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ORIGINAL ARTICLE

On the circular Sitnikov problem: the alternation of stability and instability in the family of vertical motions

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Abstract This paper is devoted to the special case of the restricted circular three-body problem, when the two primaries are of equal mass, while the third body of negligible mass performs oscillations along a straight line perpendicular to the plane of the primaries (so called periodic vertical motions). The main goal of the paper is to study the stability of these periodic motions in the linear approximation. A special attention is given to the alternation of stability and instability within the family of periodic vertical motions, whenever their amplitude is varied in a continuous monotone manner.

Keywords Sitnikov problem · Periodic motions · stability analysis

1 Introduction

The term "Sitnikov problem" appeared originally in the context of studies of oscillatory solutions in the restricted three body problem. These studies were initiated by Sitnikov (1960); they stimulated the application of symbolic dynamics in Celestial Mechanics (Alekseev 1968a,b, 1969). We recall that Sitnikov considered the case when two primaries have equal masses and rotate around their barycenter O, while the infinitesimal third body moves along a straight line normal to the plane defined by the motion of the primaries and passing through O (usually the motions of the third body perpendicularly to the plane of the primaries are called "vertical"; below we will follow this tradition).

Sitnikov concentrated his attention on phenomena taking place when the primaries move in elliptic orbits. More bibliography on "elliptic" Sitnikov problem can be found, for example, in Hagel (2009), Hagel and Lhotka (2005), Kovacs and Erdi (2009).

If the primaries move in circular orbits, then the vertical motions are integrable. The corresponding quadratures were presented at the beginning of the XX century by Pavanini

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(1907) and MacMillan (1911)—much before the start of Sitnikov's studies. Relatively simple formulae for the vertical motions, written in terms of Jacobi elliptic functions, can be found in Belbruno et al. (1994).

Since the integrability of third body motion is something extraordinary within the restricted three body problem, many specialists investigated the properties of vertical motions in the case of primaries moving on circular orbit. Very often the term "circular Sitnikov problem" is applied to describe this field of research. Taking into account its popularity, we will use it too. Nevertheless, some authors prefer terms like "Pavanini problem" or "MacMillan problem", which are probably more correct from the historical point of view.

Depending on the initial values, three types of vertical motions are possible in the circular Sitnikov problem: the hyperbolic escape (i.e., the escape of the third body with non-zero velocity at infinity), the parabolic escape (i.e., the escape of the third body with zero velocity as the limit at infinity) and, finally, the periodic motion, in which third body goes away up to a distance a from the plane defined by primaries and then returns to it.

The first stability analysis of the periodic vertical motions in the circular Sitnikov problem was undertaken by Perdios and Markellos (1988), but they drew the wrong conclusion that vertical motions are always unstable (Perdios and Markellos only analyzed the vertical motions with the initial conditions such that $a < 4$; as it was established lately it is not enough to put any hypothesis about the stability properties of the motions with larger values of a). The mistake was pointed out in Belbruno et al. (1994), where the alternation of stability and instability of vertical motions were found numerically in the case of continuous monotone variation of their amplitude a. Lately the existence of such an alternation was confirmed by the results of computations presented in Perdios (2007) and Soulis et al. (2007). Taking into account their numerical results, the authors of Soulis et al. (2007) proposed the hypothesis that the lengths of stability and instability intervals have finite limits as a increases. This hypothesis was formulated on the basis of computations in which a did not exceed the value 13. Our numerical investigations demonstrate that the rapidly decreasing difference of the stability intervals at $a \approx 13$ is a manifestation of a local maximum of their lengths; if a is increased further, then the lengths of the stability and instability intervals tend to zero.

There is one more important property of vertical motions, which can be observed only for $a \gg 1$: the intervals of "complex saddle" instability, when all eigenvalues of the monodromy matrix are complex and do not lie on the unit circle. According to our computations first such an interval begins at $a \approx 546.02624$, its length is $\approx 10^{-5}$. It means the erroneous of the statement in Belbruno et al. (1994) (p. 113), that the stability indexes of the vertical motions in circular Sitnikov problem are always real (this statement was based on the results of numerical studies in which the amplitude of the motion a was smaller 17; as one can see it was not enough for such a general conclusion).

To conclude our short review on previous investigations of vertical motions' stability in circular Sitnikov problem we would like to mention the generalization of this problem for systems of four and more bodies (Bountis and Papadakis 2009; Soulis et al. 2008). Numerical results presented in Bountis and Papadakis (2009), Soulis et al. (2008) demonstrate that in the generalized problem the absence of stability/instability alternation in the family of vertical motions persists.

The aim of our paper is to study the stability property of the periodic vertical motions at large values of the "oscillation amplitude" a, both numerically and analytically. A special attention will be given to the phenomenon of infinite alternation of stability and instability in this family.

In fact, the infinite alternation of stability and instability in the one-parameter family of periodic solutions is rather typical for Hamiltonian systems, although the general investigation was carried out only for 2 degrees of freedom systems (Churchill et al. 1980; Grotta Ragazzo 1997). Different examples can be found in Contopoulos and Zikides (1980), Heggie (1983), Neishtadt and Sidorenko (2000).

Nevertheless, an important difference exists between the circular Sitnikov problem and other systems in which the alternation of stability and instability was established earlier. In the circular Sitnikov problem the discussed family of periodic solutions possesses as a limit unbounded aperiodic motions—parabolic escapes, while in previously considered systems the corresponding families and their aperiodic limits were bounded (Contopoulos and Zikides 1980; Neishtadt and Sidorenko 2000). Due to this difference, the alternation of stability and instability in the circular Sitnikov problem can not be studied in the same way as it was done in Contopoulos and Zikides (1980), Heggie (1983), Neishtadt and Sidorenko (2000) (one could try to compactify the phase space by means of certain changes of variables, but we were unable to find any reduction to what was investigated already).

This paper is organized as follows. In Sect. 2 some general properties of the vertical motions are discussed. In Sect. 3 we present the linearized motion equations used in our studies of the vertical motions' stability. The results of the numerical investigation of the stability are reported in Sect. 4. In Sect. 5 we prepare for the analytical investigation: the approximate expression for the monodromy matrix is derived here. Using this expression, some important stability properties of vertical periodic solutions with large amplitudes a are established in Sect. 6 (in particular, the asymptotic formulae for the intervals of stability and instability are obtained). In Sect. 7 we discuss briefly the vertical motions in the generalized circular Sitnikov problem with four and more bodies. Some concluding remarks can be found in Sect. 8.

