$, then we have the third inequality

$$\frac{1}{\sin^{\alpha-1}(j-1/2)\zeta/2} > \frac{1}{\sin^{\alpha-1}(j\zeta/2)}.$$

The fact $\sigma > s$ follows from these inequalities. □

In Bang and Elmabsout (2003), one may find an integral representation which is used to prove the next corollary. In addition, a direct proof of the integral representation and of this corollary will be given in the last section.

Corollary 1 *For $\alpha \in (1, 3)$, the derivative $V_r(r, \varphi)$ is the product of $-\sin(n\varphi)$ with a positive function $\omega(r, \varphi)$,*

$$V_\varphi(r, \varphi) = -\sin(n\varphi)\omega(r, \varphi).$$

We may now prove the existence of $\mathbb{Z}_n$-orbits of equilibrium points.

Proposition 6 *For $\alpha \in (1, 3)$ and $n \geq 3$ there are three orbits of critical points. We are showing only the points of the $\mathbb{Z}_n$-orbits with $\varphi \in [0, \pi/n]$:*

(a) *If $\mu \in (0, \infty)$, there are two saddle points at r_2 and r_1, with $0 < r_2 < 1 < r_1$, and there is a minimum point at $r_3 e^{i\pi/n}$, with $r_3 > 1$.*
(b) *If $\mu = 0$, there are two saddle points at r_1 and $r_2 e^{i\pi/n}$, with $0 < r_2 < 1 < r_1$, and there is a minimum point at $r_3 e^{i\pi/n}$, with $r_3 > 1$.*

Furthermore, there are no other critical points when $\varphi \in [0, \pi/n)$.

Proof Since $V_\varphi(r, \varphi) = -\sin(n\varphi)\omega(r, \varphi)$, with a positive function $\omega(r, \varphi)$, then $V_\varphi(r, \varphi) = 0$ only for $\varphi = k\pi/n$. Furthermore, at these points we have $V_{\varphi\varphi}(r, k\pi/n) = -n\omega(r, \varphi) \cos k\pi$. Consequently, the critical points must be in $\varphi \in \{0, \pi/n\}$ with

$$V_{\varphi\varphi}(r, 0) < 0 \quad \text{and} \quad V_{\varphi\varphi}(r, \pi/n) > 0.$$

Thus, in order to find critical points, we need to look only for points where $V_r(r, \varphi) = 0$, with $\varphi = 0, \pi/n$.

Before we start finding critical points, we wish to prove that all the critical points with $\varphi = 0$ are saddle points. The trace of $D^2V(x_0)$ at a critical point is

$$T = V_{xx} + V_{yy} = V_{rr} + r^{-2}V_{\varphi\varphi}.$$

Similarly, it is easy to see that the determinant of $D^2V(x_0)$ at a critical point is

$$D = V_{rr}V_{\varphi\varphi}r^{-2}.$$

Now, since T is always positive and $V_{\varphi\varphi}(r, 0)$ is always negative, then $V_{rr}(r, 0)$ is positive. Consequently, all critical points, with $\varphi = 0$, satisfy

$$V_{rr}(r, 0) > 0 \quad \text{and} \quad V_{\varphi\varphi}(r, 0) < 0.$$

For $\mu \in [0, \infty)$, the potential $V(r, 0)$ goes to infinity when $r \to \{1, \infty\}$. Hence, the potential has a saddle point at $r_1 \in (1, \infty)$. Now, if there were another critical point r_* in $(1, \infty)$, then $V_{rr}(r_*, 0)$ would be positive. In that case there would be another critical point between r_1 and r_* with $V_{rr}(r, 0) \leq 0$. But that cannot happen, and consequently r_1 is unique in $(1, \infty)$.

For $\mu \in (0, \infty)$, the potential $V(r, 0)$ goes to infinity when $r \to \{1, 0\}$. Hence the potential has a saddle point with $r_2 \in (0, 1)$. As before with r_1, we can prove that r_2 is unique in $(0, 1)$.

For $\mu = 0$, remember that $V_r(0, \varphi) = 0$ and $V_{rr}(0, \varphi) > 0$. Then, by a similar argument to the uniqueness of r_1 we can prove that the potential $V(r, 0)$ does not have critical points in $(0, 1)$. Now, for $\varphi = \pi/n$, since $V_r(1, \pi/n)$ is negative and $V_r(0, \pi/n) = 0$ with $V_{rr}(0, \pi/n) > 0$, there must be a $r_2 < 1$ such that $V_r(r_2, \pi/n) = 0$ with $V_{rr}(r_2, \pi/n) < 0$. Consequently $r_2 e^{i\pi/n}$ is a saddle point.

For $\mu \in [0, \infty)$, since $V_r(1, \pi/n)$ is negative and since $V_r(r, \pi/n)$ goes to infinity as $r \to \infty$, there is a critical point $r_3 \in (1, \infty)$ such that $V_{rr}(r_3, \pi/n) > 0$. Therefore $r_3 e^{i\pi/n}$ is a minimum. □

In the article (Bang and Elmabsout 2004), the existence of these three orbits of equilibrium points is proven, as well as their stability. However, our proofs are simpler.

For $n = 2$ and $\mu > 0$ we can prove the previous proposition with the same argument, except for the existence of r_3. Instead, we get the existence of a $r_3 \in (0, \infty)$ because the potential $V(r, \pi/2)$ goes to infinity when $r \to 0, \infty$.

Now, in Bang and Elmabsout (2004), the question of the existence of more critical points was left open. Actually, for $n = 2$ and $\mu > 0$ we can prove the following:

Proposition 7 *For $n = 2$ and $\mu > 0$ the previous proposition is true and there are no other critical points.*

Proof It remains only to prove that r_3 is in $(1, \infty)$ and is unique. Let us define $f(r) = -2(r^2 + 1)^{-(\alpha+1)/2}$. After some computations, we find that the derivative $V_r(r, \pi/2)$ satisfies the equality

$$(s + \mu)V_r = r(f(r) + s) + \mu(r - r^{-\alpha}). \tag{8}$$

Let us denote the μ-dependence of the potential as $V(r, \varphi; \mu)$. Therefore, from the equality (8), we have that $V_r(r, \pi/2; \mu) < V_r(r, \pi/2; 0)$ for $r \leq 1$. Now, as the three body problem is the case $n = 2$ with $\mu = 0$, we know that $V_r(r, \pi/2; 0) = 0$ only at the triangular Lagrangian point $r = \sqrt{2}$. Furthermore, $V_r(r, \pi/2; 0) < 0$ for $r \leq 1$, and hence $V_r(r, \pi/2; \mu) < 0$ for $r \leq 1$.

Now, let us analyze the case $r > 1$. From (8), we see, for the second derivative, that

$$(s + \mu)V_{rr} = (rf' + f) + s + \mu(1 + \alpha r^{-(\alpha+1)}).$$

Since $rf' + f = 2\left(r^2\alpha - 1\right)\left(r^2 + 1\right)^{(\alpha+3)/2}$ is a positive function, then $V_{rr}(r, \pi/2) > 0$ for $r > 1$. From this statement, we conclude that $V_r(r, \pi/2)$ has only the critical point $r_3 \in (1, \infty)$. □

We proved that there may be more critical points only if $\varphi = \pi/n$. And indeed, for $n \geq 3$ we can find more critical points when μ is near zero.

Proposition 8 *For $n \geq 3$ and μ near zero the potential has also a minimum and a saddle point at $r_4 e^{i\pi/n}$ and $r_5 e^{i\pi/n}$ with $r_4 < r_5 < 1$. On the other hand, for μ large, $r_3 e^{i\pi/n}$ is the only critical point on that line.*