2 Preliminary. Some general properties of the vertical motions in the circular Sitnikov problem

We consider the restricted, circular, three-body problem with primaries having equal masses, say $m_1 = m_2 = m$. Let $Ox_1x_2x_3$ be a synodic (rotating) reference frame with the origin at the barycenter O; the masses m_1 and m_2 are arranged on the axis Ox_1, while the axis Ox_3 is directed along the rotation axis of the system. The coordinates of the infinitesimal third body in the synodic reference frame will be used as generalized variables:

$$q_1 = x_1, \quad q_2 = x_2, \quad q_3 = x_3.$$

Below we assume that all variables are dimensionless.

The equations of motion of the third body can be written in Hamiltonian form with Hamiltonian function (Belbruno et al. 1994)

$$\mathcal{H} = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + p_1 q_2 - p_2 q_1 - \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

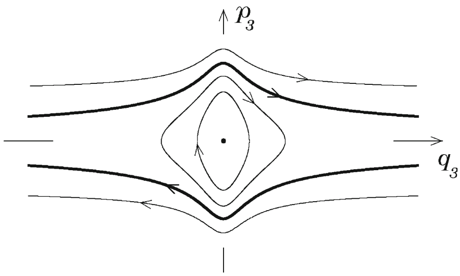
Here r_1 and r_2 denote the distance between the third body and the corresponding primary, while p_1, p_2, p_3 are the momenta conjugated to q_1, q_2, q_3.

The phase space $\mathcal{V} = \{(p, q)\}$ possesses a manifold

$$\tilde{\mathcal{V}} = \{(p, q), p_1 = p_2 = q_1 = q_2 = 0\}$$

which is invariant with respect to the phase flow. The phase trajectories lying on $\tilde{\mathcal{V}}$ correspond to vertical motions with the third body staing always on the axis Ox_3. Consequently, the vertical motions are governed by a reduced one degree of freedom system with Hamiltonian

Fig. 1 Phase flow on the manifold $\tilde{\mathcal{V}}$. *Thick lines* denote the separatrices ($\tilde{\mathcal{H}} = 0$)



$$\tilde{\mathcal{H}} = \frac{p_3^2}{2} - \frac{1}{\sqrt{q_3^2 + \frac{1}{4}}}. \tag{1}$$

The phase portrait of the system with the Hamiltonian (1) is shown in Fig. 1. It is remarkable that the separatrices (the borders between trajectories representing periodic motions and hyperbolic escapes) intersect at infinity.

The periodic solutions associated to the system with Hamiltonian $\tilde{\mathcal{H}}$ form a one-parameter family

$$p_3(t, a), \quad q_3(t, a), \tag{2}$$

where as parameter a one can choose the "amplitude" of the periodic motion (i.e., $a = \max_{t \in R^1} |q_3|$) or the absolute value of p_3 at the passage trough the barycenter O or the value $\tilde{h}$ of the Hamiltonian $\tilde{\mathcal{H}}$ in this periodic motion. The first variant is the most convenient for us, therefore a in (2) will denote the "amplitude" of the periodic motion. For definiteness we assume that

$$p_3(0, a) = 0, \quad q_3(0, a) = a.$$

There exist explicit expressions for the solutions (2) in terms of Jacobi elliptic functions (Belbruno et al. 1994). Since they are not used in the forthcoming analysis, we do not rewrite them here, except for the formula about the period of vertical motion:

$$T = \frac{\sqrt{2}}{1 - 2k^2} \left[E(k) + \frac{\pi}{2\sqrt{2(1 - 2k^2)}} \left(1 - \Lambda_0 \left(\arcsin \sqrt{\frac{1 - 2k^2}{1 - k^2}}, k \right) \right) \right]. \tag{3}$$

Here $E(k)$ is the complete elliptic integral of the second kind, $\Lambda_0(\varphi, k)$ is the Heuman Lambda Function, while the value of the modulus k is given by the formula

$$k = \frac{1}{2}\sqrt{2 + \tilde{h}},$$

where

$$\tilde{h} = -\frac{1}{\sqrt{a^2 + \frac{1}{4}}}.$$

For motions with large amplitudes ($a \gg 1$) the following approximate formula can be used in place of (3):

$$T \approx \sqrt{2}\pi a^{3/2}. \tag{4}$$

As it was mentioned before, the separatrices $S^{\pm} = \left\{\left(p_3^{\pm}(t), q_3^{\pm}(t)\right), t \in R^1\right\}$, representing the parabolic escapes, can be interpreted as a formal limit for periodic motions at $a \to \infty$. The parabolic escapes obey the approximate law

$$q_3^{\pm}(t) \approx \pm \left(\frac{3}{\sqrt{2}}\right)^{2/3} t^{2/3}. \tag{5}$$

Formulae (4) and (5) are easily obtained if one suitably relates the properties of vertical motions with the properties of rectilinear motions of a particle in a Newtonian field.

3 The stability problem for periodic vertical motions

Our efforts are concentrated on the analysis of the vertical motions' stability with respect to "horizontal" perturbations, due to which the third body leaves the axis Ox_3. Under the linear approximation, the behavior of the variables p_1, p_2, q_1, q_2 in the perturbed motion is described by the linear Hamiltonian system of equations with periodic coefficients:

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}\mathbf{H}(t)\mathbf{z}. \tag{6}$$

Here

$$\mathbf{z} = (p_1, p_2, q_1, q_2)^T,$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{0} \end{pmatrix}, \quad \mathbf{H}(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \left(\frac{1}{D^3} - \frac{3}{4D^5}\right) & 0 \\ 1 & 0 & 0 & \frac{1}{D^3} \end{pmatrix}.$$

The symbol $\mathbf{E}_k$ is used to denote the identity matrix of the k-th order. The function $D(t, a) = \left(q_3^2(t, a) + \frac{1}{4}\right)^{1/2}$ depends periodically on time with a period $T_* = \frac{T(a)}{2}$, where $T(a)$ denotes the period of the particular vertical motion whose stability is investigated.

As it is known, the restricted circular three-body problem admits several types of symmetry (for example, they are used for the numerical construction of 3D periodic solutions Perdios 2007). The consequence of these symmetries is the following property of the variational equation (6): if $\mathbf{z}(t)$ is a solution of (6), then these equations admit the solution

$$\tilde{\mathbf{z}}(t) = \mathbf{Q}\mathbf{z}(-t), \tag{7}$$

where $\mathbf{Q}$ it the 4×4-diagonal matrix, $\mathbf{Q} = \mathrm{diag}(1, -1, -1, 1)$.