Proof As before, we represent the dependence of the potential in μ as $V_r(r, \pi/n; \mu)$. Remember that $V_r(0, \varphi; 0) = 0$ with $V_{rr}(0, \varphi; 0) = \lambda > 0$ for $n \geq 3$, then there is a $r_* \in (0, \varepsilon)$ such that $V_r(r_*, \pi/n; 0) > 0$. Therefore, $V_r(r_*, \pi/n; \mu) > 0$ for μ near zero due to the continuity. Gathering data, we get $V_r(0, \pi/n) = -\infty$, $V_r(r_*, \pi/n) > 0$ and $V_r(1, \pi/n) < 0$

for μ near zero. Consequently, there are two points r_4 and r_5 where $V_r(r, \pi/n)$ is zero with $r_4 < r_5 < 1$. Moreover, the second derivative satisfies $V_{rr}(r, \pi/n) \geq 0$ for r close to r_4 and $V_{rr}(r, \pi/n) \leq 0$ for r close to r_5. Therefore, $r_4 e^{i\pi/n}$ is a minimum and $r_5 e^{i\pi/n}$ is a saddle point. On the other hand, for μ large it is easy to see that V_r is strictly increasing. □

The existence of the solutions $r_4 e^{i\pi/n}$ and $r_5 e^{i\pi/n}$ was pointed out in the paper (Arribas and Elipe 2004).

Existence of bifurcation

At the saddle points we have the following result:

Theorem 4 *The potential has two $\mathbb{Z}_n$-orbits of saddle points for $n \geq 2$, and one more when $n \geq 3$ and μ is near zero. Furthermore, each one of the saddle points has one global bifurcation of planar periodic and one bifurcation of periodic eight solutions.*

Proof The saddle point on the line $\varphi = 0$ is non-degenerate, while the critical points on the line $\varphi = \pi/n$ are isolated, since V_r is locally analytic. Hence the index at r_5 will be -1, unless r_5 and r_4 coincide, in which case the index would be 0. □

Also at the orbit of minimum points we have the following:

Theorem 5 *The potential has one $\mathbb{Z}_n$-orbit of minimum points for $n \geq 2$, and one more when $n \geq 3$ and μ is near zero. Moreover, provided μ is big enough, each minimum point has two global bifurcations of planar periodic solutions and one global bifurcation of periodic eight solutions. On the other hand, if $\alpha \geq 2$ and μ is small, the minimum $r_4 e^{i\pi/n}$ has no bifurcation of planar periodic solutions and it has a global bifurcation of spatial eight solutions.*

Proof Since the minima are isolated, with $\sigma = 1$, we only need to confirm that the bifurcation condition (b), $T < 4$ and $(2 - T/2)^2 > D > 0$, is satisfied at $r_3 e^{i\pi/n}$ provided that μ is big enough.

As r_3 is a critical point, i.e. $V_r(r_3 e^{i\pi/n}; \mu) = 0$, from (7) we can see that $r_3(\mu) \to 1$ when $\mu \to \infty$. From the definition (1) of A_j, the matrix

$$M_0(0) = I + \frac{1}{s+\mu}\sum_{j=1}^{n} A_j + \frac{\mu}{s+\mu} A_0$$

converges, when $\mu \to \infty$, to the matrix

$$I + A_0 = (\alpha + 1)\begin{pmatrix} (\cos \pi/n)^2 & \cos \pi/n \sin \pi/n \\ \cos \pi/n \sin \pi/n & (\sin \pi/n)^2 \end{pmatrix}.$$

Given that $T(\mu) \to \alpha + 1$ and $D(\mu) \to 0$ when $\mu \to \infty$, then $(2 - T/2)^2 - D \to \varepsilon > 0$ for $\alpha \in (1, 3)$. Consequently, for $\alpha \in (1, 3)$, at the minimum point the bifurcation condition (b) holds provided μ is big enough. Finally, for the minima inside the unit disc, one has that d_1 and d_n are less than 1, hence, for $\alpha \geq 2$ one has that $T > 4$. □

Remark 7 As a consequence of the previous proposition, we get that the minimum point $r_3 e^{i\pi/n}$ is linearly stable for μ big enough. This is one of the aims of the article (Bang and Elmabsout 2004) where the stability, for the system of the primaries and the satellite, is proven for $n \geq 7$ and μ big enough.