According to Floquet theory, in order to draw a conclusion about the stability or instability of the solutions of (6), one should analyze the spectral properties of the monodromy matrix $\mathbf{M} = \mathbf{W}(T_*, 0)$, where $\mathbf{W}(t, t')$ denotes the normal fundamental matrix corresponding to the system (6) (i.e., the matrix solution of (6) with the initial condition $\mathbf{W}(t', t') = \mathbf{E}_4$).

The normal fundamental matrix corresponding to the linear Hamiltonian system (6) is a symplectic one, i.e.

$$\mathbf{W}^T(t, t')\mathbf{J}\mathbf{W}(t, t') = \mathbf{J}.$$

It is also worthwhile to mention some other properties of this matrix:

$$\begin{aligned} &\mathbf{W}(t, t'') = \mathbf{W}(t, t')\mathbf{W}(t', t''), \quad \mathbf{W}(t + T_*, t' + T_*) = \mathbf{W}(t, t'), \\ &\mathbf{W}(0, -t) = \mathbf{Q}\mathbf{W}^{-1}(t, 0)\mathbf{Q} = -\mathbf{Q}\mathbf{J}\mathbf{W}^T(t, 0)\mathbf{J}\mathbf{Q}. \end{aligned} \tag{8}$$

The first two equalities in (8) are elementary, while the last one is a consequence of the symmetry property (7).

Using the relation (8) one easily obtains

$$\mathbf{M} = \mathbf{QW}^{-1}\left(\frac{T_*}{2}, 0\right)\mathbf{QW}\left(\frac{T_*}{2}, 0\right) = -\mathbf{QJW}^{T}\left(\frac{T_*}{2}, 0\right)\mathbf{JQW}\left(\frac{T_*}{2}, 0\right).$$

The characteristic equation of the system (6)

$$\det\left(\mathbf{M} - \rho\mathbf{E}_4\right) = 0 \tag{9}$$

is reciprocal and it can be written as

$$\rho^4 - c_1\rho^3 + c_2\rho^2 - c_1\rho + 1 = 0$$

where

$$c_1 = \operatorname{tr}\mathbf{M}\,, \qquad c_2 = \sum_{j=1}^{3}\sum_{k=j+1}^{4}(m_{jj}m_{kk} - m_{jk}m_{kj}).$$

The quantities m_{ij} in the last formula are the elements of the monodromy matrix $\mathbf{M}$.

It is also possible to rewrite the characteristic Eq. (9) as the product

$$(\rho^2 - 2b_1\rho + 1)(\rho^2 - 2b_2\rho + 1) = 0. \tag{10}$$

The coefficients b_1, b_2 in (10) are the roots (real or complex) of the quadratic equation:

$$4x^2 - 2c_1x + (c_2 - 2) = 0.$$

Often enough the quantities b_1, b_2 are called the stability indices (Bountis and Papadakis 2009). The periodic vertical motion is stable whenever $\{b_1, b_2\} \subset I = (-1, 1) \subset R^1$ (i.e., when b_1, b_2 are real and their absolute values are smaller than 1). In the case

$$\{b_1, b_2\} \subset \bar{I} = [-1, 1], \quad \{b_1, b_2\} \not\subset I$$

an additional investigation is needed to draw a conclusion about stability or instability. In all other cases the instability takes place.

4 Numerical results

We recall that in Belbruno et al. (1994) the alternation of the stability and instability in the family of periodic vertical motions (2) was discovered. Later on, more accurate results were published in Soulis et al. (2007): the length of the first 35 intervals of stability and of the first 34 intervals of instability was calculated. In Soulis et al. (2007) also an attempt was undertaken to establish certain regularity in the variation of these quantities: the existence of non-zero limits for the intervals' lengths was proposed.

In Figs. 2 and 3 we present the results of some calculations, when the first 700 intervals of stability and instability are considered. The graph in Fig. 2 shows that for the first 30 intervals of stability the length of the intervals increases and only afterwards the decrease of the length takes place. The hypothesis formulated in Soulis et al. (2007) was based on the wrong interpretation of the small variation of the intervals length in vicinity of the maximum. In Fig. 3 the length of the instability intervals decreases monotonically and it does not follow the empirical law derived in Soulis et al. (2007) (according to this law, the length of the instability intervals has the limit $\Delta_{inst} \approx 0.254$; evidently, it is not so).

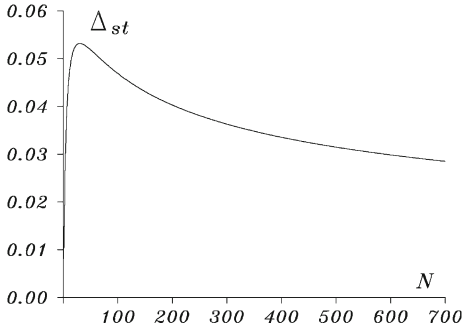


Fig. 2 Length of the stability interval as a function of its number N

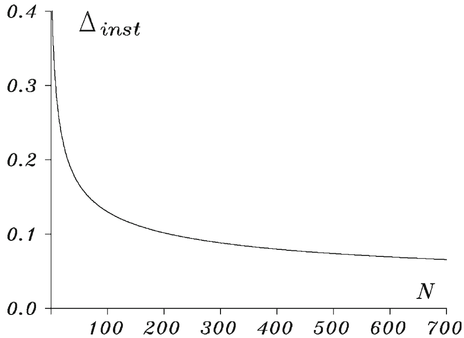


Fig. 3 Length of the instability interval as a function of its number N (the first interval is not presented: if it was shown in the same scale with all subsequent intervals, it would have been difficult to understand the behaviour of the graph for large N)

Our results allow us to propose the following approximate formulae to characterize the behavior of the stability and instability intervals' length in Figs. 2 and 3:

$$\Delta_{st} \approx 0.25\, N^{-1/3}, \quad \Delta_{inst} \approx 0.584\, N^{-1/3}. \tag{11}$$

More precisely, these formulae are valid for the periodic vertical motions with amplitude a smaller the critical value $a_* = 546.02624\ldots$ The reason of such a restriction and the situation for $a > a_*$ will be revealed a little bit later.