Remark 8 For $n \geq 3$ with $\mu = 0$, as we have seen before, at the origin $x_0 = 0$, we have $M_0(0) = \lambda I$. Actually, since the trace $T > 2$, we can prove that the condition for bifurcation (b) is not satisfied. Hence the origin is a minimum point without bifurcation of planar periodic solutions.

On the other hand, the origin is a minimum with one bifurcation of spatial eight periodic solutions. Moreover, we can prove, from the symmetries, that the bifurcating solutions satisfy $x(t) = 0$, $y(t) = 0$ and $z(t) = -z(t + \pi)$. In fact, we can find $z(t)$ by quadrature from the equation $\ddot{z} = \nabla V(z)$, with $V(z) := \frac{n}{\nu^2}\phi_\alpha(\sqrt{z^2+1})$, with ν^2 close to n. Recall that, at it is well known, that in this case the system of the primaries is linearly unstable.

Remark 9 The study of the bifurcation of periodic solutions, in the plane and also in space, for the full system of primaries, will be published in another paper.

6 Conclusion

For an arbitrary relative equilibrium of primaries in the plane, we have proved that each relative equilibrium of the satellite generates several global branches of periodic solutions: for a saddle point one gets a global branch of planar solutions and a global branch of eight-solutions, which are truly spatial if a non-resonance condition holds. For a minimum point of the potential, one gets either two global branches of planar solutions (long and short period) and a global branch of eight-solutions, or only the branch of eight-solutions which is then truly spatial.

A global branch may be non-admissible if the period or the norm of the solutions on the branch go to infinity or the branch goes to collision with one of the primaries. On the other hand, if the branch is admissible, then the sum of the jumps of the Morse indices at the critical points on the branch must be zero. In particular, a saddle point has to be connected with a short period minimum, the number of points on non-admissible planar branches is at least the number of saddle points and the number of saddle points on these non-admissible planar branches is at least one less than the number of primaries. Also, the number of saddle points on non-admissible branches of eight-solutions is at least one less than the number of primaries.

We have applied this general result in order to describe a rather complete picture of the restricted three-body problem and of the restricted Maxwell ring.

The topological degree approach, combined with the use of the orthogonality (or first integrals) and a systematic use of representation theory, gives information which is a good complement to classical analytical local calculus and allows flexible applications. In particular, one may extend easily these results to different potentials and to systems with more bodies.

For concrete situations, there are many local techniques, such as normal form theory which often requires to check some generic assumptions (this is not always done in practice), Poincaré mappings, stable and unstable manifold decomposition of the phase space and so on. For a low dimensional bifurcation equation, there is a common starting point for these analytical methods and for the computation of a topological degree, that is the linearization of the equations. Higher order approximations may give a better local picture of the bifurcated solutions. But, as soon as there are resonances or more couplings, the analytical methods become more difficult to apply, while the topological degree approach can still give a complementary information on the set of bifurcating solutions, in particular on the global properties of the branches. It is important to point out that, in many relevant applied problems, one may carry out symbolic manipulations of high order which may be even converted into a valid mathematical proof using interval arithmetics. We are fully familiar with higher order symbolic manipulations of formal power series and the use of computer assisted proofs.

With these considerations in mind, we have several papers in preparation on bifurcation of the whole arrangement of primaries, either as relative equilibria or as periodic solutions. For instance, in the case of the Maxwell ring, one gets n global branches of periodic solutions, each with different symmetries and where the central mass plays an important role, for the existence of these periodic solutions. Similar results were obtained for vortices, filaments, charged particles and nonlinear oscillators. See García-Azpeitia (2010).

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Appendix: Integral representation

Let us define the sum $S(r, \varphi)$ as

$$S(r, \varphi) = \sum_{j=1}^{n} \frac{1}{\left\| r - e^{i(j\zeta - \varphi)} \right\|^{\beta}}.$$

In Bang and Elmabsout (2003) the following integral representation of $S(r, \varphi)$ is proved. We shall give here a direct proof using Cauchy integrals.

Lemma 2 *For $\beta \in (0, 2)$ and $r \in (0, 1)$ the function $S(r, \varphi)$ has the integral representation*

$$S(r, \varphi) = \frac{n}{\pi} \sin(\pi\beta/2) \int_0^1 \frac{1}{(\tau^{-1} - 1)^{\beta/2}} f(\tau) d\tau \text{ with}$$

$$f(\tau) = \frac{1}{\tau(1 - r^2\tau)^{\beta/2}} \frac{1 - (r\tau)^{2n}}{\left\| 1 - (\tau r e^{-i\varphi})^n \right\|^2}.$$

We are now in a position of proving the corollary on V_φ.

From the integral representation we get that

$$S_\varphi = -\sin(n\varphi) \left(\frac{n^2}{\pi} \sin \frac{\pi\beta}{2} \int_0^1 \frac{1}{(\tau^{-1} - 1)^{\beta/2}} \frac{2(r\tau)^n}{\tau(1 - r^2\tau)^{\beta/2}} \frac{1 - (r\tau)^{2n}}{\left\| 1 - (\tau r e^{-i\varphi})^n \right\|^4} d\tau \right)$$

for $r \in [0, 1)$. Therefore $S_\varphi(r, \varphi)$ is the product of $-\sin(n\varphi)$ with the function between parentheses, which is positive. Moreover, since the sum $S(r)$ satisfies the equality $S(1/r) = r^\beta S(r)$, we conclude that $S_\varphi(r, \varphi)$ is the product of $-\sin(n\varphi)$ with a positive function for $\beta \in (0, 2)$ and $r \neq 1$.

We use this integral representation to prove the following: Set $\beta = \alpha - 1$, then we have $\phi_\alpha(r) = 1/(\beta r^\beta)$. Now, we can express the potential $V(r, \varphi)$ in terms of $S(r, \varphi)$ as

$$V(r, \varphi) = r^2/2 + \frac{\mu}{s + \mu} \phi_\alpha(r) + \frac{1}{s + \mu} \frac{1}{\beta} S(r, \varphi).$$

Since V depends on φ only through $S(r, \varphi)$, we conclude that $V_\varphi(r, \varphi)$ is the product of $-\sin(n\varphi)$ with a positive function for $\alpha = \beta + 1 \in (1, 3)$.

We may now prove the last lemma:

Proof Let us define the function $w(z)$ as

$$w(z) = \frac{1}{[z^{-1} - 1]^{\beta/2}}.$$

This function has an analytic extension to $\mathbb{C} - [0, 1]$. Indeed, using the principal branch of the logarithm we can extend it as

$$w(z) = e^{-(\beta/2)[\log|z^{-1}-1|+i\arg(z^{-1}-1)]}.$$

Let $w^{\pm}(r)$ be the limits $w^{\pm}(r) = \lim_{\varepsilon\to 0} w(r \pm i\,|\varepsilon|)$ for $r \in (0, 1)$, then

$$w^{+}(r) = e^{-i\beta\pi} \frac{1}{(r^{-1} - 1)^{\beta/2}} \quad \text{and} \quad w^{-}(r) = \frac{1}{(r^{-1} - 1)^{\beta/2}}.$$

Let Ω_ε be the set of points

$$\Omega_\varepsilon = \{|z| < 1/\varepsilon : \ |z - r| > \varepsilon \text{ for } r \in [0, 1]\}.$$

As the function $w(z) f(z)$ is of order $O(1/z^{1+\beta/2})$ when $z \to \infty$, if $\beta > 0$ then the integral over the circle of radius $1/\varepsilon$ goes to zero when $\varepsilon \to 0$. Moreover, since the product $w(z) f(z)$ is of order $O(z^{\beta/2-1})$ when $z \to 0$ and of order $O((1 - z)^{-\beta/2})$ when $z \to 1$, then for $\beta < 2$ the integrals over the half circles around $z = 0$ and $z = 1$ go to zero when $\varepsilon \to 0$. Consequently, we have that

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} w(z) f(z) dz &= \int_0^1 [w^{+}(\tau) - w^{-}(\tau)] f(\tau) d\tau \\ &= (e^{-i\beta\pi} - 1) \int_0^1 w^{-}(\tau) f(\tau) d\tau.\end{aligned}$$