It is also useful to discuss here in what way the length of the stability intervals Δ_{st} and the length of the instability intervals Δ_{inst} depend on the amplitude of the vertical oscillations. Under the same restriction $a < a_*$ we obtain from our numerical investigations

$$\Delta_{st} \approx 0.3\, a^{-1/2}, \quad \Delta_{inst} \approx 0.64\, a^{-1/2}.$$

Remark If one needs a rigorous definition about the meaning of the quantity a in the last formulae, one could interpret it as the boundary value between two successive intervals of stability and instability.

In Fig. 4 the behavior of the coefficients b_1, b_2 appearing in the characteristic Eq. (10) is shown. Figure 4a, b and c allow us to compare the properties of these coefficients, when the parameter a varies in different intervals. All graphs demonstrate the approximate periodicity, their period with respect to the parameter a corresponds to an increase of period of vertical oscillations T of about 8π. It is important to point out the small gaps in the Fig. 4c: for the corresponding value of the parameter a (i.e., when a belongs to the intervals where the graphs are not defined) the stability indices b_1, b_2 have complex values and the so-called "complex saddle" instability of the vertical motion takes place. The enlarged fragments of the graphs in the vicinity of the gaps are given in Fig. 5.

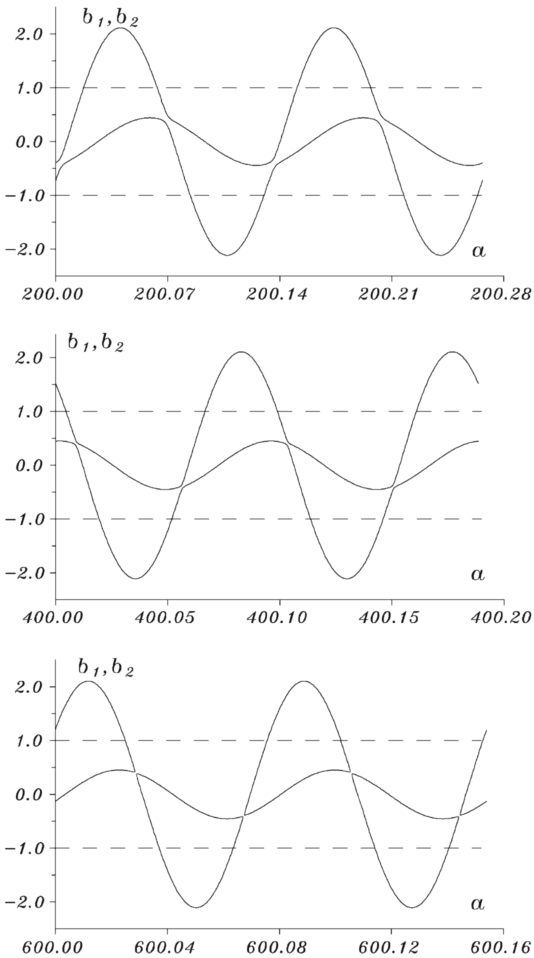


Fig. 4 The behaviour of the coefficients b_1 and b_2 appearing in the characteristic Eq. (10)

As it follows from our calculations, the first interval of "complex saddle" instability begins at $a = a_*$. Since such a value of vertical motion amplitude is large enough, it provides us with an explanation why this kind of instability of vertical motions in the circular Sitnikov problem was not recognized in previous studies where relatively small values of a were considered.

Increasing further the parameter a (i.e., for $a > a_*$), we observe a stability/instability alternation of more complicated type: "wide" interval of instability—"narrow" interval of stability—"narrow" interval of "complex saddle" instability—"wide" interval of stability—"wide" interval of instability—… An analog of the formulae (11) can be constructed in the case $a > a_*$, but we prefer to present in Sect. 6 several asymptotics written in a more convenient way.

Finally it worthwhile to mention that the Runge-Kutta-Fehlberg method of 7–8 order with variable step was used to integrate numerically the variation equation (6). The accuracy of the integration procedure (the local tolerance) was taken 10^{-10}. Since the period of vertical oscillations increases proportionally $a^{3/2}$ the variation equations should be integrated over relatively large time intervals: if we take for example $a = 500$ then the value of half-period

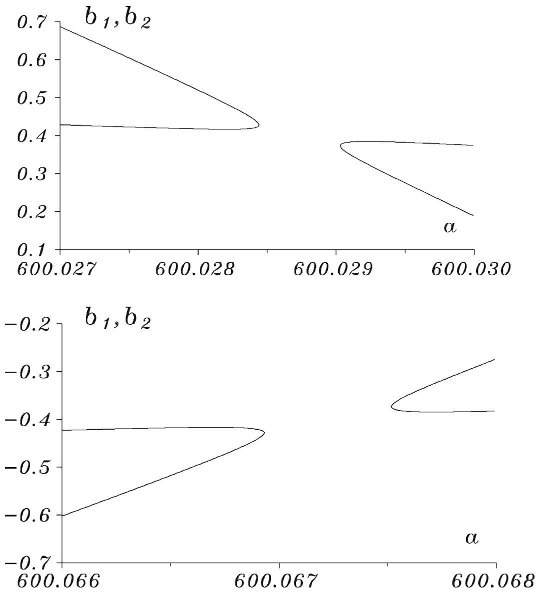


Fig. 5 The enlarged fragments of the coefficients graphs (see Fig. 4) in the vicinity of the gaps

$T_* \approx 2.4837 \cdot 10^4$. To check the influence of the round-off errors some computations were done both with double and quadruple precision arithmetic.

5 Approximate expression for monodromy matrix

In this section an approximate expression for the monodromy matrix $\mathbf{M}$ is derived. It will be used to discuss the phenomena described in Sect. 4 (the alternation of stability and instability, the decrease of stability and instability intervals by increasing the parameter a, etc).

We assume that the amplitude a of the periodic solution (2) is so large, that we can define an auxiliary quantity d such that

$$1 \ll d \ll a. \tag{12}$$

To start with we write down the monodromy matrix $\mathbf{M} = \mathbf{W}(T_*, 0)$ as the product of three fundamental matrices:

$$\mathbf{M} = \mathbf{W}(T_*, t_d^-)\mathbf{W}(t_d^-, t_d^+)\mathbf{W}(t_d^+, 0), \tag{13}$$

where $t_d^+ \in \left(0, \frac{T_*}{2}\right)$ and $t_d^- = T_* - t_d^+$ are the instants at which the third body is at distance d from the barycenter O in the periodic vertical motion (2) (at $t = t_d^+$ the third body moves away from barycenter, at $t = t_d^-$ it approaches the barycenter).