Now, the function $w(z) f(z)$ has n poles in $\mathbb{C} - [0, 1]$ and another one at $z = r^{-2}$, but the residue at $z = r^{-2}$ is zero because $\beta/2 \in (0, 1)$. The other n poles are the roots of the polynomial function

$$g(z) = \left\| 1 - (zre^{-i\varphi})^n \right\|^2 = 1 + (rz)^{2n} - 2(rz)^n \cos n\varphi.$$

Consequently, the poles are found at the points $z_j^{-1} = (re^{-i\varphi}) e^{ij\zeta}$ for $j = 0, \ldots, n - 1$. As $(rz_j)^n = e^{in\varphi}$, the derivative of g at the pole z_j is

$$g'(z_j) = 2nz_j^{-1} e^{in\varphi} (e^{in\varphi} - \cos n\varphi) = 2inz_j^{-1} e^{in\varphi} \sin n\varphi.$$

Consequently, the residue of $w(z) f(z)$ at the pole z_j is

$$\operatorname{res}_{z_j} w(z) f(z) = \frac{1}{[(z_j^{-1} - 1)(1 - r^2 z_j)]^{\beta/2}} \frac{1 - e^{2ni\varphi}}{z_j g'(z_j)}.$$

Moreover, since $r^2 z_j = \bar{z}_j^{-1}$ and $(1 - e^{2ni\varphi})/(z_j g'(z_j)) = -1/n$, then

$$\operatorname{res}_{z_j} w(z) f(z) = -\frac{1}{n} \frac{1}{(-1)^{\beta/2}} \frac{1}{\left\| z_j^{-1} - 1 \right\|^{\beta}} = -\frac{1}{n} e^{-i\pi\beta/2} \frac{1}{\left\| r - e^{i(j\zeta-\varphi)} \right\|^{\beta}}.$$

Now, from the Cauchy theorem, we obtain that

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} w(z)f(z)dz = 2\pi i \sum_{z\in\mathbb{C}-[0,1]} \mathrm{res}_z w(z)f(z).$$

Consequently, from the integral and the residues we have

$$(e^{-i\beta\pi}-1)\int_0^1 \frac{1}{(\tau^{-1}-1)^{\beta/2}} f(\tau)d\tau = -2\pi i e^{-i\pi\beta/2}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\left\|r-e^{i(j\zeta-\varphi)}\right\|^{\beta}}.$$

Finally, we conclude that

$$\sum_{j=1}^{n}\frac{1}{\left\|r-e^{i(j\zeta-\varphi)}\right\|^{\beta}} = \frac{n}{\pi}\sin(\pi\beta/2)\int_0^1 \frac{1}{(\tau^{-1}-1)^{\beta/2}} f(\tau)d\tau.$$

□

References

Arribas, M., Elipe, A.: Bifurcations and equilibria in the extended n-body ring problem. Mech. Res. Comm. **31**(1), 1–8 (2004)

Bang, D., Elmabsout, B.: Representations of complex functions, means on the regular n-gon and applications to gravitational potential. J. Phys. **36**(45), 11435–11450 (2003)

Bang, D., Elmabsout, B.: Restricted $n+1$-body problem: existence and stability of relative equilibria. Celest. Mech. Dyn. Astron. **89**(4), 305–318 (2004)

Bardin, B.: On motions near the Lagrange equilibrium point L_4 in the case of Routh's critical mass ratio. Celest. Mech. Dyn. Astron. **82**, 163–177 (2002)

Barrio, R., Blesa, F., Serrano, S.: Qualitative analysis of the $(N+1)$-body ring problem. Chaos, Solitons and Fractals **36**, 1067–1088 (2008)

Doedel, E.J., Romanov, V.A., Paffenroth, R.C., Keller, H.B., Dichmann, D.J., Galán-Vioque, J., Vanderbauwhede, A.: Elemental periodic orbits associated with the libration points in the circular restricted 3-body problem. Bifurcation and Chaos **17**(8), 2625–2677 (2007)

Efthymiopoulos, C.: Formal integrals and Nekhoroshev stability in a mapping model for the Trojan asteroids. Celest. Mech. Dyn. Astron. **92**, 29–52 (2005)

Érdi, B., Forgács-Dajka, E., Nagy, I., Rajnai, R.: A parametric study of stability and resonances around L_4 in the elliptic restricted three-body problem. Celest. Mech. Dyn. Astron. **104**, 145–158 (2009)

García-Azpeitia, C.: Aplicación del grado ortogonal a la bifurcación en sistemas hamiltonianos. UNAM. PhD thesis (2010)

García-Azpeitia, C., Ize, J.: Global bifurcation of planar and spatial periodic solutions from the polygonal relative equilibria for the n-body problem. Submitted, 2011 (2011)

Gómez, G., Simó, C., Llibre, J., Martinez, R.: Dynamics and Mission Design near Libration Points, vol. 1–4. World Scientific Monograph Series in Mathematics. World Scientific, Singapore (2000)

Ize, J.: Topological bifurcation. Prog. Nonlinear Differ. Equ. Their Appl. **15**, 341–463 (1995)

Ize, J., Vignoli, A.: Equivariant degree theory. De Gruyter Series in Nonlinear Analysis and Applications 8, Walter de Gruyter (2003)

Kalvouridis, T.J.: Particle motions in Maxwell's ring dynamical systems. Celest. Mech. Dyn. Astron. **102**(1–3), 191–206 (2008)

Kasdin, N., Gurfil, P., Kolemen, E.: Canonical modelling of relative spacecraft motion via epicyclic orbital elements. Celest. Mech. Dyn. Astron. **92**(4), 337–370 (2005)

Llibre, J., Stoica, C.: Comet and Hill-type periodic orbits in restricted $(N+1)$-body problems. J. Differ. Equ. **250**, 1747–1766 (2011)

Maciejewski, A., Rybicki, S.: Global bifurcations of periodic solutions of the restricted three body problem. Celest. Mech. Dyn. Astron. **88**, 293–324 (2004)

Marchal, C.: The Three-Body Problem, Studies in Astronautics vol. 4. Elsevier, Amsterdam (1990)

Mavraganis, A.G., Kalvouridis, T.J.: A proper choice of variables for the study of satellite's close approaches in a ring assembly of N massive bodies. Planet. Space Sci. **55**, 401–406 (2007)
Meyer, K.: Periodic Solutions of the N-Body Problem (Lecture Notes in Mathematics vol. 1719). Springer, Berlin (1999)
Meyer, K., Hall, G.R.: An Introduction to Hamiltonian Dynamical Systems. Springer, Berlin (1991)
Pinotsis, A.D.: Evolution and stability of the theoretically predicted families of periodic orbits in the N-body ring problem. Astron. Astrophys. **432**, 713–729 (2005)
Pinotsis, A.D.: Instabilities and bifurcations of the families of collision periodic orbits in the restricted three-body problem. Planet. Space Sci. **55**(4), 401–406 (2009)
Roberts, G.E.: Linear stability in the $1+n$-gon relative equilibrium. In: Delgado, J. (ed.) Hamiltonian Systems and Celestial Mechanics. HAMSYS-98. Proceedings of the 3rd International Symposium, World Sci. Monogr. Ser. Math. vol. 6, pp. 303–330. World Scientific, 2000 (2000)
Sandor, Z., Érdi, B., Efthymiopoulos, C.: The phase space structure around L_4 in the restricted three-body problem. Celest. Mech. Dyn. Astron. **78**, 113–123 (2000)
Sicardy, B.: Stability of the triangular Lagrange points beyong Gaschau's value. Celest. Mech. Dyn. Astron. **107**, 145–155 (2010)
Siegel, C., Moser, J.: Lectures on Celestial Mechanics. Springer, Berlin (1971)
Vanderbei, R.J., Kolemen, E.: Linear stability of ring systems. The Astron. J. **133**, 656–664 (2007)