Approximate expression for the matrix $\mathbf{W}\left(t_d^+, 0\right)$. If the condition (12) is satisfied the phase point $(p_3(t), q_3(t))$ moves on the manifold $\tilde{\mathcal{V}}$ in close vicinity of the separatrix S^+ at $t \in \left[0, t_d^+\right]$. Within such time interval the difference between $q_3(t, a)$ and $q_3^+(t)$ is small enough. Neglecting this difference, we replace $q_3(t, a)$ in (6) by $q_3^+(t)$; as a consequence, the normal matrix solution $\mathbf{W}_+(t, 0)$ of the obtained system provides us the suitable approximation for $\mathbf{W}(t, 0)$ at $t \in \left[0, t_d^+\right]$.

The behavior of $\mathbf{W}_+(t,0)$ at $t \to +\infty$ is described by the remarkable asymptotic formula:

$$\mathbf{W}_+(t,0) \approx \mathbf{R}(t)\Lambda\left(q_3^+(t)\right)\mathbf{U}. \tag{14}$$

Here

$$\mathbf{R}(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix},$$

$$\Lambda(q_3) = \begin{pmatrix} \frac{1}{q_3} & 0 & -\sqrt{\frac{2}{q_3}} & 0 \\ 0 & \frac{1}{q_3} & 0 & -\sqrt{\frac{2}{q_3}} \\ \sqrt{2q_3} & 0 & -q_3 & 0 \\ 0 & \sqrt{2q_3} & 0 & -q_3 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} 0.3248\ldots & 0.1020\ldots & -0.4664\ldots & 0.2228\ldots \\ 0.1302\ldots & 0.1189\ldots & 0.5296\ldots & -2.0211\ldots \\ 1.1175\ldots & 0.1718\ldots & 1.4408\ldots & 0.4791\ldots \\ 0.2113\ldots & 0.6404\ldots & 0.9414\ldots & -2.5646\ldots \end{pmatrix}.$$

The derivation of the formula (14) is based on some simple ideas. Let us take $\overline{d} \gg 1$ and write down $\mathbf{W}_+(t,0)$ as the product

$$\mathbf{W}_+(t,0) = \mathbf{W}_+\left(t, t_{\overline{d}}\right)\mathbf{W}_+\left(t_{\overline{d}}, 0\right), \tag{15}$$

where $t_{\overline{d}}$ is the moment of time when the third body is at distance $\overline{d}$ from the barycenter O in the motion corresponding to the parabolic escape $q_3 = q_3^+(t)$. As next step, we modify the Eq. (6) to find the approximate expression for $\mathbf{W}_+\left(t, t_{\overline{d}}\right)$ at $t > t_{\overline{d}}$. Since at $t > t_{\overline{d}}$ the third body is far enough from the primaries m_1 and m_2, it looks natural to replace D by $q_3^+(t)$ in the right parts of the first two equations in system (6) and to neglect the small term $\frac{3}{4D^5}$. The system (6) takes the form

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}\overline{\mathbf{H}}\left(q_3^+(t)\right)\mathbf{z}, \tag{16}$$

with

$$\overline{\mathbf{H}}(q_3) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{1}{q_3^3} & 0 \\ 1 & 0 & 0 & \frac{1}{q_3^3} \end{pmatrix}.$$

Now it is worthwhile to make the following remark. Let us consider the rectilinear parabolic escape of the material point in the field of an attracting center. Under a proper choice of units, the distance between the attracting center and the point varies as

$$\overline{q}(t) = \left(\frac{3}{\sqrt{2}}\right)^{2/3} t^{2/3}. \tag{17}$$

If the asymptotics (5) is used for $q_3^+(t)$ in the Eq. (16), then these equations coincide with the motion equations of the above mentioned material point, linearized in the vicinity of the solution (17) and written in the reference frame uniformly rotating around the line of the escape.

Taking this into account, we implement in (16) the change of variables

$$\mathbf{z} = (p_1, p_2, q_1, q_2)^T \mapsto \overline{\mathbf{z}} = (\overline{p}_1, \overline{p}_2, \overline{q}_1, \overline{q}_2)^T,$$

where

$$\overline{\mathbf{z}} = \mathbf{R}(t_{\overline{d}} - t)\mathbf{z}.$$

This change of variables can be interpreted as the transfer from the synodic reference frame $Ox_1x_2x_3$ to the sidereal (fixed) reference frame $O\overline{x}_1\overline{x}_2\overline{x}_3$ ($Ox_3 \| O\overline{x}_3$). As a result the linearized equations of motion split into two independent subsystems

$$\frac{d\overline{p}_i}{dt} = -\frac{\overline{q}_i}{\overline{q}_3^3}, \quad \frac{d\overline{q}_i}{dt} = \overline{p}_i, \qquad i = 1, 2. \tag{18}$$

It is not difficult to find partial solutions to the system (18)

$$\overline{p}_i = \dot{\overline{q}}_3 = \sqrt{\frac{2}{\overline{q}_3}}, \quad \overline{q}_i = \overline{q}_3, \quad \overline{p}_{3-i} \equiv 0, \quad \overline{q}_{3-i} \equiv 0, \quad i = 1, 2$$

and

$$\overline{p}_i = \frac{1}{\overline{q}_3}, \quad \overline{q}_i = \overline{q}_3\dot{\overline{q}}_3 = \sqrt{2\overline{q}_3}, \quad \overline{p}_{3-i} \equiv 0, \quad \overline{q}_{3-i} \equiv 0, \quad i = 1, 2.$$

Here and below the dots are used for derivatives with respect to time.

Four independent partial solutions allow us to write down the normal fundamental matrix in terms of the variables $\overline{\mathbf{z}}$:

$$\overline{\mathbf{W}}_+\left(t, t_{\overline{d}}\right) = \Lambda\left(\overline{q}_3(t)\right)\Lambda^{-1}\left(\overline{q}_3(t_{\overline{d}})\right) \approx \Lambda\left(q_3^+(t)\right)\Lambda^{-1}(\overline{d}).$$

Coming back to the initial variables, we get

$$\mathbf{W}\left(t, t_{\overline{d}}\right) = \mathbf{R}\left(t - t_{\overline{d}}\right)\overline{\mathbf{W}}_+\left(t, t_{\overline{d}}\right). \tag{19}$$

Substituting (19) into (15) we obtain the expression for the normal fundamental matrix $\mathbf{W}_+(t, 0)$ as the product of three matrices with only one of them depending on time:

$$\mathbf{W}_+(t, 0) \approx \mathbf{R}(t)\Lambda(q_3^+(t))\overline{\mathbf{U}}(\overline{d}). \tag{20}$$

Here

$$\overline{\mathbf{U}}(\overline{d}) = \Lambda^{-1}(\overline{d})\mathbf{R}\left(-t_{\overline{d}}\right)\mathbf{W}\left(t_{\overline{d}}, 0\right).$$

The formula (20) can be used to compute the elements of the matrix $\mathbf{W}_+(t, 0)$ at $t \gg 1$. Asymptotically their values should not depend on the choice of $\overline{d}$. It means that the following limit exists:

$$\mathbf{U} = \lim_{\overline{d} \to +\infty} \overline{\mathbf{U}}(\overline{d}).$$

Substituting $\mathbf{U}$ instead of $\overline{\mathbf{U}}(\overline{d})$ into (20) we arrive to the formula (14).

The fundamental matrix $\mathbf{W}_+(t,0)$ was introduced in such a way that it provides the vertical motions satisfying (12) with a "universal" (i.e., independent on a) approximation $\mathbf{W}(t,0) \approx \mathbf{W}_+(t,0)$ at $t \in \left[0, t_d^+\right]$. Using the relation (14), we finally obtain

$$\mathbf{W}\left(t_d^+, 0\right) \approx \mathbf{R}\left(t_d^+\right) \Lambda(d) \mathbf{U}. \tag{21}$$

Approximate expression for the matrix $\mathbf{W}\left(t_d^-, t_d^+\right)$. Since at $t \in \left[t_d^+, t_d^-\right]$ the third body is far enough from the primaries, we neglect again the difference between their gravity field and the gravity field of the attracting center placed at the barycenter O. To obtain the expression for $\mathbf{W}\left(t_d^-, t_d^+\right)$ within such an approximation we need to integrate the system

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}\overline{\mathbf{H}}\left(\hat{q}_3(t,a)\right)\mathbf{z}, \tag{22}$$

where $\hat{q}_3(t,a)$ describes the motion in the Newtonian field along the segment $[0,a]$ on the axis Ox_3. It is supposed that the maximum distance a from the body to the attracting center is achieved at $t = \frac{T_*}{2}$. In this case $q_3(t,a) \approx \hat{q}_3(t,a)$ at $t \in \left[t_d^+, t_d^-\right]$. Of course the motion along a segment corresponds to the singular impact orbit (Szebehely 1967), but it is used here to approximate the regular vertical motion on the time interval were the singularities are absent.

The change of variables

$$\mathbf{z} = (p_1, p_2, q_1, q_2)^T \mapsto \hat{\mathbf{z}} = \left(\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2\right)^T,$$

where

$$\hat{\mathbf{z}} = \mathbf{R}\left(\frac{T_*}{2} - t\right)\mathbf{z}, \tag{23}$$

allows us to rewrite the Eq. (22) in the more simple form:

$$\frac{d\hat{p}_i}{dt} = -\frac{\hat{q}_i}{\hat{q}_3^3}, \quad \frac{d\hat{q}_i}{dt} = \hat{p}_i, \qquad i = 1,2. \tag{24}$$

It is easy to check that the system (24) admits the following partial solutions:

$$\hat{p}_i = \dot{\hat{q}}_3, \quad \hat{q}_i = \hat{q}_3, \quad \hat{p}_{3-i} \equiv 0, \quad \hat{q}_{3-i} \equiv 0, \quad i = 1,2 \tag{25}$$

and

$$\hat{p}_i = \frac{1}{\hat{q}_3} - \frac{2}{a}, \quad \hat{q}_i = \hat{q}_3\dot{\hat{q}}_3, \quad \overline{p}_{3-i} \equiv 0, \quad \overline{q}_{3-i} \equiv 0, \quad i = 1,2. \tag{26}$$

To compute $\dot{\hat{q}}_3$ in (25) and (26) the energy integral can be used. In the case of the motion along the segment $[0,a]$ in the Newtonian field, this integral takes the form

$$\frac{\dot{\hat{q}}_3^2}{2} - \frac{1}{\hat{q}_3} = -\frac{1}{a},$$

and consequently

$$\dot{\hat{q}}_3(t) = \pm\sqrt{2\left(\frac{1}{\hat{q}_3(t)} - \frac{1}{a}\right)}.$$

Taking into account (25) and (26) we write down the fundamental matrix for the system (24) as

$$\hat{\mathbf{W}}\left(t, \frac{T_*}{2}\right) = \begin{pmatrix} 2-\frac{a}{\hat{q}_3} & 0 & \frac{\dot{\hat{q}}_3}{a} & 0 \\ 0 & 2-\frac{a}{\hat{q}_3} & 0 & \frac{\dot{\hat{q}}_3}{a} \\ -a\hat{q}_3\dot{\hat{q}}_3 & 0 & \frac{\hat{q}_3}{a} & 0 \\ 0 & -a\hat{q}_3\dot{\hat{q}}_3 & 0 & \frac{\hat{q}_3}{a} \end{pmatrix}$$

and then (taking into account the relation (23)) we write the matrix for the system (22)

$$\mathbf{W}\left(t, \frac{T_*}{2}\right) = \mathbf{R}\left(t - \frac{T_*}{2}\right)\hat{\mathbf{W}}\left(t, \frac{T_*}{2}\right). \tag{27}$$

Using the expression (27) we find

$$\mathbf{W}\left(t_d^-, \frac{T_*}{2}\right) = \mathbf{R}\left(t_d^- - \frac{T_*}{2}\right)\mathbf{N}(d,a),$$

where we denote by

$$\mathbf{N}(d,a) = \hat{\mathbf{W}}\left(t_d^-, \frac{T_*}{2}\right)$$
$$= \begin{pmatrix} 2-\frac{a}{d} & 0 & -\frac{1}{a}\sqrt{2\left(\frac{1}{d}-\frac{1}{a}\right)} & 0 \\ 0 & 2-\frac{a}{d} & 0 & -\frac{1}{a}\sqrt{2\left(\frac{1}{d}-\frac{1}{a}\right)} \\ d\sqrt{2\left(\frac{a}{d}-1\right)} & 0 & \frac{d}{a} & 0 \\ 0 & d\sqrt{2\left(\frac{a}{d}-1\right)} & 0 & \frac{d}{a} \end{pmatrix}.$$

The final step is based on the last formula in (8), namely

$$\begin{aligned} \mathbf{W}\left(t_d^-, t_d^+\right) &= \mathbf{W}\left(t_d^-, \frac{T_*}{2}\right)\mathbf{W}\left(\frac{T_*}{2}, t_d^+\right) \\ &= \mathbf{W}\left(t_d^-, \frac{T_*}{2}\right)\mathbf{Q}\mathbf{W}^{-1}\left(t_d^-, \frac{T_*}{2}\right) \approx \mathbf{R}\left(t_d^+ - t_d^-\right)\mathbf{K}(d). \end{aligned} \tag{28}$$

Here

$$\mathbf{K}(d) = \begin{pmatrix} -3 & 0 & 2\sqrt{2}d^{-3/2} & 0 \\ 0 & -3 & 0 & 2\sqrt{2}d^{-3/2} \\ 2\sqrt{2}d^{3/2} & 0 & -3 & 0 \\ 0 & 2\sqrt{2}d^{3/2} & 0 & -3 \end{pmatrix}$$

For completeness we should add the following formula:

$$\mathbf{K}(d) \approx \mathbf{N}(d,a)\mathbf{Q}\mathbf{N}^{-1}(d,a)\mathbf{Q}.$$

Approximate expression for the matrix $\mathbf{W}\left(T_*, t_d^-\right)$. Using again the relations (8) we get

$$\mathbf{W}\left(T_*, t_d^-\right) = \mathbf{W}\left(0, -t_d^+\right) = \mathbf{Q}\mathbf{W}^{-1}\left(t_d^+, 0\right)\mathbf{Q}. \tag{29}$$

Then the substitution of the previously obtained expression for $\mathbf{W}\left(t_d^+, 0\right)$ (the formula (21)) into the right part of (29) provides us with the desired approximate formula for $\mathbf{W}\left(T_*, t_d^-\right)$.

Finalizing the construction of the approximate formula for the monodromy matrix $\mathbf{M}$. The substitution of the approximate expressions for $\mathbf{W}\left(T_*, t_d^-\right)$, $\mathbf{W}\left(t_d^-, t_d^+\right)$, $\mathbf{W}\left(t_d^+, 0\right)$ into (13) yields

$$\mathbf{M}(a) \approx \left\{\mathbf{Q}\left(\Lambda(d)\mathbf{U}\right)^{-1}\mathbf{QR}\left(T_* - t_d^-\right)\right\}\left\{\mathbf{R}\left(t_d^- - t_d^+\right)\mathbf{K}(d)\right\}\left\{\mathbf{R}\left(t_d^+\right)\Lambda(d)\mathbf{U}\right\}. \quad (30)$$

Simplifying (30) the extraordinary simple result can be obtained

$$\mathbf{M}(a) \approx \mathbf{QU}^{-1}\mathbf{QR}\left(T_*(a)\right)\mathbf{U}. \quad (31)$$

This formula allow us to investigate analytically the stability properties of the periodic vertical motions in the case $a \gg 1$.

6 New insight into the stability properties of the vertical motions

As it follows from (31) the coefficients of the monodromy matrix $\mathbf{M}$ and, respectively, the coefficients of the characteristic Eq. (9) are 2π -periodic functions of the semiperiod of the vertical motion T_*. To illustrate this we present in Fig. 6 the graphs of the coefficients b_1, b_2 (only the real values) as T_* varies in the interval $[2\pi n, 2\pi(n+1)]$, where n is a large enough integer number. Using the formula (3), which defines the dependence of T_* on the amplitude a, it is not difficult to prove that in terms of a the lengths of the stability and instability intervals decrease proportionally to $a^{-1/2}$ as $a \to +\infty$.

Taking into account the approximate expression for the monodromy matrix $\mathbf{M}$, we describe in more details the repeating pattern of stable and unstable intervals mentioned at the end of Sect. 4. This pattern consists of four intervals appearing in the following order as a increases:

"Wide" interval of instability. Both coefficients b_1, b_2 are real, but one of them has absolute value greater than 1 ("saddle-center" instability). The asymptotic length of the interval in terms of the amplitude of the motion is about $0.643544 \cdot a^{-1/2}$, while the variation of the semiperiod T_* equals to about 2.144392.

"Narrow" interval of stability. The coefficients b_1, b_2 are real and belong to the interval $(-1, 1)$. The approximate length is $0.068655 \cdot a^{-1/2}$; the variation of the semiperiod equals to about 0.228768.

"Narrow" interval of instability. The coefficients b_1, b_2 are complex ("complex saddle" instability). The approximate length is $0.048166 \cdot a^{-1/2}$; the variation of the semiperiod equals to about 0.160497.

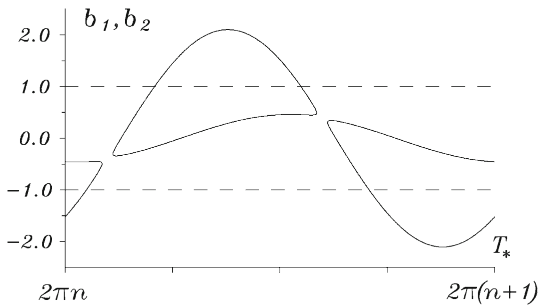


Fig. 6 Graphs of the coefficients b_1, b_2 computed on the base of the approximate formula for the monodromy matrix $\mathbf{M}$. Only real values are shown

"Wide" interval of stability. The coefficients b_1, b_2 are real and belong to the interval $(-1, 1)$ again. The approximate length is $0.182445 \cdot a^{-1/2}$; the variation of the semiperiod equals to about 0.607936.

To conclude, we recall that before the first appearance of the interval of "complex saddle" instability at $a = a_*$, a more simple pattern with only two intervals was observed. The "transient" asymptotics for the length of the stability intervals in the case $1 \ll a < a_*$ can be obtained by adding of the lengths of the "narrow" instability interval and both stability intervals in the final pattern. It yields

$$\Delta_{st}^{tr} \approx 0.299 \cdot a^{-1/2},$$

which is in good agreement with the corresponding numerical result presented in Sect. 4.

7 Stability of the vertical motions in the circular Sitnikov problem with four and more bodies

The investigation of the generalized circular Sitnikov problem with four and more bodies revealed that in contrast to the case of the three body problem there is no alternation of stability/instability in the family of vertical motions (Bountis and Papadakis 2009; Soulis et al. 2008).

For simplicity we limit our consideration to the case of the restricted four body problem. It is assumed that three primaries of equal mass rotate around the barycenter O in circular orbit with the radius $R = 1/\sqrt{3}$ (Soulis et al. 2008). Under the linear approximation the stability analysis of the fourth body periodic vertical motion $\breve{q}(t, a)$ is reduced to the study of the spectral properties of the monodromy matrix associated to the system of linear differential equations with periodic coefficients

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}\breve{\mathbf{H}}(t)\mathbf{z}, \tag{32}$$

where

$$\breve{\mathbf{H}}(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \left(\frac{1}{D^3} - \frac{1}{2D^5}\right) & 0 \\ 1 & 0 & 0 & \left(\frac{1}{D^3} - \frac{1}{2D^5}\right) \end{pmatrix},$$

$$D(t, a) = \left(\breve{q}_3^2(t, a) + \frac{1}{3}\right)^{1/2}.$$

It is remarkable that Eq. (32) possess a circular symmetry: for any real α they are invariant with respect to transformations of the form

$$\widetilde{\mathbf{z}} = \mathbf{R}(\alpha)\mathbf{z},$$

while the non-linearized equations of motion of the fourth body in the synodic reference frame admit only the rotational symmetry of the 3rd order. The possibility for the linearized equations of motion to have a larger group of symmetries in comparison to the original non-linear system was pointed out by Arnold (1989)(Sec. 23). In particular, in the case of the initial system rotational symmetry of N -th order ($N \geq 3$) the linearized equations always have circular symmetry. This is the reason why the stability analysis of the vertical motions, based on the linearized equations, yields similar results for the Sitnikov problem with four and more bodies and for the particle dynamics in the gravity field of the circular ring (Broucke and Elipe 2005) [numerically it was shown in Bountis and Papadakis (2009)].

It is convenient to rewrite the equations of motion (32) in a sidereal (fixed) reference frame by means of the transformation of variables

$$\mathbf{z} = (p_1, p_2, q_1, q_2)^T \mapsto \breve{\mathbf{z}} = (\breve{p}_1, \breve{p}_2, \breve{q}_1, \breve{q}_2)^T,$$

where

$$\breve{\mathbf{z}} = \mathbf{R}(-t)\mathbf{z}.$$

After that the equations of motion split into two identical independent subsystems:

$$\frac{d\breve{p}_i}{dt} = -\frac{\breve{q}_i}{2D^3}\left(2 - \frac{1}{D^2}\right), \quad \frac{d\breve{q}_i}{dt} = \breve{p}_i, \quad i = 1, 2. \tag{33}$$

Let $\breve{\mathbf{W}}_+\left(t, t'\right)$ denote the normal fundamental matrix for the system (33) in the case when $\breve{q}_3(t, a)$ is replaced by $\breve{q}_3^+(t)$, which corresponds to the parabolic escape. Using the same technique as in Sect. 5 we obtain the asymptotic formula

$$\breve{\mathbf{W}}_+(t, 0) \approx \Lambda\left(\breve{q}_3^+(t)\right)\breve{\mathbf{U}},$$

where

$$\breve{\mathbf{U}} = \lim_{d\to+\infty} \Lambda^{-1}(d)\breve{\mathbf{W}}_+(t_d, 0) \approx \begin{pmatrix} 0.2456 & 0 & -1.2690 & 0 \\ 0 & 0.2456 & 0 & -1.2690 \\ 0.9246 & 0 & -0.7061 & 0 \\ 0 & 0.9246 & 0 & -0.7061 \end{pmatrix}.$$

Applying the main ideas of Sect. 5, we establish the following property of the monodromy matrix $\breve{\mathbf{M}}(a)$ associated to (33): at $a \to +\infty$ the matrix $\breve{\mathbf{M}}(a) \to \breve{\mathbf{M}}_*$, where the constant matrix $\breve{\mathbf{M}}_* = \mathbf{Q}\breve{\mathbf{U}}^{-1}\mathbf{Q}\breve{\mathbf{U}}$. The eigenvalues of the matrix $\breve{\mathbf{M}}_*$ are the asymptotic limits for multiplicators (of multiplicity 2) of the system (33):

$$\lim_{a\to+\infty} \breve{\rho}_i(a) = \breve{\rho}_i^*, \quad \breve{\rho}_1^* = -0.4446\ldots, \quad \breve{\rho}_2^* = -2.2488\ldots$$

Finally, it is not difficult to derive the asymptotic formulae the for multiplicators of the original system (32):

$$\rho_{1,2} \approx \breve{\rho}_1^* \exp(\pm i T_*), \quad \rho_{3,4} \approx \breve{\rho}_2^* \exp(\pm i T_*).$$

On the complex plane $\rho_1, \ldots, \rho_4$ are placed in the small vicinity of the circles with radii $|\breve{\rho}_1^*| < 1$ and $|\breve{\rho}_2^*| > 1$. Consequently in the circular Sitnikov problem with four bodies the periodic vertical motions with large amplitudes are always unstable.

Finally we would like to note the another opportunity to introduce the generalized circular Sitnikov problem with N bodies using the appropriate straight line solution of the problem of $(N - 1)$ bodies (Moulton 1910). If in such a solution $(N - 1)$ primaries are arranged symmetrically with respect to the barycenter then the infinitesimal N th body can move periodically along an axis around which the rotation of the primaries takes place (naturally, the proposed generalization is possible for odd N only). Likely this family of periodic motions exhibits the alternation of stability and instability.

8 Conclusion

The combination of numerical and analytical approaches provided us with the opportunity to correct, clarify and extend some previously known results related to the circular Sitnikov problem (mainly about the stability of vertical motions). For the first time under the scope of this problem the possibility of the "complex saddle" instability was revealed within the family of vertical motions.

For our theoretical constructions it was essential that the phase trajectories corresponding to the solution under consideration have lengthy parts in the vicinity of the peculiar separatrices of the problem—the parabolic escapes to infinity. Often enough it is possible to introduce a suitable auxiliary mapping in the vicinity of the separatrix in order to study the local properties of the phase flow. It would be very interesting to develop similar for the circular Sitnikov problem.

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